Appendix (For Online Publication)

Past Automation and Future AI: How Weak Links Tame the Growth Explosion

Charles I. Jones and Christopher Tonetti

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The authors derived all paper results directly. The appendix was written with substantial help from LLMs.

A. Task Allocation and the Reduced-Form Production Function

A.1 The Choice of Factor and Production Procedure

Factor choice. Producing one unit of task i costs $r_t/\tilde{\psi}_{kit}$ with capital and $w_t/\tilde{\psi}_{\ell it}$ with labor. The two factors are perfect substitutes within a task, so the firm uses the one with the lower unit cost. The adjustment by $\alpha_i^{\sigma/(\sigma-1)}$ scales both productivities alike, so

$$\frac{r_t}{\tilde{\psi}_{kit}} \leq \frac{w_t}{\tilde{\psi}_{\ell it}} \quad \Longleftrightarrow \quad \frac{\psi_{kit}}{\psi_{\ell it}} \geq \frac{r_t}{w_t}, \tag{A.1}$$

which is equation (8). The body assigns ties to Ω_{kt}. This convention matters only if the indifferent tasks have positive measure. In the dynamic model of Section 6, that happens only at a cutoff where f is flat, which no simulated path reaches.

Multiple procedures. The body's equivalence between new procedures and higher task productivity follows from linear task technology and a common price for each factor. For any allocation across the N_{kt} capital procedures,

$$\sum_{n=1}^{N_{kt}} \tilde{\psi}_{kit}^n K_{it}^n \leq \left(\max_{1 \leq n \leq N_{kt}} \tilde{\psi}_{kit}^n \right) \sum_{n=1}^{N_{kt}} K_{it}^n.$$

Assigning all capital for the task to a procedure that attains the maximum achieves equality at the same cost $r_t \sum_n K_{it}^n$. The same argument applies to labor, so the effective productivities are the maxima across procedures.

A.2 Proof of Proposition 1

Step 1. Allocate each factor across its active tasks. Consider an equilibrium with positive finite output, positive factor prices, positive use of both factors, and $0 < \sigma < 1$. On a task performed entirely by capital, $Y_{it} = \tilde{\psi}_{kit} K_{it}$. Differentiating equation (3) with respect to capital on this task gives

$$\frac{\partial Y_t}{\partial K_{it}} = Z_t^{\frac{\sigma-1}{\sigma}} \alpha_i \tilde{\psi}_{kit}^{\frac{\sigma-1}{\sigma}} Y_t^{1/\sigma} K_{it}^{-1/\sigma} = (Z_t \psi_{kit})^{\frac{\sigma-1}{\sigma}} \left(\frac{Y_t}{K_{it}} \right)^{1/\sigma}. \tag{A.2}$$

The last equality uses the productivity adjustment already defined in the body. The first-order condition equates this marginal product to r_t/P_t. Raising that condition to

the power σ and solving for K_{it} yields the first equation below. The labor condition gives the second.

$$K_{it} = \psi_{kit}^{\sigma-1} Z_t^{\sigma-1} \left(\frac{r_t}{P_t}\right)^{-\sigma} Y_t, \qquad i \in \Omega_{kt}, \tag{A.3}$$

$$L_{it} = \psi_{\ell it}^{\sigma-1} Z_t^{\sigma-1} \left(\frac{w_t}{P_t}\right)^{-\sigma} Y_t, \qquad i \in \Omega_{\ell t}. \tag{A.4}$$

These conditions characterize optimal allocations because the CES aggregator is concave for $0 < \sigma < 1$ and task production is linear.

Step 2. Express task allocations as shares of total inputs. Let H_{kt} and $H_{\ell t}$ denote the two integrals in the definitions of B_t and A_t,

$$H_{kt} := \int_{\Omega_{kt}} \psi_{kit}^{\sigma-1}\, di, \qquad H_{\ell t} := \int_{\Omega_{\ell t}} \psi_{\ell it}^{\sigma-1}\, di. \tag{A.5}$$

Integrating (A.3) and (A.4) gives

$$K_t = H_{kt} Z_t^{\sigma-1} (r_t/P_t)^{-\sigma} Y_t, \qquad L_t = H_{\ell t} Z_t^{\sigma-1} (w_t/P_t)^{-\sigma} Y_t. \tag{A.6}$$

Dividing each task allocation by its aggregate allocation eliminates the common price and output terms. Thus,

$$\omega_{kit} = \frac{K_{it}}{K_t} = \frac{\psi_{kit}^{\sigma-1}}{H_{kt}}, \qquad i \in \Omega_{kt}, \tag{A.7}$$

$$\omega_{\ell it} = \frac{L_{it}}{L_t} = \frac{\psi_{\ell it}^{\sigma-1}}{H_{\ell t}}, \qquad i \in \Omega_{\ell t}. \tag{A.8}$$

Each density integrates to one over its active set. Because all units of a factor receive the same price, it is also a cost-share density, so the cost share of a set of tasks is its integral over that set.

Step 3. Substitute the allocations into total production. The productivity adjustment absorbs α_i into the task technology. Partitioning the integral in equation (3) therefore gives

$$\left(\frac{Y_t}{Z_t}\right)^{\frac{\sigma-1}{\sigma}} = \int_{\Omega_{kt}} (\psi_{kit} K_{it})^{\frac{\sigma-1}{\sigma}}\, di + \int_{\Omega_{\ell t}} (\psi_{\ell it} L_{it})^{\frac{\sigma-1}{\sigma}}\, di.$$

Using (A.7), the capital integral becomes

$$\begin{aligned}\int_{\Omega_{kt}} (\psi_{kit} K_{it})^{\frac{\sigma-1}{\sigma}}\, di &= \left(\frac{K_t}{H_{kt}}\right)^{\frac{\sigma-1}{\sigma}} \int_{\Omega_{kt}} \psi_{kit}^{\sigma-1}\, di \\ &= K_t^{\frac{\sigma-1}{\sigma}} H_{kt}^{1/\sigma}.\end{aligned}$$

The labor integral equals $L_t^{(\sigma-1)/\sigma} H_{\ell t}^{1/\sigma}$ by the same substitution. Multiplying by $Z_t^{(\sigma-1)/\sigma}$ gives

$$Y_t^{\frac{\sigma-1}{\sigma}} = (Z_t H_{kt}^{1/(\sigma-1)} K_t)^{\frac{\sigma-1}{\sigma}} + (Z_t H_{\ell t}^{1/(\sigma-1)} L_t)^{\frac{\sigma-1}{\sigma}} = (B_t K_t)^{\frac{\sigma-1}{\sigma}} + (A_t L_t)^{\frac{\sigma-1}{\sigma}}. \tag{A.9}$$

Raising both sides to the power $\sigma/(\sigma-1)$ proves the production-function representation in Proposition 1.

Step 4. Derive task and aggregate factor shares. Multiplying (A.2) by K_{it}/Y_t and using the first-order condition gives

$$\frac{r_t K_{it}}{P_t Y_t} = \left(\frac{Z_t \psi_{kit} K_{it}}{Y_t}\right)^{\frac{\sigma-1}{\sigma}}, \qquad i \in \Omega_{kt}. \tag{A.10}$$

The labor version is

$$\frac{w_t L_{it}}{P_t Y_t} = \left(\frac{Z_t \psi_{\ell it} L_{it}}{Y_t}\right)^{\frac{\sigma-1}{\sigma}}, \qquad i \in \Omega_{\ell t}. \tag{A.11}$$

Integrating the capital equation and using the integral calculated in Step 3 proves

$$s_{Kt} = \left(\frac{B_t K_t}{Y_t}\right)^{\frac{\sigma-1}{\sigma}}, \qquad s_{Lt} = \left(\frac{A_t L_t}{Y_t}\right)^{\frac{\sigma-1}{\sigma}}. \tag{A.12}$$

Equation (A.9) then implies $s_{Kt} + s_{Lt} = 1$. This completes the proof of Proposition 1.

Step 5. Relate factor shares to output elasticities. For a given allocation of tasks to factors, direct differentiation of the CES representation yields $\partial \log Y_t / \partial \log K_t = s_{Kt}$ and $\partial \log Y_t / \partial \log L_t = s_{Lt}$. The same equalities hold for the optimized aggregate production function wherever it is differentiable. In that problem, capital and labor are fixed total resources, and their shadow values are r_t/P_t and w_t/P_t. The envelope theorem therefore gives $\partial Y_t/\partial K_t = r_t/P_t$ and $\partial Y_t/\partial L_t = w_t/P_t$. Multiplication by the corresponding

input-output ratios yields the same elasticities. This proves the elasticity equality in equation (14) even when equilibrium factor prices and the task sets respond to aggregate factor quantities.

A.3 The Elasticity of Substitution and Changes in Factor Shares

Step 1. Express factor demand as a function of relative prices. For this comparative static, hold all task productivities fixed and let $q := r_t/w_t$. Write $H_k(q)$ and $H_\ell(q)$ for the integrals in (A.5) evaluated at the task sets implied by this price ratio. The aggregate demand equations give

$$\frac{K_t}{L_t} = \frac{H_k(q)}{H_\ell(q)} q^{-\sigma}. \tag{A.13}$$

As q rises, capital becomes more expensive relative to labor. Its task set contracts, so $H_k(q)$ weakly decreases. Labor's task set expands, so $H_\ell(q)$ weakly increases. These signs use positivity of the integrands and do not depend on their exponent being positive.

Step 2. Differentiate relative factor demand. At a price ratio where the two integrals are positive and differentiable, the elasticity of substitution is

$$-\frac{d\log(K_t/L_t)}{d\log(r_t/w_t)} = \sigma - \frac{qH_k'(q)}{H_k(q)} + \frac{qH_\ell'(q)}{H_\ell(q)} \geq \sigma. \tag{A.14}$$

Holding the task sets fixed makes the two derivatives zero and recovers σ. The inequality is strict when a relative-price change reallocates tasks with positive first-order cost weight. Thus, task reallocation weakly increases the elasticity relative to its value with fixed task sets.

Step 3. Isolate the changes in factor shares described in the body. Taking the ratio of the share formulas gives

$$\frac{s_{Kt}}{s_{Lt}} = \left(\frac{B_t K_t}{A_t L_t}\right)^{\frac{\sigma-1}{\sigma}}. \tag{A.15}$$

For any differentiable change in these quantities,

$$ds_{Kt} = s_{Kt}s_{Lt}\frac{\sigma-1}{\sigma}\big(d\log B_t + d\log K_t - d\log A_t - d\log L_t\big). \tag{A.16}$$

To obtain this formula, use $s_{Lt} = 1 - s_{Kt}$ to write $d\log(s_{Kt}/s_{Lt}) = ds_{Kt}/(s_{Kt}s_{Lt})$ and then differentiate (A.15). At fixed inputs and task sets, a rise in capital productivity on a positive-measure set of active capital tasks raises B_t and lowers the capital share because $\sigma - 1 < 0$. At fixed inputs and productivities, expanding the capital task set raises H_{kt} and lowers $H_{\ell t}$. Since $1/(\sigma-1) < 0$, this lowers B_t, raises A_t, and raises the capital share. These are the separate effects discussed in Section 3.4. Along an equilibrium path, the total change is the combination in (A.16).

A.4 Growth Accounting, Factor Prices, and Measurement

Step 1. Differentiate the CES representation along the observed path. At dates where the quantities are differentiable, equation (A.9) implies

$$Y_t^{\frac{\sigma-1}{\sigma}}\hat{Y}_t = (B_tK_t)^{\frac{\sigma-1}{\sigma}}(\hat{B}_t + \hat{K}_t) + (A_tL_t)^{\frac{\sigma-1}{\sigma}}(\hat{A}_t + \hat{L}_t).$$

The common factor $(\sigma-1)/\sigma$ has been canceled from both sides. Divide by $Y_t^{(\sigma-1)/\sigma}$ and use (A.12) to obtain

$$\hat{Y}_t - s_{Kt}\hat{K}_t - s_{Lt}\hat{L}_t = s_{Kt}\hat{B}_t + s_{Lt}\hat{A}_t. \tag{A.17}$$

This is equation (13). The calculation differentiates an identity along a path, so it includes changes in A_t and B_t induced by changes in the task sets.

Step 2. Recover factor-augmenting productivity from the data. Raising the labor-share formula to the power $\sigma/(\sigma-1)$ gives $s_{Lt}^{\sigma/(\sigma-1)} = A_tL_t/Y_t$. Rearranging proves equation (15). The analogous capital identity is

$$A_t = s_{Lt}^{\frac{\sigma}{\sigma-1}}\frac{Y_t}{L_t}, \qquad B_t = s_{Kt}^{\frac{\sigma}{\sigma-1}}\frac{Y_t}{K_t}. \tag{A.18}$$

Their log derivatives are

$$\hat{A}_t = \hat{Y}_t - \hat{L}_t + \frac{\sigma}{\sigma-1}\frac{\dot{s}_{Lt}}{s_{Lt}}, \qquad \hat{B}_t = \hat{Y}_t - \hat{K}_t + \frac{\sigma}{\sigma-1}\frac{\dot{s}_{Kt}}{s_{Kt}}. \tag{A.19}$$

If $s_{Kt} > 0$, one can instead recover capital-augmenting growth from (A.17) as $\hat{B}_t = (\widehat{TFP}_t - s_{Lt}\hat{A}_t)/s_{Kt}$. When factor shares are constant, their derivatives vanish. In that case $\hat{A}_t = \hat{Y}_t - \hat{L}_t$ and $\hat{B}_t = \hat{Y}_t - \hat{K}_t$, independent of σ, as stated in Section 3.4.

B. Growth of Factor-Augmenting Productivity

B.1 The Finite-Boundary Setup

The body works with finitely many points of automation. Automation then moves the boundaries β_{mt}, $m = 1, \dots, M_t$, between capital and labor tasks, adding tasks to the capital set and removing them from the labor set, so the integrals H_{vt}, $v \in \{k, \ell\}$, in equation (A.5) change both through their integrands on the current sets and through the moving endpoints. The proofs in this appendix use the following conditions on a compact time interval $[a, b]$ on which both factors are used.

Assumption A1 (Finite-boundary setup)**.** (i) *Boundaries.* Ω_{kt} is a finite union of intervals. Except at finitely many dates at which boundaries appear or merge, each boundary β_{mt} is absolutely continuous in time, up to those dates. Assumption 1 holds, so right endpoints are nondecreasing and left endpoints nonincreasing, and the set of indifferent tasks has measure zero at every date.

(ii) *Productivities near the boundaries.* Both ψ_{kit} and $\psi_{\ell it}$ are positive, finite, and jointly continuous in (i, t) in a neighborhood of each boundary path $\{(\beta_{mt}, t)\}$.

(iii) *Time paths.* For almost every task, the productivity paths of the factors it uses during $[a, b]$ are positive, finite, and absolutely continuous in time. The active integrals satisfy $0 < H_{kt}, H_{\ell t} < \infty$, and the productivity of the factor in use has integrable variation, $\int_a^b \int_{\Omega_{vt}} |\partial_t \psi_{vit}^{\sigma-1}| \, di \, dt < \infty$. The paths of r_t, w_t, and Z_t are positive and absolutely continuous, as are K_t and L_t where the output-growth identity is used.

A time path is absolutely continuous if its change over any subinterval equals the integral of an integrable time derivative, so the growth rates of these paths exist almost everywhere and integrate to changes in logarithms. Differential identities below hold almost everywhere in time. Item (ii) defines the cost-share densities $\omega_{v\beta_m t} = \psi_{v\beta_m t}^{\sigma-1} / H_{vt}$ at the boundaries, and it makes each boundary task an indifference point, because the productivity ratio is continuous there and capital is used on one side and labor on the other:

$$\frac{\psi_{k\beta_m t}}{\psi_{\ell\beta_m t}} = \frac{r_t}{w_t}, \qquad m = 1, \dots, M_t. \tag{B.1}$$

The assumption restricts a factor's productivity only on tasks it performs during the interval or near a boundary. Capital productivity may therefore be zero on tasks that are never automated, such as the essential human tasks in Section 6, even though $\psi_{kit}^{\sigma-1}$ is then infinite. Away from the boundaries, productivities may be discontinuous across tasks.

B.2 Proof of Proposition 2

Step 1. Differentiate the active integrals. Write H_{kt} as the sum of the integrals of $\psi_{kit}^{\sigma-1}$ over the intervals that make up Ω_{kt}. The Leibniz rule applies to each moving interval, because the integrand is continuous at the moving endpoints by item (ii) of Assumption A1 and its time derivative is integrable over the active set by item (iii). At almost every date,

$$\dot{H}_{kt} = \int_{\Omega_{kt}} \partial_t \psi_{kit}^{\sigma-1} \, di + \sum_{m=1}^{M_t} |\dot{\beta}_{mt}| \, \psi_{k\beta_m t}^{\sigma-1}, \tag{B.2}$$

$$\dot{H}_{\ell t} = \int_{\Omega_{\ell t}} \partial_t \psi_{\ell it}^{\sigma-1} \, di - \sum_{m=1}^{M_t} |\dot{\beta}_{mt}| \, \psi_{\ell\beta_m t}^{\sigma-1}. \tag{B.3}$$

Each moving endpoint adds the integrand at the boundary task times the outward speed to the capital integral and removes the same task measure from the labor integral. The absolute values cover left endpoints, which move leftward when a capital interval expands. Between the dates at which boundaries appear or merge, H_{kt} and $H_{\ell t}$ are absolutely continuous, and they are continuous at those dates, so the identities integrate across them.

The same calculation identifies the automation rate. The set $\Omega_{k,t+h} \setminus \Omega_{kt}$ in equation (20) is the union of the intervals between β_{mt} and $\beta_{m,t+h}$, and $\omega_{\ell it} = \psi_{\ell it}^{\sigma-1}/H_{\ell t}$ is continuous in i at each boundary, so

$$x_t = \lim_{h \downarrow 0} \frac{1}{h} \int_{\Omega_{k,t+h} \setminus \Omega_{kt}} \omega_{\ell it} \, di = \sum_{m=1}^{M_t} \omega_{\ell\beta_m t} |\dot{\beta}_{mt}| = \bar{\omega}_{\ell\beta t} \dot{\beta}_t. \tag{B.4}$$

The last equality is the definition in equation (12). The products $\bar{\omega}_{v\beta t}\dot{\beta}_t$ are the sums $\sum_m \omega_{v\beta_m t}|\dot{\beta}_{mt}|$, which are zero when every boundary is stationary, so no division by $\dot{\beta}_t$ is needed.

Step 2. Differentiate B_t and A_t. On an active task, $\partial_t \psi_{vit}^{\sigma-1} = (\sigma-1)\psi_{vit}^{\sigma-1}\dot{\psi}_{vit}/\psi_{vit}$. Dividing equations (B.2) and (B.3) by H_{kt} and $H_{\ell t}$ turns the first terms into the cost-weighted averages in Proposition 2 and the sums into the average boundary densities:

$$\frac{\dot{H}_{kt}}{H_{kt}} = (\sigma-1)\hat{\psi}_{kt} + \bar{\omega}_{k\beta t}\dot{\beta}_t, \tag{B.5}$$

$$\frac{\dot{H}_{\ell t}}{H_{\ell t}} = (\sigma-1)\hat{\psi}_{\ell t} - \bar{\omega}_{\ell\beta t}\dot{\beta}_t. \tag{B.6}$$

The definitions in Proposition 1 give $\log B_t = \log Z_t + \frac{1}{\sigma-1}\log H_{kt}$ and $\log A_t = \log Z_t + \frac{1}{\sigma-1}\log H_{\ell t}$. Substituting the two log derivatives into their time derivatives gives

$$\hat{B}_t = \hat{Z}_t + \hat{\psi}_{kt} - \frac{1}{1-\sigma}\bar{\omega}_{k\beta t}\dot{\beta}_t, \tag{B.7}$$

$$\hat{A}_t = \hat{Z}_t + \hat{\psi}_{\ell t} + \frac{1}{1-\sigma}\bar{\omega}_{\ell\beta t}\dot{\beta}_t. \tag{B.8}$$

These are equations (10) and (11). The automation component is negative for B_t and positive for A_t because $1-\sigma > 0$ and the boundary densities are nonnegative. The logarithms are absolutely continuous, so integrating these rates over any subinterval gives the corresponding changes in log productivity.

General task sets. The finite-boundary setup is an expository restriction. Proposition 2 and the TFP identity, Propositions 3 and 4, the category measurement in Appendix D.3, and the fixed-assignment counterfactual in Proposition 5 extend, each under its own economic assumptions, to automated task sets that are arbitrary measurable subsets of $[0,1]$ nested over time, with productivities that may be discontinuous across tasks and need not generate finitely many crossings. In place of item (ii), the extension requires both productivity paths to be continuous through each task's switch, finite active integrals with integrable variation, and an absolutely continuous flow of newly automated task mass. The automation rate is then still the labor cost of the tasks automated at each date, the boundary sums in equation (12) become averages over those tasks, and the aggregate formulas are unchanged.

B.3 Homogeneous Productivities on the Active Task Sets

Step 1. Evaluate the integrals and allocate inputs. If capital productivity is ψ_{kt} on the capital task set and labor productivity is $\psi_{\ell t}$ on the labor task set, then

$$H_{kt} = \beta_t \psi_{kt}^{\sigma-1}, \qquad H_{\ell t} = (1-\beta_t)\psi_{\ell t}^{\sigma-1}.$$

For $0 < \beta_t < 1$, the allocation formulas therefore give $K_{it} = K_t/\beta_t$ on capital tasks and $L_{it} = L_t/(1-\beta_t)$ on labor tasks. Substitution into production yields

$$Y_t = Z_t \left[\beta_t \left(\frac{\psi_{kt} K_t}{\beta_t} \right)^{\frac{\sigma-1}{\sigma}} + (1-\beta_t) \left(\frac{\psi_{\ell t} L_t}{1-\beta_t} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}. \tag{B.9}$$

The homogeneity applies only to the factor in use. If both productivities were the same on every task, equation (A.1) would assign every task to the same factor.

Step 2. Identify the factor-augmenting terms. Using the evaluated integrals in the definitions of B_t and A_t gives

$$B_t = Z_t \psi_{kt} \beta_t^{-1/(1-\sigma)}, \tag{B.10}$$

$$A_t = Z_t \psi_{\ell t} (1-\beta_t)^{-1/(1-\sigma)}. \tag{B.11}$$

Thus, holding productivities fixed, more automated tasks reduce capital-augmenting productivity B_t and increase labor-augmenting productivity A_t. The associated differential identities are

$$\hat{B}_t = \hat{Z}_t + \hat{\psi}_{kt} - \frac{\dot{\beta}_t}{(1-\sigma)\beta_t}, \qquad \hat{A}_t = \hat{Z}_t + \hat{\psi}_{\ell t} + \frac{\dot{\beta}_t}{(1-\sigma)(1-\beta_t)}. \tag{B.12}$$

Here the cost-share densities are $1/\beta_t$ and $1/(1-\beta_t)$ on their respective sets. In particular, homogeneous labor productivity implies $x_t = \dot{\beta}_t/(1-\beta_t)$ for the automation rate in equation (20).

C. The Bounds Implied by Complementary Tasks

Let $E \subseteq [0,1]$ be a measurable set of tasks with positive measure. Every task contributes positively to the CES integral in equation (3), so dropping the tasks outside E lowers the

integral. The exponent $\sigma/(\sigma-1)$ is negative, so

$$Y_t \leq Z_t \left(\int_E \alpha_i Y_{it}^{\frac{\sigma-1}{\sigma}} \, di \right)^{\frac{\sigma}{\sigma-1}}. \tag{C.1}$$

Output is therefore bounded by what the tasks in E alone would produce, however large the output of every other task. In the discrete, unweighted illustration of the introduction with $Z_t = 1$, the same argument applied to any task j gives

$$\left(\sum_i Y_{it}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \leq Y_{jt},$$

so output is at most the output of the scarcest task. The bound is attained in the limit as the output of every other task tends to infinity, because each such task's term in the sum tends to zero. With the body's weights and Z_t, the discrete bound is $Y_t \leq Z_t \alpha_j^{\sigma/(\sigma-1)} Y_{jt}$.

D. Automation Accounting and Identification

D.1 Proof of Proposition 3

Automation has no immediate effect on output because capital and labor have the same cost on a boundary task, so moving the boundary changes nothing at that instant. Productivity growth on the tasks already assigned to each factor remains in the growth accounting formula. The allocation formulas in equations (A.4)–(A.7) imply

$$s_{Lt}\omega_{\ell it} = (Z_t \psi_{\ell it})^{\sigma-1} \left(\frac{w_t}{P_t} \right)^{1-\sigma}, \qquad s_{Kt}\omega_{kit} = (Z_t \psi_{kit})^{\sigma-1} \left(\frac{r_t}{P_t} \right)^{1-\sigma}. \tag{D.1}$$

At each boundary β_{mt}, the densities are defined by item (ii) of Assumption A1 and costs are equal by equation (B.1), so

$$\frac{s_{Kt}\omega_{k\beta_m t}}{s_{Lt}\omega_{\ell\beta_m t}} = \left(\frac{\psi_{k\beta_m t}}{\psi_{\ell\beta_m t}} \right)^{\sigma-1} \left(\frac{r_t}{w_t} \right)^{1-\sigma} = 1. \tag{D.2}$$

Hence $s_{Lt}\omega_{\ell\beta_m t} = s_{Kt}\omega_{k\beta_m t}$ at every boundary. Multiplying by $|\dot{\beta}_{mt}|$ and summing over the boundaries gives

$$s_{Lt}\bar{\omega}_{\ell\beta t}\dot{\beta}_t = s_{Kt}\bar{\omega}_{k\beta t}\dot{\beta}_t, \tag{D.3}$$

with both sides zero when every boundary is stationary. The automation term in equation (16) therefore vanishes. Since $s_{Kt} + s_{Lt} = 1$, the remaining terms are

$$\begin{aligned}\widehat{TFP}_t &= s_{Kt}\hat{\psi}_{kt} + s_{Lt}\hat{\psi}_{\ell t} + \hat{Z}_t \\ &= \hat{Z}_t + \hat{\psi}_{\ell t} + s_{Kt}\big(\hat{\psi}_{kt} - \hat{\psi}_{\ell t}\big).\end{aligned} \tag{D.4}$$

The first line proves Proposition 3 and the second proves equation (18). The term $\hat{\psi}_{\ell t}$ in common TFP growth is the labor-cost-weighted average of productivity growth on the current labor tasks, not the growth rate of the slowest task.

D.2 Proof of Proposition 4

The automation rate identifies two combinations of task productivity growth. The first is common TFP growth, $\hat{Z}_t + \hat{\psi}_{\ell t}$, in equation (18). The second is the automation boost, $\hat{\psi}_{kt} - \hat{\psi}_{\ell t}$, capital task productivity growth relative to labor task productivity growth. The calculation uses positive capital and labor shares and $0 < \sigma < 1$.

The factor-share data recover A_t and its growth rate through equations (A.18) and (A.19). By the definition of x_t and equation (11),

$$\hat{Z}_t + \hat{\psi}_{\ell t} = \hat{A}_t - \frac{x_t}{1-\sigma}. \tag{D.5}$$

Substituting this expression into equation (D.4) and dividing by the positive capital share gives

$$\hat{\psi}_{kt} - \hat{\psi}_{\ell t} = \frac{\widehat{TFP}_t - \hat{A}_t + x_t/(1-\sigma)}{s_{Kt}}. \tag{D.6}$$

Equations (D.5) and (D.6) are the two claims in Proposition 4. They also imply

$$\hat{B}_t = \hat{Z}_t + \hat{\psi}_{kt} - \frac{1-s_{Kt}}{s_{Kt}}\frac{x_t}{1-\sigma}, \tag{D.7}$$

because equation (D.3) equates the capital and labor contributions of switching tasks before their opposite signs are applied.

The individual growth rates are not separately identified. The proposition identifies the sum and difference just derived. For any admissible function h_t, replacing the three

growth rates by

$$\hat{Z}_t + h_t, \qquad \hat{\psi}_{\ell t} - h_t, \qquad \hat{\psi}_{kt} - h_t$$

preserves both combinations. Separate identification of the three components therefore needs information beyond the two equations in the proposition.

D.3 From Task Categories to the Automation Rate

The sector automation rate is a labor-cost-weighted average of the category automation rates. Under the equal-cost assumption within each category, the category rate is the proportional decline in its measure of labor tasks.

Step 1. Aggregate category rates using labor cost shares. The fixed categories J_j form a measurable partition of the task interval. Write $|J_j| > 0$ for category j's measure. A category need not be an interval, so the boundaries β_{mt} move through it and out of it. The measure of its automated tasks, $|J_j|\beta_t^j = \int_{\Omega_{kt}} \mathbf{1}_{J_j}(i)\, di$, is the integral of a fixed indicator over the moving intervals, so by the chain rule, at almost every date,

$$|J_j|\dot{\beta}_t^j = \sum_{m \in M_t^j} |\dot{\beta}_{mt}|, \tag{D.8}$$

where M_t^j indexes the boundaries that lie in J_j at date t, as in equation (21). Thus each category's automation fraction is absolutely continuous.

Let $H_{\ell t}^j := \int_{J_j \cap \Omega_{\ell t}} \psi_{\ell it}^{\sigma-1}\, di$. For a category with positive labor use, the allocation formulas imply $\tilde{\omega}_{\ell t}^j = L_t^j / L_t = H_{\ell t}^j / H_{\ell t}$, and the normalized within-category density is $\omega_{\ell it}^j := \psi_{\ell it}^{\sigma-1} / H_{\ell t}^j$. Consequently,

$$\omega_{\ell it} = \frac{\psi_{\ell it}^{\sigma-1}}{H_{\ell t}} = \frac{H_{\ell t}^j}{H_{\ell t}} \frac{\psi_{\ell it}^{\sigma-1}}{H_{\ell t}^j} = \tilde{\omega}_{\ell t}^j \omega_{\ell it}^j, \qquad i \in J_j. \tag{D.9}$$

The category automation rate is the labor cost that the boundaries in J_j switch to capital,

$$x_t^j := \sum_{m \in M_t^j} \omega_{\ell \beta_{mt} t}^j |\dot{\beta}_{mt}|, \tag{D.10}$$

which is equation (21) and is zero when no boundary in J_j moves. Summing over the

partition and using equations (D.9) and (B.4) gives

$$x_t = \sum_{m=1}^{M_t} \omega_{\ell\beta_m t}|\dot{\beta}_{mt}| = \sum_j \tilde{\omega}^j_{\ell t} \sum_{m\in M^j_t} \omega^j_{\ell\beta_m t}|\dot{\beta}_{mt}| = \sum_j \tilde{\omega}^j_{\ell t} x^j_t. \tag{D.11}$$

This proves equation (22). A category with no remaining labor tasks has no moving boundary and contributes zero.

Step 2. Equal labor costs within a category. The equal-cost restriction is that labor costs per unit of task measure are the same across the active labor tasks of a category. Since the category's remaining labor-task measure is $|J_j|(1-\beta^j_t)$, the restriction and the allocation formulas give

$$\psi^{\sigma-1}_{\ell i t} = \frac{H^j_{\ell t}}{|J_j|(1-\beta^j_t)} \tag{D.12}$$

for every $i \in J_j \cap \Omega_{\ell t}$ at every date with $\beta^j_t < 1$. A boundary task is assigned to capital, so the restriction does not cover it directly. At a moving boundary, however, the boundary task was a labor task at every earlier date, and both sides of equation (D.12) are continuous in time, the left by item (ii) of Assumption A1 and the right because $H^j_{\ell t}$ and β^j_t are, so the equality holds at the boundary task as well. Dividing by $H^j_{\ell t}$ gives the within-category density on the active labor tasks and at the moving boundaries in J_j. With the flow identity in equation (D.8),

$$\frac{L_{it}}{L^j_t} = \frac{1}{|J_j|(1-\beta^j_t)}, \qquad i \in J_j \cap \Omega_{\ell t}, \qquad \sum_{m\in M^j_t} |\dot{\beta}_{mt}| = |J_j|\dot{\beta}^j_t. \tag{D.13}$$

Substituting the density at each moving boundary into equation (D.10), the category measure cancels, which gives

$$x^j_t = \frac{\sum_{m\in M^j_t} |\dot{\beta}_{mt}|}{|J_j|(1-\beta^j_t)} = \frac{\dot{\beta}^j_t}{1-\beta^j_t} = -\frac{d\log(1-\beta^j_t)}{dt}. \tag{D.14}$$

Step 3. Approximate the integral between observed years. For observed dates $t_a < t_b$ with $\beta^j_t < 1$ throughout the closed interval, $\log(1-\beta^j_t)$ is absolutely continuous, and

integrating equation (D.14) gives the average rate

$$x^j_{a,b} = \frac{1}{t_b - t_a} \log\left(\frac{1-\beta^j_{t_a}}{1-\beta^j_{t_b}}\right). \tag{D.15}$$

We read an LLM report of 100 percent automation as a rounded value and set it to $\beta^j_t = 0.9999$, which keeps this rate finite. In a few category-period observations a draw reports a lower automation fraction in the later year. The resulting negative category rates are retained and change no sector's automation rate by more than 0.006 percentage points per year. The continuous-time sector average is

$$\begin{aligned}\frac{1}{t_b - t_a}\int_{t_a}^{t_b} x_t\, dt &= \frac{1}{t_b - t_a}\sum_j \int_{t_a}^{t_b} \tilde{\omega}^j_{\ell t}\left[-d\log(1-\beta^j_t)\right] \\ &\approx \sum_j \frac{\tilde{\omega}^j_{\ell t_a} + \tilde{\omega}^j_{\ell t_b}}{2}\, x^j_{a,b}.\end{aligned} \tag{D.16}$$

The second line is the Tornqvist approximation used in the empirical work. It replaces a changing category share inside each integral by its endpoint average. It is exact when that share is constant, or affine in $-\log(1-\beta^j_t)$ on the interval. Under smooth paths it gives a second-order approximation to the average rate as the interval shrinks.

Step 4. Average across draws. The category automation fractions β^j_t and the labor cost shares $\tilde{\omega}^j_{\ell t}$ come from separate LLM calls, each repeated D times per model. Draw d of the automation fractions is combined with draw d of the labor cost shares. The two calls are independent, so this matching by draw index is a convention. When two models are pooled, draws are matched within each model before pooling. If draw d gives $x^{(d)}_{a,b}$, its whole-period average over the benchmark dates $t_0 < \cdots < t_N$ is

$$\bar{x}^{(d)} := \frac{\sum_{n=1}^N (t_n - t_{n-1}) x^{(d)}_{n-1,n}}{t_N - t_0}. \tag{D.17}$$

Table 4 reports the mean of $\bar{x}^{(d)}$ across the D draws of each model and the sample standard deviation,

$$\bar{x}^{\text{model}} := \frac{1}{D}\sum_{d=1}^D \bar{x}^{(d)}, \qquad \text{SD}(\bar{x}) := \left[\frac{1}{D-1}\sum_{d=1}^D (\bar{x}^{(d)} - \bar{x}^{\text{model}})^2\right]^{1/2}. \tag{D.18}$$

With the same number of draws for the two benchmark models, the pooled mean is the mean of the two model means. Replacing the labor-cost weights by equal weights across categories changes the whole-period automation rate by at most 0.21 percentage points per year across the six sectors.

Step 5. Category contributions. For a fixed interval, category j's contribution in draw d is $c_j^{(d)} := \tilde{\omega}_{\ell a,b}^{j,(d)} x_{a,b}^{j,(d)}$, where $\tilde{\omega}_{\ell a,b}^{j,(d)}$ is its endpoint-average labor cost share. Table 5 reports $\bar{c}_j$, the mean of $c_j^{(d)}$ across draws, and its share $\bar{c}_j / \sum_{j'} \bar{c}_{j'}$ of the sector's mean automation rate. Because the product is averaged within each draw, $\bar{c}_j$ is not the product of the separately averaged labor cost share and automation rate columns.

D.4 Factor Prices and the Automation Rate

Equation (23) follows from the indifference condition at each automation margin and the elasticity in equation (A.14). Its integral over 1950–2023 gives the numbers in Section 4.5.

Maintained assumptions. Retain competitive cost minimization, Assumption A1 over 1950–2023, and the measured factor shares and automation rate. With $a(i,t) := \log(\psi_{kit}/\psi_{\ell it})$, a is continuously differentiable in (i,t) near each boundary with $\partial_i a \neq 0$, and away from the boundaries relative costs stay bounded away from indifference. The capital–labor elasticity $\sigma_{KL,t}$ is the local elasticity in equation (A.14), which holds all task productivities fixed, including those of factors not in use.

Step 1. The labor costs that switch with relative prices. With $q_t = r_t/w_t$ as in Appendix A.3, let $\varrho_t := q_t H_\ell'(q_t)/H_\ell(q_t)$, the proportional change in the labor integral H_ℓ per unit change in $\log q_t$ at fixed technology. At every switching task $\psi_{kit} = q_t \psi_{\ell it}$, so the capital integral loses $q_t^{\sigma-1}$ times what the labor integral gains, $H_k'(q) = -q^{\sigma-1} H_\ell'(q)$. Since $s_{Kt}/s_{Lt} = q_t^{1-\sigma} H_{kt}/H_{\ell t}$ by equation (A.13), the capital term in equation (A.14) equals $(s_{Lt}/s_{Kt})\varrho_t$. Hence

$$\sigma_{KL,t} = \sigma + \frac{s_{Lt}}{s_{Kt}}\varrho_t + \varrho_t = \sigma + \frac{\varrho_t}{s_{Kt}}, \qquad \text{so} \qquad \varrho_t = s_{Kt}(\sigma_{KL,t} - \sigma). \tag{D.19}$$

A one percent fall in q_t therefore switches ϱ_t percent of labor costs to capital.

Step 2. The flow of tasks at each boundary. Indifference at boundary m means $a(\beta_{mt}, t) = \log q_t$. Differentiating in time gives

$$\partial_i a(\beta_{mt}, t)\, \dot{\beta}_{mt} + \partial_t a(\beta_{mt}, t) = \hat{q}_t.$$

With no de-automation, the measure of tasks switching to capital at boundary m is $|\dot{\beta}_{mt}| = [\partial_t a(\beta_{mt}, t) - \hat{q}_t]/|\partial_i a(\beta_{mt}, t)|$. The numerator is the improvement in capital's relative cost on a fixed task, and the denominator converts that improvement into a measure of tasks.

Step 3. Weight and sum the flows. Let $\varrho_{mt} := \omega_{\ell\beta_m t}/|\partial_i a(\beta_{mt}, t)|$. At fixed technology, a unit change in $\log q_t$ moves boundary m by $1/|\partial_i a(\beta_{mt}, t)|$, so $\varrho_t = \sum_m \varrho_{mt}$. Multiplying each boundary's flow by its labor cost-share density and summing gives, by the definition of x_t in equation (20),

$$x_t = \sum_m \varrho_{mt} \big[\partial_t a(\beta_{mt}, t) - \hat{q}_t\big] = \varrho_t \big[(\hat{w}_t - \hat{r}_t) + (\hat{\psi}_{k\beta t} - \hat{\psi}_{\ell\beta t})\big], \tag{D.20}$$
$$\hat{\psi}_{k\beta t} - \hat{\psi}_{\ell\beta t} := \frac{1}{\varrho_t} \sum_m \varrho_{mt}\, \partial_t a(\beta_{mt}, t).$$

With a single boundary, $\hat{\psi}_{k\beta t} - \hat{\psi}_{\ell\beta t}$ is the difference between the growth rates of ψ_{kit} and $\psi_{\ell it}$ on the fixed task $i = \beta_t$. With several, it averages these differences with weights proportional to the labor cost that switches per unit improvement in relative cost at each boundary. Substituting equation (D.19) gives equation (23). Under the constant σ_{KL} imposed in Step 5, its integral over 1950–2023 is exact for large cumulative changes in prices and technology.

Step 4. Express the rising-wages term with observed growth rates. Factor payments give $w_t/P_t = s_{Lt} Y_t/L_t$ and $r_t/P_t = s_{Kt} Y_t/K_t$. Since $s_{Kt} + s_{Lt} = 1$, their log derivatives satisfy $s_{Kt}(\hat{s}_{Lt} - \hat{s}_{Kt}) = \hat{s}_{Lt}$. Combining these equalities with $\widehat{TFP}_t = \hat{Y}_t - s_{Kt}\hat{K}_t - s_{Lt}\hat{L}_t$ gives

$$s_{Kt}(\hat{w}_t - \hat{r}_t) = s_{Kt}(\hat{K}_t - \hat{L}_t) + \hat{s}_{Lt} = \widehat{(w/P)}_t - \widehat{TFP}_t. \tag{D.21}$$

Step 5. Integrate over 1950–2023. Impose a constant σ_{KL} at every date, so that $\sigma_{KL} - \sigma$ factors out of the time integrals. Let $\bar{z} := \frac{1}{73}\int_{1950}^{2023} z_t\, dt$ be the average over the 73 years,

Table D.1: Rising Wages and Better Machines in Private Business Automation, 1950–2023

	$\sigma_{KL} = 0.3$	$\sigma_{KL} = 0.6$	$\sigma_{KL} = 1$
Automation rate, $\bar{x}$ (percent per year)	2.00	2.00	2.00
Rising wages, $\bar{x}^{\mathrm{w}}$	0.06	0.23	0.47
Better machines, $\bar{x} - \bar{x}^{\mathrm{w}}$	1.94	1.76	1.53
Share of automation from rising wages (percent)	2.9	11.7	23.3
Relative productivity growth (percent per year):			
On the marginal tasks, $\hat{\psi}_{k\beta t} - \hat{\psi}_{\ell\beta t}$	55.9	12.7	5.5
On the active tasks, $\hat{\psi}_{kt} - \hat{\psi}_{\ell t}$	4.9	4.9	4.9

Note: $\sigma = 0.2$. Both relative productivity growth rates are averages weighted by the capital share, as in equation (D.22).

and let $\bar{x}^{\mathrm{w}}$ be the average of the first term of equation (23). By equation (D.21),

$$\bar{x}^{\mathrm{w}} = \frac{\sigma_{KL} - \sigma}{73}\big[\Delta \log(w/P) - \Delta \log TFP\big], \tag{D.22}$$
$$\frac{\int_{1950}^{2023} s_{Kt}(\hat{\psi}_{k\beta t} - \hat{\psi}_{\ell\beta t})\,dt}{\int_{1950}^{2023} s_{Kt}\,dt} = \frac{\bar{x} - \bar{x}^{\mathrm{w}}}{(\sigma_{KL} - \sigma)\,\bar{s}_K}.$$

The first expression uses only endpoint changes. The second identifies the capital-share-weighted average of relative productivity growth on the marginal tasks.

Results. For private business, real wages per unit of labor input grow 1.74 percent per year and TFP 1.16 percent, so $\bar{x}^{\mathrm{w}} = (\sigma_{KL} - \sigma) \times 0.58$ percentage points per year. The measured automation rate averages 2.00 percent per year and the capital share 0.347. Table D.1 reports the decomposition. The measured averages approximate $\bar{z}$ by weighting each interval by its length and using interval means of the endpoint capital shares, as in Table 6. These are contributions along the observed equilibrium path, so a technology shock that moves equilibrium factor prices enters the rising-wages term. The better-machines term identifies growth in $\psi_{kit}/\psi_{\ell it}$ on the marginal tasks. Attributing it to capital alone would also require zero growth of labor productivity on those tasks.

Common growth across tasks. Suppose each factor's productivity grows at a common rate across tasks, so that $\hat{\psi}_{k\beta t} - \hat{\psi}_{\ell\beta t} = \hat{\psi}_{kt} - \hat{\psi}_{\ell t}$ at every date. Equation (23) then determines the capital–labor elasticity instead of the marginal growth rate. With a

constant σ_{KL}, equation (D.22) gives

$$\sigma_{KL} = \sigma + \frac{\bar{x}}{\left[\Delta \log(w/P) - \Delta \log TFP\right]/73 + \bar{s}_K \left(\hat{\psi}_{kt} - \hat{\psi}_{\ell t}\right)}, \tag{D.23}$$

where $\hat{\psi}_{kt} - \hat{\psi}_{\ell t}$ is the capital-share-weighted average in Table D.1. For private business the denominator is $0.58 + 0.347 \times 4.93 = 2.29$ percentage points per year, so $\sigma_{KL} = 0.2 + 2.00/2.29 = 1.07$.

The dynamic model's capital–labor elasticity. In the dynamic model of Section 6, $\psi_{kit}/\psi_{\ell it} = Q_t^{\theta_k - \theta_\ell} f(i)$. Each factor's productivity therefore grows at a common rate across tasks, for any $f(i)$. Under a constant σ_{KL}, equation (D.23) gives the elasticity implied by matching the historical automation rate and automation boost. For the calibrated function, the time-varying elasticity follows directly. With $\bar{f} = 0$ and the single boundary on the tasks where $f(i) = (1-i)^\mu$, $\partial_i a = -\mu/(1-\beta_t)$ and $\omega_{\ell\beta t} = 1/(1-\beta_t)$ at the boundary, so $\varrho_t = 1/\mu$. Equation (D.19) then gives

$$\sigma_{KL,t} = \sigma + \frac{1}{\mu\, s_{Kt}}. \tag{D.24}$$

At the 2023 cutoff the calibrated function implies $s_{Kt} = 0.391$ (Table J.1), so $\sigma_{KL} = 1.05$, close to the 1.07 from equation (D.23). As s_{Kt} rises to its limit $1/[\mu(1-\sigma)]$ (Appendix I), $\sigma_{KL,t}$ falls to one, the elasticity of the limiting Cobb–Douglas production function.

E. Freezing the Set of Automated Tasks

Assumption A1 applies to the actual and the counterfactual allocation. In the counterfactual the task sets are fixed, so every boundary is stationary and the switching terms are zero. The capital share is continuous and positive, so it is bounded away from zero on each compact interval and the integrals of x_t/s_{Kt} below are finite.

E.1 Proof of Proposition 5 for the Sectoral Calculation

Step 1. Find the change in relative factor productivity. Subtracting equation (11) from equation (D.7) gives

$$\hat{B}_t - \hat{A}_t = \hat{\psi}_{kt} - \hat{\psi}_{\ell t} - \frac{x_t}{(1-\sigma)s_{Kt}}. \tag{E.1}$$

In the counterfactual, the assignment is fixed and $x_t^{cf} = 0$. Assumption 2 therefore gives

$$\hat{B}_t^{cf} - \hat{A}_t^{cf} = \hat{\psi}_{kt} - \hat{\psi}_{\ell t}. \tag{E.2}$$

Subtracting the actual growth rate from the counterfactual growth rate and integrating from their common initial condition yields

$$\log\left(\frac{B_t^{cf}/A_t^{cf}}{B_t/A_t}\right) = \frac{1}{1-\sigma}\int_0^t \frac{x_\tau}{s_{K\tau}}\, d\tau. \tag{E.3}$$

Step 2. Convert relative productivity into the capital share. The conditional factor-demand formulas imply

$$\frac{s_{Kt}}{1-s_{Kt}} = \left(\frac{B_t}{A_t}\frac{w_t}{r_t}\right)^{\sigma-1}. \tag{E.4}$$

Applying the same equation to the counterfactual and taking the ratio gives

$$\frac{s_{Kt}^{cf}/(1-s_{Kt}^{cf})}{s_{Kt}/(1-s_{Kt})} = \left(\frac{B_t^{cf}/A_t^{cf}}{B_t/A_t}\right)^{\sigma-1}\left(\frac{w_t^{cf}/r_t^{cf}}{w_t/r_t}\right)^{\sigma-1}. \tag{E.5}$$

The sectoral relative-price closure makes the final factor equal to one. Since $(\sigma-1)/(1-\sigma) = -1$, substitution of equation (E.3) gives

$$\frac{s_{Kt}^{cf}}{1-s_{Kt}^{cf}} = \frac{s_{Kt}}{1-s_{Kt}}\exp\left(-\int_0^t \frac{x_\tau}{s_{K\tau}}\, d\tau\right). \tag{E.6}$$

This is equation (26). Equation (E.5) also shows the additional relative-price term when that path changes.

Step 3. Compute the lost TFP growth. There is no switching flow in the counterfactual. Its TFP growth rate is therefore the weighted average of productivity growth on its fixed task sets. Assumption 2 implies

$$\widehat{TFP}_t^{cf} = \hat{Z}_t + \hat{\psi}_{\ell t} + s_{Kt}^{cf}\left(\hat{\psi}_{kt} - \hat{\psi}_{\ell t}\right). \tag{E.7}$$

Subtracting this expression from equation (D.4) proves

$$\widehat{TFP}_t - \widehat{TFP}_t^{cf} = \left(s_{Kt} - s_{Kt}^{cf}\right)\left(\hat{\psi}_{kt} - \hat{\psi}_{\ell t}\right). \tag{E.8}$$

Together with equation (E.6), this establishes Proposition 5 under its sectoral relative-price closure.

The sign of lost growth. No de-automation implies $x_t \geq 0$. Equation (E.6) and the monotonicity of $s/(1-s)$ therefore imply $s_{Kt}^{cf} \leq s_{Kt}$, so the share lost in Proposition 5, $1 - s_{Kt}^{cf}/s_{Kt}$, lies in $[0,1]$. Lost TFP growth is nonnegative whenever $\hat{\psi}_{kt} - \hat{\psi}_{\ell t} \geq 0$.

E.2 The Aggregate Calculation with a Given Capital–Output Ratio

The private business calculation in equation (27) requires only that the two economies have the same capital–output ratio at each date,

$$\frac{K_t^{cf}}{Y_t^{cf}} = \frac{K_t}{Y_t}. \tag{E.9}$$

The common ratio may change over time.

Step 1. Compare capital-augmenting productivity. The fixed counterfactual task set and equation (D.7) give

$$\hat{B}_t^{cf} - \hat{B}_t = \frac{1}{1-\sigma}\frac{1-s_{Kt}}{s_{Kt}}x_t. \tag{E.10}$$

Integrating from the common initial condition yields

$$\log\frac{B_t^{cf}}{B_t} = \frac{1}{1-\sigma}\int_0^t \frac{1-s_{K\tau}}{s_{K\tau}}x_\tau\, d\tau. \tag{E.11}$$

Step 2. Apply the capital-share equation. Proposition 1 implies

$$\frac{s_{Kt}^{cf}}{s_{Kt}} = \left(\frac{B_t^{cf}}{B_t}\frac{K_t^{cf}/Y_t^{cf}}{K_t/Y_t}\right)^{(\sigma-1)/\sigma}. \tag{E.12}$$

The common capital–output ratio cancels. Equation (E.11) then gives

$$s_{Kt}^{cf} = s_{Kt} \exp\left(-\frac{1}{\sigma}\int_0^t \frac{1-s_{K\tau}}{s_{K\tau}} x_\tau \, d\tau\right). \tag{E.13}$$

This is the aggregate formula used in the computations. The lost-growth formula (E.8) continues to hold. For $x_t \geq 0$, equation (E.13) again gives $s_{Kt}^{cf} \leq s_{Kt}$, so the share lost again lies in $[0,1]$.

E.3 Why Faster Machines on a Fixed Set Do Not Eliminate All Weak Links

The CES representation bounds output by effective labor when tasks are complements. With positive inputs and $0 < \sigma < 1$,

$$Y_t = A_t L_t \left[1 + \left(\frac{B_t K_t}{A_t L_t}\right)^{(\sigma-1)/\sigma}\right]^{\sigma/(\sigma-1)} < A_t L_t. \tag{E.14}$$

The inequality follows because the bracket exceeds one and its exponent is negative. If $B_t K_t/(A_t L_t)$ tends to infinity, the expression in brackets tends to one, so $Y_t/(A_t L_t)$ tends to one. With automation frozen, $\hat{A}_t^{cf} = \hat{Z}_t + \hat{\psi}_{\ell t}$. Consequently, higher capital productivity on a fixed proper subset of tasks cannot raise output above the effective-labor bound. Growth in that bound depends on labor quantities and productivity on the remaining labor tasks.

F. Computation of the Accounting Tables

The tables use the observed benchmark years rather than a continuously observed path. Growth rates in this appendix are decimal rates per year. The tables multiply them by 100.

Growth rates by interval. For the benchmark dates $t_0 < \cdots < t_N$, write $\Delta_n := t_n - t_{n-1}$, $\bar{s}_{K,n} := (s_{K,t_{n-1}} + s_{K,t_n})/2$, and, for each positive level series V_t, $g_{V,n} := \log(V_{t_n}/V_{t_{n-1}})/\Delta_n$. Integrating equation (A.19) over the interval gives the growth rate of A, and the growth accounting identity gives that of B:

$$g_{A,n} = g_{Y/L,n} - \frac{\sigma}{1-\sigma} g_{s_L,n}, \qquad g_{B,n} := \frac{g_{TFP,n} - (1-\bar{s}_{K,n}) g_{A,n}}{\bar{s}_{K,n}}. \tag{F.1}$$

Multiplying endpoint-average shares by interval-average growth rates approximates the continuous-time integral of their product.

Identification by interval. With the measured interval automation rate x_n, the statistics

$$b_n := g_{A,n} - \frac{x_n}{1-\sigma}, \qquad d_n := \frac{g_{TFP,n} - b_n}{\bar{s}_{K,n}} \tag{F.2}$$

estimate $\hat{Z}_t + \hat{\psi}_{\ell t}$ and $\hat{\psi}_{kt} - \hat{\psi}_{\ell t}$. Whole-period averages weight each interval by its length, $\bar{b} := \sum_n \Delta_n b_n/(t_N - t_0)$ and likewise for $\bar{d}$, $\bar{g}_A$, $\bar{x}$, $\bar{g}_{TFP}$, and $\bar{s}_K$. Since the first formula is linear, $\bar{b} = \bar{g}_A - \bar{x}/(1-\sigma)$. In general $\bar{d} \neq (\bar{g}_{TFP} - \bar{b})/\bar{s}_K$ because capital shares vary across intervals, so the whole-period columns of Table 6 need not reproduce one another by direct substitution.

Counterfactual capital shares. For the sectoral calculation, the exponent in equation (26) through date t_n is approximated by a sum over intervals, so the counterfactual share satisfies

$$\frac{s^{cf}_{K,t_n}}{1 - s^{cf}_{K,t_n}} = \frac{s_{K,t_n}}{1 - s_{K,t_n}} \exp\left(- \sum_{n'=1}^{n} \Delta_{n'} x_{n'} \frac{s^{-1}_{K,t_{n'-1}} + s^{-1}_{K,t_{n'}}}{2} \right). \tag{F.3}$$

The aggregate calculation instead uses equation (E.13),

$$s^{cf}_{K,t_n} = s_{K,t_n} \exp\left[-\frac{1}{\sigma} \sum_{n'=1}^{n} \Delta_{n'} x_{n'} \frac{1}{2} \left(\frac{1 - s_{K,t_{n'-1}}}{s_{K,t_{n'-1}}} + \frac{1 - s_{K,t_{n'}}}{s_{K,t_{n'}}} \right) \right]. \tag{F.4}$$

Both begin at $s^{cf}_{K,t_0} = s_{K,t_0}$.

Lost growth. With $\bar{s}^{cf}_{K,n} := (s^{cf}_{K,t_{n-1}} + s^{cf}_{K,t_n})/2$, the period-average contributions of the automation boost to TFP growth in the actual and counterfactual economies are

$$\bar{\kappa} := \frac{\sum_n \Delta_n \bar{s}_{K,n} d_n}{t_N - t_0}, \qquad \bar{\kappa}^{cf} := \frac{\sum_n \Delta_n \bar{s}^{cf}_{K,n} d_n}{t_N - t_0}. \tag{F.5}$$

Lost growth is $\bar{\kappa} - \bar{\kappa}^{cf}$ and the share lost is $(\bar{\kappa} - \bar{\kappa}^{cf})/\bar{\kappa}$. The share lost is a weighted average of the interval shares $1 - \bar{s}^{cf}_{K,n}/\bar{s}_{K,n}$, with weights proportional to the interval contributions $\Delta_n \bar{s}_{K,n} d_n$. Table 7 reports the final-year capital shares, $\bar{d}$, $\bar{\kappa}$, $\bar{\kappa}^{cf}$, and the share lost.

Panels B and C of Table 6. Let $\vartheta := \max\{-\bar{b}, 0\}$. Panel B lowers the automation rate in every interval by $(1-\sigma)\vartheta$, so that $b_n^B = b_n + \vartheta$, $\bar{b}^B = \bar{b} + \vartheta = \max\{\bar{b}, 0\}$, $\bar{x}^B = \bar{x} - (1-\sigma)\vartheta$, and the interval automation boost becomes $d_n^B = d_n - \vartheta/\bar{s}_{K,n}$. The table reports its duration-weighted average $\bar{d}^B$, the same average used in Panels A and C. Panel C instead raises $\hat{Z}_t$ in every interval by ϑ. This raises $g_{A,n}$, $g_{TFP,n}$, and b_n by ϑ and leaves d_n unchanged, so its automation boost is $\bar{d}$. Attributing a share $\eta \in [0,1]$ of the correction to automation and $1-\eta$ to TFP gives $d_n^\eta = d_n - \eta\vartheta/\bar{s}_{K,n}$, so the reported automation boost is $(1-\eta)\bar{d} + \eta\bar{d}^B$.

Robustness to σ and to the automation rate. Combining equations (F.1) and (F.2) gives

$$b_n(\sigma, x_n) = g_{Y/L,n} - \frac{\sigma g_{s_L,n} + x_n}{1-\sigma}, \tag{F.6}$$

so a larger σ lowers common TFP growth when $g_{s_L,n} + x_n > 0$. Replacing x_n by $0.8x_n$ raises it by $0.2x_n/(1-\sigma)$. With observed TFP growth and capital shares held fixed, d_n moves by $-1/\bar{s}_{K,n}$ times the change in b_n. For each column of Table 8, the counterfactual calculation is repeated with the new interval growth differences and automation rates.

Implementation. The category calculations are implemented in `Programs/ExactAutomationRate.py`, with `Programs/load_exact_x.m` reading their period rates and `Programs/DecomposeAutomationRate.py` producing the category ranking. The growth accounting and counterfactual calculations are in `Programs/accounting.m`, and `Programs/MainAccountingTable.m` constructs the tables and robustness columns.

G. Infinite Productivity on a Fixed Set of Tasks

Equation (2) holds the output of the remaining tasks fixed. Its derivation therefore concerns the direct production gain from removing the specified weak links. Reallocation of the factors formerly used on those tasks is a separate change.

Since $\sigma < 1$, the exponent $(\sigma-1)/\sigma$ is negative. The two task aggregates in Section 2 satisfy

$$Y_t^{\frac{\sigma-1}{\sigma}} = Y_{\varnothing t}^{\frac{\sigma-1}{\sigma}} + Y_{\infty t}^{\frac{\sigma-1}{\sigma}}.$$

Differentiating with respect to either aggregate gives

$$\frac{\partial Y_t}{\partial Y_{jt}} = Y_t^{1/\sigma} Y_{jt}^{-1/\sigma}, \qquad \frac{P_{jt} Y_{jt}}{P_t Y_t} = \left(\frac{Y_{jt}}{Y_t}\right)^{\frac{\sigma-1}{\sigma}}, \qquad j \in \{\varnothing, \infty\}.$$

The second equality uses competitive pricing of the two aggregates. Their shares sum to one because the two terms in the production function sum to $Y_t^{\frac{\sigma-1}{\sigma}}$.

As $Y_\infty \to \infty$, its contribution $Y_\infty^{\frac{\sigma-1}{\sigma}}$ tends to zero. If $Y_\varnothing$ remains at its initial value, counterfactual output therefore tends to $Y_{\varnothing t}$. Substituting its initial share gives

$$\frac{Y^{cf}}{Y_t} = s_{\varnothing t}^{\sigma/(\sigma-1)} = (1 - s_{\infty t})^{-\sigma/(1-\sigma)}.$$

This is equation (2). It is finite whenever $0 < \sigma < 1$ and $s_{\infty t} < 1$.

The small-share approximation. The function $(1-s)^{-\sigma/(1-\sigma)}$ equals one at $s = 0$ and has derivative $\sigma/(1-\sigma)$ there. Its first-order approximation is

$$(1-s)^{-\sigma/(1-\sigma)} \approx 1 + \frac{\sigma}{1-\sigma} s.$$

Its second derivative is positive. Thus the approximation understates the exact gain for $s > 0$. At $s = 0.02$, the gains are 2.04% for $\sigma = 0.5$ and 0.51% for $\sigma = 0.2$. At $s = 1/3$ and $\sigma = 0.2$, the output ratio is $(3/2)^{1/4} = 1.10668$, or a gain of 10.67%.

For $\sigma \geq 1$, infinite productivity on a positive-measure set of tasks makes output unbounded.

H. Equilibrium Automation in the Dynamic Model

The dynamic model's static equilibrium reduces to one equation for the automation cutoff. This equation determines the cutoff from the stocks of ideas, capital, and labor. It also shows whether automation converges to completion or becomes complete at a finite state.

Step 1. Substitute the task productivities. For $0 < \beta < 1$ with $f(\beta) > 0$, let

$$J(\beta) := \int_0^\beta f(i)^{\sigma-1}\, di. \tag{H.1}$$

The function ξ in the footnote on the capital share in Section 6.2 is $J^{1/(\sigma-1)}$. Proposition 1 then gives

$$B_t = Z_t Q_t^{\theta_k} \xi(\beta_t), \qquad A_t = Z_t Q_t^{\theta_\ell} (1-\beta_t)^{-1/(1-\sigma)}. \tag{H.2}$$

Homogeneous labor productivity implies $\omega_{\ell\beta t} = 1/(1-\beta_t)$. The automation rate is therefore $x_t = \dot{\beta}_t/(1-\beta_t)$, and

$$\widehat{A}_t = \widehat{Z}_t + \theta_\ell \widehat{Q}_t + \frac{x_t}{1-\sigma}. \tag{H.3}$$

Step 2. Derive the capital-share formula. The factor-share equations and indifference at the marginal task imply

$$\frac{s_{Kt}}{1-s_{Kt}} = \left(\frac{B_t K_t}{A_t L_t}\right)^{(\sigma-1)/\sigma}, \qquad \frac{r_t}{w_t} = Q_t^{\theta_k-\theta_\ell} f(\beta_t), \qquad \frac{s_{Kt}}{1-s_{Kt}} = \frac{r_t K_t}{w_t L_t}.$$

Eliminating r_t/w_t and K_t/L_t yields

$$\frac{s_{Kt}}{1-s_{Kt}} = \frac{f(\beta_t)^{1-\sigma} J(\beta_t)}{1-\beta_t} = \left(\frac{f(\beta_t)}{\xi(\beta_t)}\right)^{1-\sigma} \frac{1}{1-\beta_t}. \tag{H.4}$$

Raising this equality to the power $1/(1-\sigma)$ gives the expression in the footnote on the capital share in Section 6.2.

Step 3. Determine the equilibrium cutoff. Let $z_t := Q_t^{\theta_k-\theta_\ell} K_t/L_t$ denote capital per person adjusted for the relative effect of ideas on the two factors. Substituting the indifference condition at the cutoff into the last share equality gives $s_{Kt}/(1-s_{Kt}) = z_t f(\beta_t)$. Combining it with equation (H.4) gives

$$z_t = \frac{J(\beta_t)}{(1-\beta_t) f(\beta_t)^\sigma}. \tag{H.5}$$

For a positive, nonincreasing f, the right side is strictly increasing in β. Its logarithmic derivative is

$$\frac{f(\beta)^{\sigma-1}}{J(\beta)} + \frac{1}{1-\beta} - \sigma \frac{f'(\beta)}{f(\beta)} > 0.$$

The right side of equation (H.5) tends to zero as $\beta \downarrow 0$ and to infinity as $\beta \uparrow 1$ when $f(1) \geq 0$, or as $\beta \uparrow \bar{\beta}$ if f first reaches zero at $\bar{\beta} < 1$. Hence every positive, finite z_t determines one interior cutoff. Equation (H.4) makes the share a function of the cutoff, but not a monotone one. Uniqueness of the equilibrium cutoff comes from the strictly increasing right side of equation (H.5).

The model's two differential equations can now be written without wages or rental rates. Evaluate β_t from equation (H.5), evaluate A_t, B_t from equation (H.2), and let Y_t be the CES output in Proposition 1. The stocks then satisfy

$$\dot{Q}_t = \bar{q}\bar{\iota}_R^{\lambda} Y_t^{\lambda} Q_t^{\phi}, \qquad \dot{K}_t = \bar{\iota}_K Y_t - \delta K_t. \tag{H.6}$$

Equation (H.6), together with the equilibrium cutoff, is the exact task model. Conditional on positive finite states and positive productivity at the cutoff, the cutoff and output are smooth wherever f is, so the differential equations have a unique local solution. The limiting function $h(u) = \min\{u^{\mu}, \bar{\mu}\}$ has a kink at $i = 1 - (\bar{\mu})^{1/\mu}$, but every simulated cutoff lies above it.

Differentiating equation (H.5) also determines the instantaneous automation rate,

$$x_t = \frac{(\theta_k - \theta_\ell)\widehat{Q}_t + \widehat{K}_t - n}{1 + (1-\beta_t) f(\beta_t)^{\sigma-1}/J(\beta_t) - \sigma(1-\beta_t) f'(\beta_t)/f(\beta_t)}. \tag{H.7}$$

The denominator is positive. Thus continued automation is equivalent to growth in z_t. It is not implied by growth in Q_t alone if capital per person declines sufficiently fast. The instantaneous output-growth identity is

$$\widehat{Y}_t = \widehat{Z}_t + [s_{Kt}\theta_k + (1 - s_{Kt})\theta_\ell]\widehat{Q}_t + s_{Kt}\widehat{K}_t + (1 - s_{Kt})n. \tag{H.8}$$

This follows by substituting the homogeneous idea elasticities into the task-productivity growth accounting identity. The marginal automation terms cancel because capital and labor have equal costs at the switching task.

The convergence proofs in Appendix I use this system without requiring β_t to be monotone. Under Assumption 1, admissible paths also have a nonnegative numerator in equation (H.7).

H.1 The Three Long-Run Automation Cases

Full automation: positive productivity on every task. If $f(1) > 0$, then $J(1)$ is finite and positive. Equation (H.4) implies $s_{Kt} \to 1$ as $\beta_t \to 1$. The capital-share equation also gives

$$\frac{Y_t}{B_t K_t} = s_{Kt}^{\sigma/(1-\sigma)} \longrightarrow 1, \qquad \frac{B_t}{Z_t Q_t^{\theta_k}} \longrightarrow \xi(1) > 0. \tag{H.9}$$

Thus production becomes proportional to $Z_t Q_t^{\theta_k} K_t$ in this limit.

At every positive finite state, equation (H.5) gives $\beta_t < 1$. In particular, its right side is asymptotic to $J(1)/[(1-\beta)f(1)^\sigma]$, which diverges as $\beta \to 1$. Appendix I.5 proves that the calibrated dynamic model has a finite singular date t_∞, the date at which output becomes infinite, with $\lim_{t \uparrow t_\infty} \beta_t = 1$ when $f(1) > 0$. Complete automation is the limit at the singular date, and $\beta_t < 1$ at every date before t_∞.

Essential human tasks: tasks that capital cannot perform. For $\bar{f} < 0$, tasks with $f(i) \leq 0$ cannot be performed by capital and are supplied with labor permanently. All powers $f(i)^{\sigma-1}$ are taken on the automated interval, where $f > 0$. Suppose the last task that capital can perform is $\bar{\beta} < 1$, and $f(\beta) \downarrow 0$ as $\beta \uparrow \bar{\beta}$. For the paper's shifted function, the zero is simple, so $J(\bar{\beta}) < \infty$ because $\sigma - 1 > -1$. Equation (H.4) then gives $s_{Kt} \to 0$, and

$$\frac{Y_t}{A_t L_t} = (1 - s_{Kt})^{\sigma/(1-\sigma)} \longrightarrow 1, \qquad \frac{A_t}{Z_t Q_t^{\theta_\ell}} \longrightarrow (1-\bar{\beta})^{-1/(1-\sigma)}. \tag{H.10}$$

With the remaining tasks performed by labor, output therefore has a finite upper bound for each fixed stock of ideas and labor, even when the effective capital available for other tasks is arbitrarily large. The growth rate of the limiting A_t is $\widehat{Z}_t + \theta_\ell \widehat{Q}_t$. Omitting $\widehat{Z}_t$ is valid when other productivity is fixed, or as a limiting growth-rate statement when $\widehat{Z}_t \to 0$.

The fraction never automated and the shift parameter. If $1 - \bar{\beta} > 0$ is the fraction of tasks never automated, the equation $f(\bar{\beta}) = 0$ gives

$$\bar{f} = -h(1-\bar{\beta}). \tag{H.11}$$

Thus $\bar{f}$ is a productivity shift rather than the fraction itself. In the calibrated limit, $h(1-\bar{\beta}) = (1-\bar{\beta})^{\mu}$ because $(1-\bar{\beta})^{\mu} < \bar{\mu}$, so for $1-\bar{\beta} = 0.03$ its magnitude is about 2.7×10^{-5}.

I. Long-Run Growth in the Dynamic Model

I.1 The Limiting Capital Share and Production Function When $f(1) = 0$

Step 1. Evaluate the share for the power function. For $f(i) = (1-i)^{\mu}$, write $u := 1-\beta$. If $\mu(1-\sigma) \neq 1$, direct integration gives

$$J(\beta) = \int_u^1 v^{-\mu(1-\sigma)}\, dv = \frac{u^{1-\mu(1-\sigma)} - 1}{\mu(1-\sigma) - 1}.$$

Substituting into equation (H.4) gives

$$\frac{s_K}{1-s_K} = u^{\mu(1-\sigma)-1} J(\beta) = \frac{1 - u^{\mu(1-\sigma)-1}}{\mu(1-\sigma) - 1}. \tag{I.1}$$

When $\mu(1-\sigma) > 1$, the last expression tends to $1/(\mu(1-\sigma)-1)$. Solving for the share gives

$$s_K^* = \frac{1}{\mu(1-\sigma)} \in (0,1). \tag{I.2}$$

The condition $\mu(1-\sigma) > 1$ is necessary for this interior limit. If $\mu(1-\sigma) = 1$, $J(\beta) = \log(1/u)$ and the share tends to one. If $\mu(1-\sigma) < 1$, equation (I.1) also tends to infinity in odds, so the share again tends to one.

Step 2. Verify the same limit for the general function. With $\bar{f} = 0$, the function in equation (29), from Benhabib, Perla, and Tonetti (2014) (hereafter BPT) satisfies

$$\frac{f(1-u)}{u^{\mu}} = \left[1 + \left(\frac{u^{\mu}}{\bar{\mu}}\right)^{\mu_1}\right]^{-1/\mu_1} \longrightarrow 1$$

as $u \to 0$. In the limit $\mu_1 \to \infty$, the ratio equals one whenever $u^{\mu} < \bar{\mu}$. Consequently its integral, for $\mu(1-\sigma) > 1$, satisfies

$$\frac{J(1-u)}{u^{1-\mu(1-\sigma)}/(\mu(1-\sigma)-1)} \longrightarrow 1. \tag{I.3}$$

To justify this integration step, fix a small positive ε. For all sufficiently small v, the integrand $f(1-v)^{\sigma-1}$ lies between $(1-\varepsilon)v^{-\mu(1-\sigma)}$ and $(1+\varepsilon)v^{-\mu(1-\sigma)}$. Integrate these bounds from u to a fixed small upper endpoint. The remaining integral is finite, whereas $u^{1-\mu(1-\sigma)}$ diverges. Dividing by $u^{1-\mu(1-\sigma)}/(\mu(1-\sigma)-1)$ and then letting $\varepsilon \downarrow 0$ proves the displayed limit. Substitution into equation (H.4) proves equation (I.2) for the BPT function.

Step 3. Derive output rather than infer it from shares alone. Equations (H.5) and (I.3) imply

$$z_t \sim \frac{(1-\beta_t)^{-\mu}}{\mu(1-\sigma)-1}, \qquad (1-\beta_t)^{-1/(1-\sigma)} \sim [(\mu(1-\sigma)-1)z_t]^{s_K^*},$$

where $X \sim W$ means $X/W \to 1$. The labor-share equation gives

$$\frac{Y_t}{L_t} = (1-s_{Kt})^{\sigma/(1-\sigma)} Z_t Q_t^{\theta_\ell} (1-\beta_t)^{-1/(1-\sigma)}.$$

Combining these equations yields

$$Y_t \sim (1-s_K^*)^{\sigma/(1-\sigma)} [\mu(1-\sigma)-1]^{s_K^*} Z_t Q_t^{(1-s_K^*)\theta_\ell + s_K^*\theta_k} K_t^{s_K^*} L_t^{1-s_K^*}. \tag{I.4}$$

Thus the limiting production function has the capital elasticity s_K^* and the idea elasticity $(1-s_K^*)\theta_\ell + s_K^*\theta_k$. The same calculation holds for the BPT function because its tail has the same coefficient as the power function.

Bounds used in the dynamic proofs. The static output function can be written as

$$\frac{Y_t}{L_t} = Z_t Q_t^{\theta_\ell} G(z_t), \qquad G(z) \sim (1-s_K^*)^{\sigma/(1-\sigma)} [\mu(1-\sigma)-1]^{s_K^*} z^{s_K^*} \quad \text{as } z \to \infty. \tag{I.5}$$

At $z=0$, using homogeneous labor on all tasks gives $G(0)=1$. The unique cutoff makes G positive and continuous. There are therefore constants $c_1, c_2 > 0$ such that

$$c_1 z^{s_K^*} \le G(z) \le c_2(1+z^{s_K^*}) \qquad (z \ge 0). \tag{I.6}$$

On any interval $z \ge z_{\min} > 0$, the upper bound can instead be written $G(z) \le c_3 z^{s_K^*}$. These bounds follow by using the limiting ratio for large z and continuity on the remaining compact interval.

I.2 Balanced-Growth Identities

The formula for Φ follows from the limiting production function and the idea production function. These growth-rate identities identify a candidate balanced-growth path. Existence and convergence for the calibrated case are established separately in Appendix I.3.

Step 1. Relate output growth to idea growth. On a balanced-growth path with positive investment and a constant capital–output ratio, $g_K = g_Y$. If other productivity is constant, equation (I.4) gives

$$g_Y = [(1 - s_K^*)\theta_\ell + s_K^*\theta_k]g_Q + s_K^* g_K + (1 - s_K^*)n.$$

Substitute $g_K = g_Y$, subtract population growth, and divide by $1 - s_K^*$ to obtain

$$g_y = \left(\theta_\ell + \frac{s_K^*}{1 - s_K^*}\theta_k\right) g_Q. \tag{I.7}$$

This is also obtained directly from automation. With constant shares, the indifference condition at the cutoff for the limiting power function implies $\mu x = (\theta_k - \theta_\ell)g_Q + g_y$. Substituting it into $g_y = \theta_\ell g_Q + x/(1 - \sigma)$ and using $1/[\mu(1 - \sigma)] = s_K^*$ gives the same equality.

Step 2. Impose constant idea growth. The idea production function gives

$$\widehat{Q}_t = \bar{q}\iota_R^\lambda \frac{Y_t^\lambda}{Q_t^{1-\phi}}.$$

If $g_Q > 0$ is constant, differentiating its logarithm gives $0 = \lambda g_Y - (1 - \phi)g_Q$. For $\phi < 1$,

$$g_Q = \frac{\lambda}{1 - \phi}(g_y + n).$$

Combining this equation with equation (I.7) gives

$$g_y = \Phi(g_y + n), \qquad \Phi = \frac{\lambda}{1 - \phi}\left(\theta_\ell + \frac{s_K^*}{1 - s_K^*}\theta_k\right),$$
$$g_y = \frac{\Phi n}{1 - \Phi} \qquad (\Phi < 1). \tag{I.8}$$

For $n > 0$ and $\Phi > 1$, this equation has no nonnegative balanced-growth solution. That fact alone does not prove a finite-time explosion. Appendix I.4 supplies a differential-inequality proof for $\lambda = 1$, the value used in both calibrations.

I.3 Existence and Convergence with Semi-Endogenous Growth

For the calibrated $\lambda = 1$, the model converges to the growth rate in equation (31) when $n > 0$ and $\Phi < 1$. The proof uses $\phi < 1$, $\theta_k > \theta_\ell > 0$, $\mu(1-\sigma) > 1$, positive initial stocks, $\bar{\iota}_K > 0$, and $\bar{q}\bar{\iota}_R > 0$. These inequalities hold in the two calibrations. To prove convergence of levels after detrending, take $Z_t \to Z_\infty \in (0, \infty)$ and $\widehat{Z}_t \to 0$, as for the simulation's exponentially decaying growth rate of other productivity (Appendix J.1).

Step 1. Transform the stock of ideas. Let

$$\tilde{Q}_t := \frac{Q_t^{1-\phi}}{1-\phi}. \tag{I.9}$$

With $\lambda = 1$, the idea equation becomes $\dot{\tilde{Q}}_t = \bar{q}\bar{\iota}_R Y_t$. The definition of Φ gives $(1 - s_K^*)\theta_\ell + s_K^*\theta_k = (1-\phi)(1-s_K^*)\Phi$, so the limiting production function in equation (I.4) has elasticity $(1-s_K^*)\Phi$ with respect to $\tilde{Q}_t$. The returns to the two stocks together, $s_K^* + (1-s_K^*)\Phi$, are below one exactly when $\Phi < 1$.

Step 2. Construct the balanced-growth path of the limiting model. Write $g := n/(1-\Phi)$ for aggregate output growth, and let

$$\mathcal{D} := (1-s_K^*)^{\sigma/(1-\sigma)}[\mu(1-\sigma)-1]^{s_K^*} Z_\infty (1-\phi)^{(1-s_K^*)\Phi} L_0^{1-s_K^*}.$$

For the limiting production function, output is $\mathcal{D}\tilde{Q}_t^{(1-s_K^*)\Phi} K_t^{s_K^*} e^{(1-s_K^*)nt}$. Detrending $\tilde{Q}$ and K at rate g gives

$$\dot{\mathsf{Q}}_t = \bar{q}\bar{\iota}_R \mathcal{D}\mathsf{Q}_t^{(1-s_K^*)\Phi}\mathsf{K}_t^{s_K^*} - g\mathsf{Q}_t, \qquad \mathsf{Q}_t := e^{-gt}\tilde{Q}_t,$$
$$\dot{\mathsf{K}}_t = \bar{\iota}_K \mathcal{D}\mathsf{Q}_t^{(1-s_K^*)\Phi}\mathsf{K}_t^{s_K^*} - (g+\delta)\mathsf{K}_t, \qquad \mathsf{K}_t := e^{-gt}K_t.$$

The time-dependent exponential cancels because $[s_K^* + (1 - s_K^*)\Phi]g + (1 - s_K^*)n = g$. The unique positive stationary point satisfies

$$\frac{\mathsf{Q}_*}{\mathsf{K}_*} = \frac{\bar{q}\bar{\iota}_R(g+\delta)}{\bar{\iota}_K g}, \qquad \mathsf{K}_*^{(1-s_K^*)(1-\Phi)} = \frac{\bar{\iota}_K \mathcal{D}}{g+\delta}\left(\frac{\bar{q}\bar{\iota}_R(g+\delta)}{\bar{\iota}_K g}\right)^{(1-s_K^*)\Phi}. \tag{I.10}$$

Both coordinates are positive and unique because $g > 0$ and $\Phi < 1$. They determine a balanced-growth path with $g_Y = g_K = g$, $g_Q = g/(1-\phi)$, and $g_y = g - n$.

Step 3. Prove convergence. In the limiting model, the stationary conditions in equation (I.10) turn the two detrended equations into

$$\frac{d}{dt}\frac{\mathsf{Q}_t}{\mathsf{Q}_*} = g\left[\left(\frac{\mathsf{Q}_t}{\mathsf{Q}_*}\right)^{(1-s_K^*)\Phi}\left(\frac{\mathsf{K}_t}{\mathsf{K}_*}\right)^{s_K^*} - \frac{\mathsf{Q}_t}{\mathsf{Q}_*}\right],$$
$$\frac{d}{dt}\frac{\mathsf{K}_t}{\mathsf{K}_*} = (g+\delta)\left[\left(\frac{\mathsf{Q}_t}{\mathsf{Q}_*}\right)^{(1-s_K^*)\Phi}\left(\frac{\mathsf{K}_t}{\mathsf{K}_*}\right)^{s_K^*} - \frac{\mathsf{K}_t}{\mathsf{K}_*}\right].$$

Consider the largest of 1, $\mathsf{Q}_t/\mathsf{Q}_*$, and $\mathsf{K}_t/\mathsf{K}_*$. When it exceeds one, the first term in each bracket is at most this largest value raised to the power $s_K^* + (1 - s_K^*)\Phi$. Because this power is below one when $\Phi < 1$ and the largest value exceeds one, that term is below the largest value. The ratio that attains the largest value therefore declines at a rate of at least g times the difference between the largest value and that power of it, so the largest value is nonincreasing. If it converged to a limit above one, this difference would stay bounded away from zero, and the largest value would fall below its limit in finite time. It therefore converges to one. By the same argument, the smallest of 1, $\mathsf{Q}_t/\mathsf{Q}_*$, and $\mathsf{K}_t/\mathsf{K}_*$ is nondecreasing and converges to one, because the first term in each bracket exceeds the smallest value when that value is below one. Both ratios therefore converge to one.

The bounds in equation (I.6) keep the task model's detrended stocks in the same kind of compact set, because the additional all-labor term in the upper bound decays at rate $n[\Phi - \theta_\ell/(1-\phi)]/(1-\Phi) > 0$. On that set $z_t = Q_t^{\theta_k - \theta_\ell} K_t/L_t$ tends to infinity uniformly, since $g - n > 0$ and $\theta_k > \theta_\ell$. Equation (I.4) and $Z_t \to Z_\infty$ then imply that the task model's detrended output differs from $\mathcal{D}\mathsf{Q}^{(1-s_K^*)\Phi}\mathsf{K}^{s_K^*}$ by a factor that tends to one uniformly.

Repeating the monotone bounds with production scaled by $1 - \varepsilon$ and $1 + \varepsilon$, and letting $\varepsilon \downarrow 0$, proves that the detrended stocks converge to equation (I.10). Hence $\widehat{K}_t \to g$, $\widehat{Q}_t \to g/(1-\phi)$, and, by equation (H.8), $\widehat{Y}_t \to g$. Equation (31) is therefore the asymptotic

growth rate of the task model, although its shares need not be constant at finite dates.

I.4 Finite-Time Explosion and the Knife Edge

With $\lambda = 1$, the criterion for infinite output in finite time is $\Phi > 1$ under asymptotic automation, $\bar{f} = 0$, and the calibrated restrictions stated in Appendix I.3. In this proof, $n \geq 0$ is allowed and Z_t is bounded above and away from zero. The simulation's path of other productivity (Appendix J.1) has these properties. The proof concerns the exact task model and does not replace large shares by full automation.

Step 1. Relate ideas to accumulated investment. The equations for $\tilde{Q}_t$ and K_t imply

$$\tilde{Q}_t = \tilde{Q}_0 + \frac{\bar{q}\bar{\iota}_R}{\bar{\iota}_K}(K_t - K_0) + \frac{\bar{q}\bar{\iota}_R\delta}{\bar{\iota}_K}\int_0^t K_v\,dv. \tag{I.11}$$

This follows by substituting $Y_t = (\dot{K}_t + \delta K_t)/\bar{\iota}_K$ into $\dot{\tilde{Q}}_t = \bar{q}\bar{\iota}_R Y_t$ and integrating. In particular, $\tilde{Q}_t \geq (\bar{q}\bar{\iota}_R/\bar{\iota}_K)K_t - C$ for a fixed constant C.

For finite-horizon upper bounds, let $E_t := e^{\delta t}K_t$. Then $\dot{E}_t = \bar{\iota}_K e^{\delta t}Y_t > 0$. On any interval $0 \leq t \leq T < \infty$,

$$\tilde{Q}_0 + \frac{\bar{q}\bar{\iota}_R}{\bar{\iota}_K e^{\delta T}}(E_t - K_0) \leq \tilde{Q}_t \leq \tilde{Q}_0 + \frac{\bar{q}\bar{\iota}_R}{\bar{\iota}_K}(E_t - K_0). \tag{I.12}$$

Thus the two stocks can be bounded in terms of each other on every finite horizon.

Step 2. Rule out finite-time explosion when $\Phi \leq 1$. Fix a finite T. Positivity gives $K_t \geq K_0 e^{-\delta T}$ and $Q_t \geq Q_0$ on $[0,T]$. Since $L_t \leq L_0 e^{nT}$ and $\theta_k > \theta_\ell$, z_t is bounded away from zero. The final bound after equation (I.6) therefore gives

$$Y_t \leq C_T \tilde{Q}_t^{(1-s_K^*)\Phi} K_t^{s_K^*}.$$

Using equation (I.12), increase C_T if needed to obtain

$$\dot{E}_t \leq C_T(1 + E_t)^{s_K^* + (1-s_K^*)\Phi}.$$

If $\Phi \leq 1$, the exponent is at most one, so the right side is at most $C_T(1 + E_t)$. Integration gives a finite exponential upper bound for E_t on $[0,T]$. Equation (I.12) bounds $\tilde{Q}_t$, and hence Q_t, as well. Smoothness at finite positive states then extends the solution beyond

T. Because T was arbitrary, output remains finite at every finite date in this case.

Step 3. Prove finite-time explosion when $\Phi > 1$. Suppose, to obtain a contradiction, that a solution with $\Phi > 1$ exists for all time. The all-labor allocation is feasible, so $Y_t/L_t \geq Z_t Q_t^{\theta_\ell}$. If Q_t were bounded, $\dot{Q}_t = \bar{q}\bar{\iota}_R Y_t Q_t^{\phi}$ would have a strictly positive lower bound. Hence $Q_t \to \infty$. Because $\theta_\ell > 0$, GDP per person is bounded below by a quantity tending to infinity. The equation

$$\frac{d}{dt}\left(\frac{K_t}{L_t}\right) = \bar{\iota}_K \frac{Y_t}{L_t} - (\delta + n)\frac{K_t}{L_t}$$

then implies $K_t/L_t \to \infty$. Thus $z_t \to \infty$ and $K_t \to \infty$.

The lower production bound and equation (I.11) now imply, for all sufficiently late dates,

$$Y_t \geq c_4 \tilde{Q}_t^{(1-s_K^*)\Phi} K_t^{s_K^*} \geq c_5 K_t^{s_K^* + (1-s_K^*)\Phi}, \qquad \dot{K}_t \geq \bar{\iota}_K c_5 K_t^{s_K^* + (1-s_K^*)\Phi} - \delta K_t \geq c_6 K_t^{s_K^* + (1-s_K^*)\Phi},$$

where c_4, c_5, c_6 are positive constants. The last inequality holds once K_t is sufficiently large. Multiplying it by $(1 - s_K^*)(1 - \Phi)K_t^{-s_K^* - (1-s_K^*)\Phi} < 0$ gives

$$\frac{d}{dt} K_t^{(1-s_K^*)(1-\Phi)} \leq -(1 - s_K^*)(\Phi - 1)c_6.$$

The positive quantity on the left would reach zero in finite additional time, which contradicts existence of a finite global solution. Therefore the maximal solution has a finite endpoint. Equation (I.12) shows that both stocks diverge there. The lower production bound implies that output also diverges. This establishes a finite-time explosion for $\Phi > 1$, including when $n = 0$.

Step 4. Distinguish the knife edge from a finite-time explosion. At $\Phi = 1$, Step 2 rules out infinite output at a finite date even if $n > 0$. With constant population, the limiting production function admits endogenous balanced growth. To determine its growth rate, let $R_t := \tilde{Q}_t/K_t$. When $n = 0$ and $Z_t = Z_\infty$, limiting output per unit of capital is

$$\frac{Y_t}{K_t} = \mathcal{D} R_t^{1-s_K^*}, \qquad \dot{R}_t = (\bar{q}\bar{\iota}_R - \bar{\iota}_K R_t)\frac{Y_t}{K_t} + \delta R_t.$$

A balanced-growth rate $g > 0$ satisfies $R = \bar{q}\bar{\iota}_R(g+\delta)/(\bar{\iota}_K g)$ and $Y/K = (g+\delta)/\bar{\iota}_K$. Substitution gives

$$g^{1-s_K^*}(g+\delta)^{s_K^*} = \bar{\iota}_K \mathcal{D}\left(\frac{\bar{q}\bar{\iota}_R}{\bar{\iota}_K}\right)^{1-s_K^*}. \tag{I.13}$$

The left side increases continuously from zero to infinity. Hence this equation determines one positive endogenous growth rate.

With $n > 0$, the ratio R_t still converges to $\bar{q}\bar{\iota}_R/\bar{\iota}_K$, and the factor $L_t^{1-s_K^*}$ in limiting output grows exponentially. Capital and output therefore grow double exponentially. Their growth rates rise without bound, while their levels remain finite at every finite date. At the knife edge with population growth, growth therefore accelerates without bound, but output does not explode in finite time.

I.5 The Other Two Endpoints with Endogenous Ideas

Positive capital productivity on every task. When $f(1) > 0$, allocating capital across every task produces $Z_t\xi(1)Q_t^{\theta_k}K_t$ before adding labor. Adding the available labor cannot reduce output, so optimized output is at least this large. Under $\lambda = 1$, equation (I.11) makes this lower bound at least a positive constant times $K_t^{1+\theta_k/(1-\phi)}$ once K_t is large. The global-solution contradiction in Step 3 of Appendix I.4 still shows that K_t must eventually become large. Since $\theta_k > 0$, the exponent is above one. This proves a finite-time explosion at some $t_\infty < \infty$. Equation (I.12) implies that both K_t and Q_t tend to infinity as $t \uparrow t_\infty$. Population remains finite at t_∞, so $z_t \to \infty$. The strictly increasing cutoff equation (H.5) then implies $\beta_t \to 1$, and equation (H.4) implies $s_{Kt} \to 1$. This proves complete automation as a limit at the finite singular date, with $\beta_t < 1$ at every earlier date.

Essential human tasks as permanent weak links. Suppose, as in Appendix H.1, that capital cannot perform tasks above $\bar{\beta}$, so $1-\bar{\beta} > 0$ is the fraction of tasks never automated. Removing the capital term from the CES expression provides the upper bound

$$Y_t \le A_t L_t \le (1-\bar{\beta})^{-1/(1-\sigma)} Z_t Q_t^{\theta_\ell} L_t.$$

For $\lambda = 1$, this gives $\dot{\tilde{Q}}_t \le C_T \tilde{Q}_t^{\theta_\ell/(1-\phi)}$ on every finite horizon. If $\theta_\ell/(1-\phi) \le 1$, as in both calibrations, ideas and output cannot diverge in finite time. Thus the calibrated

explosions do not occur with essential human tasks. This conclusion depends on the effect of ideas on labor productivity as well as on the existence of unautomated tasks.

In the limiting production function $Y_t = Z_\infty(1-\bar{\beta})^{-1/(1-\sigma)}Q_t^{\theta_\ell}L_t$, the idea equation is

$$\dot{Q}_t = \overline{q\iota}_R Z_\infty(1-\bar{\beta})^{-1/(1-\sigma)}L_0 e^{nt}Q_t^{\theta_\ell+\phi}.$$

For $1-\phi-\theta_\ell > 0$, integration gives

$$Q_t^{1-\phi-\theta_\ell} = Q_0^{1-\phi-\theta_\ell} + (1-\phi-\theta_\ell)\overline{q\iota}_R Z_\infty(1-\bar{\beta})^{-1/(1-\sigma)}L_0\frac{e^{nt}-1}{n} \qquad (n>0).$$

Consequently $g_Q \to n/(1-\phi-\theta_\ell)$ and $g_y \to \theta_\ell n/(1-\phi-\theta_\ell)$. These limits also hold in the exact task model when $Z_t \to Z_\infty > 0$ and $\widehat{Z}_t \to 0$. To establish this, note that the all-labor bound again gives $Q_t \to \infty$, $K_t/L_t \to \infty$, and $z_t \to \infty$. Equations (H.10) and (I.5) give

$$Z_t G(z_t) \longrightarrow Z_\infty(1-\bar{\beta})^{-1/(1-\sigma)}.$$

The exact idea equation is

$$\frac{d}{dt}Q_t^{1-\phi-\theta_\ell} = (1-\phi-\theta_\ell)\overline{q\iota}_R Z_t G(z_t)L_0 e^{nt}.$$

Integrating and using convergence of $Z_t G(z_t)$ gives $Q_t^{1-\phi-\theta_\ell} \sim (1-\phi-\theta_\ell)\overline{q\iota}_R Z_\infty(1-\bar{\beta})^{-1/(1-\sigma)}L_0 e^{nt}/n$. Substitution back into the differential equation gives $\widehat{Q}_t \to n/(1-\phi-\theta_\ell)$. Output levels therefore satisfy $Y_t \sim Z_\infty(1-\bar{\beta})^{-1/(1-\sigma)}Q_t^{\theta_\ell}L_t$. Integrating the linear capital equation gives $K_t/Y_t \to \bar{\iota}_K/(n+\theta_\ell n/(1-\phi-\theta_\ell)+\delta)$, so $\widehat{K}_t$ has a finite limit. Equation (H.8), together with $s_{Kt} \to 0$, then gives $\widehat{Y}_t - n \to \theta_\ell n/(1-\phi-\theta_\ell)$. With $\theta_\ell = 0.5$, $\phi = -2$, and $n = 0.01$ from Table 10, this growth rate of GDP per person is 0.2% per year. The 0.5% labor-productivity growth rate used in calibration is an initial rate, not its endogenous long-run value.

J. Calibration of the Dynamic Model

J.1 Recovering the Calibration from Its Targets

We calibrate the shape of comparative advantage separately from the effect of ideas on productivity. At each historical cutoff β_j, equation (H.4) implies a capital share. The fitted parameters $\mu, \mu_1, \bar{\mu}$ minimize

$$\sum_j \left[100\left(s_K(\beta_j;\mu,\mu_1,\bar{\mu},\bar{f}) - s_{Kj}\right)\right]^2.$$

The minimum is attained in the limit $\mu_1 \to \infty$ (Appendix J.2). For a given assumed fraction of tasks never automated, $\bar{f}$ is set consistently with the chosen shape parameters through equation (H.11). Asymptotic automation has $\bar{f} = 0$.

The initial idea-growth target is $g_{Q0} = \lambda g_{Y0}/(1-\phi)$, with $g_{Y0} = 3\%$, the sum of $g_{y0} = 2\%$ and $n = 1\%$, so $g_{Q0} = 1\%$. Output per unit of labor input in the private business sector grew 1.9% per year over 1950–2023, close to g_{y0}. The task-productivity targets are $\hat{\psi}_{\ell 0} = 0.5\%$ and $\hat{\psi}_{k0} - \hat{\psi}_{\ell 0} = 4.9\%$ (private business) or 10.7% (computers). They determine

$$\theta_\ell = \frac{\hat{\psi}_{\ell 0}}{g_{Q0}}, \qquad \theta_k = \frac{\hat{\psi}_{\ell 0} + (\hat{\psi}_{k0} - \hat{\psi}_{\ell 0})}{g_{Q0}}.$$

The target capital–output ratio is $K_0/Y_0 = \bar{\iota}_K/(g_{Y0}+\delta) = 2.5$, with $L_0 = 1$ and $y_0 = Y_0/L_0 = 100$ as an index. The saved paths divide GDP per person by its initial equilibrium level, so $y_{2020} = 1$ in the figures. The target share s_{K0} is the mean of the four benchmark capital shares, 0.357. The factor-share formulas imply

$$A_0 = \frac{y_0}{(1-s_{K0})^{\sigma/(1-\sigma)}}, \qquad B_0 = s_{K0}^{\sigma/(\sigma-1)}\frac{Y_0}{K_0}.$$

For both calibrations, we use the 2023 private business sector automation fraction as the initial cutoff target β_0 and label the initial simulation date 2020. Evaluate $\xi(\beta_0)$ at the

target cutoff. Dividing the two productivity equations in (H.2) eliminates Z_0. Thus

$$Q_0 = \left[\frac{B_0}{A_0\xi(\beta_0)}(1-\beta_0)^{-1/(1-\sigma)}\right]^{1/(\theta_k-\theta_\ell)}, \qquad Z_0 = \frac{B_0}{Q_0^{\theta_k}\xi(\beta_0)}, \tag{J.1}$$
$$\bar{q}\bar{\iota}_R^{\lambda} = g_{Q0}\frac{Q_0^{1-\phi}}{Y_0^{\lambda}}.$$

These equations recover the productivity levels and initial idea productivity from the targets.

If the fitted share function does not pass exactly through the initial target pair (β_0, s_{K0}), evaluating the competitive equilibrium from these stocks can yield a cutoff and output slightly different from the targets. The initialized growth rate of ideas then follows from the actual output in equation (H.6). It need not equal the calibration target g_{Q0} exactly. In both calibrations the solved initial equilibrium has $\beta_0 = 0.756$ against the target 0.753, $s_{K0} = 0.391$ against the target 0.357, $Y_0 = 100.06$, and an idea growth rate of 1.00%. Appendix K.1 evaluates Φ at the actual initial quantities.

The simulations use an exponentially decaying growth rate of other productivity. Both calibrations start from the private business value $\widehat{Z}_0 = -1.05\%$, the estimate of $\widehat{Z}_t + \hat{\psi}_{\ell t}$ in Table 6 less $\hat{\psi}_{\ell 0}$. They set $\zeta = -\log(0.9)/10 = 0.0105$ per year, so $\widehat{Z}_t$ falls to 90% of its value every ten years. Its level is

$$\widehat{Z}_t = \widehat{Z}_0 e^{-\zeta t}, \qquad Z_t = Z_0 \exp\left\{\frac{\widehat{Z}_0}{\zeta}(1-e^{-\zeta t})\right\}.$$

Hence Z_t tends to the positive level $Z_0 \exp\{\widehat{Z}_0/\zeta\}$ and $\widehat{Z}_t$ tends to zero. These are the properties used in the asymptotic proofs.

J.2 Estimating the Task Productivity Function

Deriving the BPT function. For $\mu_1 > 0$, the elasticity in equation (29) implies

$$\frac{dh(u)^{-\mu_1}}{d\log u} = -\mu\mu_1\left[h(u)^{-\mu_1} - (\bar{\mu})^{-\mu_1}\right].$$

Integrating gives $h(u)^{-\mu_1} = (\bar{\mu})^{-\mu_1} + Cu^{-\mu\mu_1}$ for a constant $C > 0$. Consequently $h(u)/u^{\mu} \to C^{-1/\mu_1}$ as $u \downarrow 0$. The normalization $h(u)/u^{\mu} \to 1$ sets $C = 1$, giving the function in equation (29). Setting $\mu_1 = 1$ gives the logistic function $h(u) = u^{\mu}/[1 + u^{\mu}/\bar{\mu}]$.

The limit $\mu_1 \to \infty$. Hold μ and $\bar{\mu}$ fixed. Factoring the smaller of u^μ and $\bar{\mu}$ out of equation (29) shows that $h(u)$ lies between $2^{-1/\mu_1}\min\{u^\mu, \bar{\mu}\}$ and $\min\{u^\mu, \bar{\mu}\}$, so $h(u)/\min\{u^\mu, \bar{\mu}\} \to 1$ uniformly. For $\bar{\mu} < 1$, let $u_* := (\bar{\mu})^{1/\mu}$ be the remaining task mass at the kink. With $\bar{f} = 0$, $\mu(1-\sigma) \neq 1$, and $u = 1 - \beta$, the integral in equation (H.1) is

$$J(\beta) = \begin{cases} \beta u_*^{-\mu(1-\sigma)}, & \beta \leq 1 - u_*, \\ (1-u_*)u_*^{-\mu(1-\sigma)} + \dfrac{u^{1-\mu(1-\sigma)} - u_*^{1-\mu(1-\sigma)}}{\mu(1-\sigma)-1}, & \beta > 1 - u_*. \end{cases} \tag{J.2}$$

Other BPT specifications. Replacing μ by μ_0/μ_1 in the elasticity in equation (29), with $\mu_0 > 0$, extends it to two other cases in BPT: Gompertz (the limit $\mu_1 \to 0$) and the confined exponential ($\mu_1 = -1$). Under the confined exponential, h reaches zero at a positive fraction of the hardest tasks, which are then never automated. Both fit worse than the limit $\mu_1 \to \infty$, with root-mean-square errors of 6.76 and 10.05 percentage points against 4.26 (Table J.1).

An exact aggregate CES benchmark. One task productivity function makes the aggregate production function exactly CES in effective capital and effective labor. With $\bar{f} = 0$ and $\mu(1-\sigma) < 1$, the function $h(u) = u^\mu/(1-u^{1-\mu(1-\sigma)})^{\mu/(1-\mu(1-\sigma))}$ gives the constant capital–labor elasticity $\sigma_{KL} = \sigma + 1/\mu$. Substituting it into equations (H.4) and (H.5) and eliminating the cutoff shows that the share odds are a constant power of z.

An upper bound on the capital share. Because f is nonincreasing, $f(i)^{\sigma-1} \leq f(\beta)^{\sigma-1}$ for every $i \leq \beta$, so $J(\beta) \leq \beta f(\beta)^{\sigma-1}$. Equation (H.4) then gives $s_K/(1-s_K) \leq \beta/(1-\beta)$, that is,

$$s_K \leq \beta, \tag{J.3}$$

with equality if and only if f is constant on $[0, \beta]$. The bound relies on labor productivity being the same on every task.

Estimates. Table J.1 reports the fit at the four benchmark dates, and Figure J.1 reports the lowest root-mean-square error (RMSE) of the capital share at each $\mu_1 > 0$. The RMSE falls as μ_1 rises, from 6.16 percentage points for the logistic to 4.26 in the limit. Every $\mu_1 \geq 6$ fits within 0.5 percentage points of the limit. A joint search that also admits $\mu_1 \leq 0$ selects the same limit. The estimate $1 - u_* = 0.278$ equals the 1950 cutoff, so the model's

Table J.1: Historical Fit of the Task Productivity Function (%)

	1950	1975	2000	2023	RMSE
Automation cutoff, β_t	27.8	44.8	62.7	75.3	
Capital share, s_{Kt}	34.7	33.1	32.8	42.0	
Fitted capital share:					
Limit, $\mu_1 \to \infty$	27.8	32.8	36.9	39.1	4.26
Logistic ($\mu_1 = 1$)	24.5	34.2	38.4	38.2	6.16
Gompertz ($\mu_1 = 0$)	23.1	32.9	39.2	39.4	6.76
Confined exponential ($\mu_1 = -1$)	25.8	39.2	49.6	39.7	10.05
Pure power	21.6	30.5	37.8	42.0	7.14
Exact CES ($\mu = 0.63$)	14.8	25.4	38.5	49.8	11.70

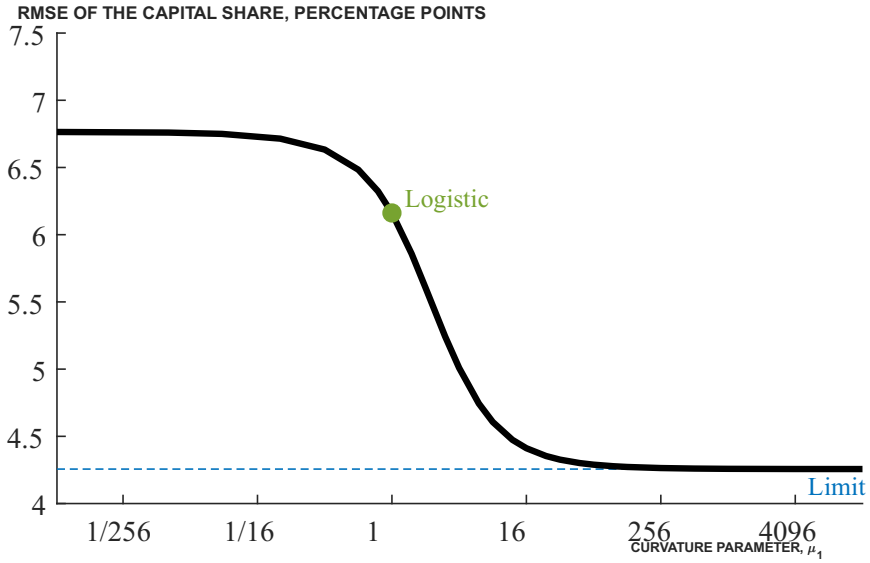
Note: The fitted rows report the capital share that each cutoff implies through equation (H.4), with $\bar{f} = 0$. The RMSE is in percentage points. The limit has $\mu = 3.00$ and $\bar{\mu} = 0.376$, and the logistic has $\mu = 3.48$ and $\bar{\mu} = 0.277$.

1950 share attains the bound in equation (J.3). The pure power function $h(u) = u^{\mu}$ has an RMSE of 7.14, and the exact CES benchmark has 11.70.

Robustness. The improvement over the logistic comes mainly from 1950. Its error falls from -10.2 to -6.9 percentage points. This fall accounts for 72 percent of the reduction in squared errors. The 2000 observation accounts for another 19 percent. Without the 1950 observation, the limit improves on the logistic by 0.19 percentage points of RMSE, against 1.91 with it. Fitted to 1950–2000 alone, the limit predicts a 2023 share of 32.7 percent and the logistic predicts 25.1 percent, against 42.0 observed. The ranking holds for the cutoffs from each LLM separately, where the RMSE falls from 6.17 to 4.40 percentage points with GPT-5.5 and from 6.15 to 4.13 with Claude Opus 4.7. It also holds for the shifted functions with $\bar{f} = \mp h(0.03)$, which leave μ at 3.00.

Simulations under the two fits. Table J.2 compares the simulations under the logistic and the limit, with every other calibration target unchanged. The limit raises the limiting capital share $s_K^* = 1/[\mu(1-\sigma)]$ from 0.36 to 0.42. Through equation (30), this raises Φ from 1.17 to 1.44 for private business and from 2.27 to 2.84 for Moore's Law Everywhere, so growth accelerates sooner and each explosion arrives earlier.

Figure J.1: Historical Fit and the Curvature Parameter μ_1



Note: At each positive μ_1, the line reports the lowest root-mean-square error of the capital share at the four benchmark dates over $\mu > 0$ and $\bar{\mu} > 0$, with $\bar{f} = 0$. The dashed line is the limit $\mu_1 \to \infty$. The dot is the logistic fit, $\mu_1 = 1$.

Table J.2: Simulations under the Two Fits of the Task Productivity Function

	Logistic	Limit
Historical RMSE of s_{Kt} (percentage points)	6.16	4.26
Limiting capital share, s_K^*	0.36	0.42
Φ_{pbs}, Φ_{mle}	1.17, 2.27	1.44, 2.84
Automation boost at which $\Phi_{pbs} = 1$	3.95%	3.00%
Private business:		
Growth in 2050, 2075, 2100	2.5, 3.2, 4.1%	3.2, 4.8, 8.3%
GDP per person over initial trend, 2050	1.07	1.15
Singular date: asymptotic, full automation	2241, 2120	2146, 2099
Moore's Law Everywhere:		
Growth in 2030, 2040	6.6, 11.1%	7.6, 16.2%
Singular date: asymptotic, full automation	2058, 2051	2051, 2046

Note: Both columns use the calibration targets in Table 10; only the task productivity function differs. The singular date is the date at which GDP per person becomes infinite.

K. Intuition for Slowly Accelerating Growth

K.1 Connecting Φ to Initial Moments

The parameter Φ can be expressed in terms of the model's actual initial growth rates without imposing stationarity of idea growth. Because $f(i)$ is fixed, $\hat{\psi}_{k0} = \theta_k \widehat{Q}_0$ and $\hat{\psi}_{\ell 0} = \theta_\ell \widehat{Q}_0$, where $\widehat{Q}_0$ is the growth rate obtained by evaluating the idea equation at the initialized equilibrium. Substituting these into the definition of Φ, and replacing $\hat{\psi}_{k0}$ using equation (10) together with $\bar{\omega}_{k\beta 0}\dot{\beta}_0 = [(1 - s_{K0})/s_{K0}]x_0$ from equation (D.3), gives the identity

$$\Phi = \frac{\lambda}{(1-\phi)\widehat{Q}_0}\left[\hat{\psi}_{\ell 0} + \frac{s_K^*}{1-s_K^*}(\widehat{B}_0 - \widehat{Z}_0) + \frac{1}{1-\sigma}\frac{s_K^*}{1-s_K^*}\frac{1-s_{K0}}{s_{K0}}x_0\right]. \tag{K.1}$$

Every growth rate and the initial capital share in this equation refer to the same initialized equilibrium, which need not coincide with the calibration targets.

The approximation for Φ in Section 6 makes three further substitutions. It replaces $\widehat{Q}_0$ by the target $g_{Q0} = \lambda g_{Y0}/(1-\phi)$ of Appendix J.1, approximates s_{K0} by s_K^*, and sets $\widehat{B}_0 - \widehat{Z}_0 \approx 0$. Equation (K.1) then becomes

$$\Phi \approx \frac{\hat{\psi}_{\ell 0} + x_0/(1-\sigma)}{g_{Y0}}.$$

Comparisons across values of x_0 or σ in this formula hold the other measured moments fixed, so they differ from a recalibration.

At the model's initial moments, with the initial automation rate from equation (H.7), the exact identity in equation (K.1) gives the calibrated values $\Phi_{pbs} = 1.444$ and $\Phi_{mle} = 2.838$. The approximate formula evaluated at the same moments gives 1.074 and 1.848, so the omitted terms contribute 0.370 and 0.990. The term in $\widehat{B}_0 - \widehat{Z}_0$ accounts for 0.27 and 0.80 of these, and replacing $s_{K0} = 0.391$ by $s_K^* = 0.417$ accounts for 0.10 and 0.19.

K.2 The One-Dimensional Explosion Calculation

The scalar calculation underlying equation (32) integrates directly. If $X_0 > 0$, $\bar{g} > 0$, and $\Phi > 1$, multiply $\dot{X}_t = \bar{g} X_t^\Phi$ by $X_t^{-\Phi}$ and integrate to obtain

$$\frac{X_t^{1-\Phi} - X_0^{1-\Phi}}{1-\Phi} = \bar{g}t.$$

Setting $X_0 = 1$ and letting $X_t \to \infty$, so that $X_t^{1-\Phi} \to 0$, gives equation (32). The scalar dates are illustrations, not explosion dates derived from the exact task model in equation (H.6).

L. Instructions for the LLMs to Measure β_t^j and $\tilde{\omega}_{\ell t}^j$

We have a three-step process for measuring the task-level inputs the growth accounting requires. First, we ask the LLM to construct a detailed list of tasks that are essential over the relevant time frame. The stored task lists contain 200 categories for private business and 100 for each other sector. Second, using that task list, we ask the LLM to estimate the fraction of each task category j that is automated in each benchmark year. This gives β_t^j, which aggregates to β_t. Third, we ask the LLM to estimate each task's share of total sector labor costs in each year. This gives $\tilde{\omega}_{\ell t}^j$. We repeat the second and third steps across several LLMs — GPT-5.5, Claude Opus 4.7, and others — as reported in Table L.1, and pool the GPT-5.5 and Claude Opus 4.7 results for the benchmark. Appendix D.3 derives the category automation rates and the averaging across draws from these inputs. The prompts below use motor vehicles as the running example. For other sectors we substitute sector-specific wording, such as "fraction of farms" for agriculture.

Here is the instruction we gave to the model to create the list of tasks:

```
**INSTRUCTIONS FOR CREATING TASK LIST**
Consider the motor vehicles sector of the U.S. economy for the
   past 75 years since 1950. I am writing an economics research
   paper at the PhD level on automation. Please do the following:
1. Construct a detailed list of 100 specific tasks that are
   essential and important in the motor vehicles industry in the
   United States over that time frame.

**CLARIFICATIONS**
(a) All tasks must in principle be able to be performed by people
   . Over time, some tasks may have been automated, which means
   they are performed without human involvement. But it is
   crucial that the tasks be things that could be performed by
   labor historically. Check carefully to ensure this is the case
   .
(b) On the other hand, do not neglect tasks that are fully
```

```
   mechanized now. For example, "Exterior painting" is surely an
   important task historically that is now fully automated.
   Prioritize tasks that are both required and essential and
   economically significant in some way, at least historically.
(c) All tasks should be things that were accomplished in motor
   vehicle production in the year 1950 as well as today. Make
   sure that all tasks were present more than 75 years ago.
(d) Please only use excellent, reputable resources that are
   specific to motor vehicles and empirical in nature. Do not use
    models or theory papers from macroeconomics or growth
   economics.
(e) Examples of tasks might include "Engine assembly" or "Tire
   attachment" or "Windshield installation." Also, "Management of
    the factory" is one possible high-level task that may have
   subtasks; we certainly want to consider management as one of
   the important categories of tasks.

**DELIVERABLES**
DELIVERABLE 1: Provide a short narrative summary of the results.

DELIVERABLE 2: Provide an Excel file containing the detailed
   results.
- The first sheet should be called "Overview". It should contain
   the date, the prompt, and the narrative summary.
- The second sheet should be called "Task Data". Report your task
    results in the form of a table with the tasks as rows. Some
   entries that explain the "category" of the task in each row
   would be helpful, with one column for the high-level category
   and another column for the detailed task description.
- The third sheet should be called "Sources". Document all
   sources used in a standard academic reference style. Include
   hyperlinks.
```

Here is the instruction we gave to the model to measure β_t:

```
**INSTRUCTIONS FOR AUTOMATION RATES**
Consider the motor vehicles sector of the U.S. economy for the
```

past 75 years since 1950. I am writing an economics research paper at the PhD level on automation. Below is a list of 100 tasks that are essential in this sector over this time frame. Please do the following:

1. Consider each task in the list below.
2. Consider the year 1957. Indicate whether the task was automated or not in 1957; for partially automated, use a fraction such as 10% or 35% or 85%. Take into account the fraction of plants that have automated the task in constructing your estimate.
3. Repeat Step 2 for 2017, the final year of our exercise, for that task.
4. Repeat Step 2 for the intermediate years 1967, 1977, 1987, 1997, and 2007 for that task.
5. Repeat Steps 1 to 4 for each task in the list.

CLARIFICATIONS

(a) Please only use excellent, reputable resources that are specific to motor vehicles and empirical in nature. Do not use models or theory papers from macroeconomics or growth economics.

TASK LIST

[The 100-task list from the spreadsheet is inlined here, one task per line in the format:
<id>. [<Category>] <Task Name> -- <Detailed description>]

OUTPUT FORMAT

Report the automation percentage (0 to 100) for each task and each year.

Return your results as a JSON object with the following structure:

{"tasks": [{"task_id": 1, "automation": {"1957": 5.0, "2017": 75.0, ...}}, ...]}

Each task must have a "task_id" (integer) and an "automation" object with keys for each year: 1957, 1967, 1977, 1987, 1997, 2007, 2017. Values are percentages from 0 to 100. Return ONLY

```
   the JSON object.
```

Finally, we ask the LLM to estimate each task's share of total sector labor costs. We split this into two stages to reduce the combinatorial burden on the model. Stage 1 asks for the *category-level* share of labor costs across a handful of broad task categories. Stage 2 asks for the *within-category* share of each task. We then multiply the normalized category share by the normalized within-category share to obtain $\tilde{\omega}_{\ell t}^{j}$ for every task. We repeat this pair of prompts for each benchmark year. Here is the Stage 1 prompt:

```
**INSTRUCTIONS FOR LABOR COST SHARES BY CATEGORY**
Consider the motor vehicles sector of the U.S. economy for the
   past 75 years since 1950. I am writing an economics research
   paper at the PhD level on automation.

I need to estimate what percentage of total labor costs (wages,
   salaries, and benefits paid to workers) in motor vehicles is
   accounted for by each category of tasks in the year 1957.

Here are the categories of tasks, with the number of individual
   tasks in each:
1. Stamping & Body Fabrication (10 tasks)
2. Paint & Sealing (10 tasks)
3. General Assembly (20 tasks)
...
11. Management/IE/H&S (4 tasks)

For each category, estimate the percentage of total labor costs
   in motor vehicles devoted to that category in 1957. The
   percentages should ideally sum to 100, but don't worry about
   exact precision -- just give your best estimate for each
   category independently.

Important: estimate labor costs (wages/salaries/benefits), not
   output value or capital costs.

**CLARIFICATIONS**
```

```
(a) Please only use excellent, reputable resources that are
    specific to motor vehicles and empirical in nature. Do not use
     models or theory papers from macroeconomics or growth
    economics.
(b) Include assembly line workers, quality inspectors,
    maintenance, office staff, and engineering.
(c) Estimate labor costs (wages, salaries, benefits paid to
    workers), not output value or capital costs.

**OUTPUT FORMAT**
Return a JSON object using the numeric category IDs shown above:
{"categories": [{"cat_id": 1, "cost_pct": 15.0}, {"cat_id": 2, "
    cost_pct": 10.0}, ...]}
Return ONLY the JSON object.
```

Here is the Stage 2 prompt, repeated once per category (shown here for "General Assembly"):

```
**INSTRUCTIONS FOR LABOR COST SHARES WITHIN A CATEGORY**
Consider the motor vehicles sector of the U.S. economy for the
    past 75 years since 1950. I am writing an economics research
    paper at the PhD level on automation.

Consider the category "General Assembly" within motor vehicles in
     the year 1957.

Below are the 20 tasks in this category. For each task, estimate
    what percentage of this category's labor costs is devoted to
    that task. The percentages should ideally sum to 100, but don'
    t worry about exact precision -- just give your best estimate
    independently.

Important: estimate labor costs (wages/salaries/benefits), not
    output value or capital costs.

**CLARIFICATIONS**
(a) Please only use excellent, reputable resources that are
```

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   specific to motor vehicles and empirical in nature. Do not use
    models or theory papers from macroeconomics or growth
   economics.
(b) Include assembly line workers, quality inspectors,
   maintenance, office staff, and engineering.
(c) Estimate labor costs (wages, salaries, benefits paid to
   workers), not output value or capital costs.

**TASK LIST**
[The tasks belonging to "General Assembly" are inlined here, one
   per line.]

**OUTPUT FORMAT**
Return a JSON object:
{"tasks": [{"task_id": 21, "cost_pct": 8.5}, {"task_id": 22, "
   cost_pct": 12.0}, ...]}
Each task must have a "task_id" (integer matching the IDs in the
   list) and "cost_pct" (a non-negative number). Return ONLY the
   JSON object.
```

Table L.1: Automation Rates by Sector and Model — All Models (%)

Sector	GPT-5.5	GPT-5.4 Thinking	GPT-5.4 No thinking	Claude Opus 4.7	Claude Opus 4.6	Claude Sonnet 4.6	Gemini 3 Flash
Private business	1.84 (0.17)	2.24 (0.20)	2.07 (0.10)	2.15 (0.07)	1.37 (0.05)	1.24 (0.08)	4.01 (0.37)
Agriculture	2.30 (0.29)	2.50 (0.30)	2.40 (0.20)	3.18 (0.26)	2.49 (0.07)	1.62 (0.10)	5.13 (1.14)
Computers	4.71 (0.78)	4.79 (0.93)	6.12 (0.68)	5.63 (0.32)	4.78 (0.35)	3.79 (0.31)	10.67 (2.76)
Motor vehicles	3.02 (0.49)	4.25 (0.40)	4.57 (0.26)	4.39 (0.15)	3.46 (0.08)	2.90 (0.27)	9.13 (1.73)
Retail trade	4.16 (0.71)	5.40 (1.16)	6.58 (0.79)	4.75 (0.35)	2.81 (0.20)	2.50 (0.29)	9.39 (1.71)
Software	3.41 (0.44)	3.15 (0.49)	2.99 (0.38)	3.97 (0.31)	1.76 (0.18)	1.96 (0.17)	7.17 (1.19)

Note: Each cell shows the mean automation rate and sample standard deviation (in parentheses). Gemini 3 Flash has 9 private-business, 5 agriculture, 25 computer, 24 motor-vehicle, 25 retail, and 20 software draws. Every other model-sector cell has 25 draws. There is typically little variation within model across draws and larger variation across models. The "Average" column in the body of the paper combines the GPT-5.5 and Claude Opus 4.7 columns.

References

Benhabib, Jess, Jesse Perla, and Christopher Tonetti, "Catch-up and Fall-back through Innovation and Imitation," *Journal of Economic Growth*, 2014, *19* (1), 1–35.