A Note on the Closed-Form Solution of the Solow Model

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This brief note presents the closed-form solution of the Solow (1956) model when the production function is Cobb-Douglas. The solution seems to have been first published by Ryuzo Sato (1963) in an analysis of neoclassical convergence to the steady state.[1] Some people seem to know this trick but others do not, so I thought it would be worth posting on my web page.

Consider a special case of the standard Solow (1956) model in which the production function is Cobb-Douglas:

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}, \quad 0 < \alpha < 1 \tag{1}$$

$$\dot{K}_t = sY_t - \delta K_t, \quad K_0 > 0, \tag{2}$$

$$L_t = L_0 e^{nt}, \quad L_0 > 0, \tag{3}$$

$$A_t = A_0 e^{gt}, \quad A_0 > 0, \tag{4}$$

where s, n, g, and δ are nonnegative parameters with $0 < s < 1$. The notation is standard: Y denotes output, K denotes capital, A is an exogenous index of technology that grows exponentially, and L is the labor force, which is also allowed to grow exponentially. Time is continuous and denoted by t, and a dot above a variable denotes its time derivative.

As is customary, we normalize variables by dividing by $A_t L_t$. We denote normalized variables with a tilde: $\tilde{y}_t = Y_t / A_t L_t$ and $\tilde{k}_t = K_t / A_t L_t$. With this normalization, the key equations of the Solow model are

$$\tilde{y}_t = \tilde{k}_t^\alpha \tag{5}$$

and

$$\dot{\tilde{k}}_t = s\tilde{k}_t^\alpha - (n + g + \delta)\tilde{k}_t \tag{6}$$

with the initial condition given by $\tilde{k}_0 = K_0 / A_0 L_0$.

[1] Kazuo Mino of Kobe University provided me with the reference to Sato's paper. I originally learned of the basic solution from Ove Granstrand of Chalmers University in Sweden. The basic idea can also be found in the appendix to Chapter 1 of Barro and Sala-i-Martin (1995) on page 53.

It turns out that equation (6) can be solved explicitly in closed form. An equation of this form is called a Bernoulli equation, and the solution is found by making the substitution $z_t = \tilde{k}_t^{1-\alpha}$, yielding

$$\dot{z}_t = (1-\alpha)s - \lambda z_t, \tag{7}$$

where $\lambda \equiv (1-\alpha)(n+g+\delta)$. This equation has a nice interpretation since z_t is easily seen to equal the capital-output ratio: the capital-output ratio in the Solow model with Cobb-Douglas production obeys a linear differential equation.[2]

Equation (7) has a standard form which is easily solved. After some substitutions, we see that the closed-form solution of the model is

$$\tilde{k}_t = \left(\frac{s}{n+g+\delta}(1-e^{-\lambda t}) + \tilde{k}_0^{1-\alpha} e^{-\lambda t} \right)^{\frac{1}{1-\alpha}}. \tag{8}$$

Therefore, the capital-output ratio $z_t = \tilde{k}_t^{1-\alpha}$ is a weighted average of its initial value and its steady-state value, where the weights are an exponential function of time. Letting $y_t = Y_t/L_t$, the solution for output per worker at any point in time is

$$y_t = \left(\frac{s}{n+g+\delta}(1-e^{-\lambda t}) + \left(\frac{y_0}{A_0}\right)^{\frac{1-\alpha}{\alpha}} e^{-\lambda t} \right)^{\frac{\alpha}{1-\alpha}} A_t. \tag{9}$$

The parameter λ is the rate at which the economy converges to its balanced growth path, familiar from, e.g., Mankiw, Romer and Weil (1992).

References

Barro, Robert J. and Xavier Sala-i-Martin, *Economic Growth*, McGraw-Hill, 1995.

Mankiw, N. Gregory, David Romer, and David Weil, "A Contribution to the Empirics of Economic Growth," *Quarterly Journal of Economics*, May 1992, *107* (2), 407–438.

Sato, Ryuzo, "Fiscal Policy in a Neo-Classical Growth Model: An Analysis of Time Required for Equilibrating Adjustment," *Review of Economic Studies*, February 1963, *30* (1), 16–23.

Solow, Robert M., "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, February 1956, *70*, 65–94.

[2] Thanks to Nicholas Rau for this interpretation.