MATH 116 FINAL

Tuesday December 12, 2017 (2.5 hours).

Name: ____

Please turn cell phones off completely and put them away. No books, notes, or electronic devices are permitted during this exam.

Complete arguments and/or proofs must be written to receive full credits.

A list of useful formulas and results are given on the last page of the papers.

You may cite without proof any result proved in class, the textbook and the handouts, as well as results in the problem sets, **unless you are warned otherwise** (which will happen if the question is about such a result itself).

This set is for practice. In the actual exam you will be given 9 to 11 problems of a similar flavor (broadly interpreted).

No more than 2 problems in the actual exam will be based solely on the material in the first five weeks. All other problems will be either based on the material in the latter five weeks, or in a combined manner.

(Do not write below this line.)

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Total	out of ???

• **Problem 1** Prove that the function $f(z) = |z|^4$ is not holomorphic.

Solution. There are many methods. One can note $f(z) = z^2\bar{z}^2$ and conclude that $\frac{\partial f}{\partial \bar{z}} = 2z^2\bar{z} \neq 0$, or one can use the open mapping theorem that if $f(z)$ was holomorphic it needs to be constant or its image has to be open - evidently neither is the case.

• **Problem 2** Explain why $\sin z := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$ defines an entire function.

Solution. This is mostly a recall about convergence radius, for which we need to compute $\frac{1}{R} = \limsup_{n\to\infty} \sqrt[2n+1]{|\frac{(-1)^n}{(2n+1)!}|}$. We have $(2n+1)! \geq (n+1)^{n+1} \implies \sqrt[2n+1]{\frac{1}{(2n+1)!}} \leq (n+1)^{\frac{-(n+1)}{2n+1}} \leq \frac{1}{\sqrt{n+1}}$. Thus the above limit is 0, and $R = +\infty$.

• **Problem 3** Compute the following integrals (and provide a complete argument for each):

(a) $\int_0^{2\pi} \frac{1}{\sin x + 2} dx$.

Solution. Substituting $z = e^{ix}$, one has $dx = \frac{dz}{iz}$ and $\sin x = (z - z^{-1})/2i$. This gives

$$\int_0^{2\pi} \frac{1}{\sin x + 2} dx = \oint_{|z|=1} \frac{2}{z^2 + 4i - 1} dz.$$

The roots of $z^2 + 4i - 1$ are $(-2 \pm \sqrt{3})i$. The root $(-2 - \sqrt{3})i$ is in the exterior of $|z| = 1$, and the root $(-2 + \sqrt{3})i$ is in the interior. By the residue formula, we have

$$\oint_{|z|=1} \frac{2}{z^2 + 4i - 1} dz = 2\pi i(\mathrm{Res}_0 f + \mathrm{Res}_{-2+\sqrt{3}} f)$$

$$= 2\pi i \left(\lim_{z\to(-2+\sqrt{3})i} \frac{2(z - (-2 + \sqrt{3}i))}{(z - (-2 - \sqrt{3}i))(z - (-2 + \sqrt{3}i))} \right) = \frac{4\pi i}{2\sqrt{3}i} = \frac{2\pi}{\sqrt{3}}.$$

This gives the answer.

(b) $\int_{-\infty}^{\infty} \frac{16x^{15} + 1000x}{x^{16} + 500x^2 + 1} dx$.

Solution. This problem in fact has a bug. The integral $\int_0^\infty \frac{16x^{15}+1000x}{x^{16}+500x^2+1} dx$ does not converge, neither does the negative part. Note however that $\lim_{R\to+\infty} \int_{-R}^{R} \frac{16x^{15}+1000x}{x^{16}+500x^2+1} dx = 0$, as the integrand is odd, so that the integral is zero even without the limit.

The original intention of the problem (which is, regrettably, designed badly) is that this function is of the form $\frac{f'}{f}$, so that if γ_1 is the segment from $-R$ to R, and $\gamma_2(t) = Re^{it}$ for $t \in [0, \pi]$. Then by argument principle, when R is large enough, $\int_{\gamma_1} \frac{f'}{f} dz + \int_{\gamma_2} \frac{f'}{f} dz$ is equal to $2\pi i$ times the number of zeroes of f within the upper half plane, which is 8 since the function $f(z)$ has no zero on the real line; $f(x) = x^{16} + 500x^2 + 1 > 0$, and the zeroes are symmetric about the real line. That is $\int_{\gamma_1} \frac{f'}{f} dz + \int_{\gamma_2} \frac{f'}{f} dz = 16\pi i$. Then one argues that $\lim_{R\to+\infty} \int_{\gamma_2} \frac{f'}{f} dz \to \int_{\gamma_2} \frac{16}{z} dz = 16\pi i$, and thus the original integral is zero.

(c) $\displaystyle\oint_{|z|=1} e^{\frac{1}{z}} dz.$

(Hint: Make a change of variable.)

Solution. Writing $w = \frac{1}{z}$ and note $\frac{dw}{w} = -\frac{dz}{z}$; $dz = -w^{-2} dw$, we have $\displaystyle\oint_{|z|=1} e^{\frac{1}{z}} dz = -\oint_{|w|=1} -w^{-2} e^{w} dw$. Here we note the important minus sign when we change from $\displaystyle\oint_{|z|=1}$ to $\displaystyle\oint_{|w|=1}$; when z goes counter-clockwise, w goes clockwise. Now by residue formula we note thst $w^{-2}e^{w} = w^{-2} + w^{-1} + 2 + 6w + \ldots$ and thus $\displaystyle -\oint_{|w|=1} -w^{-2} e^{w} dw = 2\pi i \cdot 1 = 2\pi i.$

- **Problem 4** Let $U \subset \mathbb{C}$ be open and let $f : U \to \mathbb{C}$ be a holomorphic function. Write $u = \mathrm{Re}(f)$. Prove that u is a harmonic function, i.e. $u_{xx} + u_{yy} = 0$.

  (Warning: We do not assume u_{xx} exist, etc.)

  Solution. Firstly, f is smooth (infinitely differentiable) because any holomorphic function is. In particular, all second partial derivatives of u and $v = \mathrm{Im}(f)$ exists and are continuous. Now the Cauchy-Riemann equation gives $u_{xx} = (u_x)_x = (v_y)_x = (v_x)_y = (-u_y)_y = -u_{yy}$ (the equality $(v_y)_x = (v_x)_y$ holds whenever they are continuous). Hence the result.

- **Problem 5** Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that $f(x + 2\pi) = f(x)$ for any $x \in \mathbb{R}$. Prove that $f(z + 2\pi) = f(z)$ for all $z \in \mathbb{C}$.

  Solution. The function $f(z + 2\pi i) - f(z)$, as an entire function in z, has the whole real line being its zero. Since the real line is obviously not discrete, we have $f(z + 2\pi i) - f(z) \equiv 0$, hence the result.

- **Problem 6** Find all entire functions whose image is not dense in $\mathbb{C}$.

Solution. We claim that any entire function whose image is not dense is a constant. Suppose f is such a function. By definition there exists $w \in \mathbb{C}$ and $r > 0$ such that the image of f is not "dense around w", i.e. $|f(z) - w| \geq r$ for all $z \in \mathbb{C}$. But then $\frac{1}{f(z)-w}$ is a bounded (by $\frac{1}{r}$) entire function, and thus constant by Lioville's theorem. This implies that f is constant.

- **Problem 7** Give a definition of a holomorphic function on $\tilde{\mathbb{C}} - \{1\} := \mathbb{C} \cup \{\infty\} - \{1\}$. Explain why such a holomorphic function must have a singularity (pole, or essential singularity) at 1 unless it's a constant.

(Note: You have essentially proved the second statement in the Problem Set. You are allowed to cite any result other than that very exercise itself.)

Solution. A function $f : \tilde{\mathbb{C}} - \{1\} \to \mathbb{C}$ is holomorphic if $g(z) = f(z+1)$ is holomorphic on $\tilde{\mathbb{C}} - \{0\}$, i.e. if $h(z) = g(\frac{1}{z}) = f(\frac{1}{z} + 1)$ is holomorphic on $\mathbb{C}$. If f has no singularity at 1, then f is a holomorphic function on $\tilde{\mathbb{C}}$, where we have proved in Q6 in PS5 that such an f must be a constant.

- **Problem 8** Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic. Show that if f is injective, then f does not have any zero of order 2 or higher.

(Hint: Use Rouché's Theorem.)

Solution. If f has a zero of order 2 or higher, say at z_0, then $f(z_0) = f'(z_0) = 0$. Since zeroes are discrete, we may assume no zeroes of f nor f' lie in $D_{2r}(z_0)$ for some $r > 0$. Let $m = \min_{|z-z_0|=r} |f(z)| > 0$. Choose any $c \in \mathbb{C}$ with $0 < |c| < r$, and apply Rouché's Theorem with $g(z) = c$ a constant function. We have $\operatorname{ord}_{|z-z_0|<r}(f+g) = \operatorname{ord}_{|z-z_0|<r} f = \operatorname{ord}_{z_0} f \geq 2$.

Thus $f(z) + c = 0$ has at least two zeroes within $|z - z_0| < r$, and in fact within $0 < |z - z_0| < r$ as $f(z_0) = 0$. Such zeroes must have order 1 since $f'(z) \neq 0$ at the zero by the hypothesis on r. Thus $f(z) = -c$ has two solutions, contradicting that f is injective.

- **Problem 9** Let $U \subset \mathbb{C}$ be open connected and $f_1, ..., f_n, ...$ be a sequence of holomorphic functions on U that are injective. Suppose f_j converges uniformly on compact sets to a function f. Show that f is either injective or constant.

(Hint: Use Rouché's Theorem again.)

Solution. In fact the argument principle will do. Suppose on the contrary that f is neither injective nor constant. Say $f(z_1) = f(z_2) = c$. We may subtract all f_i and f by c, so that $f(z_1) = f(z_2) = 0$. Let $r > 0$ be small enough so that no zeroes of f lies in $D_{2r}(z_1) \cup D_{2r}(z_2)$. We have by uniform continuity that $\int_{|z-z_1|=r} \frac{f_n'}{f_n} dz \to \int_{|z-z_1|=r} \frac{f'}{f} dz$ as $n \to +\infty$. Since these are integers (number of zeroes) multiplied by $2\pi i$, we see that for n large enough the number of zeroes of f_n within $|z - z_1| < r$ must be one. But then the same holds for the disk $|z - z_2| < r$, which makes f_n not injective, a contradiction.

- **Problem 10** Let $f : \overline{D_1(0)} := \{z \in \mathbb{C} \mid |z| \leq 1\} \to \mathbb{C}$ be continuous and holomorphic on the open unit disk $D_1(0)$. Show that the maximum of $\mathrm{Re}(f(z))$ must occur on the boundary of the unit disk $\partial D_1(0)$.

Solution. We know that the maximum of $|e^{f(z)}|$ must occur on the boundary by the maximum principle on $e^{f(z)}$. But $|e^{f(z)}| = e^{\mathrm{Re}(f(z))}$, and e^x is strictly increasing. Hence if the maximum of $|e^{f(z)}|$ occurs at $z_0 \in D_1(0)$, then the maximum of $\mathrm{Re}(f(z))$ also occurs at the same z_0.

Alternatively, one can also run the proof of the maximum principle again; if the maximum does not occur at the boundary, then f cannot be an open map.

- **Problem 11** Let $U \subset \mathbb{C}$ be open, $z_0 \in U$ and $f : U - \{z_0\} \to \mathbb{C}$ be a holomorphic function with an essential singularity at z_0. Show that f is not injective.

Solution. This is similar to Q4 in Pset 6. Choose any $z \in U - \{z_0\}$. Since f has an essentially singularity at z_0, for any $r > 0$ the image of $D_r(z_0)^*$ under f is dense in $\mathbb{C}$. In particular let $2r < |z - z_0|$. By the open mapping theorem, the image of $D_r(z)$ is an open set that contains $f(z)$. A dense subset of $\mathbb{C}$ must intersect an non-empty open set non-trivially, and thus the image $f(D_r(z_0)) \cap f(D_r(z)) \neq \emptyset$. But as $D_r(z_0) \cap D_r(z) = \emptyset$, this is only possible if f is not injective.

- **Problem 12** Explain why a holomorphic function $f : \mathbb{C} - \{0\} \to \mathbb{C}$ can always be written as a $f(z) = f_1(z) + f_2(1/z)$, where both f_1 and f_2 are entire. Then, prove that if f is bounded, then f must be a constant.

Solution. We have worked out in class that f always has a convergent Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$. It can then be written as $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=-\infty}^{-1} a_n z^n$. The convergence conditions says that the first defines a holomorphic function f_1 with infinite convergence radius, and such entire. For the second part, it says $f_2(z) = \sum_{n=1}^{\infty} a_{-n} z^n$ defines a holomorphic function on with infinite convergence radius, and thus entire. Hence $f(z) = f_1(z) + f_2(1/z)$.

For the second part, it suffices to show that if f is bounded, then both f_1 and f_2 are bounded. Note that both $f_1(z)$ and $f_2(z)$ are evidently bounded when $|z| \leq 1$ by continuity, which for f_2 implies that $f_2(1/z)$ is bounded for $|z| \geq 1$. As f is bounded, this implies f_1 is also bounded for $|z| \geq 1$, and thus bounded. It follows $f_2(z) = f(1/z) - f_1(1/z)$ is also bounded.

• **Problem 13** Fix $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$. Let $\Lambda = \{n_1 + n_2\tau | \; n_1, n_2 \in \mathbb{Z}\}$. Write down one entire function with a zero of exactly order 2 at each $\lambda \in \Lambda$.

(Hint: Don't think too much about $\wp(z)$ - it doesn't give an entire function!)

Solution. The infinite product $z^2 \cdot \prod_{\lambda \in \Lambda^*} \left((1 - \frac{z}{\lambda})e^{\frac{z}{\lambda} + \frac{1}{2}(\frac{z}{\lambda})^2}\right)^2$ will do.

• **Problem 14** Let $a_1, ..., a_n, ..., \in \mathbb{C}$ be such that $\sum_{n=1}^{\infty} |a_n|^2 < +\infty$. Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then so does $\prod_{n=1}^{\infty}(1 + a_n)$.

Solution. By dropping finitely many terms, it makes no harm to assume all $|a_n| < 1/2$. Let us define $\log(z)$ on the right half plane $\mathrm{Re}(z) > 1$ in the usual way, so that $\prod_{n=1}^{\infty}(1 + a_n) = \exp(\sum_{n=1}^{\infty} \log(1 + a_n))$. We have $\log(1 + a_n) = a_n + \sum_{k=2}^{\infty}(-1)^{k+1}(a_n)^k/k$, and $|\sum_{k=2}^{\infty}(-1)^{k+1}(a_n)^k/k| \leq |a_n|^2$ whenever $|a_n| < 1/2$. This implies that $\sum_{n=1}^{\infty}(\log(1 + a_n) - a_n)$ converges absolutely, and thus $\prod_{n=1}^{\infty}(1 + a_n) = \exp(\sum_{n=1}^{\infty} \log(1 + a_n)) = \exp(\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty}(\log(1 + a_n) - a_n))$ converges.

(You will be provided the same list of results in the actual exam.)

You may find the following formulas/definitions useful:

(1) Cauchy-Riemann equations: $\begin{cases} u_x = v_y \\ u_y = -v_x. \end{cases}$, or $\dfrac{\partial f}{\partial y} = i \cdot \dfrac{\partial f}{\partial x}$.

(2) $\dfrac{\partial f}{\partial z} := \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} - i\dfrac{\partial f}{\partial y}\right)$, $\dfrac{\partial f}{\partial \bar{z}} := \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y}\right)$.

(3) Convergence radius R of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is given by $\dfrac{1}{R} = \limsup_{n\to+\infty} \sqrt[n]{|a_n|}$.

(4) Integral of $f : U \to \mathbb{C}$ along a curve $\gamma : [a,b] \to \mathbb{C}$ is $\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt$

(5) A curve $\gamma : [a,b] \to \mathbb{C}$ is closed if $\gamma(a) = \gamma(b)$. It is simple if $\gamma(t_1) \neq \gamma(t_2)$ unless either $t_1 = t_2$, or $\{t_1, t_2\} = \{a, b\}$.

(6) Winding number for a closed curve γ around a point z_0 is given by $W_\gamma(z_0) := \dfrac{1}{2\pi i}\int_\gamma \dfrac{1}{z - z_0}dz$.

(7) Residue formula for a counter-clockwise simple closed curve γ with interior U_0, an open set U containing the image of γ and U_0, and a holomorphic function on $U - \{z_1, ..., z_m\}$ with a pole at each $z_j \in U$:

$$\int_\gamma f(z)dz = 2\pi i \cdot \sum_{j=1}^{m} \mathrm{Res}_{z_j} f.$$

Note the special case when $m = 0$.

(8) Residue formula for a closed curve γ, an open set U containing the image of γ with either $U = \mathbb{C}$ or $\mathbb{C} - U$ unbounded and connected, and a holomorphic function on $U - \{z_1, ..., z_m\}$ with a pole at each $z_j \in U$:

$$\int_\gamma f(z)dz = 2\pi i \cdot \sum_{j=1}^{m} (W_\gamma(z_j) \cdot \mathrm{Res}_{z_j} f).$$

Note the special case when $m = 0$.

(9) Cauchy integral formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}dz$$

for any (counter-clockwise) simple closed curve γ containing z_0 in its interior and function f that is holomorphic on an open set containing the curve as well as its interior.

(10) Property of an essential singularity: If $U \subset \mathbb{C}$ is open $f : U - \{z_0\} \to \mathbb{C}$ is holomorphic and f neither extends holomorphically to z_0 nor has a pole at z_0, then f has an essential singularity at z_0. In this case, $f(D_r(z_0)^*)$ is dense in $\mathbb{C}$ for any (arbitrarily small) $r > 0$ with $D_r(z_0) \subset U$.

(11) A sequence of functions $f_j : U \to \mathbb{C}$ is said to converge uniformly on compact (sub)sets if for any $K \subset \mathbb{C}$ compact, $f_j|_K$ converge uniformly. A sequence of holomorphic function that converge uniformly on compact set converges to a holomorphic function.

(12) Argument principle: Let $U \subset \mathbb{C}$ be open, $\gamma : [a, b] \to U$ be a simple closed curve with its interior $U_0 \subset U$, and $f : U \to \mathbb{C}$ be meromorphic with $f(\gamma(t)) \in \mathbb{C}^*$ for any $t \in [a, b]$. Then

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)} dz = \sum_{z_0 \in U_0} \operatorname{ord}_{z_0} f.$$

(13) Rouché's theorem: Let $U \subset \mathbb{C}$ be open, $\gamma : [a, b] \to U$ be a simple closed curve with its interior $U_0 \subset U$, and $f, g : U \to \mathbb{C}$ be meromorphic with $|f(\gamma(t))| > |g(\gamma(t))|$ for any $t \in [a, b]$ (and also requires f and g to not have poles at all $\gamma(t)$). Then

$$\sum_{z_0 \in U_0} \operatorname{ord}_{z_0}(f + g) = \sum_{z_0 \in U_0} \operatorname{ord}_{z_0} f$$

(14) Open mapping theorem: A non-constant holomorphic function defined on a open connected subset of $\mathbb{C}$ is an open map.

(15) Maximum principle: Let $U \subset \mathbb{C}$ be open and bounded, and $f : \bar{U} \to \mathbb{C}$ be a continuous function on its closure. Suppose f is holomorphic on U, then

$$\max_{z \in \bar{U}} |f(z)| = \max_{z \in \partial U} |f(z)|.$$

(16) Laurent series expansion: Let $R_1 > R_2$ and f be a holomorphic function on $D_{R_1}(0) - \overline{D_{R_2}(0)}$ (including the case when $R_1 = +\infty$, for which $D_{R_1}(0) = \mathbb{C}$, and when $R_2 = 0$, for which $\overline{D_{R_2}(0)} = \{0\}$). Then f has a Laurent series expansion as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

where $a_n := \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)}{z^{n+1}} dz$ for any $R_2 < r < R_1$.

(17) Weierstrass infinite product: Let $\{a_n\}$ be a sequence of non-zero complex numbers such that $a_n \to \infty$ as $n \to +\infty$. Then the infinite product

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{L_n(\frac{z}{a_n})}$$

converges to an entire function with zeroes exactly at the a_n's. Here $L_n(w) = w + \frac{w^2}{2} + ... + \frac{w^n}{n}$.