Harmonious Equations

Interpretations of some of the most important and beautiful equations in mathematics, in prose and song

sung by members of Zambra

with commentary by NPR's "Math Guy" Keith Devlin

Recorded at Bear Creek Studios, Santa Cruz California, Spring 2009

CELEBRATING THE BEAUTY OF MATHEMATICS

> "Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show."
>
> Bertrand Russell, English mathematician and philosopher
> in his book *Mysticism and Logic* (1918)

Mathematics, the so-called science of patterns, is a way of looking at the world, not only the physical, biological, and sociological world we inhabit, but also the inner world of our minds and thoughts. Mathematics' greatest success has undoubtedly been in the physical domain. Yet, as an entirely human creation, the study of mathematics is ultimately a study of humanity itself. For none of the entities that form the substrate of mathematics exist in the physical world. The numbers, the points, the lines and planes, the surfaces, the geometric figures, the functions, and so forth are pure abstractions that exist only in the mind.

As sentences express ideas and music expresses rhythmical patterns in the mind, so too equations describe patterns and relationships between the abstract entities of mathematics. The most beautiful equations are thus akin to the perfect sonnets that shake us to our soul and the harmonious melodies that, once encountered, continue to reverberate though our very being.

Mathematics and music: both are captured in dry symbols, but each comes alive when those symbols are interpreted by the human mind. As Russell observed so eloquently, both can exhibit their own beauty.

What better way to celebrate some of the most beautiful equations of all time than contrapuntally, through prose, song, music, and dance.

Keith Devlin

CREDITS

Equations discovered and verified by the mathematicians cited.

Choral interpretations composed and performed by members of Zambra

Narration written and presented by Keith Devlin

Show conceived, produced and directed by Keith Devlin

ABOUT ZAMBRA

Zambra is a Santa Cruz, CA based, global vocal group founded 1994. They have performed at the Cabrillo Music Festival, First Night Santa Cruz, First Night Monterey, concert series of The Monterey Museum of Art and Santa Cruz Public Libraries, Santa Cruz Accapellafest, Center for Spiritual Enlightenment, Voices of the People, Cabrillo College Women's Conference, and numerous other events. Their seasonal recording, *Iluminada: A Festival of Light*, features holiday selections from medieval, Sephardic, Celtic and other traditions. Website: www.zambra.org

The name Zambra has several meanings: an Andalusion Gypsy dance, a Moorish festival, a town in Italy, a Spanish celebration, an ancient Hebrew song, and of course, a celebration of women's voices! While Zambra sings in over 15 languages, "Harmonious Equations" is their first foray into the language of mathematics!

The members of Zambra singing in this performance are Janet Herman, Therese Johannesson, Susan Krivin, Victoria Phillips-Larson, Laura Reeve, and Ann Louise Wagner.

ABOUT KEITH DEVLIN

Dr. Keith Devlin is a professional research mathematician and mathematics author based at Stanford University. He is a co-founder and Executive Director of the university's H-STAR institute, a Consulting Professor in the Department of Mathematics, a co-founder of the Stanford Media X research network, and a Senior Researcher at CSLI. He is a World Economic Forum Fellow and a Fellow of the American Association for the Advancement of Science. His current research is focused on the use of different media to teach and communicate mathematics to diverse audiences. He also works on the design of information/reasoning systems for intelligence analysis. Other research interests include: theory of information, models of reasoning, applications of mathematical techniques in the study of communication, and mathematical cognition. He has written 28 books and over 80 published research articles. He is the recipient of the Pythagoras Prize, the Peano Prize, the Carl Sagan Award, and the Joint Policy Board for Mathematics Communications Award. He is "the Math Guy" on National Public Radio. *Harmonious Equations* is his first attempt to produce an artistic performance.

KEITH DEVLIN'S PROSE DESCRIPTIONS OF THE EQUATIONS

1. Euler's identity

$$e^{i\pi} = -1$$

At the supreme level of abstraction where mathematical ideas may be found, seemingly different concepts sometimes turn out to have surprisingly intimate connections. There is, surely, no greater illustration of this than this equation discovered in 1748 by the great Swiss mathematician Leonhard Euler.

Euler's equation connects the four most significant and most ubiquitous constants in mathematics: *e*, the base of the natural logarithms, *i*, the square root of –1, π, the ratio of the circumference of a circle to its diameter, and 1, the beginning of all number.

The number 1, that most concrete of numbers, is the basis of counting, the foundation of all commerce, engineering, science, and music.

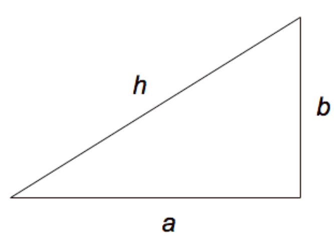
As 1 is to counting and arithmetic, π is to shape and geometry, the measure of that most perfectly symmetrical of shapes, the circle — though like an eager young debutante, π has a habit of showing up in the most unexpected of places.

As for *e*, to lift her veil you need to plunge into the depths of calculus — humankind's most successful attempt to grapple with the infinite.

And *i*, that most mysterious square root of –1, surely nothing in mathematics could seem further removed from the familiar world around us.

Four different numbers, with different origins, built on very different mental conceptions, invented to address very different issues. And yet all come together in one glorious, intricate equation, each playing with perfect pitch to blend and bind together to form a single whole that is far greater than any of the parts. A perfect mathematical composition.

2. Pythagoras' theorem



$$h^2 = a^2 + b^2$$

Pythagoras' theorem is one of the most famous equations in mathematics. It's also one of the oldest. Although named after the great mathematician Pythagoras of Samos, who lived around 500 B.C., this result was known long before his time. It says that in any right triangle, the square on the hypotenuse is the sum of the squares on the other two sides. For example, in the familiar "3,4,5 triangle", 4-squared plus 3-squared equals 5-squared.

To the mathematician, perhaps the most fascinating aspect of the theorem is not the equation itself but the fact that the identity holds for *any* right-angled triangle. There are many different ways to prove it.

Pythagoras made influential contributions to philosophy and religious teaching, but because he shrouded himself and his followers in secrecy, little is known about his life and teachings. We do know that Pythagoras and his students believed that everything was related to mathematics, that numbers were the ultimate reality, and that through mathematics everything could be predicted and measured in rhythmic patterns or cycles. Pythagoras is credited with declaring that "number is the ruler of forms and ideas and the cause of gods and demons."

3. Area of a circle

The area of a circle of radius r is given by

$$Area = \pi r^2$$

Unlike Euler's Identity or Pythagoras' theorem, which describe relationships between two quantities, this formula tells you how to calculate something, namely the area of a circle. (It's really a formula rather than an equation.)

The formula may look simple but it hides a deep issue that took many hundreds of years to resolve, namely the problem of "squaring the circle." Intuitively, area is given by multiplying length by width. This works for calculating the area of a rectangle, and by splitting up figures into rectangles and fractions of rectangles, it provides a method to calculate the area of any figure with straight edges.

But what about figures with curved edges? Circles, for instance? To calculate the area of a circle, the ancient Greeks asked if it was possible to construct a square of the same area; if so, then you can find the area of the circle by calculating the area of the equivalent square. This, however, was eventually shown to be impossible.

The familiar formula for the area of a circle resolves the problem a different way. Deriving the formula requires advanced mathematical ideas involving infinity.

The intriguing aspect for the mathematician is that the formula relates two very different kinds of shapes — rectangular ones (the term r^2 gives the area of a square of side r) and circular ones. The "magic" is provided by that mysterious number π.

4. Einstein's energy equation

$$E = mc^2$$

Undoubtedly the most famous expression in all of science, Einstein's equation, which the famous physicist announced in 1905 as a cornerstone of his Theory of Special Relativity, provides a penetrating view of the very nature of matter, and at the same time provided Mankind with the tools for our own destruction. Although it is an equation in physics, expressing the precise relationship between pure energy, E, mass, m, and the speed of light, c, Einstein discovered it simply by doing mathematics.

The equation expresses a relationship between numbers, and it is because the number c is so large, and c^2 even more so, that a small amount of matter m can yield a large quantity of energy E — a fact that leads rapidly to both nuclear power and nuclear weapons, with today's very real possibility that a device that could fit into a suitcase could power an entire city for many years, or destroy it in an eyeblink.

Yet, the real message of the equation lies not in what happens when you plug in the numbers, rather what it tells us about the stuff we and everything around us are made of. Matter, Einstein's equation tells us, is just a highly concentrated form of pure energy. Beneath the music and the poetry, beneath our minds, beneath our brains, beneath the neurons and axons that make up our brains, beneath the atoms and the molecules that make up our neurons and axons, beneath the protons and electrons that make up the atoms of our bodies, lies pure energy. Nothing more, and nothing less.

5. Leibniz's series for π

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$$

Gottfried Wilhelm Leibniz is one of the two mathematicians generally credited with the invention of the calculus in the mid seventeenth century. Calculus, as every high school student knows, or at least should know, is a method for handling infinity, and infinity appears twice in Leibniz's equation, the infinity hidden behind that symbol π — whose decimal expansion begins 3.1415 and continues indefinitely, unpredictable and always changing — and the endless sum to the right. The equation is a simple consequence of a deeper and more general result the Scottish mathematician James Gregory discovered in 1671, and for that reason it also goes under the name of Gregory's series.

The pattern of the addition is a tantalizing one. On the one hand, what could be simpler than the pattern of the denominators, being nothing other than all the odd numbers taken one after the other? What could be simpler than the pattern of the signs, minus plus, minus plus, minus plus? How could two such simple patterns give rise to that most important, most ubiquitous, and most mysterious of numbers π, the famous Greek ratio of the circumference of a circle to its diameter?

The answer lies in the infinite nature of the sum: the adding goes on for ever. Behind the equation lies the deep fact that adding together an infinite number of terms can sometimes yield a finite answer. The key is that the terms get progressively smaller, dying away like echoes bouncing back and forth across a valley until, in the end, is silence.

6**. Newton's second law of motion**

Force = mass x acceleration

Isaac Newton, child of England, mathematician extraordinaire, inventor of calculus, master of optics, founder of the science of mechanics, astronomer, natural philosopher, alchemist, and civil servant.

Newton's laws of motion are three physical laws that describe relationships between the forces acting on a body and the motion of the body. They are as far reaching and profound as they are simple. Newton first published them in 1687 in his work *Philosophiae Naturalis Principia Mathematica*. In the third volume of the text, he showed that the laws of motion, combined with his law of universal gravitation, explained the motion of the planets, as observed by Kepler.

The three laws are:

1. An object in motion will remain in motion unless a force acts on it.
2. Force equals mass multiplied by acceleration.
3. For every action there always is an equal and an opposite one.

The second is an equation, and an important one for modern life, but it is when taken together that Newton's three laws really wend their scientific magic.

7. Euler's polyhedron formula

For any convex polyhedron,

$$V - E + F = 2$$

where V is the number of vertices, E the number of edges, and F the number of faces.

The simplest example of a polyhedron is a tetrahedron, which has 4 vertices, 6 edges, and 4 faces, so the formula gives 4 – 6 + 4 = 2. The next case is a cube, with 8 vertices, 12 edges, and 6 faces, and again you get the answer 2.

The polyhedron does not have to be regular, with all its faces looking the same. The formula holds for *any* polyhedron, from the most regular and symmetrical — of which the ancient Greeks showed there are exactly five — to ugly combinations of different shapes and angles.

What is surprising about the formula is that, no matter how complicated you make a polyhedron — and in theory you can make them with billions of faces, all different — the number of vertices minus the number of edges plus the number of faces will always equal 2.

The formula thus shows us a hidden law that restricts the kinds of shapes that can exist in the world. A sculptor who sets out to carve a polyhedron has considerable freedom in how many sides it has, and how irregular it is, but that freedom is not unfettered.

Euler discovered the formula 1750, but the first person to prove it was Adrien-Marie Legendre, in 1794. Like many fundamental mathematical laws, Euler's formula has other applications. The formula also applies to the countries of the world, when you interpret faces as countries, edges as borders, and vertices as the points where borders meet. It thus tells us that, while countries may conquer one another, and politicians may change boundaries, they are not totally free to re-draw the world map to their choosing. The map must obey Euler's formula.

Even the strongest world superpower must bow to Euler's formula.

ZAMBRA'S GUIDE TO THE CHORAL INTERPRETATIONS

Interpretation Process: We're not professional mathematicians. However, we do appreciate the interconnectedness of music and mathematics. Some of us are teachers and at least one of us has even used music in her 5h grade math lessons. We look at the interpretive process as a chance for musical play. When we first consider an equation, we look for symbols—letters and numbers—that might also appear in the symbolic world of musical notation—letters that could correspond to pitches, and numbers that could be turned into intervallic relationships, or tonal progressions, or repetitive patterns. We also look at other words in the equation and see if they call to mind any musical vocabulary or qualities. We might also look at the equation's historical and cultural context for stylistic ideas. Then we just play and keep brainstorming and refining until it feels and sounds right.

Euler's Identity: E to the power i times pi equals minus one. We begin on a unison E, then build a powerful E major chord on the word "power." We express the imaginary nature of "i" as a cascade of ethereal sighs. For "pi" we musically spell out 3.14 by building a chord starting with "E," first sounding three steps up from E, then one step above that note, and four steps above that. We return in unison to E on the word "equals," singing those two syllables with equal stress and duration. To express the concept "minus one" we travel through a discord to finish one step down from E on a D. Finally we emphasize the last word, "one" by ending with only one singer sounding the note.

Pythagoras' Theorem: "In any right-angled triangle, the square on the hypotenuse equals the sum of the squares on the other two sides." For this equation we were inspired by the vision of Pythagoras at work in the Greek islands long ago. We began by composing a melody for the words of the theorem in a G natural scale, to evoke what some scholars believe might have been a common Ancient Greek musical mode. The theorem fits into two 12-beat repeating musical phrases that wind downwards through the mode, suggesting an Ancient Greek circle or spiral dance. While the middle voices sing the theorem, the sopranos sing three longer tones against the 12 beat line to suggest the three sides of the classic 3-4-5 right-angled triangle. First they sing the word "four" for 4 beats on the 4th note of the G scale. Then they sing the word "three" for 3 beats on the 3rd note. Lastly they sing the word "five" for 5 beats on the 5th note. Below these two vocal parts, the altos are singing the word "gods" on a low G and the word "demons" on a D, in a reference to the Pythagoras quote, "number is the ruler of forms and ideas and the cause of gods and demons." We also embed the triangle ratios within our overall arrangement: the altos sing their phrase, "gods and demons" three times alone, then the theorem is sung four times over that, and then the sopranos and altos finish the piece by singing their parts, in combination, another five times.

Area of a Circle: Area equals pi r squared. For this formula we "draw" a musical circle by connecting together five musical intervals, each the size of a fifth, and repeating this sequence over and over again while singing the word "area." This circling of fifths is a play on the phrase, "circle of fifths," a fundamental concept in western music theory. (It's an imaginary geometrical space that depicts relationships among 12 equal-tempered pitch classes, or the 12 diatonic scales, in chromatic music—start on any pitch class and ascend by a perfect fifth, and you'll eventually land on that same pitch class again (CGDAEBF#DbAbEbBbFC.)

Musically—and visually, if you were to see the notation of the composition—what the fifths are circling around, is middle C (the center note on the piano keyboard), which represents the "focus" or hub of our circle, and we have middle C droning throughout the piece, literally running right through the center of it. The third fifth that we sing—one that goes between middle C and G above it—represents the radius of our circle. To convey the concept of "radius squared" two people sing that C/G radius phrase two times. Against that, one person sings the words "equals pi" on the third, first and fourth degrees up from middle C to signify 3.14.

Einstein's equation: E equals m c squared. For the word "mass," we composed a melody suggestive of a medieval chant, which to us conveys a deep, heavy, weighty sound. Since Gregorian chants were sung in Latin, we sing various words in Latin that have English translations such as "mass" "heavy" "dense" "great" and the like. To represent "c", the speed of light, squared, we chime a resonating c note. Then, of course, we finish with a burst of energy!

Leibniz's series for pi. For the left side of this equation, we put pi to music in a four-four time signature—pi over a swinging beat in four. We spell out an abbreviated form of pi in the musical phrase using the third, first, fourth, first and fifth degrees of the scale, and then land on the word "four" for four beats. Meanwhile we also convey the infinite nature of pi by having one singer continuing on with the pi sequence, chanting steadily on beneath all the harmonies. We render the plus one/minus one pattern of the right side of the equation by having one person sing and move up and down, and we express the infinite nature of the pattern through a gradual fade out into silence.

Newton's laws of motion. Because Newton first published his laws of motion in England in 1687, we wanted to create a musical setting for them with an English Baroque sensibility. We selected the traditional tune commonly known as "Lillybolero," which was used for various popular songs and dances in the 17th century in Britain. It is associated with the greatest English composer of that era, Henry Purcell, who published a setting of the tune for virginal and spinet in 1689, just two years after Newton's great work *Principia Mathematica*. The tune naturally has three sections to it, which correspond delightfully with the three laws of motion. We've added our own high and low harmony parts here, and play with a faux-Baroque opera style. We emphasize force, mass, and acceleration by strongly articulating the word "force" in the alto line, and by accelerating during the course of the piece.

Euler's polyhedron formula: V–E+F=2. As we began to interpret this equation, we imagined a polyhedron with many faces, which will resemble a globe, whose faces are individual countries. We draw on this global image and have selected music from around the world to demonstrate it.

For simplicity's sake, we return to the most basic polyhedron of all, the tetrahedron, with its 4 faces, 6 edges, and 4 vertices. To depict the faces, we've select 4 songs, from 4 unique cultures and musical styles, from 4 continents and quadrants of the globe. The 6 edges become borders, where 2 countries and cultures intermingle. Our 4 songs can be sung in 6 different combinations of 2, so as we move from one song to the next, we overlap them to relate to this cultural convergence.

The 4 vertices can be thought of as places of intersection and multi-cultural exchange. To represent this idea we use a drum, a common instrument in many cultures, beating in 4/4 time. And, by combining all 4 songs near the end of the composition.

Finally, 2, the other side of Euler's equation, relates to the pairings of our songs, and also to the way our piece ends.

To further our global metaphor, the four songs we've chosen are all rounds (as in "Row, row, row your boat").. We begin in African with a song used to gather people together. Following human migration patterns we then move into Asia, with a children's song from Japan. In North American, our tune comes from the Shape Note tradition, where each note is transcribed as a geometric shape, and musical syllables are substituted for words. The last song, from Brazil, has recognizable African roots, tying us back to the first piece.

HOW IT ALL BEGAN

Keith Devlin looks back. In 2006, broadcaster Robert Pollie interviewed me for KUSP's *Talk of the Bay* radio show. The topic was beauty in mathematics, and as always when that question comes up — and it often does — I mentioned Euler's identity, that links together the three fundamental mathematical constants e, i, π. To add variety to the piece, Pollie asked his friend Susan Krivin if the choral group she was a member of, Zambra, could compose and perform a short choral piece, based on the famous identity. They did so, and when the interview was broadcast on May 1, 2006, it was accompanied by Zambra's rendition. (You can hear the entire interview, plus the choral interpretation, on the Zambra website at http://zambra.org/math.html.)

I loved the Zambra song, and when I found myself talking on the radio once again about Euler's identity, this time in my "Math Guy" slot on NPR's *Weekend Edition*, on April 16, 2007, I arranged for the choral interpretation to be played at the end. That interview turned out to be the most emailed piece on NPR that weekend, in large part, I believe, because of the Zambra choral piece.

The moment I heard the Euler piece, I thought about mounting a complete show, in which Zambra would perform choral interpretations of various famous mathematical equations. I emailed them, and they jumped at the chance. What was to become the show *Harmonious Equations* was born — at least in concept.

The (very easy) next step, for me, was to choose a collection of equations that could justifiably be billed as "the most important and beautiful equations in mathematics," and to write descriptions of them that would explain what they say, in language that would both be accessible to a lay audience and (I hoped) have a prose structure that reflected the equations' own beauty.

Then came the hard part — Zambra's challenge. With only the symbolic equations and my prose descriptions to go on, the singers had to come up with musical interpretations of the mathematics. Over the course of the next few months, choral interpretations of four more equations slowly came to life. That would give us a five-equation show that we could road test to a select (and at this stage carefully selected, enthusiastic "math insiders") audience: namely the attendants at the annual winter meeting of the California Mathematics Council, to be held in Asilomar, California, at the start of December.

Although I made a number of trips down to Santa Cruz to spend the evening with Zambra, as they developed their ideas and rehearsed through the fall of 2007, my only real input during this process was to steer them away from the occasional musical interpretation that ran contrary to the mathematics the equation represents. The interpretations they came up with were theirs and theirs alone. Those interpretations amazed me, in their ingenuity, their depth, their insights, and their occasional humor. I knew we were onto a winner.

And indeed we were. The show was a big success, and as a result we arranged with the conference organizers to return the following year, 2008, with an extended show, featuring more equations. We weren't sure how many equations would fill an hour, but after some experimentation we settled on seven, up two from the number we had focused on the year before, and got to work on the new ones I selected.

Keith Devlin