LECTURE NOTES ON GEOMETRIC FEATURES OF THE ALLEN–CAHN EQUATION (PRINCETON, 2019)

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Contents

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1. Overview

These are my notes for a mini-course that I gave at the Princeton RTG summer school in 2019 on geometric features of the Allen–Cahn equation. A "+" marks the exercises which are used subsequently in the lectures. The others can safely be skipped without subsequent confusion, but are probably more interesting/difficult. There are additional problems in Appendix B, which explore some of the theory not discussed in the lectures. I am very grateful to be informed of any inaccuracies, typos, incorrect references, or other issues.

2. Introduction to the Allen–Cahn equation

We will consider throughout (M^n, g) a complete Riemannian manifold.

Definition 2.1. We define the Allen–Cahn energy[1] by

$$E_\varepsilon(u;\Omega) := \int_\Omega \left(\frac{\varepsilon}{2}|\nabla_g u|^2 + \frac{1}{\varepsilon}W(u)\right) d\mu_g.$$

Here $W(\cdot)$ is a "double well potential," which we will take as $W(t) = \frac{1}{4}(1-t^2)^2$ (more general functions are also possible). We will often drop Ω (e.g. when M is compact).

It is clear that E_ε is well defined for $u \in H^1(\Omega) \cap L^4(\Omega)$. It is convenient to extend E_ε to functions $u \notin H^1(\Omega) \cap L^4(\Omega)$ by $E_\varepsilon(u) = \infty$.

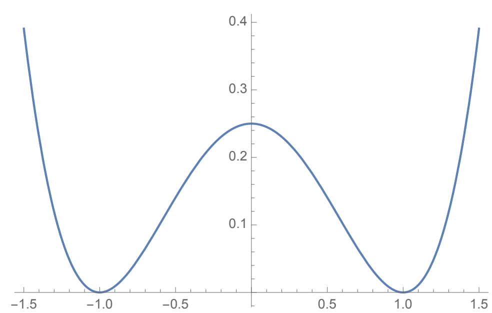

Figure 1. The double well potential $W(t) = \frac{1}{4}(1-t^2)^2$.

[1]The Allen–Cahn energy was first considered as a model for matter of a non-uniform composition by Van der Waals (see the translated article [vdW79]) in 1893 and then rediscovered by Cahn–Hillard [CH58] in 1958. Allen–Cahn [AC79] observed in 1978 that there is a basic link between the location of the interface between the two phases and the mean curvature of the interface.

2.1. **Critical points and the Allen–Cahn equation.**

Definition 2.2. A function $u : M \to \mathbb{R}$ is a *critical point* of E_ε if for any $\varphi : M \to \mathbb{R}$ smooth with support compactly contained in a precompact open set $\Omega \subset M$, we have $u \in H^1(\Omega) \cap L^\infty(\Omega)$ and

$$\frac{d}{dt}\Big|_{t=0} E_\varepsilon(u + t\varphi; \Omega) = 0.$$

Note that

$$\frac{d}{dt}\Big|_{t=0} E_\varepsilon(u + t\varphi; \Omega) = \int_\Omega \left(\varepsilon g(\nabla_g u, \nabla_g \varphi) + \frac{1}{\varepsilon} W'(u)\varphi \right) d\mu_g,$$

so it follows that u (weakly) solves the Allen–Cahn equation

$$\varepsilon \Delta_g u = \frac{1}{\varepsilon} W'(u) = \frac{1}{\varepsilon} u(u^2 - 1)$$

in Ω if and only it is a critical point of E_ε on Ω.

Exercise 2.1 (+).
(a) Prove that if $u : M \to \mathbb{R}$ is a critical point of the Allen–Cahn functional u, then u is smooth.
(b) For a smooth critical point of the Allen–Cahn functional u on a closed manifold (M, g), show that $u \in [-1, 1]$.

Observe that $u \equiv \pm 1$ and $u \equiv 0$ are critical points for the Allen–Cahn equation, since $W'(\pm 1) = W'(0) = 0$. If (M, g) is compact, then

$$E_\varepsilon(v) \geq 0 = E_\varepsilon(\pm 1).$$

Because ± 1 are the unique global minima for $W(t)$, we see that ± 1 are the unique global minimizers for E_ε in the sense that $E_\varepsilon(v) = 0$ implies that $v = \pm 1$.

Are there other solutions to the Allen–Cahn equation?

2.2. **One dimensional solution.** Let us begin by considering the Allen–Cahn equation on $\mathbb{R}$. The Allen–Cahn equation becomes

$$\varepsilon u''(t) = \frac{1}{\varepsilon} W'(u(t)). \tag{2.1}$$

Rescaling by ε allows us (in this case) to study only $\varepsilon = 1$. Set $\tilde{u}(t) = u(\varepsilon t)$, so

$$\tilde{u}''(t) = \varepsilon^2 u''(\varepsilon t) = W'(u(\varepsilon t)) = W'(\tilde{u}(t)).$$

Thus, we will begin by considering $\varepsilon = 1$ (and then rescale the coordinate function t to return to arbitrary ε).

Dropping the tilde, we seek (other than $u \equiv \pm 1$) solutions to the ODE

$$u''(t) = W'(u(t)) = u(t)^3 - u(t). \tag{2.2}$$

Observe that we have "conservation of energy," i.e.,

$$\frac{d}{dt}\left(u'(t)^2 - 2W(u(t)) \right) = 2u'(t)u''(t) - 2u'(t)W'(u(t)) = 0.$$

Thus $u'(t)^2 = 2W(u(t)) + \lambda$ for some $\lambda \in \mathbb{R}$. Let us first try to find any solution to the equation. Take $\lambda = 0$ and suppose that $u'(t) > 0$ for all $t \in \mathbb{R}$. Then,

$$\frac{du}{dt} = \frac{1}{\sqrt{2}}(1 - u^2).$$

We can solve this as $u(t) = \mathbb{H}(t - t_0)$ for $\mathbb{H}(t) = \tanh(t/\sqrt{2})$ and $t_0 \in \mathbb{R}$ arbitrary. You should check (by differentiating) that $\mathbb{H}(t)$ is indeed a solution to (2.2).

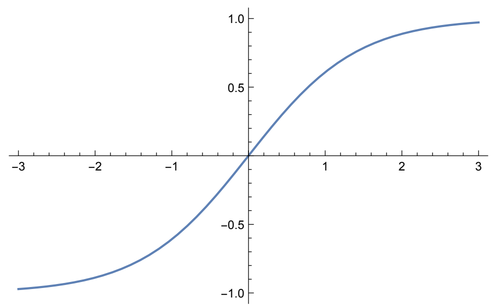

FIGURE 2. The heteroclinic solution $\mathbb{H}(t) = \tanh(t/\sqrt{2})$.

Observe that $\mathbb{H}(t)$ has finite energy in the sense that

$$\int_{-\infty}^{\infty} \frac{1}{2}\mathbb{H}'(t)^2 + W(\mathbb{H}(t))dt < \infty.$$

In Exercise 2.3 below you will be asked to compute this integral.

Lemma 2.3. *Suppose that $u(t)$ solves (2.2) for all $t \in \mathbb{R}$. Then $u(t)$ has finite energy if and only if $u(t) = \pm\mathbb{H}(t - t_0)$ or ± 1.*

Proof. We have seen that $u'(t)^2 = 2W(u(t)) + \lambda$ for $\lambda \in \mathbb{R}$. Because

$$\int_{-\infty}^{\infty} \left(\frac{1}{2}u'(t)^2 + W(u(t))\right) dt < \infty$$

and since both terms are non-negative, there is $t_k \to \infty$ with $u'(t_k) \to 0$ and $W(u(t_k)) \to 0$. Thus,

$$u'(t_k)^2 - 2W(u(t_k)) \to 0.$$

Hence, $\lambda = 0$. Now, by considering the cases $|u(0)| < 1$, $|u(0)| = 1$, and $|u(0)| > 1$ arguing as above, we find that in the first case $u(t) = \pm\mathbb{H}(t - t_0)$, in the second case $u(t) = \pm 1$, and in the third case $u(t)$ cannot be defined for all $t \in \mathbb{R}$. □

Exercise 2.2 (+)**.** (a) Do solutions to (2.2) exist that have infinite energy?

(b) Suppose that $u(t)$ solves (2.2) and satisfies $u'(t) > 0$ for all $t \in \mathbb{R}$. Show that u must be one of the solutions in Lemma 2.3 (and thus have finite energy).

Rescaling back to the general $\varepsilon > 0$ equation we've seen

$$\mathbb{H}_\varepsilon(t) := \tanh\left(\frac{t}{\varepsilon\sqrt{2}}\right)$$

is the unique (up to sign and translation) non-trivial solution with finite energy to

$$\varepsilon\mathbb{H}_\varepsilon''(t) = \frac{1}{\varepsilon}W'(\mathbb{H}_\varepsilon(t)).$$

2.2.1. *First glimpse of the $\varepsilon \to 0$ limit.* Note that for $t > 0$,

$$\lim_{\varepsilon\to 0}\mathbb{H}_\varepsilon(t) = 1$$

and for $t < 0$

$$\lim_{\varepsilon\to 0}\mathbb{H}_\varepsilon(t) = -1$$

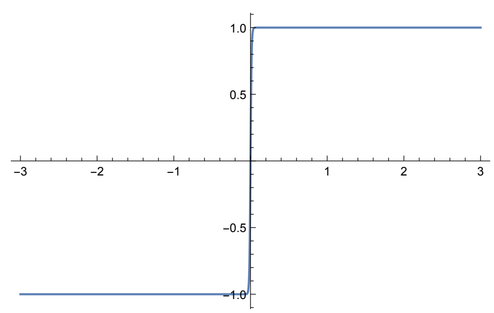

FIGURE 3. The heteroclinic solution $\mathbb{H}_\varepsilon(t)$ with $\varepsilon = .01$ is converging to a step function.

Thus, $\mathbb{H}_\varepsilon$ converges a.e., to the step function

$$\mathbb{H}_0(t) := \begin{cases} +1 & t > 0 \\ -1 & t < 0 \end{cases}.$$

Note that

$$\{0\} = \partial\{\mathbb{H}_0(t) = 1\}.$$

This somewhat trivial observation is the first hint of the connection between the singular limit $\varepsilon \to 0$ for solutions to the Allen–Cahn equation and hypersurfaces (in this case, just a point).

2.3. **Solutions on $\mathbb{R}^2$.** We have seen that the set of (finite energy) solutions to the Allen–Cahn ODE on $\mathbb{R}$ is rather simple (although the solution $\mathbb{H}(t)$ is very important). We turn to solutions on $\mathbb{R}^2$. Observe that as above, if

$$\Delta u = W'(u), \tag{2.3}$$

then $u_\varepsilon(x) := u(x/\varepsilon)$ solves

$$\varepsilon\Delta u_\varepsilon = \frac{1}{\varepsilon}W'(u_\varepsilon)$$

2.3.1. *The one-dimensional solution on* $\mathbb{R}^2$. We first observe that the one dimensional solution $\mathbb{H}(t)$ we considered before provides a solution on $\mathbb{R}^2$ as well. To that end, fixing $a \in \partial B_1(0) \subset \mathbb{R}^2$ and $b \in \mathbb{R}$, we consider the function

$$u(x) = \mathbb{H}(\langle a, x\rangle - b).$$

It is clear that u solves (2.3). Note that this u has flat level sets and defining $u_\varepsilon(x) = u(x/\varepsilon)$,

$$\lim_{\varepsilon\to 0} u_\varepsilon(x) = u_0(x) := \begin{cases} 1 & \langle a, x\rangle > b \\ -1 & \langle a, x\rangle < b \end{cases}$$

Note that $\partial\{u_0 = 1\} = \{\langle a, x\rangle = b\}$, is a straight line.

Exercise 2.3 (+)**.** Recall that $\mathbb{H}(t) = \tanh(t/\sqrt{2})$ is the 1-dimensional solution. Show that

$$\sigma := \int_{-\infty}^{\infty} \left(\frac{1}{2}\mathbb{H}'(t)^2 + W(\mathbb{H}(t))\right) dt = \int_{-\infty}^{\infty} \mathbb{H}'(t)^2 dt = \int_{-1}^{1} \sqrt{2W(s)}ds$$

and compute the value of σ.

Exercise 2.4. For $u_\varepsilon(x) = \mathbb{H}(\varepsilon^{-1}\langle a, x\rangle)$ on $\mathbb{R}^2$ considered above, compute the value of $\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon; B_1(0))$. Hint: First compute the limit with $B_1(0)$ replaced by an appropriately chosen square. Check that $\mathbb{H}(t)$ is exponentially small as $t \to \pm\infty$ to show that the associated error is small.

2.3.2. *The saddle solution and other four ended solutions.* On $\mathbb{R}^2$, there are other solutions to Allen–Cahn besides $\mathbb{H}(\langle a, x\rangle)$. The following saddle solution was first discovered by Dang–Fife–Peletier [DFP92]. It is an entire solution on $\mathbb{R}^2$ with $\{u = 0\} = \{xy = 0\}$.

Exercise 2.5. Consider

$$\Omega_R := \{(x, y) \in \mathbb{R}^2 : x, y > 0, x^2 + y^2 < R^2\}.$$

Choose u_R a smooth function with Dirichlet boundary conditions minimizing $E_1(\cdot)$ among functions in $H_0^1(\Omega_R)$ (or equivalently smooth functions on Ω_R that vanish on the boundary). Show that:

(a) The function u_R exists, is smooth, satisfies the Allen–Cahn equation, and does not change sign. Argue that u_R is either identically zero or (possibly replacing u by $-u$) $u \in (0, 1)$ in the interior of Ω_R.

(b) Show that $E_1(u_R; \Omega_R) \leq CR$ for some $C > 0$ independent of R. Conclude that u_R is strictly positive in the interior of Ω_R, for R large.

(c) Using odd reflections across the coordinate axes, construct $\tilde{u}_R$ solving the Allen–Cahn equation on $B_R(0) \subset \mathbb{R}^2$. Using elliptic regularity, check that $\tilde{u}_R$ is smooth across the axes and at 0 and has $E_1(\tilde{u}_R; B_R(0)) \leq \tilde{C}R$.

(d) Using elliptic theory, take a subsequential limit as $R \to \infty$ to find an entire solution u to Allen–Cahn.

(e) Show that $\{u = 0\} = \{xy = 0\}$. *Hint: if not, the maximum principle implies that $u \equiv 0$; this function is not minimizing on balls compactly contained in the quadrant.*

There are many other related solutions. For example:

Theorem 2.4 (Kowalczyk–Liu–Pacard [KLP12]). *Given any two lines $\ell_1, \ell_2 \subset \mathbb{R}^2$ intersecting precisely at the origin, there is a solution u on $\mathbb{R}^2$ whose nodal set $\{u = 0\}$ is asymptotic at infinity to $\ell_1 \cup \ell_2$.*

3. Convergence of (local) minimizers of the Allen–Cahn functional

The $\varepsilon \searrow 0$ limit of the Allen–Cahn functional (and associated critical points) turns out to be intimately related with the area functional for hypersurfaces (and associated critical points, minimal surfaces). This relationship was first described in works of Modica and Mortola [MM77] based on the framework of "Γ-convergence" defined by De Giorgi [DGF75].

Definition 3.1. For $\Omega \subset (M^n, g)$ an open set, a function $u \in L^1(\Omega)$ is of *bounded variation*, $u \in BV(\Omega)$, if its distributional gradient is a Radon measure, i.e. if there is a TM valued Radon measure Du so that for any vector field $X \in C^1_c(\Omega; TM)$

$$\int_\Omega u \operatorname{div}_g X \, d\mu_g = -\int_\Omega g(X, Du).$$

We write

$$\int_\Omega |Du| = \sup \left\{ \int_\Omega u \operatorname{div}_g X \, d\mu_g : X \in C^1_c(\Omega; TM), \|X\|_{L^\infty} \leq 1 \right\}$$

for the total variation norm.

Proposition 3.2 (BV compactness, see Appendix A). *If $u_k \in BV(\Omega)$ satisfies*

$$\sup_k \left(\|u_k\|_{L^1(\Omega')} + \int_{\Omega'} |Du_k| \right) < \infty,$$

for all Ω' precompact open set in Ω, then after passing to a subsequence, there is $u \in BV_{loc}(\Omega)$[2] so that $u_k \to u$ in $L^1_{\text{loc}}(\Omega)$[3] and

$$\int_{\Omega'} |Du| \leq \liminf_{k \to \infty} \int_{\Omega'} |Du_k|$$

for all Ω' precompact in Ω.

Remark 3.3. If Ω has Lipschitz boundary, then we can drop the "loc," i.e. we could replace Ω' by Ω throughout. We will not address this further.

[2] That is, $u \in BV(\Omega')$ for all Ω' compactly contained in Ω.

[3] That is, L^1 convergence on precompact open sets.

Definition 3.4. For a Borel set $E \subset \Omega$, we say that E has *finite perimeter* if $\chi_E \in BV(\Omega)$. In this case, we define the *perimeter* of E by

$$P(E;\Omega) = \int_\Omega |D\chi_E|.$$

Often, sets of finite perimeter are called *Caccioppoli* sets.

See Appendix A for further results on BV functions and sets of finite perimeter.

3.1. **Γ-convergence.** The following computation is rather simple but it underlies the theory of limits of minimizers. Define

$$\Phi(t) := \int_0^t \sqrt{2W(s)}ds.$$

Then, we compute, using AM-GM[4] and the chain rule:

$$\begin{aligned} E_\varepsilon(u;\Omega) &= \int_\Omega \left(\frac{\varepsilon}{2}|\nabla_g u|^2 + \frac{1}{\varepsilon}W(u)\right) d\mu_g \\ &\geq \int_\Omega \sqrt{2W(u)}|\nabla_g u|\, d\mu_g \\ &= \int_\Omega |\nabla_g(\Phi(u))|\, d\mu_g. \end{aligned} \tag{3.1}$$

Combined with BV compactness and some measure theoretic arguments we find the following result that loosely speaking says that the behavior of the Allen–Cahn energy is "controlled from below" as $\varepsilon \to 0$ by the perimeter functional.

Proposition 3.5 ([Mod85, MM77, Ste88, FT89])**.** *For $\Omega \subset (M,g)$ a precompact open set, suppose that u_ε satisfy $E_\varepsilon(u_\varepsilon;\Omega) \leq C$. Then, there is a subsequence $\varepsilon_k \to 0$ and $u_0 \in BV_{loc}(\Omega)$ with $u_0 \in \{\pm 1\}$ a.e., and*

$$u_{\varepsilon_k} \to u_0$$

in $L^1_{loc}(\Omega)$. Moreover,

$$\sigma P(\{u_0 = 1\};\Omega') \leq \liminf_{k\to\infty} E_{\varepsilon_k}(u_{\varepsilon_k};\Omega'),$$

where $\sigma = \Phi(1) - \Phi(-1) = \int_{-1}^1 \sqrt{2W(s)}ds$, for any Ω' compactly contained in Ω.

Sketch of the proof. We we can check that $|\Phi(t)| \leq \alpha + \beta W(t)$; thus, the uniform energy bounds $E_\varepsilon(u_\varepsilon;\Omega) \leq C$ imply that $\|\Phi(u_\varepsilon)\|_{L^1(\Omega)} \leq C$. Thus, we can use (3.1) and BV compactness to find $v_0 \in BV_{\mathrm{loc}}(\Omega)$ so that a subsequence of $\Phi(u_\varepsilon)$ converges in $L^1_{\mathrm{loc}}(\Omega)$ to v_0 and

$$\int_{\Omega'} |Dv_0| \leq \liminf_{k\to\infty} E_{\varepsilon_k}(u_{\varepsilon_k};\Omega').$$

[4] i.e., $2xy \leq x^2 + y^2$.

The function Φ is invertible, and indeed Φ^{-1} is uniformly continuous. Moreover, because $W(t) \geq ct^4$ for t sufficiently large, we see that $\|u_\varepsilon\|_{L^4(\Omega)} \leq C$. These facts suffice for us to find a further subsequence so that $u_{\varepsilon_k} \to u_0 := \Phi^{-1}(v_0)$ in $L^1_{\text{loc}}(\Omega)$ and that $u_0 \in BV_{\text{loc}}(\Omega)$ with $u_0 \in \{\pm 1\}$ a.e. in Ω. The fact that $u_0 \in \{\pm 1\}$ follows from:

$$\frac{\delta^2}{4}|\{x \in \Omega' : |u_{\varepsilon_k}(x)^2 - 1| > \delta\}| \leq \int_{\Omega'} W(u_\varepsilon) d\mu_g \leq C\varepsilon$$

for any $\delta > 0$.

Now, we compute

$$\begin{aligned}\Phi(u_0) &= \Phi(1)\chi_{\{u_0=1\}} + \Phi(-1)\chi_{\{u_0=-1\}}\\ &= (\Phi(1) - \Phi(-1))\chi_{\{u_0=1\}} + \Phi(-1)(\chi_{\{u_0\}=1\}} + \chi_{\{u_0=-1\}})\\ &= (\Phi(1) - \Phi(-1))\chi_{\{u_0=1\}} + \Phi(-1)\chi_\Omega\end{aligned}$$

a.e. in Ω. Hence,

$$\int_{\Omega'} |D\Phi(u_0)| = (\Phi(1) - \Phi(-1))P(\chi_{\{u_0=1\}}; \Omega') = (\Phi(1) - \Phi(-1))\int_{\Omega'} |Du_0|.$$

This completes the proof. □

Exercise 3.1. Fill in the details missing in the previous sketch:

(a) Show that $|\Phi(t)| \leq \alpha + \beta W(t)$.
(b) Show that Φ^{-1} exists and is uniformly continuous.
(c) Show that (after passing to a further subsequence) u_{ε_k} converges to $u_0 := \Phi^{-1}(v_0)$ in measure on Ω.[5]
(d) For (X, μ) a measure space with $\mu(X) < \infty$, if f_i are measurable functions converging to f in measure, and $\|f_i\|_{L^p(X)} \leq C$ for some $C > 0, p > 1$, show that $f_i \to f$ in $L^1(X)$.
(e) Conclude that u_{ε_k} converges to u_0 in $L^1(\Omega')$ and thus (passing to a further subsequence via a diagonal argument) a.e. in Ω.
(f) Check that $u_0 \in BV_{\text{loc}}(\Omega)$ and $u_0 \in \{\pm 1\}$ a.e. in Ω.

The counterpart to the previous result is the following "recovery" result. It says that Proposition 3.5 is sharp along certain sequences. We emphasize that the sequences u_ε constructed below are not critical points (but more on this later).

Proposition 3.6 ([Mod85, MM77, Ste88])**.** *If $E \subset \Omega$ is a set of finite perimeter, then there is a sequence $u_\varepsilon \in H^1(\Omega) \cap L^4(\Omega)$ with*

$$\sigma P(E; \Omega) = \lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon; \Omega)$$

and $u_\varepsilon \to \chi_E - \chi_{\Omega \setminus E}$ in $L^1(\Omega)$.

[5] Recall that $f_i \to f$ in measure if $\lim_{i\to\infty} \mu(|f_i - f| \geq \delta) = 0$ for all $\delta > 0$.

Sketch of the proof. Assume that ∂E is smooth and can be extended slightly beyond $\partial\Omega$. In particular, the signed distance $d_{\partial E}(\cdot)$ is smooth near ∂E. Recall that $|\nabla d_{\partial E}| = 1$. We consider $u_\varepsilon = \varphi(\varepsilon^{-1} d_{\partial E}(\cdot))$ for φ to be chosen. We assume that $\varphi \equiv \pm 1$ outside of $[-K, K]$. Then, writing $\Sigma_t = \{d_{\partial E}(\cdot) = t\}$ for t sufficiently close to 0, we find that

$$
\begin{aligned}
E_\varepsilon(u_\varepsilon; \Omega) &= \int_\Omega \left(\frac{1}{2\varepsilon} \varphi'(\varepsilon^{-1} d_{\partial E}(x))^2 + \frac{1}{\varepsilon} W(\varphi(\varepsilon^{-1} d_{\partial E}(x))) \right) d\mu_g \\
&= \int_{-K\varepsilon}^{K\varepsilon} \int_{\Sigma_t} \left(\frac{1}{2\varepsilon} \varphi'(\varepsilon^{-1} t)^2 + \frac{1}{\varepsilon} W(\varphi(\varepsilon^{-1} t)) \right) d\mu_{\Sigma_t} dt \\
&\approx \operatorname{area}(\partial E) \int_{-K\varepsilon}^{K\varepsilon} \left(\frac{1}{2\varepsilon} \varphi'(\varepsilon^{-1} t)^2 + \frac{1}{\varepsilon} W(\varphi(\varepsilon^{-1} t)) \right) dt \\
&\approx \operatorname{area}(\partial E) \int_{-K}^{K} \left(\frac{1}{2} \varphi'(t)^2 + W(\varphi(t)) \right) dt.
\end{aligned}
$$

Choosing $\varphi(t) = \mathbb{H}(t)$ (cut off to ± 1 outside of $[-K, K]$), we find that

$$
E_\varepsilon(u_\varepsilon; \Omega) \approx \operatorname{area}(\partial E) \int_{-\infty}^{\infty} \left(\frac{1}{2} \mathbb{H}'(t)^2 + W(\mathbb{H}(t)) \right) dt = \sigma P(E; \Omega)
$$

(see Exercise 2.3). □

Remark 3.7. The combination of the "lim-inf" lower bound from Proposition 3.5 for general sequences, with the "recovery" result from Proposition 3.6, means that "the Allen–Cahn functional Γ-converges to the perimeter functional (times σ)." This is a rather general phenomenon (first suggested by De Giorgi [DGF75, DG79]) that is very powerful for the study of (local) minimizers of functionals (as we will briefly discuss below). The main downside to using Γ-convergence comes when considering more general critical points (in particular, it does not seem to handle the issue of "multiplicity" well).

3.2. **Consequences for (local) minimizers.** We will consider (M, g) a closed Riemannian manifold unless stated otherwise.

Definition 3.8. We say that a function $u \in H^1(M) \cap L^4(M)$ is a *strict local minimizer* of $E_\varepsilon(\cdot)$ if there is $\delta > 0$ so that $E_\varepsilon(v) > E_\varepsilon(u)$ for any $v \in L^1(M)$ with $0 < \|u - v\|_{L^1(M)} \leq \delta$. We will also write strict δ-local minimizer to emphasize the size of δ.

Exercise 3.2 (+)**.** Show that a local minimizer is a critical point of $E_\varepsilon(\cdot)$ and thus satisfies the Allen–Cahn equation.

Definition 3.9. For (M, g) not necessarily compact and $\Omega \subset (M, g)$ precompact, we say that a set $E \subset \Omega$ *minimizes perimeter* in Ω (we will also drop Ω when $\Omega = M$) if for any $E' \subset \Omega$ with $E \Delta E'$ compactly contained in Ω, then

$$
P(E; \Omega) \leq P(E'; \Omega).
$$

We say that E is a *(strict) local minimizer* if there is $\delta > 0$ so that the previous holds (with the strict inequality) for E' with $0 < \|\chi_E - \chi_{E'}\|_{L^1(\Omega)} \leq \delta$.

Proposition 3.10. *Suppose that u_ε is a sequence of δ-local minimizers of $E_\varepsilon(\cdot)$ in a compact manifold (M, g). Assume that $E_\varepsilon(u_\varepsilon) \leq C$. Then, after passing to a subsequence $\varepsilon_k \to 0$, $u_{\varepsilon_k} \to u_0$ in $L^1(M)$, with $u_0 \in BV(M)$ and $u_0 \in \{\pm 1\}$ a.e. in M. The set $E := \{u_0 = 1\}$ is a local minimizer of perimeter.*

Proof. We only need to prove that E is a local minimizer of perimeter. If not, there is $\tilde{E}$ with

$$\|\chi_E - \chi_{\tilde{E}}\|_{L^1(M)} < \frac{\delta}{2}$$

and $P(\tilde{E}) < P(E)$. Using Proposition 3.6, we can find $\tilde{u}_{\varepsilon_k}$ with $\tilde{u}_{\varepsilon_k} \to \chi_{\tilde{E}}$ in $L^1(M)$ and

$$\lim_{k \to \infty} E_{\varepsilon_k}(\tilde{u}_{\varepsilon_k}) = \sigma P(\tilde{E})$$

On the other hand, we compute

$$\begin{aligned}
\|\tilde{u}_{\varepsilon_k} - u_{\varepsilon_k}\|_{L^1(M)} &\leq \|\tilde{u}_{\varepsilon_k} - (\chi_{\tilde{E}} - \chi_{M \setminus \tilde{E}})\|_{L^1(M)} + \|u_{\varepsilon_k} - (\chi_E - \chi_{M \setminus E})\|_{L^1(M)} \\
&\qquad + \|(\chi_E - \chi_{M \setminus E}) - (\chi_{\tilde{E}} - \chi_{M \setminus \tilde{E}})\|_{L^1(M)} \\
&= \|\tilde{u}_{\varepsilon_k} - (\chi_{\tilde{E}} - \chi_{M \setminus \tilde{E}})\|_{L^1(M)} + \|u_{\varepsilon_k} - (\chi_E - \chi_{M \setminus E})\|_{L^1(M)} \\
&\qquad + 2\|\chi_E - \chi_{\tilde{E}}\|_{L^1(M)} \\
&< \delta + o(1)
\end{aligned}$$

as $k \to \infty$. Thus, for k sufficiently large, we find that

$$E_{\varepsilon_k}(\tilde{u}_{\varepsilon_k}) \geq E_{\varepsilon_k}(u_{\varepsilon_k})$$

Thus, we find that

$$\sigma P(E) \leq \liminf_{k \to \infty} E_{\varepsilon_k}(u_{\varepsilon_k}) \leq \liminf_{k \to \infty} E_{\varepsilon_k}(\tilde{u}_{\varepsilon_k}) = \sigma P(\tilde{E})$$

This is a contradiction. This completes the proof. $\square$

The following result is a sort of strengthening of the "recovery" result in Proposition 3.6 in the sense that it finds (for local minimizers of perimeter) recovery sequences that are again themselves local minimizers.

Proposition 3.11 (Kohn–Sternberg [KS89])**.** *Suppose that $E \subset (M, g)$ is a strict local minimizer of perimeter. Then, for ε sufficiently small, there exists u_ε solving the Allen–Cahn equation and locally minimizing $E_\varepsilon(\cdot)$ so that $u_\varepsilon \to (\chi_E - \chi_{M \setminus E})$ in $L^1(M)$ and $\lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = \sigma P(E)$.*

Exercise 3.3. Prove this. Hint: minimize $E_\varepsilon(\cdot)$ in a (closed) L^1-ball centered at $\chi_E - \chi_{M \setminus E}$. Is the minimizer in the interior of the ball or at the boundary?

3.3. **Regularity of minimal surfaces.** We state without proof the regularity of local-minimizers of perimeter. See [Sim83] for details.

Proposition 3.12 (De Giorgi, Flemming, Almgren, Federer, Simons)**.** *If a set $E \subset (M^n, g)$ is a local minimizer of perimeter for $3 \le n \le 7$, then, after changing E by a set of measure zero, the topological boundary of E, ∂E, is a smooth hypersurface.*

We note that the *mean curvature* of ∂E vanishes

$$H_{\partial E} = \operatorname{tr}_{T\partial E} A_{\partial E}(\cdot, \cdot) = 0,$$

for $A_{\partial E}$ the second fundamental form. and that ∂E is *stable* in the sense that $\mathcal{Q}_\Sigma(\varphi, \varphi) \ge 0$ for all $\varphi \in C^\infty(\Sigma)$ where $\mathcal{Q}_\Sigma$ is the second variation operator defined below.

4. Non-minimizing solutions to Allen–Cahn

For $\Sigma^{n-1} \subset (M^n, g)$ an arbitrary (closed) minimal (two-sided[6]) surface, recall that if we vary Σ with normal speed φ then

$$\frac{d^2}{dt^2}\Big|_{t=0} \operatorname{area}_g(\Sigma_t) = \int_\Sigma \varphi J \varphi \, d\mu := \mathcal{Q}_\Sigma(\varphi, \varphi).$$

is the second variation, where

$$J\varphi = -\Delta_\Sigma \varphi - (|A_\Sigma|^2 + \operatorname{Ric}_g(\nu, \nu))\varphi. \tag{4.1}$$

The *Morse index* of Σ is the largest dimension of a linear subspace $W \subset C^\infty(\Sigma)$ so that for $\varphi \in W \setminus \{0\}$, $\mathcal{Q}_\Sigma(\varphi, \varphi) < 0$.

Exercise 4.1. For $M = \mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ and $\Sigma = \{x^n = 0\} \cap \mathbb{S}^n$, check that Σ is minimal and has Morse index 1.

Similarly, for a function u_ε on (M^n, g) solving the Allen–Cahn equation $\varepsilon^2 \Delta_g u_\varepsilon = W'(u_\varepsilon)$, we define

$$\frac{d^2}{dt^2}\Big|_{t=0} E_\varepsilon(u_\varepsilon + t\psi) := \mathcal{Q}_{u_\varepsilon}(\psi, \psi).$$

As with minimal surfaces, we say that u_ε is *stable* if $\mathcal{Q}_{u_\varepsilon}(\psi, \psi) \ge 0$ for $\psi \in C^\infty(M)$. Similarly, the *Morse index* of u_ε is the largest dimension of a linear subspace $W \subset C^\infty(M)$ so that for $\psi \in W \setminus \{0\}$, $\mathcal{Q}_{u_\varepsilon}(\psi, \psi) < 0$.

We now analyze the Allen–Cahn second variation expression further. Observe that

$$\mathcal{Q}_{u_\varepsilon}(\psi, \psi) = \int_M \left(\varepsilon |\nabla \psi|^2 + \frac{1}{\varepsilon} W''(u_\varepsilon) \psi^2 \right) d\mu_g.$$

[6]Two-sided means that there is a consistent choice of unit normal. Compare to $\mathbb{R}P^2 \subset \mathbb{R}P^3$. Note that if $\Sigma = \partial E$, then Σ is necessarily two-sided.

It turns out to be crucial to rewrite this in a somewhat different form. Recall the Bochner formula

$$\frac{1}{2}\Delta_g|\nabla f|^2 = |D^2 f|^2 + g(\nabla\Delta_g f, \nabla f) + \operatorname{Ric}_g(\nabla f, \nabla f),$$

valid for any smooth function f on a Riemannian manifold. In particular we find that

$$\frac{\varepsilon}{2}\Delta_g|\nabla u_\varepsilon|^2 = \varepsilon|D^2 u_\varepsilon|^2 + \frac{1}{\varepsilon}W''(u_\varepsilon)|\nabla u_\varepsilon|^2 + \varepsilon\operatorname{Ric}_g(\nabla u_\varepsilon, \nabla u_\varepsilon),$$

Now, formally plugging in $\psi|\nabla u_\varepsilon|$ into the second variation of energy, we we find

$$\begin{aligned}
&\mathcal{Q}_{u_\varepsilon}(\psi|\nabla u_\varepsilon|, \psi|\nabla u_\varepsilon|)\\
&= \int_M \left(\varepsilon|\nabla(|\nabla u_\varepsilon|\psi)|^2 + \frac{1}{\varepsilon}W''(u_\varepsilon)\psi^2|\nabla u_\varepsilon|^2\right) d\mu_g\\
&= \int_M \left(\varepsilon|\nabla|\nabla u_\varepsilon||^2\psi^2 + \frac{\varepsilon}{2}g(\nabla\psi^2, \nabla|\nabla u_\varepsilon|^2) + \varepsilon|\nabla\psi|^2|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W''(u_\varepsilon)\psi^2|\nabla u_\varepsilon|^2\right) d\mu_g\\
&= \int_M \left(\varepsilon|\nabla|\nabla u_\varepsilon||^2\psi^2 - \frac{\varepsilon}{2}(\Delta_g|\nabla u_\varepsilon|^2)\psi^2 + \varepsilon|\nabla\psi|^2|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W''(u_\varepsilon)\psi^2|\nabla u_\varepsilon|^2\right) d\mu_g\\
&= \varepsilon\int_M \left(|\nabla\psi|^2|\nabla u_\varepsilon|^2 - ((|D^2 u_\varepsilon|^2 - |\nabla|\nabla u_\varepsilon||^2) + \operatorname{Ric}_g(\nabla u_\varepsilon, \nabla u_\varepsilon))\psi^2\right) d\mu_g
\end{aligned}$$

Note that $|\nabla u_\varepsilon|$ might not be smooth. However, we can justify the previous computation to prove the following:

Proposition 4.1. *If u_ε is a stable solution to the Allen–Cahn equation, then*

$$\int_{\{|\nabla u_\varepsilon|\neq 0\}} \left(|\nabla\psi|^2|\nabla u_\varepsilon|^2 - ((|D^2 u_\varepsilon|^2 - |\nabla|\nabla u_\varepsilon||^2) + \operatorname{Ric}_g(\nabla u_\varepsilon, \nabla u_\varepsilon))\psi^2\right) d\mu_g \geq 0.$$

Exercise 4.2. Prove this. Hint: For $\delta > 0$, show that $\sqrt{|\nabla u_\varepsilon|^2 + \delta^2}$ is smooth so stability implies that $\mathcal{Q}_{u_\varepsilon}(\psi\sqrt{|\nabla u_\varepsilon|^2 + \delta^2}, \psi\sqrt{|\nabla u_\varepsilon|^2 + \delta^2}) \geq 0$. Then send $\delta \to 0$. It might be useful to observe that $|\nabla|\nabla u_\varepsilon||^2 \leq |D^2 u|^2$ when $|\nabla u_\varepsilon| \neq 0$, which follows from $2|\nabla u_\varepsilon|\nabla|\nabla u_\varepsilon| = \nabla|\nabla u_\varepsilon|^2 = 2D^2 u_\varepsilon(\nabla u_\varepsilon, \cdot)$.

The observation that $|\nabla|\nabla u_\varepsilon||^2 \leq |D^2 u|^2$ when $|\nabla u_\varepsilon| \neq 0$ yields the following non-existence result.

Exercise 4.3. Suppose now that (M^n, g) has positive Ricci curvature.

(a) Show that there are no stable minimal (two-sided) hypersurfaces.
(b) Show that there are no stable (non-trivial) solutions to Allen–Cahn.

4.1. **The Pacard–Ritoré construction.** It turns out that solutions to Allen–Cahn exist near minimal surfaces Σ beyond just local minimizers, e.g. for unstable Σ. We say that Σ is *non-degenerate* if $\ker J = \{0\}$.

Theorem 4.2 (Pacard–Ritoré [PR03], cf. [JS09, Pac12])**.** *If $\Sigma^{n-1} \subset (M^n, g)$ is a smooth non-degenerate minimal hypersurface that divides M into two pieces, then for $\varepsilon_0 = \varepsilon_0(\Sigma, M, g) > 0$ sufficiently small, there exists $\{u_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ solving*

$$\varepsilon^2 \Delta_g u_\varepsilon = W'(u_\varepsilon) \tag{4.2}$$

and so that u_ε approximates Σ in the sense that u_ε converges to 1 on one side of Σ and -1 on the other side of Σ and so that

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = \sigma \operatorname{area}_g(\Sigma).$$

At a high level, the proof consists of perturbing the function $\mathbb{H}(\varepsilon^{-1} d_\Sigma(\cdot))$ to a solution of (4.2). See Problems 2 and 3 for important elements of the proof.

4.2. **Gaspar–Guaraco existence theory.** Theorem 4.2 shows how to find a solution to Allen–Cahn given a (non-degenerate) minimal surface. However, recently there has been a lot of activity regarding the other direction, i.e., using the Allen–Cahn equation to find (unstable) minimal surfaces. As such, one needs an existence result that does not depend on the existence of a minimal surface. We describe such a result below.

Consider (M^n, g) closed Riemannian manifold. Recall that ± 1 are the (unique) global minimizers for $E_\varepsilon(\cdot)$ (and they both have the same energy 0). This suggests the use of a mountain-pass theorem. Because $E_\varepsilon(\cdot)$ is defined on $H^1(M)$ an infinite dimensional space, one must check the so-called *Palais–Smale* condition. The idea of this (in reality, one must slightly modify this argument) is contained in the following exercise:

Exercise 4.4. For $\varepsilon > 0$ fixed, consider $u_k \in H^2(M)$ with $|u_k| \leq 1$, $E_\varepsilon(u_k) \leq C$ and $\|\Delta u_k - W'(u_k)\|_{L^2(M)} \to 0$. Show that a subsequence of u_k converges *strongly* in $H^1(M)$ to u which is a solution to the Allen–Cahn equation on (M, g).

Hint: Show that $\|u_k\|_{H^1(M)}$ is uniformly bounded and then for a weak subsequential limit u of the u_k, check that u solves the Allen–Cahn equation. Finally, relate $\int |\nabla(u_k - u)|^2 d\mu_g$ to $\langle \Delta u_k - W'(u_k), (u_k - u)\rangle_{L^2(M)}$ (up to other terms tending to zero) to prove strong convergence in $H^1(M)$.

This implies (when combined with energy bounds for paths betwen $+1$ and -1) the following result. In fact, higher index critical points exist as well, see the work of Gaspar–Guaraco [GG19].

Theorem 4.3 (Guaraco [Gua18])**.** *For (M^n, g) a closed Riemannian manifold and $\varepsilon > 0$ sufficiently small, there exists u_ε solving the Allen–Cahn equation with* $\operatorname{index}(u_\varepsilon) \leq 1$ *and $C^{-1} \leq E_\varepsilon(u_\varepsilon) \leq C$, for C independent of $\varepsilon > 0$.*

The function u_ε is constructed via a mountain pass result, by considering Υ, the set of all paths $[-1, 1] \mapsto u_\varepsilon^{(s)} \in H^1(M)$ with $u_\varepsilon^{(\pm 1)} = \pm 1$. Then u_ε is found by minimizing the quantity

$$\sup_{s\in[0,1]} E_\varepsilon(u_\varepsilon^{(s)}).$$

over all paths in Υ.

By taking $\varepsilon \to 0$ and using the regularity results of Hutchinson–Tonegawa, Tonegawa–Wickramasekera [HT00, TW12], this yields a new proof of the existence of minimal hypersurfaces in a closed Riemannian manifold (M^n, g), originally proved by Almgren, Pitts, and Schoen–Simon [Pit81, SS81].

5. Entire solutions to the Allen–Cahn equation

Exercise 5.1 (+)**.** Suppose that u_ε is a sequence of functions solving the Allen–Cahn equation on (M, g). Choose $x_j \in M$ and $\varepsilon_j \to 0$, and consider for K fixed (large) the ball $B_{K\varepsilon_j}(x_j) \subset M$.

(a) Make sense of what it means to "zoom in" by scale ε_j^{-1} by defining a new metric g_j on B_K and a rescaled function $\tilde{u}_j$.
(b) Show that g_j converges smoothly to the flat Euclidean metric on $B_K(0) \subset \mathbb{R}^n$ and (after passing to a subsequence) $\tilde{u}_j$ converges smoothly to $\tilde{u}$ solving the Allen–Cahn equation on B_K with $\varepsilon = 1$.
(c) If u_{ε_j} was stable in $B_\rho(x_j)$ for some $\rho > 0$ fixed, show that $\tilde{u}$ is stable in $B_K(0)$ for compactly supported variations.
(d) If $\operatorname{index}(u_{\varepsilon_j}; B_\rho(x_j)) \leq I_0$ show the same for $\tilde{u}$.
(e) If u_{ε_j} was δ-locally minimizing, what property does $\tilde{u}$ satisfy?
(f) Writing $\tilde{u}_K$ to emphasize the choice of K, show that we can send $K \to \infty$ to find an entire solution to the Allen–Cahn equation on $\mathbb{R}^n$ with $\varepsilon = 1$. What happens in cases (c)-(e)?

Exercise 5.1 motivates the study of entire solutions to Allen–Cahn with $\varepsilon = 1$ on $\mathbb{R}^n$ with various additional conditions (e.g., stability, bounded index, minimizing). However, this is not the original motivation for the study of entire solutions to Allen–Cahn. Indeed, a motivating problem in the study of the Allen–Cahn equation has been the following conjecture of De Giorgi made in 1978:

Conjecture 5.1 (De Giorgi [DG79])**.** *Consider $u \in C^2(\mathbb{R}^n)$ solving the Allen–Cahn equation*

$$\Delta u = W'(u) = u^3 - u$$

so that $|u| \leq 1$ and $\frac{\partial u}{\partial x^n} > 0$. At least for $n \leq 8$, is it true that $u(x) = \mathbb{H}(\langle a, x\rangle - b)$ is a one-dimensional solution?

This conjecture (and particularly the monotonicity $\frac{\partial u}{\partial x^n} > 0$ condition) here is motivated by the classical Bernstein conjecture for minimal surfaces.

Theorem 5.2 (Bernstein [Ber27], Fleming [Fle62], De Giorgi [DG65], Almgren [Alm66], Simons [Sim68], Bombieri–De Giorgi–Giusti [BDGG69])**.** *Suppose that*

$u : \mathbb{R}^{n-1} \to \mathbb{R}$ has the property that $\operatorname{graph}(u) \subset \mathbb{R}^n$ *is a minimal surface. Equivalently,*

$$\sum_{i=1}^{n-1} D_i \left(\frac{D_i u}{\sqrt{1 + |Du|^2}} \right) = 0.$$

Then, for $n \leq 8$, $u(x) = \langle x, a \rangle + b$ is an affine function. For $n > 8$, non-flat minimal graphs exist.

Unlike the Bernstein conjecture for minimal surfaces, the De Giorgi conjecture is not entirely resolved. It is completely understood in low dimensions

Theorem 5.3 (Ghoussoub–Gui [GG98] ($n = 2$), Ambrosio–Cabre [AC00] ($n = 3$))**.** *For $n = 2, 3$, consider $u \in C^2(\mathbb{R}^n)$ solving the Allen–Cahn equation with $|u| \leq 1$ and $\frac{\partial u}{\partial x^n} > 0$. Then, $u(x) = \mathbb{H}(\langle a, x \rangle - b)$.*

It is also completely understood in high dimensions (here, the dimensional restriction is expected to be sharp).

Theorem 5.4 (del Pino–Kowalczyk–Wei [dPKW11])**.** *For $n \geq 9$, there is $u \in C^\infty(\mathbb{R}^n)$ solving the Allen–Cahn equation with $|u| \leq 1$ and $\frac{\partial u}{\partial x^n} > 0$ that does not have flat level sets.*

For $n \in \{4, 5, \ldots, 8\}$, the De Giorgi conjecture is still open. However, Savin has shown that it is true under an additional hypothesis.

Theorem 5.5 (Savin [Sav09])**.** *For $n \leq 8$, consider $u \in C^2(\mathbb{R}^n)$ solving the Allen–Cahn equation with $|u| \leq 1$ and $\frac{\partial u}{\partial x^n} > 0$. Assume in addition that*

$$\lim_{x^n \to \pm 1} u(x) = \pm 1.$$

Then, $u(x) = \mathbb{H}(\langle a, x \rangle - b)$.

See also [Wan17a].

5.1. **Stability and minimizing properties of monotone solutions.** Recall that $u \in C^2(\mathbb{R}^n)$ solving the Allen–Cahn equation

$$\Delta u = W'(u)$$

is *stable* if it is stable on compact sets, and *minimizing* if it minimizes $E_1(\cdot)$ on compact sets.

We now discuss the role of the De Giorgi monotonicity condition as well as Savin's limit condition.

Lemma 5.6. *If u solves the Allen–Cahn equation on $\mathbb{R}^n$ and satisfies the De Giorgi monotonicity condition $\frac{\partial u}{\partial x^n} > 0$, then u is stable.*

Proof. Set $v = \frac{\partial u}{\partial x^n} > 0$. Differentiating the Allen–Cahn equation in the x^n-direction, we find that

$$\Delta v = W''(u) v.$$

In other words, v is a positive solution to the linearized Allen–Cahn equation. We use this to show that u is stable.

Consider φ the first eigenfunction associated to $\mathcal{Q}_u(\cdot,\cdot)$ with Dirichlet boundary conditions on $B_R(0)$, i.e., φ achieves the Rayleigh quotient

$$\lambda = \inf_{\varphi\in H_0^1(B_R)\setminus\{0\}} \frac{\mathcal{Q}_u(\varphi,\varphi)}{\|\varphi\|_{L^2}^2}$$

We have that

$$\Delta\varphi + \lambda\varphi = W''(u)\varphi.$$

If we can prove that $\lambda \geq 0$ (for all R) then u must be stable.

We know that (after possibly replacing φ by $-\varphi$) $\varphi > 0$; indeed, if not we could lower the Rayleigh quotient by replacing φ by $|\varphi|$. Choose $\mu > 0$ so that $v - \mu\varphi \geq 0$ with equality for some $x \in B_R(0)$ (we know that $x \notin \partial B_R$ because φ has Dirichlet boundary conditions).

We have that

$$0 \leq \Delta(v-\mu\varphi) - W''(u)(v-\mu\varphi) = \mu\lambda\varphi < 0$$

at x. Hence, $\lambda \geq 0$, completing the proof. ☐

We emphasize that the previous result shows that $\mathbb{H}(\langle x, a\rangle)$ is a stable solution to Allen–Cahn on $\mathbb{R}^n$. Before proving that the Savin condition implies a minimizing property, we first prove the following comparison result:

Lemma 5.7. *Suppose that u, v are solutions to the Allen–Cahn equation on $B_\delta(0) \subset \mathbb{R}^n$ with $u \leq v$ and $u(0) = v(0)$. Then $u = v$ on $B_\delta(0)$.*

Proof. Write $w = u - v$ on B_δ. Note that $w \leq 0$ with $w = 0$. We would like to apply the maximum principle to w. To this end, we must find a linear PDE satisfied by w. Note that

$$\begin{aligned}\Delta w &= W'(u) - W'(v)\\ &= \int_0^1 \frac{d}{dt} W'(tu+(1-t)v)dt\\ &= \underbrace{\left(\int_0^1 W''(tu+(1-t)v)dt\right)}_{:=\Xi} w\end{aligned}$$

Thus w satisfies the *linear* PDE $\Delta w = \Xi w$. The maximum principle thus applies to prove that $w = 0$ on $B_\delta(0)$. ☐

Proposition 5.8. *Suppose that u solves the Allen–Cahn equation on $\mathbb{R}^n$, and that we have $|u| < 1$, the De Giorgi monotonicity condition $\frac{\partial u}{\partial x^n} > 0$, and Savin's condition*

$$\lim_{x^n\to\pm\infty} u(x', x^n) = \pm 1$$

for all $x' \in \mathbb{R}^{n-1}$. Then, u minimizes $E_1(\cdot)$ on compact subsets of $\mathbb{R}^n$.

Proof. Fix $R > 0$ and choose $v \in C^\infty(B_R)$ achieving (see Exercise 5.2 for the existence of v)

$$\inf\{E_1(v) : v \in C^\infty(B_R), v|_{\partial B_R} = u|_{\partial B_R}\} \tag{5.1}$$

We claim that $|v| < 1$. First, observe that truncating v at ± 1, i.e.,

$$\tilde{v} := \mathbf{1}_{\{v\in[-1,1]\}} v + \mathbf{1}_{\{v>1\}} - \mathbf{1}_{\{v<-1\}}$$

does not increase the Dirichlet energy or the potential term, i.e., $E_1(\tilde{v}) \leq E_1(v)$. Thus, $|v| \leq 1$ by elliptic theory (if not, $\tilde{v}$ would be a non-smooth minimizer). Because ± 1 solve the Allen–Cahn equation, Lemma 5.7 implies that either $|v| < 1$ on $B_R(0)$ or $v = \pm 1$ on B_R, which cannot happen since we assumed that $v = u$ on ∂B_R and $|u| < 1$.

Now, let τ denote the smallest number so that for

$$u_\tau(x', x^n) := u(x', x^n + \tau),$$

we have $u_\tau \geq v$ on B_R. Observe that $\lim_{\tau\to\infty} u_\tau = 1$ uniformly on B_R, so for τ sufficiently large the desired condition is satisfied. By Exercise 5.2, $\tau \geq 0$ exists and there is $\hat{x} \in \overline{B_R}$ with $u_\tau(\hat{x}) = v(\hat{x})$.

If $\tau > 0$, then $\hat{x} \notin \partial B_R$, since on ∂B_R, $u_\tau > u = v$ by the De Giorgi monotonicity condition. Hence, in this case we can apply Lemma 5.7 to conclude that $u_\tau = v$ in a neighborhood of $\hat{x}$ and thus all of B_R by unique continuation. This is a contradiction, since $u_\tau \neq v$ on ∂B_R. Thus $\tau = 0$, so $u \geq v$ on B_R.

A similar argument proves that $v \leq u$ on B_R, so $u = v$ on B_R. This implies that u attains the minimum in (5.1), completing the proof. $\square$

Exercise 5.2 (+)**.** Fill in the following steps of the proof of Proposition 5.8.

(a) Check that a smooth function v achieving the infimum in (5.1) exists.

(b) Check that τ exists and there is some point $\hat{x} \in \overline{B_R}$ so that $u_\tau(\hat{x}) = v(\hat{x})$.

Exercise 5.3. For u solving the Allen–Cahn equation with $|u| < 1$, suppose that u satisfies De Giorgi's monotonicity condition $\frac{\partial u}{\partial x^n} > 0$ but not necessarily Savin's condition. Show that $u_\pm(x') := \lim_{x^n\to\pm\infty} u(x', x^n)$ exist (and solve the Allen–Cahn equation). Show that u minimizes $E_1(\cdot)$ on B_R among functions $v(x', x^n)$ with $u_-(x') \leq v(x', x^n) \leq u_+(x')$ for all $(x', x^n) \in B_R$.

Exercise 5.4. Suppose that $|u| \leq 1$ minimizes $E_1(\cdot)$ on compact sets. Show that there is $C > 0$ so that $E_1(u; B_R) \leq CR^{n-1}$ for all $R > 0$.

5.2. **Classifying stable entire solutions.** As such, we see that to solve De Giorgi's conjecture it makes sense to study entire stable solutions.[7] The following results represent the state of affairs of the classification of stable solutions in $\mathbb{R}^n$.

[7] Additionally, the understanding of entire stable solutions is intimately linked with the local behavior of stable/bounded index solutions on a manifold, cf. [WW19, CM18, WW18].

Theorem 5.9 (Ghoussoub–Gui [GG98]). *Consider $u \in C^2(\mathbb{R}^2)$ a stable solution to the Allen–Cahn equation with $|u| < 1$. Then $u(x) = \mathbb{H}(\langle a, x\rangle - b)$ is the 1-dimensional solution.*

See also [FMV13], who give a slightly different strategy of proof (this is the basis for the proof we give below).

Theorem 5.10 (Ambrosio–Cabre [AC00]). *Consider $u \in C^2(\mathbb{R}^3)$ a stable solution to the Allen–Cahn equation with $|u| \leq 1$ and $E_1(u; B_R) \leq CR^2$ for some $C > 0$ independent of R. Then $u(x) = \mathbb{H}(\langle a, x\rangle - b)$ is the 1-dimensional solution.*

Theorem 5.11 (Pacard–Wei [PW13]). *For $n \geq 8$, there exists $u \in C^\infty(\mathbb{R}^n)$ a stable solution to the Allen–Cahn equation with $E_1(u; B_R) \leq CR^{n-1}$ but the level sets of u are not flat.*

Liu–Wang–Wei [LWW17] have recently extended this result to construct *minimizers* in $\mathbb{R}^n$ for $n \geq 8$.

5.3. **Stable solutions in $\mathbb{R}^2$.** We prove Theorem 5.9. The beginning of the argument will work in all dimensions. We will indicate where we specialize to $n = 2$ below. Assume that u is a stable solution to Allen–Cahn on $\mathbb{R}^n$ with $|u| \leq 1$. By Proposition 4.1, stability implies that we have

$$\int_{\mathbb{R}^n} |\nabla\varphi|^2 |\nabla u|^2 d\mu \geq \int_{\mathbb{R}^n} |\mathbf{B}|^2 \varphi^2 |\nabla u|^2 d\mu$$

for any compactly supported smooth function φ, where

$$|\mathbf{B}|^2 = |\nabla u|^{-2}(|D^2 u|^2 - |\nabla|\nabla u||^2)$$

at points where $\nabla u \neq 0$ and $|\mathbf{B}|^2 = 0$ when $\nabla u = 0$.

Exercise 5.5 (+). This problem concerns the quantity $|\mathbf{B}|^2$.

(a) Show that a solution to Allen–Cahn on $\mathbb{R}^n$ with $|u| < 1$ cannot have $\nabla u = 0$ everywhere.

(b) Show that $|D^2 u|^2 - |\nabla|\nabla u||^2 \geq 0$ at a point with $\nabla u \neq 0$.

(c) Suppose a solution to Allen–Cahn with $|u| < 1$ has $|\mathbf{B}|^2 = 0$ on all of $\mathbb{R}^n$. Show that $\frac{\nabla u}{|\nabla u|}$ is a parallel vector field on $\{\nabla u \neq 0\}$. Hint: Compute $|D(\nabla u / |\nabla u|)|^2$.

(d) Use unique continuation to prove that a solution to Allen–Cahn with $|u| < 1$ and $|\mathbf{B}|^2 = 0$ on all of $\mathbb{R}^n$ must be one-dimensional, i.e.,

$$u(x) = \tilde{u}(\langle a, x - x_0\rangle).$$

(e) Assume that $|\mathbf{B}|^2 \equiv 0$ and $|u| < 1$ is stable. Show that $u(x) = \mathbb{H}(\langle a, x - x_0\rangle)$ for some $|a| = 1$, $x_0 \in \mathbb{R}^n$. Is this true without the stability condition? What if stability is replaced by $E_1(u; B_R) \leq CR^{n-1}$?

(f) The quantity $|\mathbf{B}|^2$ is often thought of as the (square) norm of the "second fundamental form" of u. Justify this heuristic.

As such, this problem shows that the 1-dimensional solution is the unique stable solution on $\mathbb{R}^n$ with vanishing second fundamental form and $|u| < 1$.

Exercise 5.6 (+)**.** Using interior Schauder estimates and $|u| < 1$, prove that $|\nabla u| \leq C$ on $\mathbb{R}^n$, for $C = C(n)$ (note that this does not depend on the stability condition).

Now, we specialize to $n = 2$. We would like to choose cutoff functions φ_i that tends to 1 pointwise on $\mathbb{R}^2$ and so that

$$\int_{\mathbb{R}^2} |\nabla \varphi_i|^2 \to 0.$$

The function ψ_R cutting linearly (perhaps with a bit of smoothing) off between R and $2R$ only gives

$$\int_{\mathbb{R}^2} |\nabla \psi_R|^2 \lesssim R^{-2} R^2 \leq C,$$

which is just barely failing what we want. It turns out that the solution is to use the *log-cutoff trick* which appears all over the place in similar problems (e.g. stable minimal surfaces in $\mathbb{R}^3$). Motivated by the fundamental solution to the Laplacian on $\mathbb{R}^2$, we set

$$\varphi_R(x) := \begin{cases} 1 & |x| \leq R \\ 2 - \frac{\log |x|}{\log R} & R < |x| < R^2 \\ 0 & |x| \geq R^2 \end{cases}$$

for $R > 1$. As usual, φ is only Lipschitz, but we can justify plugging it into the stability inequality by an approximating argument. We thus find that

$$\int_{\mathbb{R}^2} |\nabla \varphi_R|^2 = \int_{B_{R^2}(0) \setminus B_R(0)} \frac{1}{|x|^2 \log^2 R} \lesssim \frac{1}{\log^2 R} \int_R^{R^2} r^{-1} dr = \frac{\log R}{\log^2 R} = \frac{1}{\log R} \to 0$$

as $R \to \infty$. We now use the previous two exercises: because $|\nabla u| \leq C$ we get that

$$\liminf_{R \to \infty} \int_{\mathbb{R}^2} |\nabla \varphi_R|^2 |\nabla u|^2 \leq C^2 \liminf_{R \to \infty} \int_{\mathbb{R}^2} |\nabla \varphi_R|^2 = 0.$$

Moreover, combining $\varphi \to 1$ pointwise and the fact that the right hand side of the stability inequality is non-negative, Fatou's lemma implies that

$$\int_{\mathbb{R}^2} |\mathbf{B}|^2 |\nabla u|^2 \leq 0.$$

Thus $|\mathbf{B}|^2 = 0$, so the proof is finished.

5.4. **Stable solutions in $\mathbb{R}^3$.** The strategy we used before has no hope of working if we try to repeat the steps verbatim.

Exercise 5.7. Show that if $\varphi \equiv 1$ on $B_R(0) \subset \mathbb{R}^3$ and φ has compact support, then $\int_{\mathbb{R}^3} |\nabla\varphi|^2 \geq 4\pi R$.

However, under the energy growth assumption $E_1(u; B_R) \leq CR^2$, we can do better by not using $|\nabla u| \leq C$. Using φ_R (the log-cutoff function), we find

$$\int_{\mathbb{R}^3} |\mathbf{B}|^2 \varphi_R^2 |\nabla u|^2 \leq \int_{\mathbb{R}^3} |\nabla\varphi_R|^2 |\nabla u|^2 = \frac{1}{\log^2 R} \int_{B_{R^2}(0)\setminus B_R(0)} |x|^{-2} |\nabla u|^2.$$

For simplicity, assume that $R = 2^k$ for some $k = \frac{\log R}{\log 2} \in \mathbb{N}$. Then, write

$$R_0 = 2^k, R_1 = 2^{k+1}, \ldots, R_k = 2^{2k} = R^2.$$

so

$$\begin{aligned}
\int_{B_{R^2}(0)\setminus B_R(0)} |x|^{-2}|\nabla u|^2 &= \sum_{j=0}^{k-1} \int_{B_{R_{j+1}}\setminus B_{R_j}} |x|^{-2}|\nabla u|^2 \\
&\leq \sum_{j=0}^{k-1} R_j^{-2} \int_{B_{R_{j+1}}\setminus B_{R_j}} |\nabla u|^2 \\
&\leq \sum_{j=0}^{k-1} R_j^{-2} \int_{B_{R_{j+1}}} |\nabla u|^2 \\
&\leq C \sum_{j=0}^{k-1} R_j^{-2} R_{j+1}^2 \\
&\leq C \sum_{j=0}^{k-1} 4 \\
&= Ck \\
&= \frac{C \log R}{\log 2}.
\end{aligned}$$

Because this integral is multiplied by $(\log R)^{-2}$, we find that

$$\int_{\mathbb{R}^3} |\nabla\varphi_R|^2 |\nabla u|^2 \to 0$$

as before. The proof is then completed as for $n = 2$.

5.5. **Area growth of monotone solutions in $\mathbb{R}^3$.** Finally, we present the proof of De Giorgi's conjecture in $\mathbb{R}^3$ by Ambrosio–Cabre (Theorem 5.3). Note that in $n = 2$, because monotone solutions are stable, the $n = 2$ classification of stable solutions of Ghoussoub–Gui (Theorem 5.9) automatically resolves the problem.

In $\mathbb{R}^3$, to apply the classification of stable solutions we must verify that monotone solutions have the quadratic area growth $E_1(u; B_R) \leq CR^2$.

Define $u^t(x) = u(x_1, x_2, x_3 + t)$. By monotonicity,

$$u^{\pm\infty}(x) := \lim_{t\to\pm\infty} u^t(x)$$

exists and is independent of x^3. Moreover, by using Schauder estimates, we see that the limit occurs smoothly on compact subsets of $\mathbb{R}^3$.

Exercise 5.8 (+)**.** Show that $u^{\pm\infty}(x_1, x_2)$ is a stable solution to Allen–Cahn on $\mathbb{R}^2$. Thus, the classification of stable solutions on $\mathbb{R}^2$ shows they are 1-dimensional. Use this to show that (as functions on $\mathbb{R}^3$) we have

$$E_1(u^{\pm\infty}; B_R \subset \mathbb{R}^3) \leq CR^2$$

for some $C > 0$ independent of R. Thus, conclude that

$$\lim_{t\to\pm\infty} E_1(u^t; B_R \subset \mathbb{R}^3) \leq CR^2$$

for some $C > 0$ independent of R.

Note that because u is monotone, $\partial_t u^t > 0$. Now, consider

$$E_1(u^t; B_R) := \int_{B_R} \frac{1}{2}|\nabla u^t|^2 + W(u^t)$$

The idea is to differentiate this with respect to t and use the information just gained as $t \to \infty$. Recall that $|\nabla u^t| \leq C$. We now compute:

$$\begin{aligned}
\partial_t E_1(u^t; B_R) &= \int_{B_R} \left\langle \nabla \partial_t u^t, \nabla u^t \right\rangle + W'(u^t)\partial_t u^t \\
&= \int_{B_R} -(\partial^t u^t)\Delta u^t + W'(u^t)\partial_t u^t + \int_{\partial B_R} \partial_t u^t \partial_\nu u^t \\
&= \int_{\partial B_R} \partial_t u^t \partial_\nu u^t \\
&\geq -C \int_{\partial B_R} \partial_t u^t.
\end{aligned}$$

In the final inequality we crucially used (again) the monotonicity property of u.

Thus, integrating this with respect to t, we find that

$$E_1(u^{+\infty}; B_R) - E_1(u; B_R) \geq -C \int_{\partial B_R} (u^{+\infty} - u) \geq -CR^2.$$

In the final inequality, we used $|u| \leq 1$ and $|\partial B_R| = 4\pi R^2$. Putting this together, we find that

$$E_1(u; B_R) \leq CR^2.$$

This completes the proof.

Appendix A. BV basics

We discuss some basic facts about BV functions. Further references for BV functions and sets of finite perimeter include [Sim83, Giu84, AFP00, Mag12, Leo17].

Recall that for $\Omega \subset (M^n, g)$ an open set, $u \in L^1(\Omega)$ is in $BV(\Omega)$ if Du is a TM-valued Radon measure, i.e., for any vector field $X \in C^1_c(\Omega; TM)$

$$\int_\Omega u \operatorname{div}_g X \, d\mu_g = -\int_\Omega g(X, Du).$$

Recall that $|Du|$ is then a (usual) Radon measure defined by

$$\int_{\Omega'} |Du| = \sup \left\{ \int_{\Omega'} u \operatorname{div}_g X \, d\mu_g : X \in C^1_c(\Omega'; TM), \|X\|_{L^\infty} \leq 1 \right\}.$$

Then, we set

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + \int_\Omega |Du|.$$

This is clearly a norm.

Lemma A.1. *The space $BV(\Omega)$ is a Banach space.*

Proof. We only need to check completeness. A Cauchy sequence $u_k \in BV(\Omega)$ converges in L^1 to $u \in L^1(\Omega)$, since $L^1(\Omega)$ is complete. Now,

$$\int_\Omega u_k \operatorname{div}_g X d\mu_g \to \int_\Omega u \operatorname{div}_g X d\mu_g,$$

so $u \in BV(\Omega)$. Finally, for X as in the definition of the BV norm, we have

$$\begin{aligned}\int_\Omega (u - u_k) \operatorname{div}_g X d\mu_g &= \int_\Omega (u - u_j) \operatorname{div}_g X d\mu_g + \int_\Omega (u_j - u_k) \operatorname{div}_g X d\mu_g \\ &= \int_\Omega (u - u_j) \operatorname{div}_g X d\mu_g - \int_\Omega g(D(u_j - u_k), X).\end{aligned}$$

For any $X \in C^1_c(\Omega; TM)$ there is j sufficiently large so that

$$\int_\Omega (u - u_j) \operatorname{div}_g X d\mu_g < \frac{\delta}{2},$$

since $u_j \xrightarrow{L^1} u$. Moreover, for all j, k sufficiently large,

$$\sup_{X \in C^1_c(\Omega; TM), \|X\|_\infty \leq 1} \int_\Omega g(D(u_j - u_k), X) < \frac{\delta}{2},$$

by the Cauchy sequence property. Putting this together, we find that

$$\int_\Omega |Du - Du_k| < \delta$$

for k large. This completes the proof. $\square$

Exercise A.1. In single-variable real analysis one often says that a function f defined on an interval, say $[0,1]$ is of bounded variation to mean

$$V(f) := \sup_{P\in\mathcal{P}} \sum_{i=0}^{|P|-1} |f(x_{i+1}) - f(x_i)| < \infty$$

where $\mathcal{P}$ is the set of finite partitions $0 = x_0 < x_1 < \cdots < x_{|P|} = 1$ of $[0,1]$. Show that $V(f) < \infty$ implies that $f \in BV((0,1))$ (in the sense described above) and

$$\int_{[0,1]} |Df| \leq V(f).$$

You might use the following steps (although there are several proofs):

(a) Assume that $V(f) < \infty$. For $\varphi \in C^1_c((0,1))$ and $\varepsilon > 0$ given, show that there exists a partition $P \in \mathcal{P}$ so that

$$\int_0^1 f(x)\varphi'(x)dx \leq \sum_{i=0}^{|P|-1} f(x_i)(\varphi(x_{i+1}) - \varphi(x_i)) + \varepsilon.$$

You should check that the left-hand side makes sense as a Riemann integral.

(b) Conclude that

$$\int_0^1 f(x)\varphi'(x)dx \leq \|\varphi\|_{C^0_c((0,1))} V(f)$$

(c) Use this and the dual characterization of Radon measures to conclude that $f \in BV(\Omega)$ with $\int_{[0,1]} |Df| \leq V(f)$.

Conversely, it is possible to show that if $f \in BV((0,1))$ then there is a right continuous representative $\bar{f}$ of f on $[0,1]$ with $V(\bar{f}) = \int_{[0,1]} |D\bar{f}| < \infty$, but this is slightly more involved.

Exercise A.2. (a) Show that $W^{1,1}(\Omega) \subset BV(\Omega)$ is a continuous embedding.

(b) Show that if E is an open set in Ω with ∂E a C^2-hypersurface, then $\chi_E \in BV(\Omega)$ but $\chi_E \notin W^{1,1}(\Omega)$. You might begin by considering $\Omega = (0,2)$ and $E = (0,1)$, so $\partial E = \{1\}$.

(c) Conclude that smooth functions are *not* dense in $BV(\Omega)$.

In spite of the previous exercise, it is possible to approximate BV functions by smooth functions in a certain sense. For simplicity, we restrict to $\Omega \subset \mathbb{R}^n$ although this can easily be extended to a manifold by using local coordinate charts.

Fix $\psi \in C^\infty_c$ a symmetric mollifier, i.e., $\psi \geq 0, \operatorname{supp}\psi \subset B_1(0), \int_{\mathbb{R}^n} \psi(x)dx = 1$, and $\psi(-x) = \psi(x)$. Set $\psi_\sigma(x) = \sigma^{-n}\psi(x/\sigma)$. For $u \in BV_{\mathrm{loc}}(\Omega)$, consider the mollified function $u^{(\sigma)} := u * \psi_\sigma$ (where we consider $u = 0$ outside of Ω, so $u^{(\sigma)}$ is well defined on $\mathbb{R}^n$). Set $\Omega_\sigma := \{x \in \Omega : d(x, \Omega^c) > \sigma\}$.

Lemma A.2. *The function $u^{(\sigma)}$ is smooth on $\mathbb{R}^n$ and as $\sigma \to 0$, we have $u^{(\sigma)} \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ and $|Du^{(\sigma)}| \to |Du|$ as Radon measures on Ω.*

Proof. We only prove that $|Du^\sigma| \to |Du|$ as Radon measures, the L^1 convergence is standard. For $f \in C^0_c(\Omega)$, we want to show that (for $\sigma < d(\operatorname{supp} f, \Omega^c)$)

$$\int_\Omega f|Du^{(\sigma)}| \to \int_\Omega f|Du|.$$

We can clearly restrict to $f \geq 0$. Note that

$$\int_\Omega f|Du^{(\sigma)}| = \sup_{X \in C^1_c(\Omega:\mathbb{R}^n), |X| \leq f} \int_\Omega X \cdot \nabla u^{(\sigma)} d\mu$$

Choose X as above, and note that

$$\begin{aligned}\int_\Omega X \cdot \nabla u^\sigma d\mu &= -\int_\Omega u^{(\sigma)} \operatorname{div} X \, d\mu \\ &= -\int_{\Omega_\sigma} u^{(\sigma)} \operatorname{div} X \, d\mu \\ &= -\int_{\Omega_\sigma} u * \psi_\sigma \operatorname{div} X \, d\mu \\ &= -\int_{\Omega_\sigma} u \operatorname{div}(X * \psi_\sigma) \, d\mu \\ &\leq \int_\Omega (f + o(1))|Du|\end{aligned}$$

as $\sigma \to 0$. In the last step, we used the fact that $|X * \psi_\sigma| \leq |X| * \psi_\sigma \leq f * \psi_\sigma$ and $f * \psi_\sigma \to f$ locally uniformly in Ω. Thus,

$$\limsup_{\sigma \to 0} \int_\Omega f|Du^{(\sigma)}| \leq \int_\Omega f|Du|.$$

On the other hand, since $X * \psi_\sigma \to X$ uniformly on Ω,

$$\begin{aligned}\int_\Omega X \cdot Du \, d\mu &= \lim_{\sigma \to 0} \int_\Omega (X * \psi_\sigma) \cdot Du \, d\mu \\ &= \lim_{\sigma \to 0} \int_\Omega X \cdot \nabla u^\sigma d\mu \\ &\leq \liminf_{\sigma \to 0} \int_\Omega f|Du^\sigma|\end{aligned}$$

so the opposite inequality holds. This completes the proof. ∎

Exercise A.3. Note that we are not claiming that $u^{(\sigma)} \to u$ in BV. Convince yourself that for $\Omega = (0,2)$, $u = \chi_{(0,1)}$, $u^{(\sigma)}$ converges to u in $L^1((0,2))$ but $u^{(\sigma)}$ is not Cauchy in $BV((0,2))$.

Proposition A.3. *Choose a constant $c(\Omega') > 0$ for each $\Omega' \Subset \Omega$. Then the set*

$$\{u \in C^\infty(\Omega) : \|u\|_{W^{1,1}(\Omega')} \leq c(\Omega'), \forall \Omega' \Subset \Omega\}$$

is precompact in $L^1_{\mathrm{loc}}(\Omega)$.

Proof. This is just a version of the Rellich–Kondrachov compactness result, i.e., that $W^{1,1}(\Omega') \subset L^1(\Omega'')$ is a compact embedding for $\Omega'' \Subset \Omega'$ (or we could consider Ω' with Lipschitz boundary). See e.g., [GT01, Theorem 7.22]. □

Exercise A.4. Combine Lemma A.2 with Proposition A.3 to prove BV compactness (Proposition 3.2), i.e., if $u_k \in BV(\Omega)$ has

$$\sup_k \|u_k\|_{BV(\Omega')} < \infty$$

for all $\Omega' \Subset \Omega$, then a subsequence (not relabeled) converges in $L^1_{\mathrm{loc}}(\Omega)$ to $u \in BV_{\mathrm{loc}}(\Omega)$ and

$$\int_{\Omega'} |Du| \leq \liminf_{k\to\infty} \int_{\Omega'} |Du_k|.$$

Exercise A.5. Note that we mostly use BV compactness on a manifold. For (M, g) a closed Riemannian manifold, and $u_k \in BV(M, g)$ with $\sup_k \|u_k\|_{BV} < \infty$, explain how to modify the results of this section to show that a subsequence converges in $L^1(M)$ to $u \in BV(M, g)$.

Now recall as well that an open set $E \subset \Omega \subset (M, g)$ is a *set of finite perimeter* (or *Caccioppoli* set) if $\chi_E \in BV(\Omega)$. In this case,

$$P(E; \Omega) := \int_\Omega |D\chi_E|.$$

Exercise A.6. For $E \subset \mathbb{R}^n$ a set with ∂E a C^2-hypersurface, show that $P(E; \Omega) = \mathcal{H}^{n-1}(\partial E \cap \Omega)$. Hint: $P(E; \Omega) \leq \mathcal{H}^{n-1}(\partial E \cap \Omega)$ follows from the divergence theorem. To prove the other direction, first show that the unit normal ν may be extended appropriately to a vector field on Ω.

Exercise A.7. Show that if (M, g) is a closed Riemannian manifold, and $V \in (0, \mathrm{vol}_g(M))$ is fixed, then there exists a set of finite perimeter $E \subset (M, g)$ attaining

$$\inf\{P(E') : E' \text{ has finite perimeter and } \mathcal{L}^n_g(E') = V\}.$$

The set E is said to be an *isoperimetric* region of volume V.

After learning the Γ-convergence theory for the Allen–Cahn functional (Section 3) you might figure out how to relate the Allen–Cahn functional to isoperimetric regions by considering minimizers of the Allen–Cahn energy with a constraint $\int_M u = \lambda$. How does λ relate to the enclosed volume as $\varepsilon \to 0$?

Appendix B. Additional problems

Problem 1. This problem concerns Modica's inequality [Mod85] and the related monotonicity formula.

Suppose that $u \in C^2_{\text{loc}}(\mathbb{R}^n)$ has $|u| \leq 1$ and solves the Allen–Cahn equation $\Delta u = W'(u)$. Define

$$P = |\nabla u|^2 - 2W(u).$$

This part of this problem asks you to show that $P \leq 0$.

(a) Show that $\inf_{\mathbb{R}^n} |\nabla u| = 0$.

(b) Show that

$$|\nabla u|^2 \Delta P \geq \frac{1}{2}|\nabla P|^2 + 2W'(u)\nabla u \cdot \nabla P.$$

(c) Assume that $\sup_{\mathbb{R}^n} P = P(0) > 0$. Use (b) to show that P is constant and then use (a) to obtain a contradiction. This proves that $P \leq 0$ when $\sup_{\mathbb{R}^n} P$ is attained.

(d) Show that $P \leq 0$ still holds even when the supremum is not attained. Hint: Assume that $P(x_i) \to \sup_{\mathbb{R}^n} P > 0$ and set $u_i(x) := u(x - x_i)$. Pass to a C^2_{loc} subsequence and use (c) in the limit.

(e) Suppose that $P = 0$ somewhere. Conclude that $u(x) = \mathbb{H}(\langle a, x - x_0\rangle)$ for some $x_0 \in \mathbb{R}^n, a \in \mathbb{S}^{n-1}$.

We now use $P \leq 0$ to derive a monotonicity formula for the Allen–Cahn energy.

(f) Denote by

$$E_R := R^{1-n} \int_{B_R(0)} \left(\frac{1}{2}|\nabla u|^2 + W(u)\right) d\mu$$

the (rescaled) Allen–Cahn energy on B_R, check that

$$\frac{dE_R}{dR} = -(n-1)R^{-1}E_R + R^n \int_{\partial B_R(0)} \left(\frac{1}{2}|\nabla u|^2 + W(u)\right) \langle x, \nu\rangle \, d\mu,$$

for ν the outwards pointing unit normal to ∂B_R. Integrate by parts to conclude that

$$\frac{dE_R}{dR} = R^{-n} \int_{B_R(0)} (-P) d\mu + R^{1-n} \int_{\partial B_R(0)} (\partial_\nu u)^2 d\mu \geq 0.$$

(g) For $u = \mathbb{H}(x^n)$ compute $\lim_{R\to 0} E_R$ and $\lim_{R\to\infty} E_R$. Compare your answer with the behavior of the monotonicity formula for minimal surfaces when applied to a flat plane.

For (M, g) a closed manifold, we now consider $u \in C^\infty(M)$ solving

$$\varepsilon \Delta_g u = \varepsilon^{-1} W'(u)$$

Set

$$P = \varepsilon|\nabla u|^2 - 2\varepsilon^{-1}W(u)$$

(h) Compute an equation for P analogous to the one in (b).

(i) If $\mathrm{Ric}_g \geq 0$ show that $P \leq 0$.

(Note: Without assuming $\mathrm{Ric}_g \geq 0$, it is still possible to prove $P \leq C = C(M,g)$ independent of ε. For example, one can apply a similar argument to $P_G = \varepsilon|\nabla u|^2 - 2\varepsilon^{-1}W(u) - G(u)$ for a well-chosen function G. See [HT00, Proposition 3.3]. Using this, we can prove that in (M,g) the expression $e^{\Lambda R}E_R$ is monotone non-decreasing for R small and Λ large, just as minimal surfaces satisfy a weighted monotonicity formula in Riemannian manifolds.)

Problem 2. This problem asks you to prove the following result (used in the proof of Theorem 4.2 and also in the regularity theory for stable solutions [WW19, CM18, WW18], among other places), see [PR03, Corollary 7.5] (cf. [Pac12]).

Suppose that $w \in L^\infty(\mathbb{R}^{n-1} \times \mathbb{R})$ is in the kernel of the linearized Allen–Cahn operator at the heteroclinic solution:

$$L_* w := \Delta w - W''(\mathbb{H}(x^n))w = 0.$$

We claim that $w(x', x^n) = c\mathbb{H}'(x^n)$ for some $c \in \mathbb{R}$. Prove this as follows:

(a) Check that $\mathbb{H}'(x^n) \in L^\infty(\mathbb{R}^n)$ satisfies $L_*(\mathbb{H}'(x^n)) = 0$.

(b) Prove the claim for $n = 1$. Hint, consider $(\log \mathbb{H}'(t))''$.

(c) Still when $n = 1$, argue that there is some $\mu > 0$ so that if $u(t)$ satisfies $\int_{-\infty}^{\infty} u(t)\mathbb{H}'(t)dt = 0$, then

$$\int_{-\infty}^{\infty} u'(t)^2 + W''(\mathbb{H}(t))u(t)\, dt \geq \mu \int_{-\infty}^{\infty} u(t)^2\, dt.$$

(d) For $n \geq 2$, write a solution to $L_* w = 0$ as $w(x', x^n) = c(x')\mathbb{H}'(x^n) + \bar{w}(x', x^n)$ where

$$\int_{-\infty}^{\infty} \bar{w}(x', t)\mathbb{H}'(t)dt = 0$$

for each $x' \in \mathbb{R}^{n-1}$. Show that c is bounded and harmonic and thus constant.

(e) Show that for $\sigma \in (0, \sqrt{2})$, $\delta \in (0,1)$, and all $\eta > 0$, the function

$$\Upsilon(x) := e^{-\sigma|x^n|} + \eta \cosh(\delta x^n) \sum_{i=1}^{n-1} \cosh(\delta x^i)$$

satisfies $L_*\Upsilon < 0$ when $|x^n| \geq \Lambda$ for $\Lambda = \Lambda(\sigma, \delta)$ large. Conclude that for all $\eta > 0$,

$$|w(x', x^n)| \leq \|w\|_{L^\infty(\mathbb{R}^n)} \left(e^{\sigma(\Lambda - |x^n|)} + \eta \cosh(\delta x^n) \sum_{i=1}^{n-1} \cosh(\delta x^i) \right)$$

for $|x^n| \geq \Lambda$. Let $\eta \to 0$ to find some C depending on w (and σ) but not $|x^n|$ so that

$$|w(x', x^n)| \leq Ce^{-\sigma|x^n|}.$$

Show that $|\nabla w|$ satisfies a similar inequality.

(f) Consider,

$$V(x') := \int_{-\infty}^{\infty} \bar{w}(x', t)^2 dt.$$

Use (e) to justify differentiating under the integral sign to conclude that

$$\Delta_{\mathbb{R}^{n-1}} V - \mu V \geq 2 \int_{-\infty}^{\infty} |\nabla \bar{w}(x', t)|^2 dt \geq 0.$$

(g) Conclude that $V \equiv 0$ and thus prove the claim.

Problem 3. This problem can be concerns a formal power series expansion in ε of the Allen–Cahn equation around a minimal surface $\Gamma \subset \mathbb{R}^3$. We will make finer and finer approximations for a solution to the Allen–Cahn equation. Understanding of these calculations are the first step towards perturbing these functions to be exact solutions (see Theorem 4.2) as well as understanding arbitrary solutions under certain hypothesis (see [WW19, CM18, WW18]).

The problem requires some knowledge of Riemannian geometry, particularly the Gauss equations for the Levi–Civita connection induced on a hypersurface. The problem might look very long, but this is because you will build up the approximate solution piece-by-piece, rather than all at once.

Consider $\Gamma^2 \subset \mathbb{R}^3$ a smooth embedded surface with unit normal $p \mapsto N(p)$. For $z \in \mathbb{R}$, set

$$X_z : \Gamma \ni p \mapsto p + zN(p) \in \mathbb{R}^3.$$

We begin by computing various quantities associated to this map.

(a) Choose coordinates y^1, y^2 near $p \in \Gamma$ and compute $\partial_{y^i} X_z$ for z fixed. Show that (with the convention that $\partial_{y^i} N(p) = \sum_{j=1}^{2} A_{ij}(p)\partial_{y^j}$) we have

$$\partial_{y^i} X_z = \partial_{y^i} + zA_{ij}\partial_{y^j}$$

Assuming that $A_{ij}(p) = \lambda_i \delta_{ij}$ is diagonal at p determine the largest interval $(\underline{z}(p), \overline{z}(p))$ containing 0 so that $dX_z(p)$ injective for all $z \in (\underline{z}(p), \overline{z}(p))$. In the remainder of this problem, assume that $z \in (\underline{z}(p), \overline{z}(p))$.

(b) Assume that $(-\delta, \delta) \subset (\underline{z}(q), \overline{z}(q))$ for all q in some neighborhood U of p. Consider the map $U \times (-\delta, \delta) \ni (q, z) \mapsto X_z(q)$. Using the coordinates y^1, y^2 for q and map yields *Fermi coordinates* y^1, y^2, z around Γ (a generalization of normal coordinates). Show that

$$g_{\mathbb{R}^3} = dz^2 + g_z$$

where $(g_z)_{ij} = (\partial_{y^i} X_z) \cdot (\partial_{y^j} X_z)$ is the induced metric on the smooth surface $\Gamma_z := X_z(U)$.

(c) Show that the $\mathbb{R}^3$-Laplacian satisfies

$$\Delta = \partial_z^2 + H_{\Gamma_z}\partial_z + \Delta_{\Gamma_z}$$

where Δ_{Γ_z} is the g_z-Laplacian.

Hint: you can do this by computing in local coordinates, but it might be easier to prove the general formula for the Laplacian of functions restricted to a hypersurface $\Sigma^{n-1} \subset (M^n, g)$ with unit normal ν: $\Delta_g f = D^2 f(\nu,\nu) + H_\Sigma \nabla_\nu f + \Delta_\Sigma f$. To prove this, recall how the connections on Σ and M are related by the second fundamental form.

(d) If Γ is a minimal surface, show that

$$H_z(X_z(p)) = -z|A_\Gamma(p)|^2 + O(z^3).$$

Note that is easier to prove that the error term is $O(z^2)$; this weaker estimate will suffice until part (m) below.

Similarly, show that for a function $w(y)$ on Γ,

$$\Delta_{\Gamma_z} w = \Delta_\Gamma w + O(|z|(|Dw| + |D^2 w|)).$$

for $z \in (-\delta, \delta)$.

Define an ansatz u on $X_z(U \times (-\delta,\delta))$ by $u(y,z) = \mathbb{H}(\varepsilon^{-1}z)$.

(e) Show that $\varepsilon^2 \Delta u - W'(u) = O(\varepsilon)$ on $U \times (-\delta,\delta)$ as $\varepsilon \to 0$. Show that for general surfaces Γ, this cannot be improved.

(f) Assuming that Γ is a minimal surface, show that $\varepsilon^2\Delta u - W'(u) = O(\varepsilon^2)$ as $\varepsilon \to 0$. Hint: note that $\mathbb{H}'(t) = O(e^{-\sqrt{2}|t|})$ as $|t| \to \infty$, so $t\mathbb{H}'(t) \in L^\infty(\mathbb{R})$.

(g) Suppose that $u(y,z) = \mathbb{H}(\varepsilon^{-1}z)$ is an exact solution to Allen–Cahn on $U \times (-\delta,\delta)$. Show that $\Gamma \cap U$ is a piece of a flat plane.

From now on, we assume that $H_\Gamma = 0$. For Γ non-planar, (g) shows that we must consider a more flexible ansatz. To this end, set $u(y,z) = \mathbb{H}(\varepsilon^{-1}z) + \varepsilon u_1(y, \varepsilon^{-1}z)$ where $u_1(y,\zeta)$ is a smooth function with all derivatives bounded on $U \times \mathbb{R}$.

(h) Compute $\varepsilon^2\Delta u - W'(u)$ and show that $\varepsilon^2 \Delta u - W'(u) = O(\varepsilon)$.

(i) Show that $\partial_\zeta^2 u_1(y,\zeta) - W''(\mathbb{H}(\zeta))u_1(y,\zeta) = 0$ for each $y \in U$ implies that $\varepsilon^2\Delta u - W'(u) = O(\varepsilon^2)$.

(j) Show that $\partial_\zeta^2 u_1(y,\zeta) - W''(\mathbb{H}(\zeta))u_1(y,\zeta) = 0$ is equivalent to $u_1(y,\zeta) = -h(y)\mathbb{H}'(\zeta)$ for some smooth function $h(y)$ (see Problem 2).

(k) Show that

$$\mathbb{H}(\varepsilon^{-1}z) - \varepsilon h(y)\mathbb{H}'(\varepsilon^{-1}z) = \mathbb{H}(\varepsilon^{-1}z - \varepsilon h(y)) + O(\varepsilon^2)$$

uniformly as $\varepsilon \to 0$ (along with all derivatives).

We now the next term in the expansion, i.e.,

$$u(y,z) = \mathbb{H}(\varepsilon^{-1}z - \varepsilon h(y)) + \varepsilon^2 u_2(y, \varepsilon^{-1}z).$$

for $u_2(y,\zeta)$ a smooth function with all derivatives bounded on $U \times \mathbb{R}$.

(l) Show that

$$\varepsilon^2\Delta u - W'(u) = (\varepsilon H_{\Gamma_z} - \varepsilon^3\Delta_{\Gamma_z} h)\mathbb{H}' + \varepsilon^2(\varepsilon^2\Delta u_2 - W''(\mathbb{H})u_2) + O(\varepsilon^4)$$

(m) Assume that

$$\partial_\zeta^2 u_2(y,\zeta) - W''(\mathbb{H}(\zeta))u_2(y,\zeta) = |A_\Gamma|^2(y)\zeta\mathbb{H}'(\zeta)$$

Show that $\varepsilon^2\Delta u - W'(u) = O(\varepsilon^3)$.

(n) Show that $\mathbb{J}(t) = \mathbb{H}'(t)\int_0^t\int_{-\infty}^s \tau\mathbb{H}'(s)^{-2}\mathbb{H}'(\tau)^2 d\tau ds$ is the unique solution of

$$\mathbb{J}''(t) - W'(\mathbb{H}(t))\mathbb{J}(t) = t\mathbb{H}'(t), \qquad \mathbb{J}(0) = 0,$$

so we can take $u_2(y,\zeta) = |A_\Gamma|^2(y)\mathbb{J}(\zeta)$.

Now, consider one more final ansatz. We assume that $y = 0$ corresponds to $p \in \Gamma$.

$$u(y,z) = \mathbb{H}(\varepsilon^{-1}z - \varepsilon h(y)) + \varepsilon^2|A_\Gamma|^2(y)\mathbb{J}(\varepsilon^{-1}z) + \varepsilon^3 u_3(\varepsilon^{-1}y, \varepsilon^{-1}z)$$

for $u_3(\omega,\zeta)$ a smooth function on $\mathbb{R}^2\times\mathbb{R}$. Note the difference between u_1 and u_2.

(o) Show that

$$\begin{aligned}
&\varepsilon^2\Delta u - W'(u)\\
&= \varepsilon^3(\Delta_{\mathbb{R}^3} u_3 - W''(\mathbb{H}(\zeta))u_3)\\
&- \varepsilon^3\left(\mathbb{H}'(\zeta)\Delta_\Gamma h(y) - (W'''(\mathbb{H}(\zeta))\mathbb{H}'(\zeta)\mathbb{J}(\zeta) + \zeta\mathbb{H}''(\zeta))\,|A_\Gamma|^2(y)h(y)\right)\\
&+ O(\varepsilon^4)
\end{aligned}$$

as $\varepsilon \to 0$.

(p) The first term is $\varepsilon^3 L_* u_3$, studied in Problem 2. Compute

$$\int_{-\infty}^{\infty}\Big(\mathbb{H}'(\zeta)\Delta_\Gamma h(y) - (W'''(\mathbb{H}(\zeta))\mathbb{H}'(\zeta)\mathbb{J}(\zeta) + \zeta\mathbb{H}''(\zeta))\,|A_\Gamma|^2 h(y)\Big)\mathbb{H}'(\zeta)d\zeta$$

How does the answer compare to the hypothesis in Theorem 4.2?

Having finished this problem, you can now understand much more of the proof of Theorem 4.2. The $\mathbb{H}'$-directions in the error $O(\varepsilon^4)$ can be handled by modifying h appropriately (as shown in (p)) and the other error can be handled by adjusting u_3. See [Pac12] for a proof along these lines.

Problem 4. This problem concerns the appearance of "multiplicity" in non-minimizing limits of Allen–Cahn.

(a) Show that the following situation is not possible: for $\delta > 0$ fixed, u_ε are δ-minimizers of the Allen–Cahn energy on a compact manifold (M,g) and $u_\varepsilon \to -1$ in $L^2(M)$ but $\varepsilon|\nabla u_\varepsilon|^2 d\mu_g$ converges weakly to the measure $2\sigma\mathcal{H}^{n-1}|_\Sigma$ for some smooth closed hypersurface Σ.

(b) Show that the previous phenomenon (multiplicity) can occur for limits of stable solutions. Hint: consider (M^n, g) a closed manifold containing a region isometric to a warped product on $(-1,1) \times \mathbb{S}^{n-1}$ with metric $dt^2 + f(t)^2 g_{\mathbb{S}^{n-1}}$. Choose f so that there is a sequence $\tau_i \to 0$ with $\{\pm\tau_i\} \times \mathbb{S}^{n-1}$ are non-degenerate stable minimal surfaces. Use Theorem 4.2. Alternatively, you can use the fact that non-degenerate stable surfaces are locally L^1-minimizing [Whi94, MR10] and apply Proposition 3.11.[8]

(c) Check that $\{0\} \times \mathbb{S}^{n-1}$ is a degenerate minimal surface.

(d) What is the L^1-limit of the solutions u_ε constructed in (b)?

Problem 5. The parabolic version of the Allen–Cahn equation

$$\partial_t u = \Delta u - \varepsilon^{-2} W'(u)$$

is known to approximate mean-curvature flow in a similar manner to the static situation discussed in these notes (see [Ilm93]). Assume that (M, g) is closed. Standard parabolic theory implies that for initial conditions with $|u_0| < 1$, there is a unique solution (which exists for all time).

(a) Show that $|u| < 1$ for all $t > 0$.

(b) Show that

$$\partial_t E_\varepsilon(u(\cdot, t)) = -\varepsilon^{-1} \int_M \left(-\varepsilon \Delta u + \frac{1}{\varepsilon} W'(u)\right)^2 d\mu_g \leq 0.$$

(c) Show that any sequence $t_k \to \infty$ has a further subsequence $t_{k'}$ so that $u(\cdot, t_{k'}) \to u_\infty$ in $C^\infty(M)$.

(d) Show that

$$\Delta u - \varepsilon^{-2} W'(u) > 0$$

(or similarly < 0) is preserved along the flow. This is analogous to the preservation of mean convexity along mean curvature flow.

(e) Suppose that

$$\Delta u - \varepsilon^{-2} W'(u) > 0$$

along the flow. For $t_{k'} \to \infty$ and u_∞ as in (c), show that u_∞ is stable (see Section 4).

(f) If (M, g) has positive Ricci curvature show that if u is a solution to the Allen–Cahn equation, then there is u_t solving the parabolic Allen–Cahn equation with

$$\lim_{t \to -\infty} u_t = u, \qquad \lim_{t \to \infty} u_t = 1.$$

[8]The latter option shows that the resulting solutions are stable (why?), while if one applies Theorem 4.2, then to prove that the solutions are stable one can refer to e.g. [CM18].

Conclude that for any u solving the Allen–Cahn equation on a manifold of positive Ricci curvature, there is a continuous map $[-1,1] \mapsto u_s \in H^1(M)$ with $u_{\pm 1} = \pm 1$, $u_0 = u$ and

$$\max_{s\in[-1,1]} E_\varepsilon(u_s) = E(u_0).$$

Compare with Theorem 4.3.

Problem 6. Higher co-dimension analogues of Allen–Cahn are also studied. For example, the Allen–Cahn equation is naturally associated with hypersurfaces, since they (roughly) correspond to $u_\varepsilon^{-1}(0)$ as $\varepsilon \to 0$; hence, a natural way to study higher-codimension minimal surfaces via this approach is to increase the dimension of the range of u. To this end, it is natural to consider the Ginzburg–Landau equation, coming from the energy

$$E_\varepsilon(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2$$

for $u : \Omega \to \mathbb{C}$. We will consider $\Omega \subset \mathbb{R}^n$ (but one could also consider subsets of a Riemannian manifold as we have done above). Note that the scaling in $\varepsilon > 0$ differs from the Allen–Cahn setting considered above, the reasons for this will become apparent from this problem.

(a) Compute the PDE satisfied by a critical point of E_ε.

(b) Take $\Omega = B_1(0) \subset \mathbb{R}^2$. Let α_ε denote the infimum of $E_\varepsilon(u)$ over all smooth functions with $u(x) = x$ on ∂B. Show that

$$\alpha_\varepsilon \leq \pi \log \frac{1}{\varepsilon} + C$$

as $\varepsilon \to 0$. How does this compare to the case of Allen–Cahn? (Note the shift in powers of ε).

(c) Show that there exists $u_\varepsilon \in C^\infty(B_1;\mathbb{C})$ attaining α_ε.

(d) Show that given $\varepsilon_k \to 0$, there is a subsequence (not relabeled) so that $|u_{\varepsilon_k}| \to 1$ a.e. in B_1 as $k \to \infty$.

(e) Use the fact[9] that there is no $u \in H^1(B_1;\mathbb{C})$ with $|u| = 1$ a.e. and $u(x) = x$ on ∂B_1 to conclude that $\alpha_\varepsilon \to \infty$ as $\varepsilon \to 0$.

In fact, one can prove that $\alpha_\varepsilon = \pi \log \frac{1}{\varepsilon} + O(1)$ and $u_\varepsilon(x) \to \frac{x}{|x|}$ a.e. in B_1 (and there is a rich story for other domains Ω). See e.g., [BBH17].

[9]Loosely speaking, this says that the identity map $\mathbb{S}^1 \to \mathbb{S}^1$ is not null-homotopic along a continuous family of $H^{1/2}$-maps (normally one would consider a continuous family of C^0-maps).

Appendix C. Further reading

We give (a non-exhaustive) list of some references (in addition to those given above):

- The varifold theory of solutions to Allen–Cahn and related results: [Ilm93, HT00, TW12, Hie17, Gas17]
- De Giorgi conjecture and classification stable solutions (besides those discussed above, there are several other related results): [GG03, JM04, FS17].
- Gibbons conjecture (the De Giorgi conjecture with a stronger condition as $x^n \to \pm\infty$ is proven in all dimensions): [Far99, BBG00, BHM00].
- Existence/classification of solutions on $\mathbb{R}^2$ (our understanding of entire solutions is best in dimension 2; however, many uniqueness questions are still open): [KL11, Gui12, KLP13, dPKP13, KLPW15, GLW16, Wan17b, WW19].
- Other entire solutions in $\mathbb{R}^n$ (in higher dimensions there are many interesting entire solutions; only a few classification results are known): [dP10, dPMP12, dPKW13, AdPW15, LWW19, GWW19].
- The Allen–Cahn equation on manifolds (further existence and qualitative results not discussed above): [Man17, GG19, CM18]
- Higher co-dimension analogues of Allen–Cahn, including the Ginzburg–Landau equation: [LR99, BBO01, LR01, BOS05, Ste19, Che17, PS19]

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