Second-Order *Tâtonnement* for Market Equilibria: Methods, Complexity, and Extensions

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Introduction

The Competitive Market Equilibrium Problem

- Goods $j = 1,\ldots,n$ (with ***unit*** supplies)

  Prices $\mathbf{p} = [p^{(1)}, \ldots, p^{(n)}] \in \mathbb{R}^n_+$
- Agent $i = 1,\ldots,m$

  Preferences/Utility u_i budget $w_i \in \mathbb{R}_+$

Utility Maximization Problem (UMP)

- $\mathbf{x}_i(\mathbf{p})$: the (Walrasian) demand

$$\mathbf{x}_i(\mathbf{p}) = \operatorname{argmax} \quad u_i(\mathbf{x}_i)$$
$$\text{s.t.} \quad \langle \mathbf{p}, \mathbf{x}_i \rangle \le w_i$$

The Market Equilibrium Problem

$$\mathbb{R}^n_+ \ni \mathbf{p} \perp \mathbf{1} - \sum_i \mathbf{x}_i(\mathbf{p}) \in \mathbb{R}^n_+$$

Find $\mathbf{p}$ so demand ***clears*** the market

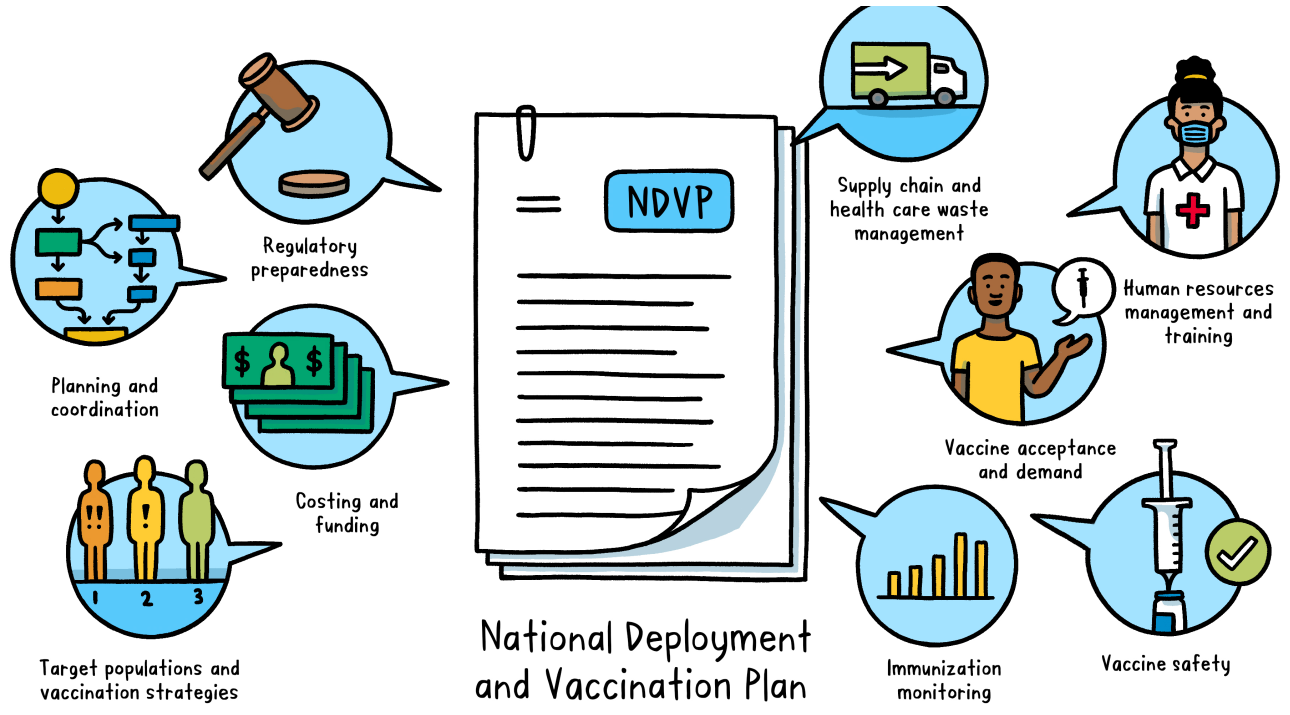
- Equilibrium Price: $\mathbf{p}$
- Optimal Allocation: $\mathbf{x}_1(\mathbf{p}), \ldots, \mathbf{x}_m(\mathbf{p})$

- Pareto efficiency/envy-freeness
- Strategy-proof-at-large
- ...

Applications

Fairness, Equity, Pareto optimality

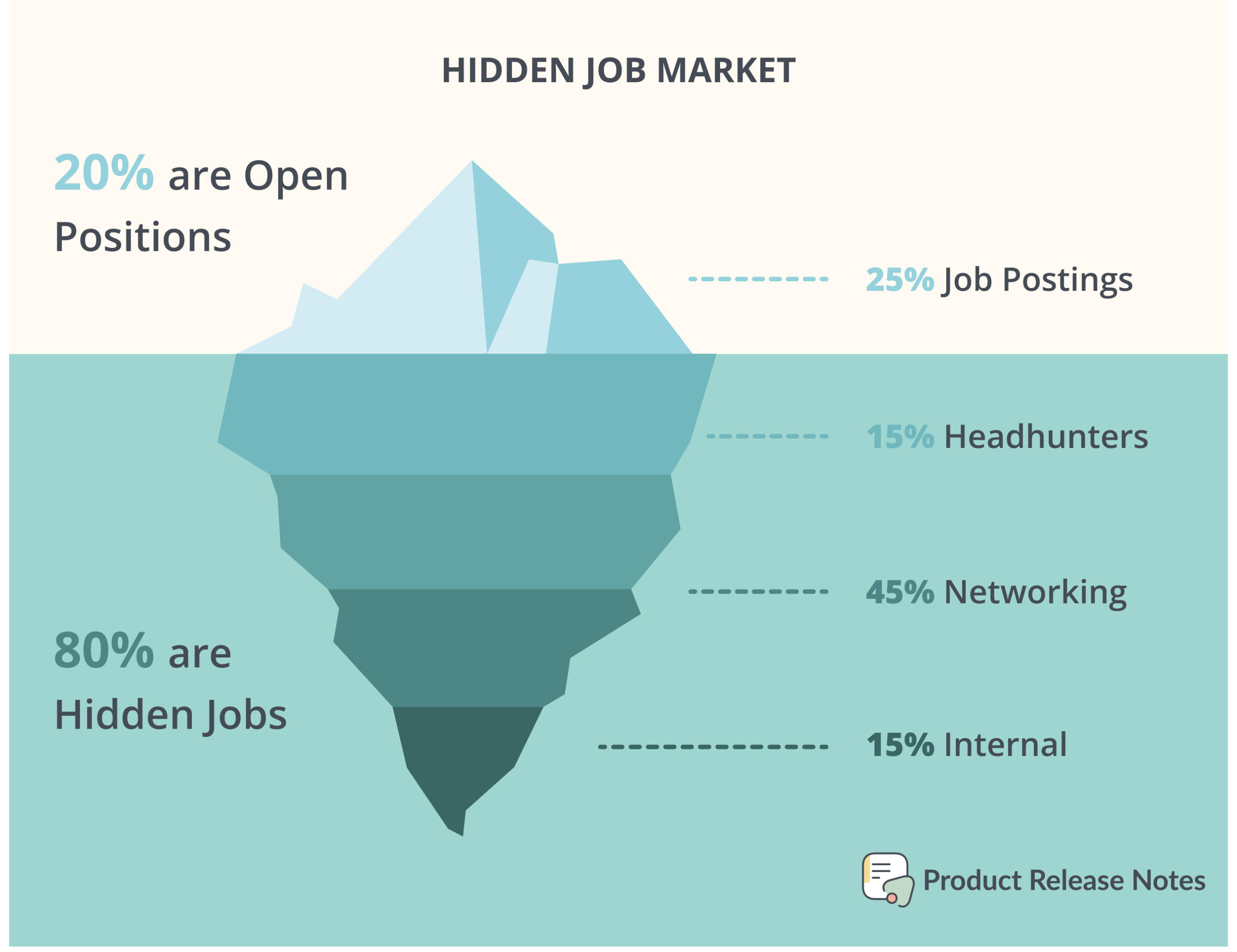
Course Allocation
Budish, 2011, J. Pol. Econ.



Vaccine Allocation
World Health Organization, 2021

Balance/congestion-free/service quality

Online Display Ad
Bateni, 2022, OR; Li and Ye, 2022, OR;
Conitzer et al MS, OR, 2022.



Job Market/Allocation
Bulow and Levin, 2006, AER; Coles et al. 2006

How to find the ***equilibrium***?

Use *first-order* information: the *Tâtonnement* (1874)

The Tâtonnement Process *(Walras, 1874)*

- Compute ***market excess demand***

$$\mathbf{z}(\mathbf{p}) = \sum_i \mathbf{x}_i(\mathbf{p}) - \mathbf{1}$$

- Update price ($\mathscr{G}$ is some customized rule)

$$\mathbf{p}_+ = \mathbf{p} + \mathscr{G}(\mathbf{z}(\mathbf{p}))$$

- $\pi_i(\cdot)$: utility at the demand

$$\pi_i(\mathbf{p}) = \max \quad u_i(\mathbf{x}_i) \qquad \text{s.t.} \quad \langle \mathbf{p}, \mathbf{x}_i \rangle \leq w_i$$

- The Lyapunov function

$$\min_{\mathbf{p}\in\mathbb{R}^n_+} \varphi(\mathbf{p}) = \langle \mathbf{1}, \mathbf{p} \rangle + \sum_i \log(\pi_i(\mathbf{p}))$$

- Tâtonnement is a ***first-order*** scheme

$$\nabla \varphi(\mathbf{p}) = \mathbf{1} - \sum_i \mathbf{x}_i(\mathbf{p}) = -\mathbf{z}(\mathbf{p})$$

- For linear economy: $\mathcal{O}(m\frac{1}{\varepsilon})$ slow!

- *Informationally* cheap:

  Only use the demand $\mathbf{x}_1(\mathbf{p}), \ldots, \mathbf{x}_m(\mathbf{p})$

  "the revealed preferences"

Use *second-order* information?

- Steve Smale (*Fields Medal, 1966*), **"Global Newton"**, *1976*

$$\mathbf{p}_{+} \leftarrow \mathbf{p} - \kappa \left[\nabla \mathbf{z}(\mathbf{p}) \right]^{-1} \mathbf{z}(\mathbf{p})$$

- Resolve a ***counterexample*** of tâtonnement (*Scarf, 1960*)
- Actually "not global": Only local convergence (*Keenan, 1981*)
- ***Informationally*** expensive: requires marginal demand changes

  Less information? Generally no! (*Saari and Simon, 1978*)

- Computational difficulty: matrix factorizations for $\left[\nabla \mathbf{z}(\mathbf{p}) \right]^{-1}$

  Need $\mathcal{O}(n^3)$ operations

Use *full* information for allocation? early 2000s

- Eisenberg-Gale program (*1959*)

$$\max_{\mathbf{x}_1,\ldots,\mathbf{x}_m} \quad \sum_{i=1}^{m} \log(u_i(\mathbf{x}_i))$$

$$\text{s.t.} \quad \sum_{i}^{m} \mathbf{x}_i \le \mathbf{1},\ \mathbf{x}_i \ge 0, \forall i$$

- The dual variable: price $\mathbf{p}$

- Convex optimization

  Use ellipsoid / use exp cones and IPMs
- Combinatorial (*Devanur et al. 2002*)
- IPMs/Weighted LCP: (*Ye 2008*)

- Need ***full*** information
- Centralization

  You act like a “**central**” planner
- Unrealistic for Amazon, eBay…

This work

Method	Style	Information	Convergence	Comp. Cost
Convex allocation Combinatorial: Devanur et al. 2002 IPMs: Ye 2008	Centralized	Full	Linear (polynomial-time)	Matrix factorizations Max-flows …
Tâtonnement Arrow and Hurwitz 1958, Cheung et al. 2013	Price-update	Demand	Sublinear	Collect *m* responses
Smale's process (second-order) Smale 1976, Saari and Simon 1978	Price-update	Demand Jacobian	?	Collect *m* responses Collect the Jacobian Matrix factorizations
Second-order tâtonnement (Decentralized IPMs)	**Price-update**	**Demand (free Jacobian)**	**Linear Superlinear**	**Collect *m* responses Cheap linear system ≅ Tâtonnement**

Agenda

- Use the Lypanov function

$$\min_{\mathbf{p}\in\mathbb{R}^n_+} \varphi(\mathbf{p}) = \langle \mathbf{1}, \mathbf{p} \rangle + \sum_i \log(\pi_i(\mathbf{p}))$$

- Goal: find the equilibrium price $\mathbf{p}$

$$\|\nabla\varphi(\mathbf{p})\|_\infty < \epsilon \quad \text{and} \quad \nabla\varphi(\mathbf{p}) = \mathbf{1} - \sum_i \mathbf{x}_i(\mathbf{p})$$

- Use high-order information as a **free lunch**

$$\nabla^2\varphi, \text{ i.e., } \nabla\mathbf{z}, \nabla\mathbf{x}_1, \ldots, \nabla\mathbf{x}_m$$

- Develop interior-point **price-update mechanisms**
- Solve the Newton-type equation “cheaply”

$$(\nabla^2\varphi)^{-1}\nabla\varphi$$

The ***Scaled Lipschitzness*** of Utility Maximization

The Utility Maximization

- Recall UMP,

$$\mathbf{x}_i(\mathbf{p}) = \text{argmax} \quad u_i(\mathbf{x}_i) \quad \text{s.t.} \quad \langle \mathbf{p}, \mathbf{x}_i \rangle \le w_i$$

- Tasks / Questions for demand $\mathbf{x}_i(\mathbf{p})$
- Calculus/How to compute $\nabla \mathbf{x}_i$?
- Lipschitz properties?

- KKT system:

$$-\nabla u_i(\mathbf{x}_i) + \lambda \mathbf{x}_i = 0$$
$$\langle \mathbf{p}, \mathbf{x}_i \rangle \le w_i$$

- Differentiation by $\mathbf{p}$

$$\Longrightarrow \begin{aligned} -\nabla^2 u_i(\mathbf{x}_i)\nabla \mathbf{x}_i + \mathbf{x}_i \nabla \lambda^\top + \lambda \nabla \mathbf{x}_i &= 0 \\ \mathbf{p}^\top \nabla \mathbf{x}_i + \mathbf{x}_i^\top &= 0 \end{aligned}$$

- Perturbation analysis (*Bonnans & Shapiro, 1998*)
- Easier way?

The Utility Maximization

Additively Homogeneous Utility

$$u(\mathbf{x}) = \left(\sum_j \theta_j(x^{(j)}) \right)^k$$

- $\theta_j(x^{(j)})$: component $r-$homogeneous

$$\forall \alpha > 0,\ \theta_j(\alpha \cdot x) = \alpha^r \theta_j(x)$$

- $u(\mathbf{x})$: overall $(r \times k)-$homogeneous

$$u(\mathbf{x}) = \langle \theta(\mathbf{x}), \mathbf{1} \rangle^k$$

- Linear:

$$u(\mathbf{x}) = \sum_j c^{(j)} x^{(j)}$$

- Constant-elasticity of substitution (CES)

$$u(\mathbf{x}) = \left(\sum_j c^{(j)} (x^{(j)})^r \right)^{1/r}$$

- Leontief:

$$u(\mathbf{x}) = \min_j x^{(j)} / c^{(j)}$$

- ...

The Logarithmic Utility Maximization

- For homogeneous utility systems

  $u(\mathbf{x}) = \langle \theta(\mathbf{x}), \mathbf{1} \rangle^k,\ d$-homogeneous

Logarithmic Utility Maximization

- Negative Logarithmic utility:

  $v(\mathbf{x}) = -\log(u(\mathbf{x}))$

- The log-UMP and dual function $f(\mathbf{p})$

  $$f(\mathbf{p}) = \max \quad -v(\mathbf{x}) \quad \text{s.t.} \quad \langle \mathbf{p}, \mathbf{x} \rangle \leq w$$

- The demand $\mathbf{x}$ also solves regular UMP
- Fenchel conjugacy: (v, f)

  "logarithmically homogeneity"

  $f(\mathbf{p}) = v_*(\mathbf{p}) + d\log(w) - C\,.$

  (Nesterov & Todd, 1996)

- Reinterpretation of Lyapunov function

  $$\min_{\mathbf{p} \in \mathbb{R}^n_+} \varphi(\mathbf{p}) = \langle \mathbf{1}, \mathbf{p} \rangle + \sum_i \log(\pi_i(\mathbf{p}))$$
  $$= \langle \mathbf{1}, \mathbf{p} \rangle + \sum_i w_i f_i(\mathbf{p})$$

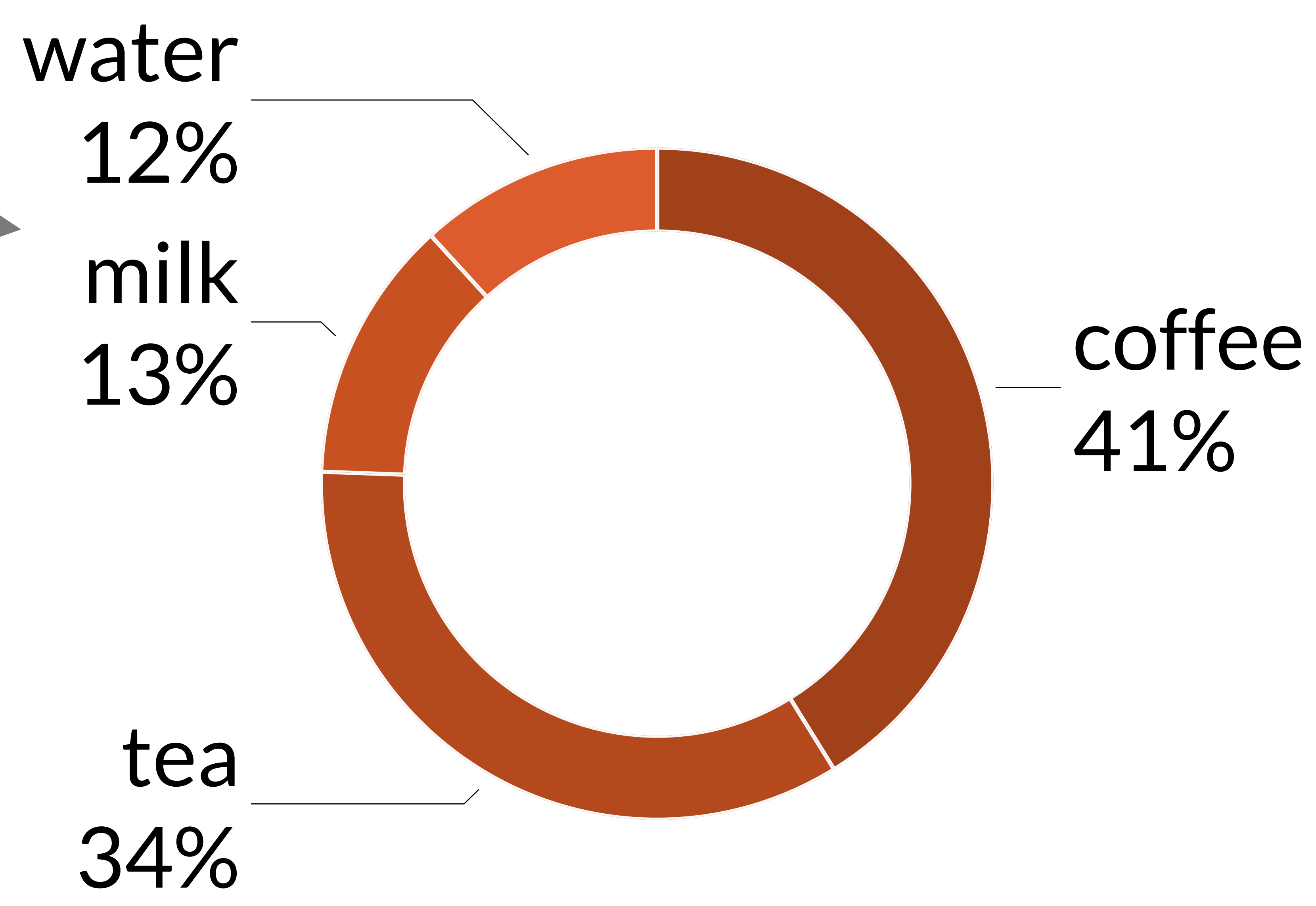
- Super-easy calculus!

The Logarithmic Utility Maximization

- First-order derivative is the “bidding vector”:

$$\frac{1}{d}\mathbf{P}\nabla f(\mathbf{p}) = \gamma = \frac{1}{w}\mathbf{P}\mathbf{x}(\mathbf{p}) \in \Delta_n$$

γ - distribution of spent money



- Second-order derivatives:

$$\mathbf{P}\nabla^2 f(\mathbf{p})\mathbf{P} \propto \mathbf{P}\nabla \mathbf{x}(\mathbf{p})\mathbf{P}$$

$$= \frac{d}{1-r}\mathrm{diag}(\gamma) - \frac{dr}{1-r}\gamma\gamma^\top$$

- High-order info is ***free***!

**params* (d, r) *can be evaluated by using several prices*

The Logarithmic Utility Maximization

Scaled Lipschitz Continuity (SLC)

- Primal (allocation) space: $\mathbf{x} \geq 0, \quad \rho = \|\mathbf{h}\|_{\mathbf{x}} = \|\mathbf{X}^{-1}\mathbf{h}\| < 1$

$$|\, v(\mathbf{x}+\mathbf{h}) - \underbrace{v(\mathbf{x}) - \nabla v(\mathbf{x})[\mathbf{h}] - \tfrac{1}{2}\nabla^2 v(\mathbf{x})[\mathbf{h},\mathbf{h}]}_{2^{\text{nd}}\text{order Taylor model}} \,| \leq \frac{T_u}{2}\frac{\rho^3}{3(1-\rho)}.$$

- Dual (price) space: $\mathbf{p} \geq 0, \quad \varrho = \|\mathbf{q}\|_{\mathbf{p}} = \|\mathbf{P}^{-1}\mathbf{q}\| < 1$

$$\left| f(\mathbf{p}+\mathbf{q}) - f(\mathbf{p}) - \nabla f(\mathbf{p})[\mathbf{q}] - \tfrac{1}{2}\nabla^2 f(\mathbf{p})[\mathbf{q},\mathbf{q}] \right| \leq \frac{T_f}{2}\frac{\varrho^3}{3(1-\varrho)}.$$

New property, similar to Monteiro and Adler 1990, Kortanek and Zhu 1993, den Hertog et al. 1995, Nesterov and Nemirovskii 1994, Andersen and Ye 1999

The Logarithmic Utility Maximization

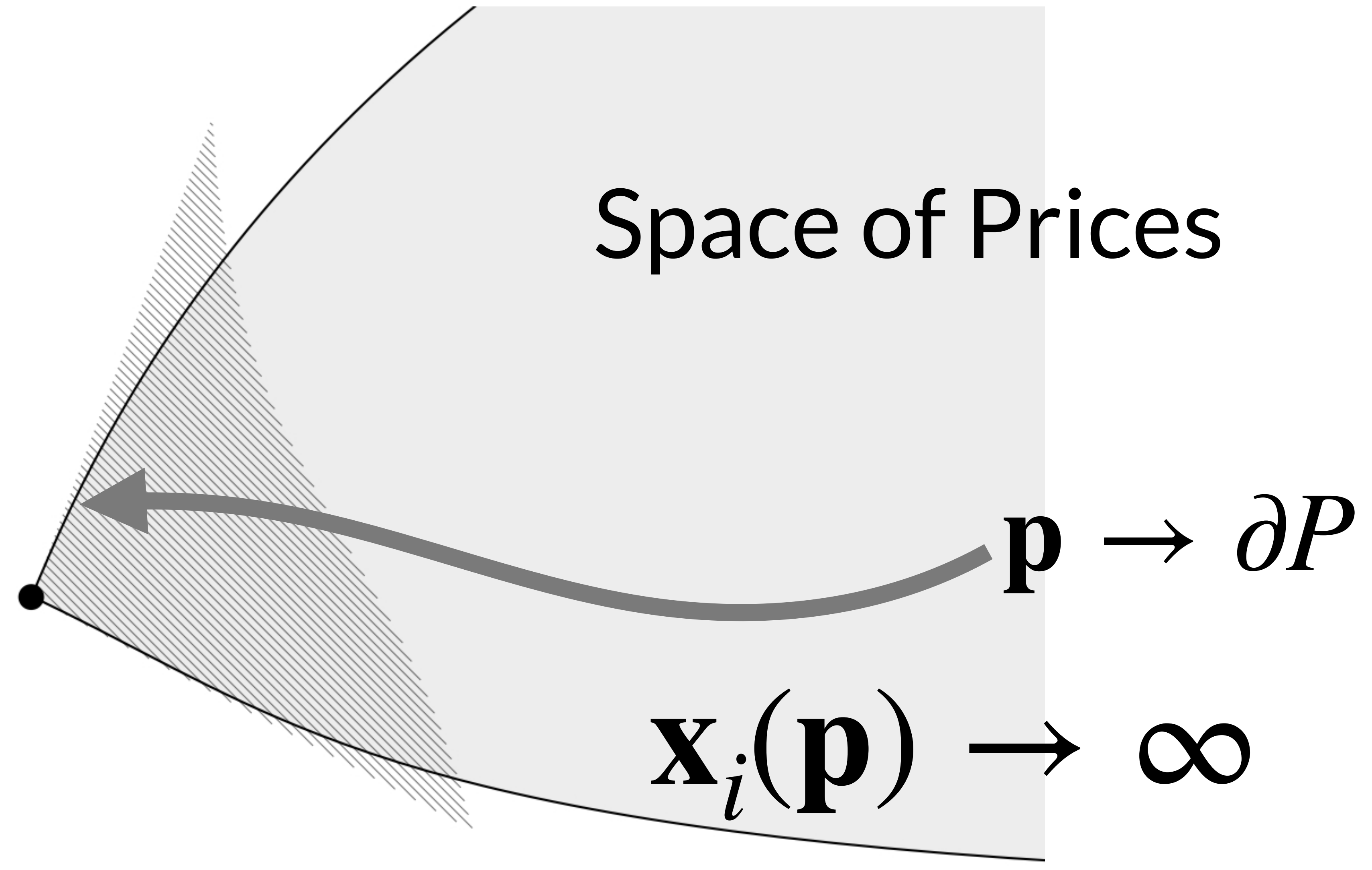
Scaled Lipschitz Continuity (SLC)

- The Dikin ball

$$\varrho = \|\mathbf{q}\|_{\mathbf{p}} = \|\mathbf{P}^{-1}\mathbf{q}\| < 1 \quad \Longrightarrow \quad \mathbf{p} + \mathbf{q} \in \mathbb{R}^n_+$$

- Demand function

$$\left|\mathbf{P}\mathbf{x}(\mathbf{p}+\mathbf{q}) - \mathbf{x}(\mathbf{p}) - \nabla \mathbf{x}(\mathbf{p})[\mathbf{q}]\right| \leq \frac{T_f}{2}\frac{w}{d}\frac{\varrho^2}{2(1-\varrho)}$$



- *SLC* **of** φ: Inheritance from $\mathbf{x}_1, \ldots, \mathbf{x}_m$

The Log-UMP: What We Gain

- The Lyapunov inherits all from $f_1, \ldots, f_m$

$$\min_{\mathbf{p} \in \mathbb{R}^n_+} \varphi(\mathbf{p}) = \langle \mathbf{1}, \mathbf{p} \rangle + \sum_i w_i f_i(\mathbf{p})$$

- Calculus: $\nabla \varphi$, $\nabla^2 \varphi$ can be easily computed

$$\nabla^2 \varphi_i = \sum_i w_i \nabla^2 f_i = -\sum_i \nabla \mathbf{x}_i$$
$$= \mathbf{P}^{-1} \left(\sum_{i \in I} \frac{w_i}{(1 - r_i)} \boldsymbol{\Gamma}_i - \sum_{i \in I} \frac{w_i r_i}{1 - r_i} \boldsymbol{\gamma}_i \boldsymbol{\gamma}_i^\top \right) \mathbf{P}^{-1}$$

- Scaled Lipschitz continuity of $f_1, \ldots, f_m$ and hence φ

The IPMs and Worst-case Complexity

Interior-Point Strategy I: `PathFol1`

`PathFol1` (*Log-barrier Path-Following*)

- At current $\mathbf{p}_k$
- Compute all $\mathbf{x}_1(\mathbf{p}_k), \ldots, \mathbf{x}_m(\mathbf{p}_k)$
  $$\widetilde{\mathbf{H}}(\mathbf{p}_k) \approx \mathbf{P}_k \nabla^2 \varphi(\mathbf{p}_k) \mathbf{P}_k$$
- Solve
  $$\left(\widetilde{\mathbf{H}}(\mathbf{p}_k) + \mu_k \mathbf{I}\right) \mathbf{d} = -\left(\mathbf{P}_k \nabla \varphi(\mathbf{p}_k) - \mu_k \mathbf{1}\right)$$
- Update $\mathbf{p}_{k+1} = \mathbf{p}_k + \mathbf{P}_k \mathbf{d}_k$
- Decrease $\mu_{k+1} = \mu_k \sigma_k, \sigma_k < 1$

- Consider a Log-barrier model:
  $$\varphi_\mu(\mathbf{p}) = \varphi(\mathbf{p}) - \mu \langle \ln(\mathbf{p}), \mathbf{1} \rangle$$
- Decrease $\mu \to 0$

- For single-valued $\mathbf{x}_1, \ldots, \mathbf{x}_m$ (like in CES)
  $\mathcal{O}(\ln(1/\varepsilon))$
- For set-valued $\mathbf{x}_1, \ldots, \mathbf{x}_m$ (like in linear)
  $\mathcal{O}(\ln(1/\varepsilon))$
  *Solve an approximate optimal play

- Use *any* reasonable approximation
  $$\widetilde{\mathbf{H}}(\mathbf{p}_k) \approx \mathbf{P}_k \nabla^2 \varphi(\mathbf{p}_k) \mathbf{P}_k$$

Interior-Point Strategy I: `PathFol1`

- Path-following IPMs will need an *initial* **analytic center** $(\mu_0, \mathbf{p}_0)$:

$$\left\| \nabla \varphi_{\mu_0}(\mathbf{p}_0) \right\|_{\mathbf{p}_0} \leq Q\mu_0 \qquad Q \in (0,1)$$

Nontrivial in the general case; need auxiliary steps

$$\mathbf{P}_0 \nabla \varphi_{\mu_0}(\mathbf{p}_0) - \mu_0 \mathbf{1} = \mathbf{p}_0 - \mu_0 \mathbf{1} - \sum_i \mathbf{P}_0 \mathbf{x}(\mathbf{p}_0)$$

- ***Setting*** $\mathbf{p}_0 = \mu_0 \mathbf{1}$ and μ_0 ***sufficiently large!***

$$\left\| \nabla \varphi_{\mu_0}(\mathbf{p}_0) \right\|_{\mathbf{p}} \leq \left\| \sum_i \mu_0 \mathbf{x}_i(\mu_0 \mathbf{1}) \right\|_1 \leq 1 \qquad \Longrightarrow \mu_0 \geq \frac{1}{Q}$$

- ***AC is free:*** Q ***could be arbitrarily small!***

Interior-Point Strategy II: `PathFol2`

`PathFol2` ***(t-Path Following method)***

- At current $\mathbf{p}_k$
- Compute all $\mathbf{x}_1(\mathbf{p}_k), \ldots, \mathbf{x}_m(\mathbf{p}_k)$

  $$\widetilde{\mathbf{H}}(\mathbf{p}_k) \approx \mathbf{P}_k \nabla^2 \varphi(\mathbf{p}_k) \mathbf{P}_k$$
- Solve

  $$\widetilde{\mathbf{H}}(\mathbf{p}_k)\mathbf{d} = -\mathbf{P}_k(\nabla \varphi(\mathbf{p}_k) - t_k \nabla \varphi(\mathbf{p}_0))$$
- Update $\mathbf{p}_{k+1} = \mathbf{p}_k + \mathbf{P}_k \mathbf{d}_k$
- Decrease

  $$t_{k+1} = \max \left\{ t_k - \frac{\sigma}{C_f \|\mathbf{P} \nabla \varphi(\mathbf{p}_0)\|_{\widetilde{\mathbf{H}}(\mathbf{p}_k)}}, 0 \right\}$$

- Consider a t-homotopy model:

  $$\varphi_\mu(\mathbf{p}) = \varphi(\mathbf{p}) - t\langle \nabla \varphi(\mathbf{p}_0), \mathbf{p} \rangle$$

  (Dvurechensky and Nesterov, 2025)
- Decrease $t \to 0$
- Non-asymptotic superlinear:

  $$t_k \leq \left(1 - \frac{\gamma(\gamma - 2\beta)(k+1)}{2C_\varphi^2 \left(\varphi(\mathbf{p}_0) - \varphi(\mathbf{p}^*)\right)} \right)^{k+1}$$
- $t_k \searrow 0$ ***superlinearly*** fast!
- Once $t_k = 0$, ***quadratic*** convergence:

  $$\left\| \mathbf{P} \nabla \varphi(\mathbf{p}) \right\|_{[\widetilde{\mathbf{H}}]^{-1}} \leq \epsilon$$

  in $\mathcal{O}(\log \log(1/\varepsilon))$

Interior-Point Strategy II: `PathFol2`

Active-Set Boundedness (ASB)

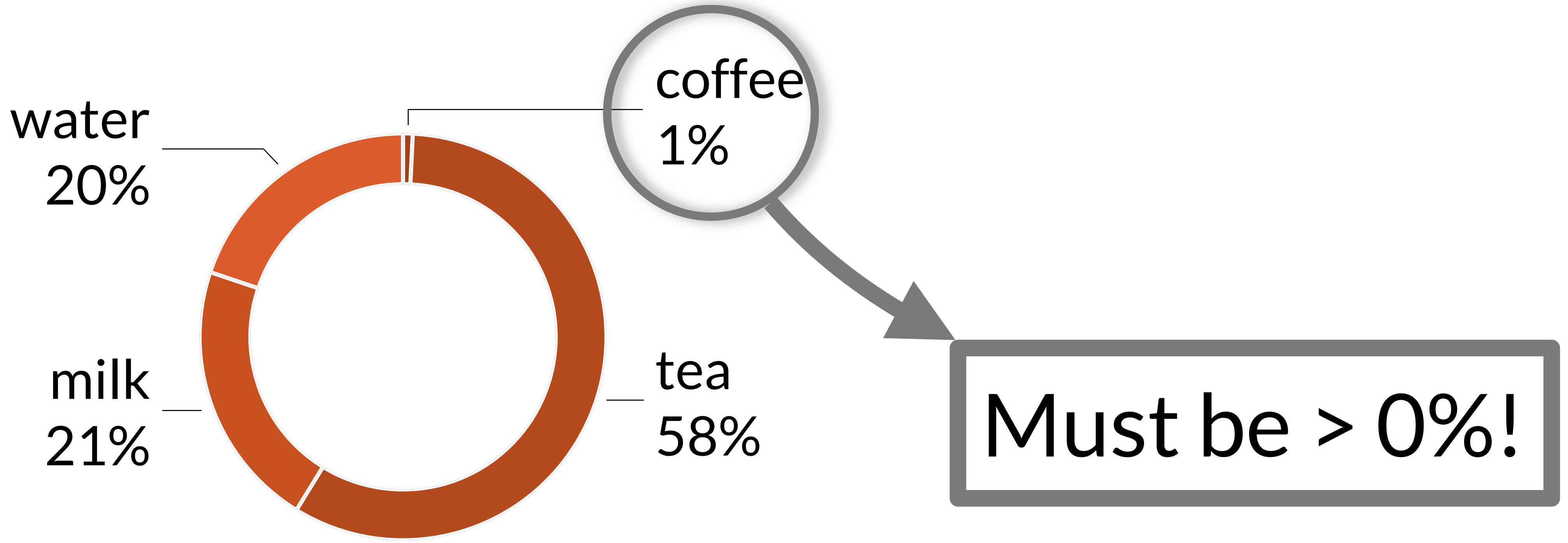
- Money in the bundle is “non-trivial”

$$\kappa_u(\mathbf{x}) = \frac{1}{\min_{j \in B(\mathbf{x})} \{\gamma^{(j)}\}} < \infty$$

- $B(\mathbf{x})$ is the set with “interested goods”:

$$B(\mathbf{x}) = \{j = 1, \ldots n \mid \gamma^{(j)} > 0\}$$

γ - distribution of spending



***Example (single-valued CES economy)*:**

$$u(\mathbf{x}) = \left[\sum_{j \in J} c^{(j)} \left(x^{(j)}\right)^{\rho}\right]^{\frac{1}{\rho}} = \langle c, \mathbf{x}^{\rho} \rangle^{\frac{1}{\rho}},$$

The “***interested***” set:

$$B = \{j = 1, \ldots, n \mid c^{(j)} \neq 0\}$$

- *ASB* for CES:

$$\min_{j \in B} c^{(j)} > 0, \quad u(\mathbf{x}) \leq D_u, \quad \|\mathbf{p}\|_{\infty} \leq D_P,$$

- Easy to guarantee in an iterative method
- An analogue of ***non-degeneracy*** in LP

Compared to First-order Schemes

Method	Type	Utility	
		Linear	CES
Tâtonnement (Arrow and Hurwitz 1958, Cheung et al. 2013)	FOM	Sublinear	Linear
Proportional response (Wu and Zhang 2007, Zhang 2011, Cheung et al. 2018)	FOM	Sublinear	Linear
IPM I: `PathFol1` This work	SOM/IPM	***Linear***	***Linear***
IPM II: `PathFol2` `(Under ASB)` This work	SOM/IPM	***Superlinear***	***Superlinear***

Decentralized IPMs:

- Each iteration requires ***m*** responses
- Same information for Tâtonnement
- Linear under ***Scaled Lipschitz Continuity***
- Superlinear under ***Active-Set Boundedness***

Up next: "***lightweight***" strategies

for $\left[\mathbf{P}\,\nabla^2\varphi\mathbf{P}\right]^{-1}$

Lightweight Iterations and Numerics

“Lightweight” IPMs: towards dimension-free iterations

- IPMs solve linear equations: $\mathcal{O}(n^3)$ ***cost***

$$\left(\widetilde{\mathbf{H}}(\mathbf{p}) + \mathbf{D}\right)\mathbf{d} = -\mathbf{r}$$

- Use approximate Hessian $\widetilde{\mathbf{H}}(\mathbf{p})$:

$$\widetilde{\mathbf{H}}(\mathbf{p}) \approx \mathbf{H}(\mathbf{p}) = \mathbf{P}\nabla^2\varphi(\mathbf{p})\mathbf{P}$$

$$= \sum_{i \in I} \frac{w_i}{(1 - r_i)} \boldsymbol{\Gamma}_i - \sum_{i \in I} \frac{w_i r_i}{1 - r_i} \boldsymbol{\gamma}_i \boldsymbol{\gamma}_i^\top$$

- Structure: sum-of diagonal + rank-1

Randomized Linear Algebra

- Sub-sampling: pick a subset of i
- ...

Krylov/Subspace Iterations

- Conjugate gradient method
- Evolving subspace

$$\mathscr{L} = \left\{\mathbf{d}_0, \mathbf{H}\mathbf{d}_0, \ldots, \mathbf{H}^\tau \mathbf{d}_0\right\}$$

- Stop at some τ when acceptable

Invertible approximation with $\mathcal{O}(n)$ cost

- Exact Hessian is ***Diagonal + Rank**-m*

$$\mathbf{H}(\mathbf{p}) = \mathbf{P}\nabla^2\varphi(\mathbf{p})\mathbf{P} = \sum_i \frac{w_i}{(1-r_i)}\Gamma_i - \sum_i \frac{w_i r_i}{1-r_i}\gamma_i\gamma_i^\top$$

Diagonal + Rank-1 approximation

- Compute the mean "bidding":

$$\boldsymbol{\xi} = \sum_i \omega_i\gamma_i, \quad \omega_i = \frac{w_i r_i}{1-r_i}\frac{1}{\Omega}, \quad \Omega = \sum_i \omega_i$$

- Construct

$$\widetilde{\mathbf{H}}(\mathbf{p}) = \sum_{i\in I} \frac{w_i}{(1-r_i)}\Gamma_i - \boldsymbol{\xi}\boldsymbol{\xi}^\top$$

This $\widetilde{\mathbf{H}}$ is ***invertible***!
(by SMW formula)

Invertible approximation with $\mathcal{O}(n)$ cost

- Performance guarantees

Theorem (informal) If $\gamma_1, \ldots, \gamma_m$ are iid, variance $\lambda_{\max}(\Sigma) = \mathcal{O}(1/n^g)$, ***then if***

$$n > \left(\frac{2\Omega}{\epsilon_H}\right)^{1/g}; \quad m \geq \frac{8\sqrt{2}(\Omega^2 \lambda_{\max}(\Sigma) + 0.5\Omega\epsilon_H)}{\epsilon_H^2} \ln\left(\frac{2n}{\delta}\right),$$

$\widetilde{\mathbf{H}}$ will be good enough in prob $1 - \delta$: $\quad 0 \preceq \widetilde{\mathbf{H}}(\mathbf{p}) - \mathbf{H}(\mathbf{p}) \preceq \epsilon_H \mathbf{I}$

- In practice, holds "at large" 10^2 agents/goods

Optimal Preconditioning for Krylov Iterations

- Use PCG (***preconditioned-CG***):

$$\mathbf{H}(\mathbf{p})\mathbf{d} = -\mathbf{r}$$

- In theory, stopping at error ϵ_H needs

$$\tau \leq \mathcal{O}(\sqrt{\kappa(\mathbf{H})}\ln(1/\epsilon_H)), \qquad \kappa(\mathbf{H}) = \frac{\lambda_{\max}(\mathbf{H})}{\lambda_{\min}(\mathbf{H})}$$

- Find some $\mathbf{K}_c$ to reduce "κ":

$$\mathbf{K}_c^{-0.5}\mathbf{H}(\mathbf{p})\mathbf{K}_c^{-0.5}\mathbf{d} = -\mathbf{K}_c^{-0.5}\mathbf{r}$$

- "Optimal" $\mathbf{K}_c$ is a quasi-convex program

$$\min_{\mathbf{K}_c} \quad \kappa\left(\mathbf{K}_c^{-0.5}\mathbf{H}(\mathbf{p})\mathbf{K}_c^{-0.5}\right)$$

Solvable via SDP
(Qu et al. 2024, Oper. Res. ; Gao et al. 2025)

Optimal Preconditioning for Krylov Iterations

- An ***analytic*** optimal preconditioner

$$\mathbf{K}_c = \mathrm{diag}\Big(\sum_i w_i \boldsymbol{\gamma}_i\Big)$$

Theorem (informal): For market equilibrium, use the ***optimal*** $\mathbf{K}_c$

The following holds for $\mathbf{H}_c := \mathbf{K}_c^{-0.5}\mathbf{H}(\mathbf{p})\mathbf{K}_c^{-0.5}$,

- The condition number is,

$$\kappa(\mathbf{H}_c) \leq \begin{cases} \frac{1}{1-r}, & 0 \leq r < 1, \\ 1 - r, & r < 0. \end{cases}$$

- κ is ***unimprovable*** for any diagonal matrix

Optimal Preconditioning: Proof sketch and insights

- Show the condition number bound:

$$\mathbf{H}_{\mathrm{c}} = \left(\sum_i w_i \boldsymbol{\Gamma}_i\right)^{-0.5} \left(\sum_i \frac{w_i}{(1-r_i)}\boldsymbol{\Gamma}_i - \sum_i \frac{w_i r_i}{1-r_i}\gamma_i \gamma_i^\top\right)\left(\sum_i w_i \boldsymbol{\Gamma}_i\right)^{-0.5}$$

$$= \frac{1}{1-r}\left(\mathbf{I} - r\mathbf{S}\right) \qquad 0 \leq \lambda_{\min}(\mathbf{S}) \leq \lambda_{\max}(\mathbf{S}) \leq 1$$

- Upper bound:

$$\kappa(\mathbf{H}_c) \leq \begin{cases} \frac{1}{1-r}, & 0 \leq r < 1, \\ 1-r, & r < 0. \end{cases}$$

- How to show the reverse direction?

Optimal Preconditioning: Proof sketch and insights

- A 2D counter example, $m = n = 2, \mathbf{x}_1 = \mathbf{x}_2$
- Space of diagonal preconditioner
  $\mathscr{K} = \{\mathbf{K}(a) \,|\, a \geq 1\}$

Any $$\mathbf{K}(a) = \begin{bmatrix} 1 & \\ & a^{-1} \end{bmatrix}, \qquad a \geq 1.$$

$$\Longrightarrow \widehat{\mathbf{H}}(a) := \begin{bmatrix} 2 - r & -r\sqrt{a} \\ -r\sqrt{a} & (2-r)a \end{bmatrix}$$

- At $a^\star = 1$, two eigenvalues become $(2,\ 2(1-r))$
- The upper bound is ***tight***!

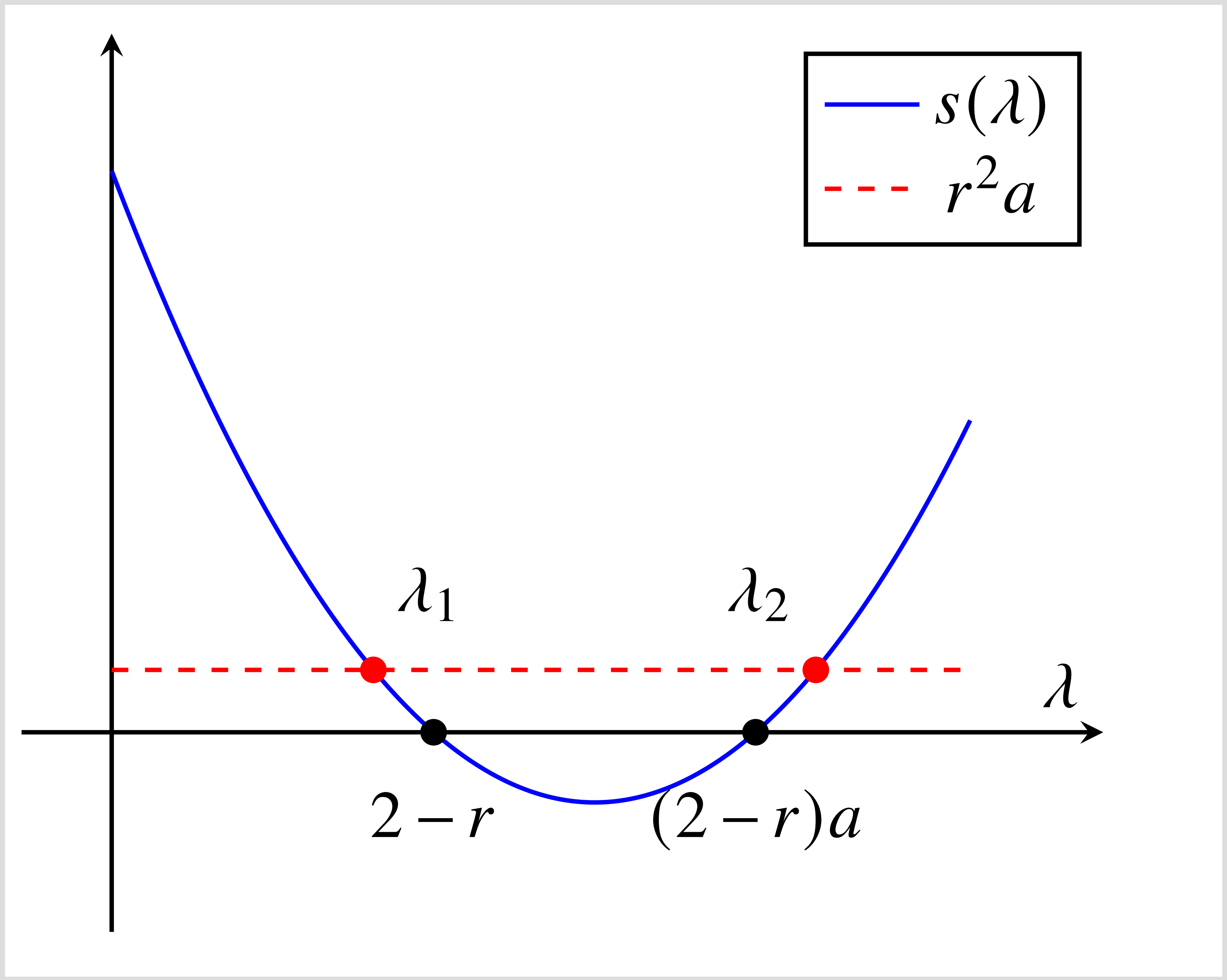


Illustration of spetrum of $\widehat{\mathbf{H}}(a)$:
minimized λ_2/λ_1 when $a^\star = 1$

$$\widehat{\mathbf{H}}(a) \geq \begin{cases} \frac{1}{1-r}, & 0 \leq r < 1, \\ 1 - r, & r < 0. \end{cases}$$

Quality of Approx. for *one* Linear System

Fig. (a) Use DR1 approx. to compute $\widetilde{\mathbf{H}}$
10^2 players suffice (***under iid assumption***)

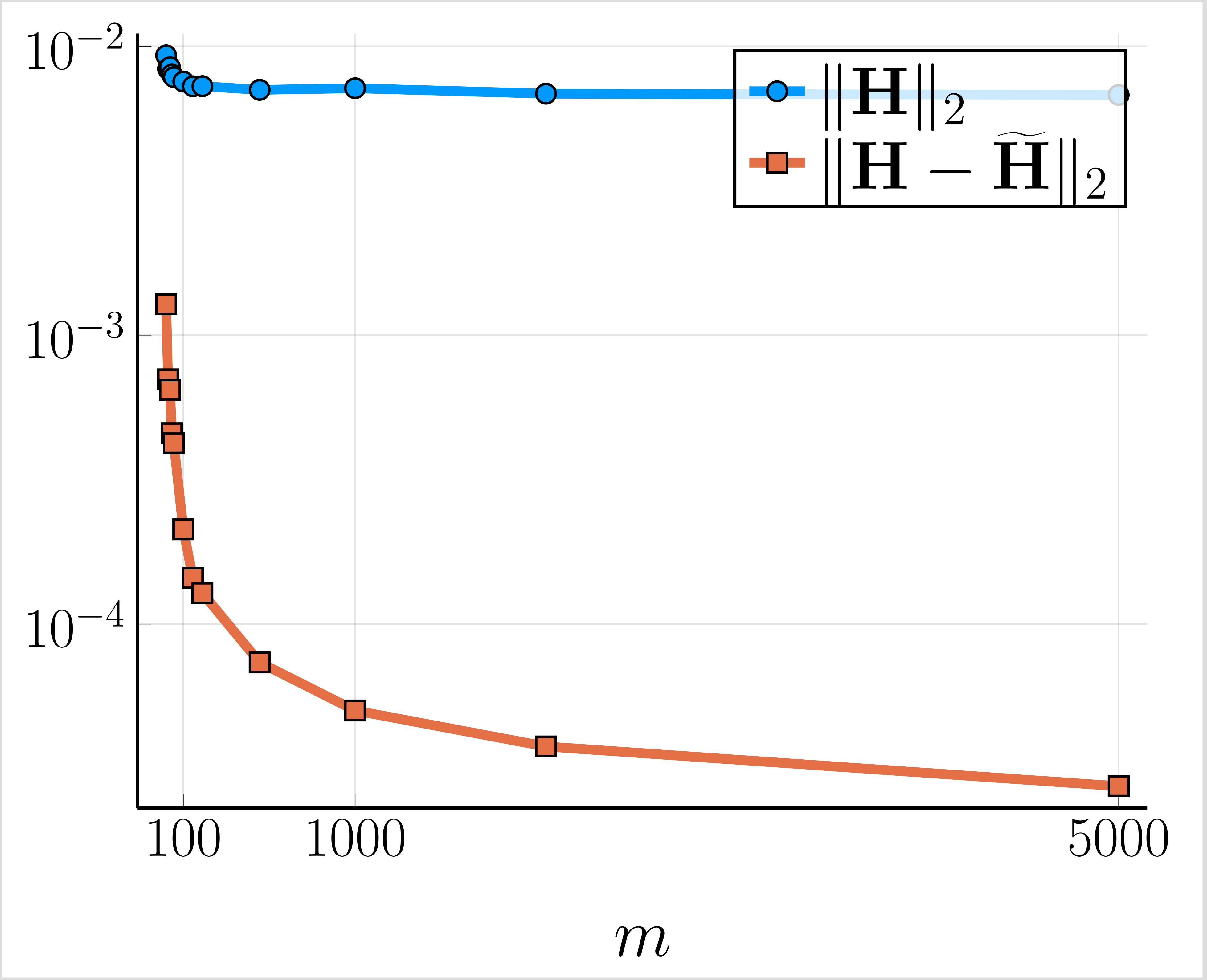
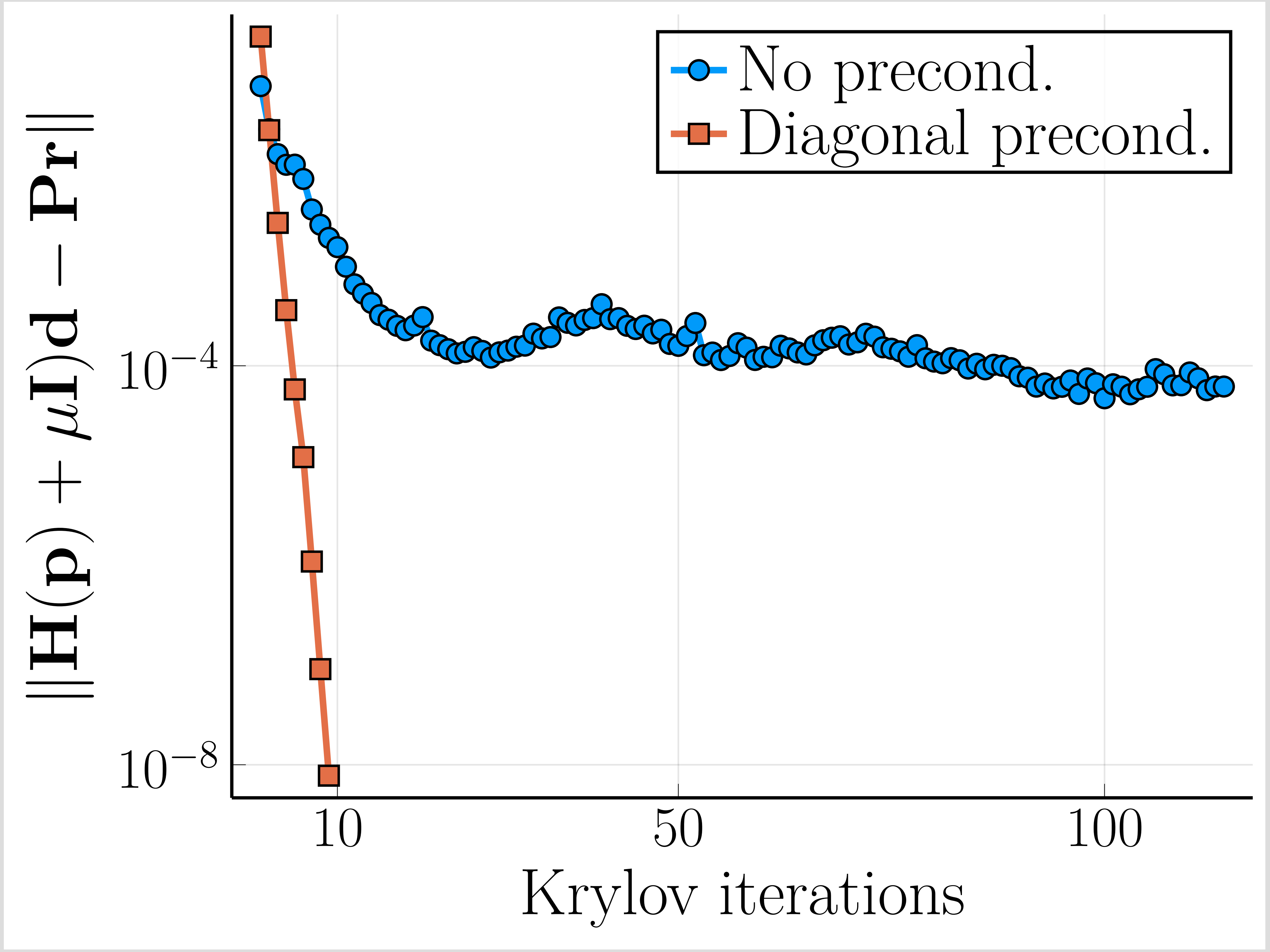


Fig. (b) Optimal precond. for PCG
Dimension-free! $10^0 - 10^1$ iterations



Test case: $n = 5000$ ***(goods)***

How good are the "*lightweight*" iterations?

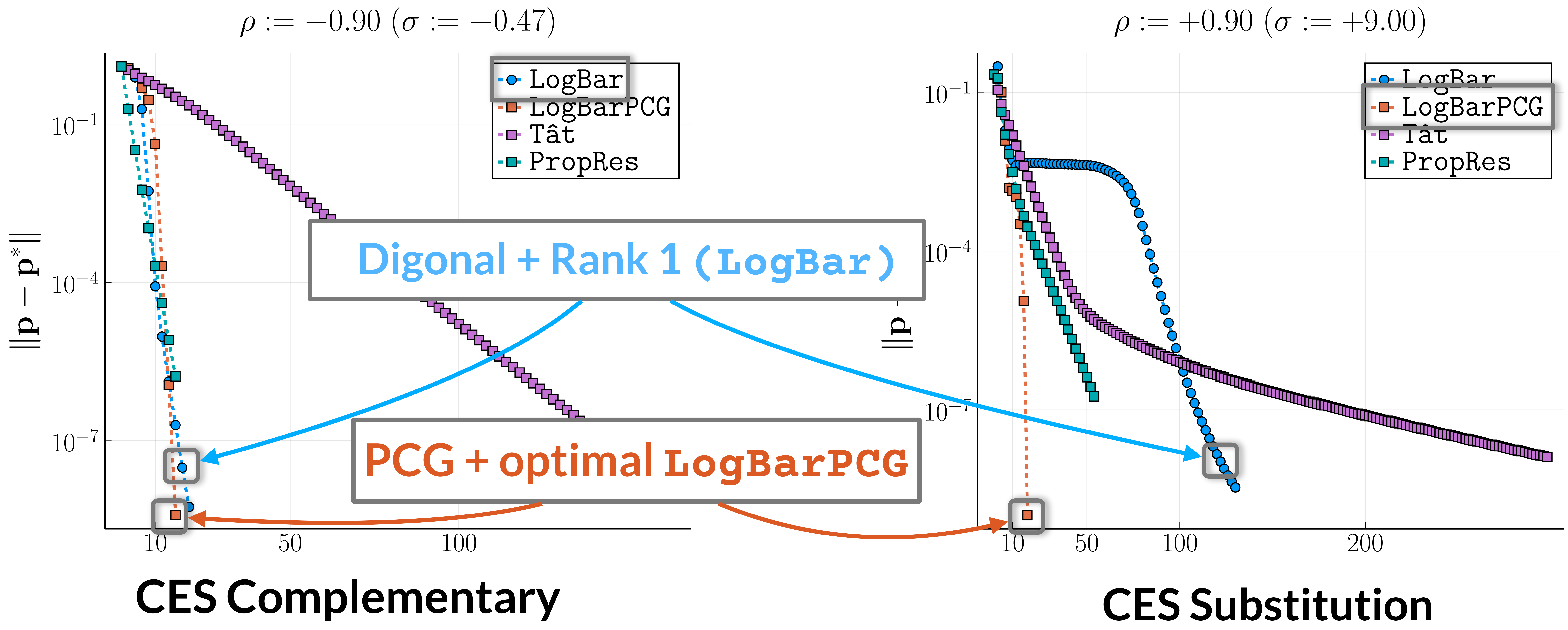
n	m	ρ	t DR1	t PCG	t PropRes	t Tât	$\lVert \mathbf{p} - \mathbf{p}^* \rVert$ DR1	$\lVert \mathbf{p} - \mathbf{p}^* \rVert$ PCG	$\lVert \mathbf{p} - \mathbf{p}^* \rVert$ PropRes	$\lVert \mathbf{p} - \mathbf{p}^* \rVert$ Tât
1,000	10,000	-0.9	0.66	1.19	1.81	8.19	7.27e-09	8.81e-09	9.73e-06	9.94e-06
		0.9	1.05	1.44	3.54	17.52	6.96e-10	1.80e-10	9.45e-06	9.80e-06
	50,000	-0.9	2.84	4.33	7.97	36.60	1.20e-09	8.81e-09	9.00e-06	9.39e-06
		0.9	4.08	4.17	15.44	58.35	5.24e-10	8.81e-09	8.23e-06	9.69e-06
	100,000	-0.9	5.80	8.46	16.82	70.98	6.48e-10	8.81e-09	6.70e-06	9.74e-06
		0.9	8.00	8.26	29.06	111.53	2.17e-10	8.81e-09	9.48e-06	9.66e-06
2,000	10,000	-0.9	1.22	1.91	3.64	16.69	1.76e-09	4.41e-09	9.82e-06	9.95e-06
		0.9	1.87	2.49	6.81	85.97	1.99e-10	8.98e-11	9.66e-06	9.93e-06
	50,000	-0.9	5.77	7.92	15.52	73.08	2.37e-10	4.41e-09	9.44e-06	9.77e-06
		0.9	7.85	7.65	28.68	-	6.00e-11	4.40e-09	8.30e-06	1.54e-05†
	100,000	-0.9	11.64	15.98	31.82	135.02	1.13e-10	4.41e-09	6.72e-06	9.71e-06
		0.9	13.52	15.54	52.87	-	1.12e-09	4.41e-09	9.11e-06	8.37e-05†
5,000	10,000	-0.9	3.17	4.76	10.40	44.68	3.89e-10	1.76e-09	4.86e-06	9.28e-06
		0.9	5.03	6.45	16.64	-	3.80e-11	3.62e-11	9.46e-06	9.49e-05†
	50,000	-0.9	17.15	26.36	44.65	195.73	2.20e-11	1.76e-09	9.50e-06	9.79e-06
		0.9	19.77	25.11	83.69	-	5.19e-10	1.76e-09	8.15e-06	4.52e-04†
	100,000	-0.9	32.94	59.79	121.84	-	1.36e-08	1.76e-09	6.55e-06	5.18e-04†
		0.9	49.76	62.41	190.05	-	1.23e-10	1.76e-09	1.07e-05	7.53e-04†

Decentralized IPMs:

- PropRes: Proportional response
- Tât: Tâtonnement process

- Fast convergence on large problems
- $\mathcal{O}(10^1)$ price adjustments
- Less CPU time, higher precision

Some real dataset: Movie Lens



- Real-world reveal preferences are super sparse, only $0.18\ \%$ (agent, movie) pairs are nonzero

  Invertible Digonal + Rank 1 (`LogBar`) is slow in the beginning
- PCG + optimal conditioning (`LogBarPCG`) is fast in both regimes

Takeaways

- Interior-point Renaissance
  - × Information-dependent Smale's process
  - ✓ Free high-order information: "***interior-point 2nd-order tâtonnement*** method"
- Structural property breaks the algorithmic barriers
  - Easy calculus / lightweight iterations / complexity guarantees
- It is **beneficial** and **economical** to incorporate observable second-order information in iterative optimization methods for real-world applications!

On-going

- Second-order tâtonnement ***for any*** utility class.
- Better price mechanisms for online Fisher market (Jalota and Ye, OR, 2024)

log-homogeneity gives $\mathcal{O}(\log T)$ regret (Zhang, He, Ye, working paper)

Further Applications

Linear Fisher market

- Linear utility

$$u(\mathbf{x}) = \langle \mathbf{c}, \mathbf{x} \rangle$$

- $\mathbf{x}(\mathbf{p})$ is not unique and piecewise smooth

$$\max_{\mathbf{x} \in \mathbb{R}^n_+} \langle \mathbf{c}, \mathbf{x} \rangle, \text{ s.t. } \langle \mathbf{p}, \mathbf{x} \rangle \leq w$$

$$= w \max_{j \in [n]} \left\{ \frac{c_j}{p_j} \right\}.$$

- Apply *smoothing*, Log-UMP:

$$f^\sigma(\mathbf{p}) = \max_{\mathbf{x} \in \mathbb{R}^n_+} \log(\langle \mathbf{c}, \mathbf{x} \rangle) + \sigma \langle \log(\mathbf{x}), \mathbf{1} \rangle$$
$$\text{s.t. } \langle \mathbf{p}, \mathbf{x} \rangle \leq w\,.$$

- Smoothed potential. $\sigma \approx \mathcal{O}(\epsilon/n)$

$$\varphi^\sigma(\mathbf{p}) = \langle \mathbf{p}, \mathbf{1} \rangle + \sum_i f^\sigma(\mathbf{p})$$

Scaled Lipschitz constant in $\mathcal{O}(\frac{1}{\varepsilon^3})$

- This needs a very small neighborhood but the initial center is ***free***
- Still, $\mathcal{O}(\log(\frac{1}{\varepsilon}))$ complexity, matching centralized IPMs (*Ye 2008, MP*)

Allocation with linear constraints

- "Restricted" $\mathbf{x}$ instead of "full-dimensional" bundle

Example 1: Flow Market

- Any player's allocation $\mathbf{x}_i$ should be a **s-t flow** in a network $G = (V, E)$, that is

$$
\begin{aligned}
x_{i,0} + \sum_{e \in \delta^+(s)} x_{i,e} - \sum_{e \in \delta^-(s)} x_{i,e} &= 0, \\
-x_{i,0} + \sum_{e \in \delta^+(t)} x_{i,e} - \sum_{e \in \delta^-(t)} x_{i,e} &= 0, \\
\sum_{e \in \delta^-(v)} x_{i,e} - \sum_{e \in \delta^+(v)} x_{i,e} &= 0, v \in V - \{s, t\}.
\end{aligned}
\quad \Longrightarrow \mathbf{A}\mathbf{x} = 0
$$

Example 2: Demand aggregation/group substitution/charaterization

- Two-stage budgeting and nested demand: Armington, 1969
- Charateristics/Household production: Becker 1965
- Demand with connected substitution: Berry et al. 2013

$$\Longrightarrow \mathbf{A}\mathbf{x} = 0$$

Allocation with linear constraints

- Affine-Constrained Log-UMP:

$$f(\mathbf{p}) := \max_{\mathbf{x}\in\mathscr{X}} -v(\mathbf{x}) = \log(u(\mathbf{x})) \quad \text{s.t. } \langle \mathbf{p}, \mathbf{x} \rangle \le w\,.$$

and $\mathscr{X} = \left\{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{A}\mathbf{x} = \mathbf{b}\right\}$

- If $\mathbf{b} \neq 0$ (non-homogeneous),

  PPAD-hard (Jalota et al. 2023), and a PTAS is unlikely to exist

- If $\mathbf{b} = 0$, $f(\mathbf{p})$ is still ***SLC***,

  with a coefficient ***no bigger than*** that from an unconstrained log-UMP

Allocation with linear constraints

- Information dependency:

$$\nabla f(\mathbf{p}) = -\frac{d}{w}\mathbf{x}, \quad \nabla^2 f(\mathbf{p}) = \frac{d^2}{w^2}(\mathbf{W}^{-1} - \mathbf{W}^{-1}\mathbf{A}^\top(\mathbf{A}\mathbf{W}^{-1}\mathbf{A}^\top)^{-1}\mathbf{A}\mathbf{W}^{-1})$$

- $\mathbf{W} = [\nabla^2 v(\mathbf{x})]^{-1}$ as before constructed from of the ***distribution of money***…
- Similar cost of obtaining an agent's response

Decentralized IPMs are still applicable

- $\mathcal{O}(\log(\frac{1}{\varepsilon}))$ complexity by using **`PathFol1`**

Towards Arrow-Debreu Market

Arrow-Debreu market

The Market Equilibrium Problem

$$\mathbb{R}^n_+ \ni \mathbf{p} \;\perp\; \mathbf{1} - \sum_i \mathbf{x}_i(\mathbf{p}) \in \mathbb{R}^n_+$$

Find $\mathbf{p}$ so demand ***clears*** the market

UMP (Arrow-Debreu case)

- $\mathbf{x}_i(\mathbf{p})$: the (Walrasian) demand

$$\mathbf{x}_i(\mathbf{p}) = \operatorname{argmax} \quad u_i(\mathbf{x}_i)$$
$$\text{s.t.} \quad \langle \mathbf{p}, \mathbf{x}_i \rangle \leq w_i = \langle \mathbf{p}, \mathbf{b}_i \rangle$$

- Endogeneity: sell $\mathbf{b}_i$ as the budget
- No Lyapunov function/convex optimization anymore!
- If $\mathbf{p} > 0$, a root-finding/fixed-point problem

$$\mathbf{1} - \sum_i \mathbf{x}_i(\mathbf{p}) = \mathbf{0}$$

- In 1994, Papadimitriou introduced the class PPAD (including LCP and Fixed-Point Problems)
- Most Arrow-Debreu market equilibrium are ***hard***.

Arrow-Debreu market (CES economy)

- Consider CES economy with same elasticity

$$u_i(\mathbf{x}_i) = \left(\sum_j c_i^{(j)} (x_i^{(j)})^r \right)^{1/r}$$

- $r = 1$ (linear) strongly polynomial
- $r = -\infty$ (Leontief $\equiv$ LCP), PPAD-complete

  determine the existence is NP-complete
- $r = (-1,1]$ convex feasibility, polynomial
- $r < -1$ ***PPAD-hard.***

- Relative hardness?

  Is $r = -1$ easier than $r = -100$?
- Convergence of price-adjustment?

Arrow-Debreu market

- PPAD-hardness (Chen et al. 2009, 2017)

$$\|\mathbf{z}(\mathbf{p})\| = \left\| \sum_i \mathbf{x}_i(\mathbf{p}) - \mathbf{1} \right\| \leq \epsilon$$

ϵ-approx. clearance price (ϵ-ACP)

- A more natural goal from ***Walras' Law***

$$\mathbb{R}^n_+ \ni \mathbf{p} \;\perp\; \mathbf{1} - \sum_i \mathbf{x}_i(\mathbf{p}) \in \mathbb{R}^n_+$$

$$\|\mathbf{P}\mathbf{z}(\mathbf{p})\| < \epsilon$$

ϵ-approx. Walrasian price (ϵ-AWP)

- We show (ϵ-AWP) is ***polynomial-time solvable*** (Zhang, He, Ye, working paper)

  By a potential-reduction method

Backup slides

Summary of Combinatorial Algorithms

Fisher Model (n goods, m players)

- Devanur et al. (FOCS 2002); later corrected at J. ACM, 2008
  $\mathcal{O}\left(n^4(\log n + n \log U + \log M)\right)$ **Max-Flows** iterations;
  state-of-art: $\mathcal{O}(|E|^{1+o(1)})$, $|E|$ depends on $O(m+n)$
- Garg and Kapoor (Math. OR., 2006);
  $\mathcal{O}\left((1/\epsilon)\left(nm^2 + mn^2\right)\log(m/\epsilon)\right)$ for $(1+\epsilon)$ - approximation.
- Strongly polynomial: Orlin (STOC. 2010), Végh (STOC 2012),
  $\mathcal{O}((m+n)^4 \log(m+n))$

Arrow-Debreu Model

- Strongly polynomial, Garg and Végh (STOC. 2019, OR. 2023) …

All these methods are based on network flows with (separable) convex objective functions