Further Development on the Interior Algorithm for Convex Quadratic Programming

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Abstract

The interior trust region algorithm for convex quadratic programming is further developed. This development is motivated by the barrier function and the "center" path-following methods, which create a sequence of primal and dual interior feasible points converging to the optimal solution. At each iteration, the gap between the primal and dual objective values (or the complementary slackness value) is reduced at a global convergence ratio $(1 - \frac{1}{4\sqrt{n}})$, where n is the number of variables in the convex QP problem. A safeguard line search technique is also developed to relax the small-step-size restriction in the original path- following algorithm.

Key words: Convex Quadratic Programming, Primal and Dual, Complementarity Slackness, Polynomial Interior Algorithm.

Abbreviated title: Interior Algorithm for Convex Quadratic Programming

Since Karmarkar proposed the new polynomial algorithm (Karmarkar [19]), several developments have been made to the growing literature on interior algorithms for LP. Adler, Karmarkar, Resende and Veiga [1], Barnes [3], Cavalier and Soyster [6], Kortanek and Shi [22], Sherali, Skarpness and Kim [34], and Vanderbei, Meketon and Freedman [40] developed the "affine scaling" method solving LP problems in standard form. Bayer and Lagarias [4] and Megiddo and Shub [26] studied the solution trajectories and boundary behavior of interior algorithms. Gill, Murray, Saunders, Tomlin and Wright [13], and Iri and Imai [17] analyzed the Newton barrier method and its similarity to Karmarkar's algorithm. Goldfarb and Mehrotra [14], Mitchell and Todd [27], Nazareth [30], and Shanno and Marsten [33] proposed relaxed variants of Karmarkar's algorithm to improve computational efficiency. Anstreicher [2], Gay [11], Ghellinck and Vial [12], Todd and Burrell [38], and Ye and Kojima [45] proposed a primal and dual polynomial-time variant using an objective lower bound updating technique.

Another popular polynomial-time interior algorithm without using Karmarkar's projective transformation, the new centering method, was introduced by Renegar [32] and Sonnevend [36]. Renegar obtained a convergence ratio $(1 - O(\frac{1}{\sqrt{n}}))$, comparing to the one $(1 - O(\frac{1}{n}))$ in Karmarkar's algorithm. Recently, using the rank-one updating scheme, Gonzaga [15] and Vaidya [39] further reduced the solution time for LP by factor $n^{0.5}$.

While interior algorithms for LP are intensely studied, several authors put their attention on interior algorithms for quadratic programming (QP) or linear complementarity problem (LCP). Megiddo [25] studied the barrier pathway to the optimal set for LCP. Kapoor and Vaidya [18], Ye and Tse [44] proposed an extension of Karmarkar's projective LP algorithm for solving convex quadratic programs with a global convergence ratio $(1 - O(\frac{1}{n}))$. Ye [43] also analyzed convergence behavior of the interior ellipsoidal trust region method, an extension of the "affine scaling" and trust region method, for convex QP problems.

In this paper, by adjoining the work of the new centering method, the barrier function method, and the interior trust region method, I develop an improved interior algorithm for QP. This algorithm creates a sequence of primal and dual interior feasible points converging to the optimal solution. At each iteration, the complementary slackness value, i.e., the objective gap between the primal and dual, is reduced at a global ratio $(1 - \frac{1}{4\sqrt{n}})$. Therefore, the algorithm solves QP in $O(Ln^{3.5})$ arithmetic operations, where n is the number of variables and L is the size of the problem (Papadimitriou and Steiglitz [31]) in the convex QP problem.

I have learnt that Kojima, Mizuno and Yoshise [21] and Monteiro and Adler [28] also independently developed a very similar algorithm for convex LCP or QP. The convergence ratio obtained in their approach is $(1 - \frac{1}{8\sqrt{n}})$. Moreover, they use the rank-one updating scheme to achieve the algorithm complexity of $O(Ln^3)$ arithmetic operations, which is the best known complexity for solving LCP. The major differences of Kojima et al-Monteiro and Adler's and my approaches are: they symmetrically work in the primal-dual spaces, and I primely work in the primal space; they scale the feasible region by using geometric mean of the primal and dual interior solutions, and I scale the region by using only the primal interior solution; the two approaches have different setting to initialize the algorithm; and our convergence ratio is slightly better than the one of Kojima et al's. Furthermore, instead of employing the rank-one scheme, I propose a safeguard line search technique to relax the small-step-size restriction. This results a faster convergence from my computational experience.

1. Convex Quadratic Programming

QP and its dual have been intensely studied and analyzed by Cottle and Dantzig [8][9], Eaves [11], Murty [29]. Several significant approaches for solving QP problems were introduced by Beale [5], Conn and Sinclair [7], Hildreth [16], Lemke [24], Van Der Heyden [41], and Wolfe [42]. However, so far as I know, none of these approaches is proved to be a polynomial-time algorithm, although they work well in practice.

The first proved polynomial-time algorithm for convex QP is the ellipsoid method (Khachiyan [20], Shor [35]). Kozlov, Tarasov and Khachiyan [23] extended the ellipsoid method to solving convex quadratic programs in $O(Ln^4)$ arithmetic operations. Unfortunately, the ellipsoid method behaves similar to its worst case complexity bound, and the method's significance remains theoretical.

The convex quadratic program is usually identified in the following standard form:

$$\text{QP} \qquad \begin{array}{ll} \text{minimize} & f(x) = \frac{x^TQx}{2} + c^Tx \\ \text{subject to} & Ax = b,\ x \geq 0, \end{array}$$

where $Q \in R^{n\times n}$, c and $x \in R^n$, $A \in R^{m\times n}$, $b \in R^m$, Q is a positive semi-definite matrix, and superscript ^{T} denotes the transpose operation. The strong dual to QP can be written by

$$\text{QD} \qquad \begin{array}{ll} \text{maximize} & d(x,y) = yb - \frac{x^TQx}{2} \\ \text{subject to} & Ax = b,\ x \geq 0,\ Qx + c - A^Ty \geq 0, \end{array}$$

where vector $y \in R^m$. for all x and y that are feasible for QD,

$$d(x,y) \leq z^* \leq f(x), \tag{1}$$

where z^* designates the optimal objective value of QP.

Based on the Kuhn-Tucker conditions, x^* is an optimal feasible solution for QP if and only if the following three optimality conditions hold:

1) Primal feasibility: x^* is feasible for QP
2) Dual feasibility: $\exists\ y^*$, such that x^*, y^* are feasible for QD
3) Complementary slackness:

$$X^*(Qx^* + c - A^T y^*) = 0 \quad \text{or} \quad f(x^*) = d(x^*, y^*) \tag{2}$$

Here, upp-case letter X^* designates the diagonal matrix of the vector x^* of lower-case. This notion will be used through this paper.

In this paper, I reserve the same assumptions as the ones in the interior trust region method (Ye's [43]) for QP, i.e.,

A1 the interior of the feasible region of QP is nonempty

A2 the feasible region is bounded

2. The Interior Trust Region Method

The basic concept of the interior (ellipsoidal) trust region method was borrowed from the trust region method for unconstrained optimization problems (see Sorensen [37] for the literature on the trust region method): we replace the nonnegative constraints $x \geq 0$ with an interior ellipsoid centered at the starting point and contained in the feasible region. Then, the objective function can be minimized over this interior ellipsoid to generate the next interior solution point. A series of such interior ellipsoids can thus be constructed to generate a sequence of points converging to the optimal solution point. This process can be represented by the following suboptimization problem:

$$
\begin{array}{lll}
\text{QP1} & \text{minimize} & f(x) = \frac{x^T Q x}{2} + c^T x \\
 & \text{subject to} & Ax = b \\
 & & \|(X^k)^{-1}(x - x^k)\|^2 \leq \rho^2 < 1
\end{array}
$$

where $X^k = \operatorname{diag}(x^k)$, and x^k is the interior feasible solution at the kth iteration. The last constraint, $\|(X^k)^{-1}(x - x^k)\|^2 \leq \rho^2$, corresponds to an ellipsoid embedded in the positive orthant. Therefore, $\{x : Ax = b,\ \|(X^k)^{-1}(x - x^k)\| \leq \rho\}$ is an algebraic representation of the interior ellipsoid centered at x^k of QP. The radius ρ characterizes the size of the ellipsoid, and X^k affects the orientation and the shape of the ellipsoid. Let x^{k+1} be the optimal solution for QP1. Then, x^{k+1} and a y^{k+1} meet the following equation:

$$
\begin{pmatrix} Q + \mu(X^k)^{-2} & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x^{k+1} - x^k \\ y^{k+1} \end{pmatrix} = \begin{pmatrix} -(Qx^k + c) \\ 0 \end{pmatrix} \tag{3}
$$

which can be solved by approximating the multiplier $\mu(\geq 0)$ such that

$$
\|(X^k)^{-1}(x^{k+1} - x^k)\| \leq \rho.
$$

If x^{k+1} and y^{k+1} are feasible for QD, then we have the following convergence ratio [43]

$$
f(x^{k+1}) - z^* \leq (1 - \frac{\rho}{\sqrt{n}})(f(x^k) - z^*).
$$

Unfortunately, this feasibility condition is not necessarily satisfied at each iteration. Hence, the question arises: how to force y^{k+1} to be a feasible solution at each iteration and to maintain the above ratio effective through the course of the algorithm?

3. The "Center" Path-Following Algorithm

One motivation from the barrier function method is to solve the following suboptimization problem instead of QP1.

BP1 minimize $\frac{x^TQx}{2} + c^Tx - \lambda e^T(X^k)^{-1}(x - x^k)$

subject to $Ax = b$

$\|(X^k)^{-1}(x - x^k)\|^2 \le \rho^2 < 1,$

where $e^T(X^k)^{-1}(x - x^k)$ is the linear approximation of the barrier function $\sum_1^n \ln(x_i)$ at x^k, and $\lambda > 0$ is the barrier parameter. Let x^{k+1} be the optimal solution for BP1. Then again, x^{k+1} and y^{k+1} meet the following equation:

$$\begin{pmatrix} Q + \mu(X^k)^{-2} & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x^{k+1} - x^k \\ y^{k+1} \end{pmatrix} = \begin{pmatrix} \lambda(X^k)^{-1}e - (Qx^k + c^T) \\ 0 \end{pmatrix} \tag{4}$$

From the idea of the "analytical" center (Renegar [32], Sonnevend [36]) and the centering pathway (Megiddo [25], Kojima et al [21], Monteiro and Adler [28]), at the beginning of the kth iteration we assume that not only initial x^k and y^k are interior feasible solutions for QD:

$$Ax^k = b, x^k > 0 \quad \text{and} \quad s^k = Qx^k + c - A^Ty^k > 0, \tag{5}$$

but also they are close to the "center" pathway:

$$\|X^ks^k - z^ke\| \le \alpha z^k, \quad \alpha < 1, \tag{6}$$

where s^k is called the slackness vector and

$$z^k = \frac{e^TX^ks^k}{n} = \frac{(x^k)^TQx^k - b^Ty^k}{n} = \frac{f(x^k) - d(x^k, y^k)}{n}. \tag{7}$$

As we can see, z^k is the mean value and $\|X^ks^k - z^ke\|$ is the standard deviation of the complementary slackness vector X^ks^k at x^k.

In the following, we show that by appropriately selecting μ, λ and α, x^{k+1} and y^{k+1} still satisfy equations (5) and (6), and

$$z^{k+1} \le (1 - \frac{1}{4\sqrt{n}})z^k. \tag{8}$$

(8) shows that the gap between the primal and dual objective values, or the mean value of the complementary slackness vector is reduced at a fixed ratio $1 - \frac{1}{4\sqrt{n}} < 1$.

Let

$$\triangle x = x^{k+1} - x^k, \quad \triangle y = y^{k+1} - y^k, \quad \text{and} \quad \triangle s = Q\triangle x - A^T\triangle y,$$

and

$$S^k = \mathrm{diag}(s^k) \quad \text{and} \quad \triangle X = \mathrm{diag}(\triangle x).$$

Then (4) can be rewritten as

$$\begin{pmatrix} Q + \mu(X^k)^{-2} & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} \triangle x \\ \triangle y \end{pmatrix} = \begin{pmatrix} \lambda(X^k)^{-1}e - s^k \\ 0 \end{pmatrix}, \tag{9}$$

or

$$X^k\triangle s + \mu(X^k)^{-1}\triangle x = \lambda e - X^k s^k \tag{10}$$

and

$$A\triangle x = 0. \tag{11}$$

Furthermore, we select

$$\mu = z^k \quad \text{and} \quad \lambda = (1 - \frac{\alpha}{\sqrt{n}})z^k. \tag{12}$$

Then we can establish the following three lemmas.

Lemma 1

$$\begin{aligned} \|(X^k)^{-1}\triangle x\| &\le \sqrt{2}\alpha; \\ \|(S^k)^{-1}\triangle s\| &\le \frac{\sqrt{2}\alpha}{1-\alpha}; \\ \|\triangle X \triangle s\| &\le \alpha^2 z^k. \end{aligned} \tag{13}$$

Proof. Since Q is positive-semi definite,

$$\begin{aligned} \triangle x^T\triangle s &= \triangle x^T Q\triangle x - \triangle x^T A^T\triangle y \\ &= \triangle x^T Q \triangle x \ge 0. \end{aligned}$$

Thus,

$$\begin{aligned} \|\lambda e - X^k s^k\|^2 &= \|X^k\triangle s + \mu(X^k)^{-1}\triangle x\|^2 \\ &= \|X^k\triangle s\|^2 + 2\mu\triangle x^T\triangle s + \|\mu(X^k)^{-1}\triangle x\|^2 \\ &\ge \|X^k\triangle s\|^2 + \|\mu(X^k)^{-1}\triangle x\|^2. \end{aligned} \tag{14}$$

Hence,

$$\|\mu(X^k)^{-1}\triangle x\| \le \|\lambda e - X^k s^k\| \tag{15}$$

$$\|X^k \triangle s\| \le \|\lambda e - X^k s^k\| \tag{16}$$

However, using (6) and (12), we have

$$\begin{aligned}
\|\lambda e - X^k s^k\|^2 &= \|z^k e - X^k s^k - \frac{\alpha z^k}{\sqrt{n}} e\|^2 \\
&= \|z^k e - X^k s^k\|^2 + \|\frac{\alpha z^k}{\sqrt{n}} e\|^2 \\
&\le (\alpha z^k)^2 + (\alpha z^k)^2 \\
&= 2(\alpha z^k)^2. \qquad (17)
\end{aligned}$$

Therefore, from (15) and (17)

$$\|(X^k)^{-1}\triangle x\| \le \frac{\sqrt{2}\alpha z^k}{\mu} = \sqrt{2}\alpha;$$

from (6), (16) and (17)

$$\begin{aligned}
\|(S^k)^{-1}\triangle s\| &= \|(S^k X^k)^{-1} X^k \triangle s\| \\
&\le \frac{\|X^k \triangle s\|}{(1-\alpha) z^k} \\
&= \frac{\sqrt{2}\alpha}{1-\alpha};
\end{aligned}$$

and from (14) and (17)

$$\begin{aligned}
\|\triangle X \triangle s\| &= \|\triangle X (X^k)^{-1} X^k \triangle s\| \\
&\le \frac{1}{\mu}\|\mu(X^k)^{-1}\triangle x\| \|X^k \triangle s\| \\
&\le \frac{1}{\mu}\frac{\|\mu(X^k)^{-1}\triangle x\|^2 + \|X^k \triangle s\|^2}{2} \\
&\le \frac{1}{2\mu}\|\lambda e - X^k s^k\|^2 \\
&\le \frac{1}{2z^k}(\sqrt{2}\alpha z^k)^2 = \alpha^2 z^k. \qquad Q.E.D.
\end{aligned}$$

Lemma 1 essentially claims that x^{k+1} and y^{k+1} remain interior solutions for QP and QD and that the second order term $\|\triangle X \triangle s\|$ of the new complementary slackness vector $X^{k+1} s^{k+1}$ is relatively small if α is small enough.

The second lemma establishes a global convergence ratio in minimizing the primal-dual objective gap.

Lemma 2

$$(1-\frac{\alpha}{\sqrt{n}}-\frac{\sqrt{2}\alpha^2}{n})z^k \le z^{k+1} \le (1-\frac{\alpha}{\sqrt{n}}+\frac{\alpha^2}{4n})z^k.$$

Proof. Note from (10) that

$$\begin{aligned} X^{k+1}s^{k+1} &= (X^k+\triangle X)(s^k+\triangle s) \\ &= X^k s^k + \triangle X s^k + X^k \triangle s + \triangle X \triangle s \\ &= X^k s^k + X^k \triangle s + \mu (X^k)^{-1}\triangle x - \mu (X^k)^{-1}\triangle x + \triangle X s^k + \triangle X \triangle s \\ &= \lambda e + \triangle X(s^k + \triangle s - \mu (X^k)^{-1} e). \end{aligned} \tag{18}$$

Again from (10), (18) can also be written by

$$X^{k+1}s^{k+1} = \lambda e + \triangle X (X^k)^{-1}(\lambda e - \mu (X^k)^{-1}\triangle x - \mu e). \tag{19}$$

From (19),

$$\begin{aligned} nz^{k+1} &= e^T X^{k+1}s^{k+1} \\ &= n\lambda + (\lambda-\mu)e^T(X^k)^{-1}\triangle x - \mu\|(X^k)^{-1}\triangle x\|^2 \\ &= n\lambda - \frac{\alpha z^k}{\sqrt{n}} e^T (X^k)^{-1}\triangle x - \mu\|(X^k)^{-1}\triangle x\|^2 \\ &\le n\lambda + \frac{\alpha z^k}{\sqrt{n}} |e^T (X^k)^{-1}\triangle x| - \mu\|(X^k)^{-1}\triangle x\|^2 \\ &\le n\lambda + \frac{\alpha z^k}{\sqrt{n}} \|e^T\| \|(X^k)^{-1}\triangle x\| - \mu\|(X^k)^{-1}\triangle x\|^2 \\ &= n\lambda + \alpha z^k \|(X^k)^{-1}\triangle x\| - \mu\|(X^k)^{-1}\triangle x\|^2 \\ &= n\lambda + z^k(\alpha\|(X^k)^{-1}\triangle x\| - \|(X^k)^{-1}\triangle x\|^2) \\ &\le n\lambda + \frac{\alpha^2 z^k}{4}. \end{aligned}$$

The last inequality holds since the quadratic term achieves maximum at $\|(X^k)^{-1}\triangle x\| = \alpha/2$. Thus, via (12)

$$z^{k+1} \le (1-\frac{\alpha}{\sqrt{n}}+\frac{\alpha^2}{4n})z^k.$$

From (18)

$$\begin{aligned} nz^{k+1} &= e^T X^{k+1} s^{k+1} \\ &= n\lambda + \triangle x^T (X^k)^{-1}(X^k s^k - \mu e) + \triangle x^T \triangle s \\ &\geq n\lambda + \triangle x^T (X^k)^{-1}(X^k s^k - \mu e) \\ &\geq n\lambda - |\triangle x^T (X^k)^{-1}(X^k s^k - \mu e)| \\ &\geq n\lambda - \|\triangle x^T (X^k)^{-1}\| \|(X^k s^k - \mu e)\| \\ &\geq n\lambda - \sqrt{2}\alpha\alpha z^k. \end{aligned}$$

Thus, via (12)

$$z^{k+1} \geq (1 - \frac{\alpha}{\sqrt{n}} - \frac{\sqrt{2}\alpha^2}{n}) z^k. \qquad Q.E.D.$$

The third lemma confirms that x^{k+1} and y^{k+1} are still close to the "center" path.

Lemma 3

$$\|X^{k+1}s^{k+1} - z^{k+1}e\| \leq (1+\sqrt{2})\alpha^2 z^k$$

Proof.

$$\begin{aligned} \|X^{k+1}s^{k+1} - z^{k+1}e\| &= \|X^{k+1}s^{k+1} - \lambda e + \lambda e - z^{k+1}e\| \\ &= \|(X^{k+1}s^{k+1} - \lambda e) - (z^{k+1} - \lambda)e\| \\ &\leq \|(X^{k+1}s^{k+1} - \lambda e)\|. \end{aligned}$$

The above inequality hold since the standard deviation is less than the square-root of the second order moment. From (10), (14), (18), and the above inequality,

$$\begin{aligned} \|X^{k+1}s^{k+1} - z^{k+1}e\| &\leq \|(X^{k+1}s^{k+1} - \lambda e)\| \\ &= \|\triangle X(s^k + \triangle s - \mu (X^k)^{-1} e)\| \\ &\leq \|\triangle X(s^k - \mu (X^k)^{-1} e)\| + \|\triangle X \triangle s\| \\ &= \|(X^k)^{-1}\triangle x\| \|(X^k s^k - \mu e)\| + \|\triangle X \triangle s\| \\ &\leq \sqrt{2}\alpha\alpha z^k + \alpha^2 z^k \\ &= (1+\sqrt{2})\alpha^2 z^k. \end{aligned} \qquad Q.E.D.$$

Based on the above three lemmas, we derive

Theorem

Let $\alpha = 1 - \frac{\sqrt{2}}{2}$ and $n \geq 2$. Then

$$Ax^{k+1} = b, x^{k+1} > 0, \quad \text{and} \quad s^{k+1} = Qx^{k+1} + c^T - A^T y^{k+1} > 0;$$

$$\|X^{k+1}s^{k+1} - z^{k+1}e\| \leq \alpha z^{k+1};$$

and

$$z^{k+1} \leq (1 - \frac{1}{4\sqrt{n}})z^k.$$

Proof. The first two inequalities hold due to (11) and Lemma 1; the third inequality holds due to the left inequality of Lemma 2 and Lemma 3; and the fourth inequality is true due to the right inequality of Lemma 2. Q.E.D.

Now the "center" path-following algorithm can be described as follows.

Algorithm 1:

Given $Ax^0 = b, x^0 > 0$, $s^0 = Qx^0 + c^T - A^T y^0 > 0$,

$\|X^0 s^0 - z^0 e\| \leq \alpha z^0$, and $z^0 = \frac{f(x^0) - d(x^0, y^0)}{n}$, where $\alpha = 1 - \frac{\sqrt{2}}{2}$;

set $k = 0$;

while $z^k \geq \epsilon$ **do**

begin

let $\mu = z^k$ and $\lambda = (1 - \frac{\alpha}{\sqrt{n}})z^k$;

let $\triangle x$ and $\triangle y$ solve (9);

let $x^{k+1} = x^k + \triangle x$ and $y^{k+1} = y^k + \triangle y$;

$k = k + 1$;

end.

The performance of Algorithm 1 results from the following corollary.

Corollary

The Algorithm terminates in $4n^{0.5}|\log(\frac{\epsilon}{z^0})|$ iterations and each iteration uses n^3 arithmetic operations.

Now we need to focus on addressing the question: how to obtain the initial solution pair x^0 and y^0 satisfying (5) and (6)?

4. Setting Initial Solution Pair x^0 and y^0

We noticed that several methods to obtain the "analytical center" [36] of a polyhedron are well illustrated in Todd and Burrell [38], Gonzaga [15], Renegar [32], and Vaidya [39]. Via those methods, QP can always be augmented to a related new QP problem with known "analytical center". Therefore, without losing generality, let the center x^0 of the feasible region of QP be known. Then x^0 should be positive and feasible for QP. However, the most attractive property of x^0 is that there exists a $\overline{y}$ such that

$$X^0 A^T \overline{y} = -e. \tag{20}$$

Now let

$$y^0 = \eta \overline{y} \quad \text{for some} \quad \eta,$$

then from (20)

$$s^0 = Qx^0 + c^T - A^T y^0,$$

$$z^0 = \frac{e^T X^0 (Qx^0 + c^T)}{n} + \eta,$$

and

$$\begin{aligned} X^0 s^0 - z^0 e &= X^0(Qx^0 + c^T) + \eta e - z^0 e \\ &= X^0(Qx^0 + c^T) - \frac{e^T X^0 (Qx^0 + c^T)}{n} e. \end{aligned}$$

Therefore, choose η such that

$$\eta e > -X^0(Qx^0 + c^T)$$

and

$$\eta \geq \frac{\|X^0(Qx^0 + c^T) - (e^T X^0 (Qx^0 + c^T)/n)e\|}{\alpha} - \frac{e^T X^0 (Qx^0 + c^T)}{n}.$$

Then it can be verified that

$$\|X^0 s^0 - z^0 e\| \leq \alpha z^0, \quad \text{and} \quad s^0 > 0.$$

Therefore, x^0 and y^0 can be used to initialize Algorithm 1.

5. A Safeguard Line Search Technique

Theoretically, both the Karmarkar's projective and the path-following algorithms allow solutions moving in a small step-size. This restriction severely slows down the convergence of these algorithms in practice. In Karmarkar's projective algorithm, one can use a line search technique to minimize the potential function, which significantly improves the practical efficiency of the projective algorithm [38]. Similarly, I propose a safeguard line search technique to overcome the small step-size difficulty in the original path-following algorithm.

From the derivation of Lemma 2, we can see that the smaller the λ, the faster the solution convergence. Moreover, λ only appears on the right-hand side of the system equation (9). These observations lead to the following safeguard line search technique to minimize z^{k+1} at each iteration of Algorithm 1.

Algorithm 2:

Given $Ax^0 = b, x^0 > 0$, $s^0 = Qx^0 + c^T - A^T y^0 > 0$,

$\|X^0 s^0 - z^0 e\| \leq \alpha z^0$, and $z^0 = \frac{f(x^0) - d(x^0, y^0)}{n}$, where $\alpha = 1 - \frac{\sqrt{2}}{2}$;

set $k = 0$;

while $z^k \geq \epsilon$ **do**

  begin

    let $\mu = z^k$;

    via system (9), select $\lambda^* \in [0 \quad (1 - \alpha/\sqrt{n}) z^k]$ to minimize z^{k+1}

    subject to the safeguard inequalities:

    $x^{k+1} > 0, s^{k+1} > 0$, and $\|X^{k+1} s^{k+1} - z^{k+1} e\| \leq \alpha z^{k+1}$;

    $k = k + 1$;

  end.

Each step of the line search costs at most $O(n)$ arithmetic operations after solving system (9) against two right-hand vectors $(X^k)^{-1} e$ and s^k. More interestingly, since λ^* makes the safeguard inequality $\|X^{k+1} s^{k+1} - z^{k+1} e\| \leq \alpha z^{k+1}$ binding, λ^* can *analytically* be obtained by solving a quadratic equation

$$\|X^{k+1} s^{k+1} - z^{k+1} e\| = \alpha z^{k+1}.$$

Obviously, Algorithm 2 remains a polynomial-time algorithm with the same worst case bound of Algorithm 1. However, from my computational experience the line search technique significantly improves performance of the "center" path-following algorithm in practice. λ can be chosen as a constant less than 1, resulting a constant convergence ratio in solving most of convex QP problems.

6. Summary

The interior ellipsoid algorithm for convex quadratic programming is further enhanced. This development is motivated by the recent "center" path-following algorithm for linear programming. The enhanced IE method for QP creates a sequence of dual as well as primal interior feasible points converging to the optimal solution point. At each iteration, the gap between the primal and dual objective values (or the complementary slackness value) is reduced at a global convergence ratio $(1-\frac{1}{4\sqrt{n}})$, where n is the number of variables in the convex QP problem. A line search technique is also incorporated into this algorithm to achieve practical efficiency.

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