An $O(\sqrt{n}L)$-Iteration Combined Phase I-Phase II Potential Reduction Algorithm for Linear Programming

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Abstract

We show that a modification of the combined Phase I-Phase II interior-point algorithm for linear programming, due to Anstreicher, de Ghellinck and Vial, Fraley, and Todd, terminates in $O(\sqrt{n}L)$ iterations from a suitable initial (interior but infeasible) solution. The algorithm either detects infeasibility, or approaches feasibility and optimality simultaneously, or generates a feasible primal-dual pair whose initial potential value is properly bounded, so that any $O(\sqrt{n}L)$-iteration algorithm can complete the rest of the work.

Key words: Linear programming, Phase I and Phase II, potential function.

Abbreviated title: A combined Phase I-Phase II algorithm for LP

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1. Introduction

By definition, interior-point algorithms for linear programming (LP) generate a sequence of points that are interior to the feasible region. Thus, techniques have to be developed to deal with the case that an initial interior feasible point is unknown or the feasible region is empty. From the theoretical point of view, this presents no difficulty since the big $\mathcal{M}$ method or the reformulation of the primal and dual LP as a single linear feasibility problem can be always used to retain the polynomial complexity result (e.g. Anstreicher and Bosch [4], Gonzaga [9], Karmarkar [10], Kojima et al. [11], Monteiro and Adler [17], Renegar [18], and Vaidya [24]). The best complexity bound is an $O(\sqrt{n}L)$ worst-case iteration count. However, the selection of $\mathcal{M}$ remains a challenging problem in practice: if $\mathcal{M}$ is chosen too small, the feasibility goal may never be met; if $\mathcal{M}$ is too large, the algorithm will be severely slowed. Furthermore, the reformulation doubles the problem size such that each iteration solves two systems of linear equations.

Kojima et al. [] and Kaliski and Ye [] used a dynamic updating scheme to increase the big $\mathcal{M}$ during the process of interior-point algorithms. The idea is to start at a small $\mathcal{M}$ to run the algorithm. Then, if $\mathcal{M}$ is too small, the algorithm will automatically report so and increase $\mathcal{M}$ accordingly. Thus, this scheme still belongs to big $\mathcal{M}$ approaches.

Several theoretical combined Phase I-Phase II methods, which avoid the big $\mathcal{M}$ dilemma, have been proposed. de Ghellinck and Vial [5] developed a projective method, which generates a sequence of interior but not necessarily feasible points that approach feasibility and optimality simultaneously. Later, Anstreicher [3], Fraley [6], and Todd [21] developed similar projective algorithms with the same feature. Theoretical and computational comparisons of these methods are given by Todd and Wang [22]. Anstreicher [2] and Todd [20] also used this idea in an affine potential reduction algorithm and developed a very simple combined Phase I-Phase II algorithm. All of these approaches have an $O(nL)$ worst-case iteration bound.

The combined Phase I-Phase II approach is widely used in practice, e.g., see Adler et al. [1], Lustig et al. [14], McShane et al. [15], and Mehrotra [16]. These approaches are generally not proved to be polynomial-time algorithms yet. Some of them even have no proof of convergence but reportedly work very well in practice. Kojima et al. [12,13,] recently proposed and analyzed several primal-dual path-following (homotopy) algorithms for solving the combined Phase I-Phase II linear complementarity problem. So far as I know, their algorithms converge under certain rules but not necessarily in polynomial-time.

Therefore, a theoretical question remains open: Is there a combined Phase I-Phase II algorithm, without using the big $\mathcal{M}$ technique and the reformulation, achieves $O(\sqrt{n}L)$ iteration bound starting from a suitable initial (interior but infeasible) solution? In this paper, we provide a final answer to this question. More specifically, we modify the recent combined Phase I-Phase II algorithm of Anstreicher and Todd, and show that in $O(\sqrt{n}L)$ iterations the algorithm terminates in one of the following three cases:

1. discovering that the LP problem is infeasible,
2. generating a feasible and optimal solution simultaneously,

or

3. detecting the LP problem is unbounded.

2. The Combined Phase I-Phase II Linear Program

Consider a pair of primal and dual linear programs

(LP1) minimize $c_1^T x$

subject to $Ax = b$

$x \geq 0,$

(LD1) maximize $b^T y$

subject to $s_1 = c_1 - A^T y \geq 0,$

where $c_1 \in R^n$, $A \in R^{m \times n}$ and $b \in R^m$ are given, $x \in R^n$, and ^{T} denotes the transpose. We assume that a positive feasible solution of (LP1) is known and $c_1^T x \geq 0$ for every feasible solution of (LP1).

Now, consider another pair of primal and dual linear programs

(LP2) minimize $c_2^T x$

subject to $Ax = b$

$c_1^T x = 0$

$x \geq 0,$

(LD2) maximize $b^T y \lambda$

subject to $s_2 = c_2 - A^T y - c_1 \lambda \geq 0.$

Let z_2^* designate the optimal value of (LP2) and (LD2) if it exists.

(LP1) and (LD1) can be interpreted as the primary problem, and (LP2) and (LD2) the secondary problem. Consequently, we call $c_1^T x$ and $c_2^T x$ the *primary* and *secondary* objective functions, respectively. If the primary problem is the Phase I of the standard linear program:

(LP) minimize $c^T x$

subject to $Ax = b$

$x \geq 0,$

then it can be written as

$$
\begin{array}{lll}
\text{(LPP)} & \text{minimize} & t \\
& \text{subject to} & Ax + at = b \\
& & (x, t) \geq 0.
\end{array}
$$

In this case, the secondary problem is

$$
\begin{array}{lll}
\text{(LPS)} & \text{minimize} & c^T x \\
& \text{subject to} & Ax + at = b \\
& & t = 0 \\
& & (x, t) \geq 0.
\end{array}
$$

Note that an interior dual feasible solution (y, λ) to (LPS) satisfies

$$c - A^T y > 0 \quad \text{and} \quad - a^T y - \lambda > 0.$$

Obviously, y is also an interior dual feasible solution to (LP). This observation plays an important role in our analysis.

Now let us focus on solving (LP1) and (LP2). For (LP1) and (LD1) we consider the potential function

$$\phi(x, s_1) = \rho \ln(x^T s_1) - \sum_{j=1}^{n} \ln(x_j (s_1)_j) \tag{2.1}$$

where $n \leq \rho < \infty$. Note that

$$x^T s_1 = c_1^T x - b^T y = c_1^T x - z_1$$

for some y with $s_1 = c_1 - A^T y > 0$ and $z_1 = b^T y < 0$. This primal-dual potential function was introduced by Todd and Ye [23], and used in an affine potential reduction algorithm (e.g., Ye [25]). Again, we let $\rho = n + \sqrt{n}$ in this paper.

Similar to [2] and [20], we assume

(A1) x^0, a positive vector, is a known feasible solution for (LP1). Moreover, a strict objective lower bounds z_1^0 for (LP1) is known, and $\phi(x^0, s_1^0) \leq O(\sqrt{n}L)$, where

$$s_1^0 = \arg\{\min_{s_1} \phi(x^0, s_1), \quad \text{subject to} \quad s_1 = c_1 - A^T y > 0,\ (x^0)^T s_1 = c_1^T x^0 - z_1^0\}. \tag{2.2}$$

Note that s_1^0 does not need to be known explicitly.

(A2) Strict objective lower bounds z_2^0 for (LP2), if both (LP2) and (LD2) are feasible , is known.

It is well known that both

$$c_1^T x^0 - z_1^0 \leq 2^L \quad \text{and} \quad c_2^T x^0 - z_2^0 \leq 2^L,$$

where L is the input length of the LP data. The assumption of the existence of z_1^0 may even be removed, see Todd [].

The algorithm described in the next section generates a sequence of points x^k, s_1^k, and z_2^k, where x^k remains an interior feasible solution for (LP1), s_1^k remains an interior feasible solution for (LD1), and z_2^k remains an objective lower bound for (LP2) if if both (LP2) and (LD2) are feasible. For any $K \geq 0$, one of the following two cases occurs:

1. We have $(x^K)^T s_1^K < c_1^T x^K$ or $z_1^K > 0$, which indicates that the *primary* goal can never be met or (LP2) is infeasible.

2. We have
$$\phi(x^{K+1}, s_1^{K+1}) \leq \phi(x^K, s_1^K) - 1/20. \tag{2.3}$$

   Moreover, if $c_2^T x^K - z_2^K > 0$, then $c_2^T x^k - z_2^k > 0$ for all $0 \leq k \leq K$ and
$$\frac{c_2^T x^{K+1} - z_2^{K+1}}{c_2^T x^K - z_2^K} \leq \frac{c_1^T x^{K+1}}{c_1^T x^K}; \tag{2.4}$$

otherwise,

$$c_2^T x^k - z_2^k \leq 0 \quad \text{for all} \quad k \geq K. \tag{2.5}$$

If Case 1 happens, then (LP2) is infeasible. In the following we assume that Case 1 never occurs. Then from relation (2.2) and (2.3) in at most $O(\sqrt{n}L)$ iterations

$$\phi(x^k, s_1^k) \leq -\sqrt{n}L \quad \text{so that} \quad c_1^T x^k \leq (x^k)^T s_1^k \leq 2^{-O(L)},$$

which indicates that the *primary* goal has been achieved, i.e., x^k becomes a feasible solution for (LP2).

If $c_2^T x^k - z_2^k$ remains positive, from (2.4) we have

$$c_2^T x^k - z_2^k \leq \frac{c_2^T x^0 - z_2^0}{c_1^T x^0}(c_1^T x^k) \leq 2^{-O(L)}.$$

If $c_2^T x^k - z_2^k$ becomes nonpositive, then from (2.5)

$$c_2^T x^k \leq z_2^k.$$

Two outcomes may happen. If z_2^k is never updated or $z_2^k = z_2^0$, then (LP2) is feasible but its objective function is unbounded from below or (LD2) is infeasible; otherwise, there exists a feasible solution (s_2^k, y^k, λ^k) for (LD2) such that

$$z_2^k = b^T y^k.$$

Thus, if $c_2^T x^k - z_2^k > 0$ then x^k and (y^k, λ^k) become the optimal solution pair for (LP2) and (LD2); otherwise

$$z_2^* - 2^{-O(L)} \leq c_2^T x^k \leq z_2^k = b^T y^k \leq z_2^*,$$

where the first inequality is due to the fact that x^k is an "almost" feasible solution for (LP2). Thus, regardless the sign of $c_2^T x^k - z_2^k$, x^k and (s_2^k, y^k, λ^k) simultaneously become the optimal solution pair for (LP2) and (LD2).

In the case of $c_2^T x^k - z_2^k \leq 0$, Anstreicher and Todd called x^k the "super-optimal" solution for (LP2). In this case, the algorithm will generate x^k that remains "super-optimal" and moves towards feasibility. Otherwise, x^k remains "inferior-optimal" and moves towards feasibility.

3. Modification of the Combined Phase I-Phase II Method

Let

$$\Delta_1^k = (x^k)^T s_1^k = c_1^T x^k - z_1^k, \quad \bar{\Delta}_1^k = c_1^T x^k \quad \text{and} \quad \Delta_2^k = c_2^T x^k - z_2^k.$$

Assume that $(x^k, s_1^k) > 0$ are feasible solutions for (LP1) and (LD1). The gradient vector of the potential function with respect to x at x^k is

$$\nabla_x \phi(x^k, s_1^k) = \frac{\rho}{\Delta_1^k} c_1 - (X^k)^{-1} e, \tag{3.1}$$

where the upper-case letter X designates the diagonal matrix of the vector x in lower-case. Let g_1 be the scaled gradient

$$g_1 = \frac{\rho}{\Delta_1^k} X^k c_1 - e, \tag{3.2.1}$$

and d_1 be the projection vector of g_1 onto the null space of AX^k, that is,

$$d_1 = P_{AX^k} g_1 = \frac{\rho}{\Delta_1^k} X^k (c_1 - A^T y^k) - e \tag{3.2.2}$$

where

$$P_{AD} = I - DA^T (AD^2 A^T)^{-1} AD$$

and

$$y^k = \frac{\Delta_1^k}{\rho} (A(X^k)^2 A^T)^{-1} AX^k g_1. \tag{3.2.3}$$

The kth iteration of the algorithm consists of the following three procedures.

Procedure 1: *primary* **Dual Updating**. In this procedure, we evaluate $\|d_1\|$. If $\|d_1\| > 1/2$, then we set

$$s_1^{k+1} = s_1^k \quad \text{and} \quad z_1^{k+1} = z_1^k,$$

that is, no update is necessary for the *primary* dual; otherwise, from Lemma 2 and Theorem 1 of [25], a new interior feasible solution, say $\bar{s}_1$, for (LD1) is at hand, and

$$(x^k)^T \bar{s}_1 < \Delta_1^k \quad \text{and} \quad \phi(x^k, \bar{s}_1) \leq \phi(x^k, s_1^k) - 1/20. \tag{3.3}$$

Now, we assign

$$s_1^{k+1} = \bar{s}_1 \quad \text{and} \quad z_1^{k+1} = c_1^T x^k - (x^k)^T s_1^{k+1} > z_1^k.$$

Next, we use $(x^k)^T s_1^{k+1}$ to replace Δ_1^k in (3.2)s to update g_1 and d_1. Hence, we shall have

$$\|d_1\| > 1/2.$$

Also, if $z_1^{k+1} > z_1^*$, then stop: the *primary* goal can never be met.

Procedure 2: Primal Direction Finding. In this procedure we solve Anstreicher's auxiliary problem [2] with a real parameter Δ_2

$$\begin{array}{lll} \mathrm{AP}(\Delta_2) & \text{maximize} & g_1^T d \\ & \text{subject to} & AX^k d = 0 \\ & & (\bar{\Delta}_1^k c_2 - \Delta_2 c_1)^T X^k d \geq 0 \\ & & \|d\| \leq 1. \end{array}$$

From Todd's analysis [20], this problem can be simplified as

$$\begin{array}{lll} \mathrm{AP}(\Delta_2) & \text{maximize} & d_1^T d \\ & \text{subject to} & (\bar{\Delta}_1^k c_2 - \Delta_2 c_1)^T X^k P_{AX^k} d \geq 0 \\ & & \|d\| \leq 1. \end{array}$$

The solution to the auxiliary problem is given by $d(\Delta_2)/\|d(\Delta_2)\|$ where

$$d(\Delta_2) = \begin{cases} d_1, & \text{if } (\bar{\Delta}_1^k c_2 - \Delta_2 c_1)^T X^k d_1 \geq 0 \\ d_p(\Delta_2), & \text{otherwise} \end{cases},$$

where $d_p(\Delta_2)$ is the projection of d_1 onto the null space of $(\bar{\Delta}_1^k c_2 - \Delta_2 c_2)^T X^k P_{AX^k}$.

We now present two lemmas to analyze a new primal feasible solution for (LP1)

$$x(\Delta_2) = X^k(e - \alpha \frac{d(\Delta_2)}{\|d(\Delta_2)\|}) \quad \text{for some constant} \quad 0 < \alpha < 1. \tag{3.4}$$

Lemma 3.1. (Anstreicher) Let $d = d(\Delta_2^k)$. If $\Delta_2^k > 0$, then

$$\frac{c_2^T x(\Delta_2^k) - z_2^k}{\Delta_2^k} \leq \frac{c_1^T x(\Delta_2^k) - z_1^*}{\bar{\Delta}_1^k};$$

otherwise,

$$c_2^T x(\Delta_2^k) - z_2^k \leq 0$$

Proof. First consider the case $\Delta_2^k > 0$. This is directly from the inequality constraint in $\mathrm{AP}(\Delta_2^k)$

$$(\bar{\Delta}_1^k c_2 - \Delta_2^k c_1)^T X^k d \geq 0,$$

or

$$\frac{\alpha}{\|d\|} \frac{c_2^T X^k d}{\Delta_2^k} \geq \frac{\alpha}{\|d\|} \frac{c_1^T X^k d}{\bar{\Delta}_1^k},$$

or

$$1 - \frac{\alpha}{\|d\|} \frac{c_2^T X^k d}{\Delta_2^k} \leq 1 - \frac{\alpha}{\|d\|} \frac{c_1^T X^k d}{\bar{\Delta}_1^k},$$

that is,

$$\frac{c_2^T x(\Delta_2^k) - z_2^k}{\Delta_2^k} \leq \frac{c_1^T x(\Delta_2^k)}{\bar{\Delta}_1^k}.$$

Similarly, in the second case (while $\Delta_2^k = 0$ is trivial) we have

$$\frac{c_2^T x(\Delta_2^k) - z_2^k}{\Delta_2^k} \geq \frac{c_1^T x(\Delta_2^k)}{\bar{\Delta}_1^k} \geq 0,$$

which implies $c_2^T x(\Delta_2^k) - z_2^k \leq 0$. $\square$

Lemma 3.2. If $\|d(\Delta_2^k)\| \geq 1/2$, then

$$\phi(x(\Delta_2^k), s_1^{k+1}) \leq \phi(x^k, s_1^{k+1}) - 1/20 \quad \text{for} \quad \alpha = .3$$

Proof. Note that $g_1^T d(\Delta_2^k) = d_1^T d(\Delta_2^k) = \|d(\Delta_2^k)\|^2$. The proof then follows from the potential reduction theorem, see for example, Theorem 1 of [25]. $\square$

If the condition in Lemma 3.2 holds, we assign $x^{k+1} = x(\Delta_2^k)$ of (3.4). Lemma 3.2 indicates that the primal feasible solution x^{k+1} produces a guaranteed decrease in the primal-dual potential function, while Lemma 3.1 indicates that the *secondary* "gap" $c_2^T x^{k+1} - z_2^k$ is well controlled by the *primary* gap to optimality $\bar{\Delta}_1^{k+1} = c_1^T x^{k+1}$. In this case, let $z_2^{k+1} = z_2^k$, i.e., no update is made for the *secondary* dual, we then go to the $(k+1)$th iteration with

$$\phi(x^{k+1}, s_1^{k+1}) \leq \phi(x^k, s_1^k) - 1/20$$

from Lemma 3.2, or possibly

$$\phi(x^{k+1}, s_1^{k+1}) \leq \phi(x^k, s_1^k) - 1/10$$

from (3.3) and Lemma 3.2 if s_1^{k+1} is updated from s_1^k. Furthermore, if $\Delta_2^k > 0$ then

$$\frac{\Delta_2^{k+1}}{\Delta_2^k} \leq \frac{\bar{\Delta}_1^{k+1}}{\bar{\Delta}_1^k} \tag{3.5.1}$$

else

$$\Delta_2^{k+1} = c_2^T x^{k+1} - z_2^{k+1} \leq 0. \tag{3.5.2}$$

Note that we have maintained $z_1^{k+1} \leq z_1^*$ and $z_2^{k+1} \leq z_2^*$.

To complete the description of the algorithm, we need to consider the case of $\|d(\Delta_2^k)\| < 1/2$ in Procedure 2. This results in the third procedure.

Procedure 3: *secondary* **Dual Updating**. The procedure is based on the following lemma.

Lemma 3.3. Suppose that $\|d(\Delta_2^k)\| < 1/2$, then there exist an interior feasible solution $(\bar{y}, \bar{\lambda})$ for (LD2) such that

$$b^T \bar{y} > z_2^k.$$

Proof. Since d_1 resulted from Procedure 1 guarantees $\|d_1\| > 1/2$, we must have

$$d(\Delta^k) = d_p(\Delta^k)$$

where $d_p(\Delta)$ is the projection of d_1 onto the null space of $\bar{c}^T P_{AX^k}$. Thus

$$d(\Delta^k) = d_1 + \lambda P_{AX^k}\bar{c} \quad \text{with} \quad \lambda = \frac{-\bar{c}^T d_1}{\|P_{AX^k}\bar{c}\|^2} > 0.$$

Recall (10) and (11),

$$e + d(\Delta^k) = (\rho/\Delta_1^k)X^k c_1 - \lambda\Delta^k X^k c_1 + \lambda\Delta_1^k X^k c - X^k A^T(y_1 + \lambda y_2)$$

where

$$y_1 = (A(X^k)^2 A^T)^{-1} A X^k g_1$$

and

$$y_2 = (A(X^k)^2 A^T)^{-1} A X^k \bar{c}.$$

This relation can be rewriten as

$$e + d(\Delta^k) = (\rho/\Delta_1^k)(X^k c_1 - A^T y^{k+1}) - \lambda\Delta^k X^k c_1 + \lambda\bar{\Delta}_1^k X^k (c - A^T\bar{y}) \qquad (13)$$

where

$$\bar{y} = (y_1 + \lambda y_2 - (\rho/\Delta_1^k) y^{k+1})/(\lambda\Delta_1^k).$$

Since $\|d(\Delta^k)\| < 1$, we have $e + d(\Delta^k) > 0$. From (13)

$$c - A^T\bar{y} + \frac{(\rho/\Delta_1^k) - \lambda\Delta^k}{\lambda\Delta_1^k}c_1 > 0,$$

i.e., $\bar{y}$ is a dual feasible solution for (LP2) and $b^T\bar{y} \le z^*$.

Summing up both sides of the equality (13) we have

$$e^T(e + d(\Delta^k)) = \rho - \lambda\Delta^k\Delta_1^k + \lambda\Delta_1^k(c^Tx^k - b^T\bar{y}).$$

However,

$$e^T(e + d(\Delta^k)) = n + e^Td(\Delta^k) \le n + \sqrt{n}\|d(\Delta^k)\| < n + \sqrt{n} \le \rho.$$

Therefore,

$$-\lambda\Delta^k\Delta_1^k + \lambda\Delta_1^k(c^Tx^k - b^T\bar{y}) < 0$$

or

$$c^Tx^k - z^k = \Delta^k > c^Tx^k - b^T\bar{y}$$

or

$$\underline{z} := b^T\bar{y} > z^k. \quad \square$$

the proof then follows from Lemma 3.5 of Anstreicher [2]. $\square$

In Lemma 3.3, a new increased lower bound $z_2^{k+1} (> z_2^k)$ can always be found such that $\|d(\Delta_2^k)\| \ge 1/2$ with $\Delta_2^k = c_2^Tx^k - z_2^{k+1}$, then we can return to Procedure 2 to update x^k to $x^{k+1} = x(\Delta_2^k)$ given in (3.4). If $c_2^Tx^k - z_2^{k+1} > 0$, then $c_2^Tx^k - z_2^k > 0$ and from Lemma 3.1

$$\frac{c_2^Tx^{k+1} - z_2^{k+1}}{c_2^Tx^k - z_2^{k+1}} \le \frac{\bar{\Delta}_1^{k+1}}{\bar{\Delta}_1^k},$$

which implies

$$\frac{c_2^Tx^{k+1} - z_2^{k+1}}{c_2^Tx^k - z_2^k} \le \frac{\bar{\Delta}_1^{k+1}}{\bar{\Delta}_1^k}.$$

If $c_2^T x^k - z_2^k \leq 0$, then $c_2^T x^k - z_2^{k+1} \leq 0$, which by Lemma 3.1 implies $c_2^T x^{k+1} - z_2^{k+1} \leq 0$. Thus, (3.5) still holds even if z_2^{k+1} is updated from z_2^k.

We clearly see the following theorem.

Theorem. The modification will terminate in $O(\sqrt{n}L)$ iterations under assumptions (A1) and (A2), either detects that the primal (LP2) or the dual (LD2) is infeasible, or simultaneously generates an optimal feasible pair for the LP problem with error less than 2^{-L}.

4. Further Remarks

The choice of ρ can be $n + \theta\sqrt{n}$ for any fixed θ to achieve $O(\sqrt{n}L)$-iteration convergence for a suitable (x^0, z_1^0). This choice may only have theoretical value. In general, ρ can be selected as $n + \theta n^\beta$ for some constants $\theta > 0$ and $\beta > 1/2$. The algorithm has a worst-case complexity bound of $O(n^\beta L)$-iteration. The larger ρ is usually favored in practice.

Although assumption (A1) is used as a standard assumption in complexity analysis, we like to present the following homogenenizing procedure to remove it for solving the combined LP Phase I and Phase II problem. In this case, we like to sove

$$
\begin{aligned}
\text{(LPP)} \qquad & \text{minimize} && t \\
& \text{subject to} && Ax - bx_0 + at = 0 \\
& && e^T x + x_0 = 1 \\
& && (x, x_0, t) \geq 0.
\end{aligned}
$$

Obviously, we have a primal $x^0 = e$, $x_0^0 = 1$ and $t^0 = 1$, and a dual $y^0 = 0$, $\lambda^0 = -1$ and $s^0 = e$. The secondary probelem is

$$
\text{(LPS)} \qquad \text{minimize} \quad c^T x - z_2^* x_0
$$

$$
\begin{aligned}
\text{subject to} \quad & Ax - bx_0 + at = 0 \\
& e^T x + x_0 = 1 \\
& \qquad t = 0 \\
& x_0 > 0, (x, t) \geq 0.
\end{aligned}
$$

Similarly, iteration K of the modification will have either

1. $(x^K)^T s_1^K = t^K - \lambda^K < t^K$ or $\lambda^K > 0$, which indicates that the LP problem is infeasible, or
2. $$\phi(x^{K+1}, s_1^{K+1}) \leq \phi(x^K, s_1^K) - 1/20.$$

Moreover, if $c_2^T x^K - z_2^K x_0^K > 0$, then $c_2^T x^k - z_2^k x_0^k > 0$ for all $0 \leq k \leq K$ and

$$\frac{c_2^T x^{K+1} - z_2^{K+1} x_0^{K+1}}{c_2^T x^K - z_2^K x_0^K} \leq \frac{t^{K+1}}{t^K};$$

otherwise,

$$c_2^T x^k - z_2^k x_0^k \leq 0 \quad \text{for all} \quad k \geq K.$$

The lower bound z_2^k for the LP problem can be similarly increased as it was the case in Procedure 3. Moreover, x_0^k will bounded above from 1 and below from zero as discussed in Güler and Ye []. In $O(\sqrt{n}L)$ iterations, we shall have

$$|c^T x^k - z_2^k x_0^k| \leq 2^{-L}$$

or

$$|c^T (x^k / x_0^k) - z_2^k| \leq 2^{-L},$$

where x^k / x_0^k is a feasible solutuion for the LP problem with error less than 2^{-L}, and z_2^k is the LP's objective lower bound.

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