# An Improved Rounding Method and Semidefinite Programming Relaxation for Graph Partition *

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**Abstract**

Given an undirected graph $G = (V, E)$ with $|V| = n$ and an integer $k$ between 0 and $n$, the maximization graph partition (MAX-GP) problem is to determine a subset $S \subset V$ of $k$ nodes such that an objective function $w(S)$ is maximized. The MAX-GP problem can be formulated as a binary quadratic program and it is NP-hard. Semidefinite programming (SDP) relaxations of such quadratic programs have been used to design approximation algorithms with guaranteed performance ratios for various MAX-GP problems. Based on several earlier results, we present an improved rounding method using an SDP relaxation, and establish improved approximation ratios for several MAX-GP problems, including Dense-Subgraph, Max-Cut, Max-Not-Cut, and Max-Vertex-Cover.

**Key words.** Dense-$k$-Subgraph, Max-Cut, Max-Not-Cut, Max-Vertex-Cover, polynomial approximation algorithm, performance ratio, semidefinite programming.

# 1 Introduction

Given an undirected graph $G = (V, E)$ with $|V| = n$, non-negative weights $w_{ij}$ on edges $(i, j) \in E$, and an integer $k$ $(1 < k < n)$, the maximization graph partition (MAX-GP) problem is to determine a subset $S \subset V$ of $k$ nodes such that an objective function $w(S)$ is maximized. Some examples of MAX-GP are: *Dense-k-Subgraph (DSP)*, where the total edge weights of the subgraph induced by $S$ is maximized; *Max-Cut with size k (MC)*, where the total edge weights of the edges crossing between $S$ and $V \setminus S$ is maximized; *Max-Not-Cut with size k (MNC)*, where the total edge weights of the non-crossing edges between $S$ and $V \setminus S$ is maximized; *Max-Vertex-Cover with size k (MVC)*, where the total edge weights of the edges covered by $S$ is maximized; etc.

Since these MAX-GP problems are NP-hard (e.g., see [7] for DSP, [1] for MC, [19] for MNC, and [15] for MVC), one should not expect to find polynomial time algorithms for computing their optimal solutions. Therefore, we are interested in how close to optimality one can approach in polynomial time. A (randomized) polynomial time approximation algorithm for a maximization problem has a performance guarantee or worst case ratio $0 < r \leq 1$, if it outputs a feasible solution whose (expected) value is at least $r$ times the maximal value for all instance of the problem. Such an algorithm is often called (randomized) $r$-approximation algorithm. A key step in designing a good approximation algorithm for such a maximization problem is to establish a good upper bound on the maximal objective value. Linear programming (LP) and semidefinite programming (SDP) have been frequently used to provide such upper bounds for many NP-hard problems.

There are several approximation algorithms for DSP. Kortsarz and Peleg [12] devised an approximation algorithm which has a performance guarantee $O(n^{-0.3885})$. Feige, Kortsarz and Peleg [8] improved it to $O(n^{-1/3+\epsilon})$, for some $\epsilon > 0$. The other approximation algorithms have performance guarantees which are the function of $k/n$, e.g., a greedy heuristic by Asahiro et.al. [3], and SDP relaxation based algorithms developed by Feige and Langberg [6] and Feige and Seltser [7], and Srivastav and Wolf [16]. The previously best performance guarantee, $k/n$ for general $k$ and $k/n+\epsilon_k$ for $k \sim n/2$, was obtained by Feige and Langberg [6]. Moreover, for the case $k = n/2$, Ye and Zhang [19], using a new SDP relaxation, obtained an improved 0.586 performance guarantee from 0.517 of Feige and Langberg [6].

For approximating the MC problem with size $k$, both LP and SDP based approximation algorithms have a performance ratio $1/2$ for all $k$, see Ageev and Sviridenko [1] and Feige and Langberg [6]. For $k = n/2$, i.e., the Max-Bisection, Frieze and Jerrum [9] obtained a 0.651-approximation algorithm (the same bound was also obtained by

Andersson [2] in his paper for max-$p$-section). Subsequently, this ratio has been improved to 0.699 by Ye [17]. Both of their approximation algorithms are based on SDP relaxations.

For approximating the MNC problem with size $k$, the LP-based approximation algorithm has a performance ratio $1 - \frac{2k(n-k)}{n(n-1)}$ for all $k$, and the SDP-based algorithm has a ratio $.5 + \epsilon_k$ for $k \sim n/2$, see Feige and Langberg [6]. Again, Ye and Zhang [19] obtained a 0.602-approximation algorithm for MNC when $k = n/2$, comparing to .541 of Feige and Langberg [6].

The main contribution of this paper, motivated from the results of all above, is to present an improved method to round an optimal solution of the SDP relaxation of the MAX-GP problem for general $k$. This rounding technique is related to the well-known rounding method introduced by Goemans and Williamson [10] in their seminal work for MAX-CUT, Feige and Goemans [5] for MAX-DICUT and MAX 2-SAT, Zwick [20] for constraint satisfaction problems, and Nesterov[14] and Zwick [21] for MAX-CUT. And this kind of randomized algorithm can be de-randomized by the technique of Mahajan and Ramesh[13].

What complicates matters in the MAX-GP problem, comparing to the MAX-CUT problem, is that two objectives are sought—the objective value of $w(S)$ and the size of $S$. Therefore, in any (randomized) rounding method, we need to balance the (expected) quality of $w(S)$ and the (expected) size of $S$. We want high $w(S)$; but, at the same time, zero or small difference between $|S|$ and $k$, since otherwise we have to either add or subtract nodes from $S$, resulting in a deterioration of $w(S)$ at the end. Our improved rounding method is built upon this balance need.

As consequences of the improved rounding method, we have yielded improved approximation performance ratios for DSP, MC, MNC and MVC, on a wide range of $k$. On approximating DSP, for example, our algorithm has guaranteed performance ratios .648 for $k = 3n/5$, 0.586 for $k = n/2$, 0.486 for $k = 2n/5$ and 0.278 for $k = n/4$. For MC and MNC, the performance guarantees are also much better than 0.5 for a wide range of $k$. On approximating MVC, our algorithm has guaranteed performance ratios .845 for $k = 3n/5$, 0.811 for $k = n/2$, and 0.733 for $k = 2n/5$.

This paper is organized as follows. In the next two sections, we present an SDP relaxation of the MAX-GP problem, the method of rounding an optimal SDP solution, and preliminary analyses of the method. In sections 4, 5, 6, and 7 we prove specific approximation ratios for DSP, MC, MNC, and MVC, respectively.

## 2 SDP Relaxation and Rounding Method for MAX-GP

By introducing a binary variable $x_i \in \{-1, 1\}$ to each node $i \in V$, where $i \in S$ if and only if $x_i = 1$, the MAX-GP problem can be formulated as a binary quadratic program with a linear constraint:

$$\begin{array}{rll} w^* := & \text{Maximize} & \displaystyle\sum_{1 \le i < j \le n} w_{ij}(c_0 + c_1 \cdot x_i + c_2 \cdot x_j + c_3 \cdot x_i x_j) \\ \text{(MAX-GP)} & & \\ & \text{subject to} & \displaystyle\sum_{i=1}^{n} x_i = 2k - n, \\ & & x_i \in \{-1, 1\}, \quad i = 1, \ldots, n, \end{array}$$

where coefficients $c_0, c_1, c_2, c_3$ are

| MAX-GP Problem | $c_0$ | $c_1$ | $c_2$ | $c_3$ |
|---|---|---|---|---|
| DSP | 1/4 | 1/4 | 1/4 | 1/4 |
| MC | 1/2 | 0 | 0 | -1/2 |
| MNC | 1/2 | 0 | 0 | 1/2 |
| MVC | 3/4 | 1/4 | 1/4 | -1/4 |

To construct an SDP relaxation of (MAX-GP), we introduce a "reference" binary variable $x_0$, where $i \in S$ if and only if $x_i = x_0$. Then, (MAX-GP) can be rewritten as a homogeneous quadratic program:

$$\begin{array}{rll} w^* := & \text{Maximize} & \displaystyle\sum_{1 \le i < j \le n} w_{ij}(c_0 + c_1 \cdot x_i x_0 + c_2 \cdot x_j x_0 + c_3 \cdot x_i x_j) \\ \text{(MAX-GP)} & & \\ & \text{subject to} & \displaystyle\sum_{i=1}^{n} x_i x_0 = 2k - n, \\ & & x_i^2 = 1, \quad i = 0, 1, \cdots, n. \end{array}$$

Noting further that

$$\sum_{1 \le i,j \le n} x_i x_j = \left( \sum_{i=1}^{n} x_i x_0 \right)^2 = (2k - n)^2,$$

we have an SDP relaxation of (MAX-GP) as:

$$
\begin{array}{lll}
w^{SDP} := & \text{Maximize} & \displaystyle\sum_{1\le i<j\le n} w_{ij}(c_0 + c_1 \cdot X_{i0} + c_2 \cdot X_{j0} + c_3 \cdot X_{ij}) \\
\text{(SDP-GP)} & & \\
& \text{subject to} & \displaystyle\sum_{1\le i,j\le n} X_{ij} = (2k-n)^2, \\
& & \displaystyle\sum_{i=1}^{n} X_{i0} = 2k-n, \\
& & X_{ii} = 1, \quad i = 0, 1, \cdots, n, \\
& & X \succeq 0.
\end{array}
\tag{1}
$$

where $X \in R^{(n+1)\times(n+1)}$ is the unknown symmetric matrix, and $X \succeq Z$ means that $X - Z$ is positive semidefinite. Obviously, $w^{SDP} \ge w^*$, since, for any feasible solution $x = (x_0, x_1, \cdots, x_n)$ of (MAX-GP), $X = xx^T$ is feasible for (SDP-GP).

Our generic approximation method for (MAX-GP) consists of the following three steps:

**Algorithm SDP-GP**

**1. SDP-GP Solving** Solve (SDP-GP) to obtain an optimal semidefinite matrix $\bar{X}$.

Repeat the following two steps many times, and output the best subset.

**2. Randomized Rounding** Using the multivariate normal distribution with 0 mean and the covariance matrix $Y := \theta\bar{X} + (1-\theta)P$, for a given $0 \le \theta \le 1$, to generate $u, \hat{x}$ and $S$ as:

$$u \in N(0, Y);$$

$$\hat{x} = \text{sign}(u),$$

i.e.,

$$\hat{x}_i = \begin{cases} 1 & \text{if } u_i \ge 0 \\ -1 & \text{if } u_i < 0; \end{cases}$$

$$S = \{i \ge 1 : \ \hat{x}_i = \hat{x}_0\};$$

where $P$ is the positive semidefinite matrix

$$P = \begin{pmatrix} 1 & \kappa & \kappa & \cdots & \kappa \\ \kappa & 1 & \kappa^2 & \cdots & \kappa^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \kappa & \kappa^2 & \cdots & \kappa^2 & 1 \end{pmatrix},$$

and $\kappa = 2\sigma - 1$ and $\sigma = k/n$.

**3. Size Adjusting** Using the greedy algorithm, either adding or removing one node at a time while causing the least deterioration in $w(S)$, to resize $S$ to $k$, and output $S$. ■

We will specify the detailed size adjusting procedure later for each individual problem.

Our rounding method is identical to that of Goemans and Williamson [10] when $\theta = 1$, and identical to that of Zwick [21] and Ye [17] when $\kappa = 0$. The design of $\kappa$ in the above rounding step is new; and it is this design, together with improved analyses, enable us to improve approximation ratios for these MAX-GP problems.

Goemans and Williamson [10] and Bertsimas and Ye [4] have showed that this type of rounding with covariance matrix $Y$, where $Y$ is positive semidefinite with diagonal values all 1, has expectation

$$\mathrm{E}[\hat{x}_i\hat{x}_j] = \frac{2}{\pi}\arcsin(Y_{ij}).$$

Thus, our rounding steps generates $\hat{x}$ who has the following expectations:

$$\mathrm{E}[\hat{x}_0\hat{x}_j] = \frac{2}{\pi}\arcsin(\theta\bar{X}_{0j} + (1-\theta)\kappa), \quad \text{for} \quad j = 1, \ldots, n,$$
$$\mathrm{E}[\hat{x}_i\hat{x}_j] = \frac{2}{\pi}\arcsin(\theta\bar{X}_{ij} + (1-\theta)\kappa^2), \quad \text{for} \quad i, j = 1, \ldots, n.$$

For clarity, we denote the subset after size adjusting by $\tilde{S}$, that is, $S$ denotes the selected subset before size adjusting; and let $|S| = \delta \cdot n$ $(0 \leq \delta \leq 1)$. To analyze the quality of the algorithm, we need to establish a bound on

$$\frac{\mathrm{E}[w(\tilde{S})]}{w^*}.$$

Let the greedy algorithm in the size adjusting procedure possess

$$w(\tilde{S}) \geq r(\delta) \cdot w(S), \tag{2}$$

where $r(\delta) \leq 1$ is a "deterioration" function ($r(\delta) = 1$ if $\delta = k/n$, i.e., no deterioration occurred), and

$$w(S) = \sum_{1 \leq i < j \leq n} w_{ij}\left(c_0 + c_1\hat{x}_i\hat{x}_0 + c_2\hat{x}_j\hat{x}_0 + c_3\hat{x}_i\hat{x}_j\right).$$

Again, the overall performance of the algorithm is determined by two factors: the expected quality of $w(S)$ and how much the size of $S$ need to be adjusted. To make

sure that $|S|$ is close to $k$ before resizing, we have added two seemingly redundant constraints on the top of (SDP-GP) of (1), which will be used to bound the expected value as well as the "variance" of $|S|$. In addition, the convex combination parameter $\theta$ used in $\theta\bar{X} + (1-\theta)P$ provides a balance between these two factors. Typically, the more use of $\bar{X}$ in the combination results in higher expected quality $w(S)$ but larger expected difference between $|S|$ and $k$; and the more use of $P$ results in less $w(S)$ and more accurate $|S|$.

To see this, one can easily verify that if we choose $\theta = 0$ and $\kappa = \sin(\frac{\pi}{2} \cdot (2\frac{k}{n} - 1))$, our rounding method would generate an $S$ whose expected size is exactly $k$.

# 3 Analysis of the SDP-GP Algorithm

To analyze an expected lower bound for $w(\tilde{S})$, from (2) we now wish to lower bound

$$\frac{\mathrm{E}[r(\delta) \cdot w(S)]}{w^*}.$$

However, since $r(\delta)$ and $w(S)$ are both random numbers and they are multiplied together, it is typically hard to calculate its expected value. The idea, initially due to Frieze and Jerrum [9] and later extended in [17], is to construct a family of artificial or auxiliary random variables, say $z_S$, whose expected value would be easily estimated and bounded; moreover, whenever $z_S$ meets its expectation (meaning $z_S$ is equal to or greater than its expectation), we have

$$r(\delta) \cdot w(S) \geq R(z_S) \cdot w^*.$$

This would establish a performance ratio $R(z_S)$ for the SDP-GP algorithm. Naturally, we like to select $z_S$ that yields the largest $R(z_S)$.

The family of artificial random variables we used throughout this paper are linear combinations of $w(S)/w^*$, $|S|/n$ and $|S|(n-|S|)/n^2$:

$$z(\gamma, \eta) := \frac{w(S)}{w^*} + \gamma\eta\frac{|S|}{n} + \gamma\frac{|S|(n-|S|)}{n^2}, \tag{3}$$

where $\eta \in [-1, 1]$ and $\gamma \geq 0$ are two constant parameters. As we will see later in Corollaries 1, 2, 3 and 4,

$$0 \leq \frac{w(S)}{w^*} \leq c(n/k)$$

for some constant $c(n/k)$, if $n/k$ (for DSP and MVC) and $n/(n-k)$ (for MC and MNC) are bounded above by a constant. (Note that for MC and MNC, we only

consider $k \geq n/2$.) Then

$$-\gamma|\eta| \leq z(\gamma,\eta) \leq C := c(n/k) + \frac{\gamma(1+\eta)^2}{4},$$

i.e., $z(\gamma,\eta)$ is always bounded from both sides by a constant for a certain range of $k$.

Thus, if we could show

$$\mathrm{E}[\frac{w(S)}{w^*}] \geq \alpha, \quad \mathrm{E}[\eta\frac{|S|}{n}] \geq \eta\beta_1, \quad \text{and} \quad \mathrm{E}[\gamma\frac{|S|(n-|S|)}{n^2}] \geq \gamma\beta_2,$$

then

$$\mathrm{E}[z(\gamma,\eta)] \geq \mu_{\eta,\gamma} := \alpha + \gamma(\eta\beta_1 + \beta_2).$$

Following the approach of [9], for any $\epsilon > 0$, we run algorithm SDP-GP independently $T = O(\frac{1}{\epsilon}\log(\frac{1}{\epsilon}))$ times, and denote the maximum value of $z(\gamma,\eta)$ in these $T$ runs by $Z_T$. Then we have,

$$\Pr(Z_T \leq \mu_{\eta,\gamma} - \epsilon(C - \mu_{\eta,\gamma})) \leq \left(\frac{C - \mu_{\eta,\gamma}}{C - \mu_{\eta,\gamma} + \epsilon(C - \mu_{\eta,\gamma})}\right)^T = \left(\frac{1}{1+\epsilon}\right)^T \leq \epsilon,$$

which is based on the following proposition:

**Proposition 1** *Let $z$ be a random variable and bounded above by $C$, and let $b \leq a \leq \mathrm{E}[z]$. Then,*

$$\Pr(z \leq b) \leq \frac{C-a}{C-b}.$$

**Proof.** Let the density function be $f(z)$. Then,

$$\begin{aligned}
C - a &\geq C - \mathrm{E}[z] \\
&= C - \int_{-\infty}^{C} z f(z) dz \\
&= C - b + b - \int_{-\infty}^{b} z f(z) dz - \int_{b}^{C} z f(z) dz \\
&= C - b + \int_{-\infty}^{b} (b - z) f(z) dz - \int_{b}^{C} (z - b) f(z) dz \\
&\geq C - b - \int_{b}^{C} (z - b) f(z) dz \\
&\geq C - b - (C - b) \int_{b}^{C} f(z) dz \\
&= (C - b)(1 - \int_{b}^{C} f(z) dz) \\
&= (C - b) \Pr(z \leq b).
\end{aligned}$$

■

Since $\epsilon$ can be an arbitrarily small positive constant, we may assume that $Z_T \geq \mu_{\eta,\gamma} - \epsilon(C - \mu_{\eta,\gamma})$ with probability almost one. For simplicity, and since $(C - \mu_{\eta,\gamma})$ is a constant as well, in the rest of the paper we remove $\epsilon(C - \mu_{\eta,\gamma})$ in our analysis and assume that $z(\gamma, \eta)$ meets its expectation, i.e., $Z_T \geq \mu_{\eta,\gamma}$ almost surely. Then, we must have

$$\frac{w(\tilde{S})}{w^*} \geq r(\delta)\frac{w(S)}{w^*} \geq r(\delta)(\alpha + \gamma\eta\beta_1 + \gamma\beta_2 - \gamma\eta\delta - \gamma\delta(1-\delta)) \geq R(\eta, \gamma, \theta)$$

where

$$\begin{array}{rlll} R(\eta,\gamma,\theta) := & \text{Minimize} & r(\delta)(\alpha + \gamma\eta\beta_1 + \gamma\beta_2 - \gamma\eta\delta - \gamma\delta(1-\delta)) \\ & \text{subject to} & 0 \leq \delta \leq 1. \end{array} \tag{4}$$

Note that $R(\eta, \gamma, \theta)$ is also a function of $\theta$, since $\alpha$, $\beta_1$ and $\beta_2$ depend on the balance combination parameter $\theta$.

Finally, we select

$$\begin{array}{rlll} R_k^* := & \text{Maximize} & R(\eta,\gamma,\theta) \\ & \text{subject to} & 0 \leq \theta \leq 1, \\ & & \gamma \geq 0, \\ & & -1 \leq \eta \leq 1. \end{array} \tag{5}$$

and we have our main result

**Theorem 1** *The worst-case performance ratio of the SDP-GP algorithm for approximating the MAX-GP problem with size requirement $k$ is at least $R_k^*$.*

■

We now establish bounds on $\mathrm{E}[\frac{|S|}{n}]$ and $\mathrm{E}[\frac{|S|(n-|S|)}{n^2}]$, because they are universal for solving all MAX-GP problems. In later sections, we will establish an explicit $r(\delta)$ and a bound on $\mathrm{E}[\frac{w(S)}{w^*}]$ for each individual MAX-GP problem.

Generally, for a given $0 \leq \theta \leq 1$, let

$$\begin{aligned}
\tau^+ = \tau^+(\theta) := &\quad \min_{-1\leq y\leq 1} \frac{2}{\pi}\frac{\arcsin(\theta-(1-\theta)\kappa)+\arcsin(\theta y+(1-\theta)\kappa)}{1+y}\\
\beta^+ = \beta^+(\theta) := &\quad 1-\tfrac{2}{\pi}\arcsin(\theta-(1-\theta)\kappa)+2\sigma\tau^+(\theta)\\
\\
\tau^- = \tau^-(\theta) := &\quad \min_{-1\leq y\leq 1} \frac{2}{\pi}\frac{\arcsin(\theta+(1-\theta)\kappa)-\arcsin(\theta y+(1-\theta)\kappa)}{1-y}\\
\beta^- = \beta^-(\theta) := &\quad 1-\tfrac{2}{\pi}\arcsin(\theta+(1-\theta)\kappa)+2(1-\sigma)\tau^-(\theta)\\
\\
\tau = \tau(\theta) := &\quad \min_{-1\leq y\leq 1} \frac{2}{\pi}\frac{\arcsin(\theta+(1-\theta)\kappa^2)-\arcsin(\theta y+(1-\theta)\kappa^2)}{1-y}\\
\beta = \beta(\theta) := &\quad 1-\tfrac{2}{\pi}\arcsin(\theta+(1-\theta)\kappa^2)+4\sigma(1-\sigma)\tau(\theta).
\end{aligned}$$

Then, we have the following lemma regarding $|S|$:

**Lemma 1** *Let*

$$\beta_1 = \begin{cases} \frac{\beta^+}{2} & \text{if} \quad \eta \geq 0, \\ \frac{2-\beta^-}{2} & \text{otherwise;} \end{cases}$$

*and*

$$\beta_2 = \frac{\beta - (\beta - 4\sigma(1-\sigma)\tau)/n}{4}.$$

*Then, the approximation algorithm yields $S$ satisfying the following inequalities:*

$$\begin{aligned}
\mathrm{E}[\eta\frac{|S|}{n}] &\geq \eta\beta_1,\\
\mathrm{E}[\frac{|S|(n-|S|)}{n^2}] &\geq \beta_2.
\end{aligned}$$

**Proof.** We need to prove

$$\beta^+\frac{n}{2} \leq \mathrm{E}[|S|] \leq (2-\beta^-)\frac{n}{2},$$

which can be seen from:

$$\mathrm{E}[|S|]$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}[Pr(x_i = 1, x_0 = 1) + Pr(x_i = -1, x_0 = -1)] \\
&= \sum_{i=1}^{n}[Pr(u_i \geq 0, u_0 \geq 0) + Pr(u_i < 0, u_0 < 0)] \\
&= \sum_{i=1}^{n}[\frac{1}{2} + \frac{1}{\pi}\arcsin(\theta\bar{X}_{0i} + (1-\theta)\kappa)] \\
&= \sum_{i=1}^{n}[\frac{1}{2} - \frac{1}{\pi}\arcsin(\theta - (1-\theta)\kappa) + \\
&\quad \frac{1}{\pi}\arcsin(\theta - (1-\theta)\kappa) + \frac{1}{\pi}\arcsin(\theta\bar{X}_{0i} + (1-\theta)\kappa)] \\
&\geq \frac{1}{2}\sum_{i=1}^{n}[1 - \frac{2}{\pi}\arcsin(\theta - (1-\theta)\kappa) + (1 + \bar{X}_{0i})\tau^{+}] \\
&= \frac{1}{2}\left((1 - \frac{2}{\pi}\arcsin(\theta - (1-\theta)\kappa))n + 2k\tau^{+}\right) \\
&= \beta^{+}\frac{n}{2},
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{E}[|S|] \\
&= \sum_{i=1}^{n}[Pr(x_i = 1, x_0 = 1) + Pr(x_i = -1, x_0 = -1)] \\
&= \sum_{i=1}^{n}[Pr(u_i \geq 0, u_0 \geq 0) + Pr(u_i < 0, u_0 < 0)] \\
&= \sum_{i=1}^{n}[\frac{1}{2} + \frac{1}{\pi}\arcsin(\theta\bar{X}_{0i} + (1-\theta)\kappa)] \\
&= n - \frac{1}{2}\sum_{i=1}^{n}[1 - \frac{2}{\pi}\arcsin(\theta\bar{X}_{0i} + (1-\theta)\kappa)] \\
&= n - \frac{1}{2}\sum_{i=1}^{n}[1 - \frac{2}{\pi}\arcsin(\theta + (1-\theta)\kappa) + \\
&\quad \frac{2}{\pi}\arcsin(\theta + (1-\theta)\kappa) - \frac{2}{\pi}\arcsin(\theta\bar{X}_{0i} + (1-\theta)\kappa)] \\
&\leq n - \frac{1}{2}\sum_{i=1}^{n}[1 - \frac{2}{\pi}\arcsin(\theta + (1-\theta)\kappa) + (1 - \bar{X}_{0i})\tau^{-}] \\
&= n - \frac{1}{2}\left((1 - \frac{2}{\pi}\arcsin(\theta + (1-\theta)\kappa))n + 2(n-k)\tau^{-}\right) \\
&= (2 - \beta^{-})\frac{n}{2}.
\end{aligned}
$$

Noting that

$$\sum_{i\neq j} \bar{X}_{ij} = (2k-n)^2 - n,$$

we derive

$$\begin{aligned}
& 4\mathrm{E}[|S|(n-|S|)] \\
&= n^2 - \mathrm{E}[(e^T\hat{x})^2] \\
&= \sum_{i\neq j}\left(1-\frac{2}{\pi}\arcsin(\theta\bar{X}_{ij}+(1-\theta)\kappa^2)\right) \\
&= \sum_{i\neq j}\left(1-\frac{2}{\pi}\arcsin(\theta+(1-\theta)\kappa^2)+\right. \\
&\quad \left.\frac{2}{\pi}\arcsin(\theta+(1-\theta)\kappa^2)-\frac{2}{\pi}\arcsin(\theta\bar{X}_{ij}+(1-\theta)\kappa^2)\right) \\
&\geq \sum_{i\neq j}\left(1-\frac{2}{\pi}\arcsin(\theta+(1-\theta)\kappa^2)+(1-\bar{X}_{ij})\tau\right) \\
&= (n^2-n)\left(1-\frac{2}{\pi}\arcsin(\theta+(1-\theta)\kappa^2)\right)+(n^2-(2k-n)^2)\tau \\
&= \left(\beta-\frac{1-\frac{2}{\pi}\arcsin(\theta+(1-\theta)\kappa^2)}{n}\right)\cdot n^2
\end{aligned}$$

■

# 4 Approximation of DSP

The detailed greedy algorithm used in the size-adjusting procedure for DSP is:

**3. Size Adjusting for DSP** Let $\tilde{S} = S$, where $S$ is the subset determined in Step 2.

• If $|\tilde{S}| > k$, then for each $i \in \tilde{S}$, let $\zeta(i) = \sum_{j\in\tilde{S}} w_{ij}$ and $\tilde{S} = \{i_1, i_2, ..., i_{|\tilde{S}|}\}$, where $\zeta(i_1) \geq \zeta(i_2) \geq ... \geq \zeta(i_{|\tilde{S}|})$. Then, remove node $i_{|\tilde{S}|}$ from $\tilde{S}$ and reassign $\tilde{S} := \{i_1, i_2, ..., i_{|\tilde{S}|-1}\}$. Repeat this swapping process till $|\tilde{S}| = k$.

• If $|\tilde{S}| < k$, arbitrarily add $k - |\tilde{S}|$ nodes to $\tilde{S}$. ■

Note that in the case of $|\tilde{S}| < k$, some greedy procedures may be used, e.g., a node with the largest total weight of edges between itself and nodes in $\tilde{S}$ is added. However, the worst case ratio stays the same.

The following lemma is due to Srivastav and Wolf [16].We include the proof here since the method can be used to derive the deterioration functions for MC, MNC, and MVC in Lemmas 5, 6, and 7, respectively.

**Lemma 2** *The total weights of edges in the subgraph induced by $\tilde{S}$ resulted from the SDP-GP algorithm satisfies*

$$|\tilde{S}| = k$$

*and*

$$w(\tilde{S}) \geq \begin{cases} \frac{k(k-1)}{|S|(|S|-1)} w(S) & \text{if} \quad |S| > k; \\ w(S) & \text{if} \quad |S| \leq k. \end{cases} \tag{6}$$

**Proof.** The edge disappears in the swapping procedure if and only if one of its endpoints is removed. It follows that

$$\sum_{i \in \tilde{S}} w(\tilde{S} \setminus \{i\}) = (|\tilde{S}| - 2) w(\tilde{S}),$$

since each edge is counted $|\tilde{S}| - 2$ times. Suppose $v$ is the node that is removed in the swapping procedure, then

$$\sum_{j \in \tilde{S}} w_{vj} \leq \sum_{j \in \tilde{S}} w_{ij} \qquad \text{for any other} \quad i \in \tilde{S}.$$

Thus,

$$w(\tilde{S} \setminus \{v\}) \geq \frac{1}{|\tilde{S}|} \sum_{i \in \tilde{S}} w(\tilde{S} \setminus \{i\}) = \frac{|\tilde{S}| - 2}{|\tilde{S}|} \cdot w(\tilde{S}).$$

By induction we get that the final $\tilde{S}$ satisfies

$$w(\tilde{S}) \geq \frac{k(k-1)}{|S|(|S|-1)} \cdot w(S).$$

■

**Corollary 1** $\frac{w^*}{w(S)} \geq \frac{w^*}{w^{tol}} \geq \frac{k(k-1)}{n(n-1)}$, *where $w^{tol}$ is the total weight of the edges of $E$.*

**Proof.** In the proof of Lemma 2, if we let $S = V$, then

$$w^* \geq w(\tilde{S}) \geq \frac{k(k-1)}{n(n-1)} w(V) = \frac{k(k-1)}{n(n-1)} w^{tol}.$$

■

For simplicity, in what follows we treat $\frac{k(k-1)}{|S|(|S|-1)}$ as $\frac{k^2}{|S|^2}$, this is because, for $n$ and $k$ sufficiently large, they are virtually identical. Then, based on Lemma 2, the deterioration function $r(\delta)$ of (2) for DSP is

$$r(\delta) = \begin{cases} \frac{\sigma^2}{\delta^2} & \text{if} \quad \delta \geq \sigma, \\ 1 & \text{otherwise.} \end{cases} .$$

Now we analyze $\mathrm{E}[w(S)]/w^*$ for DSP. Recall

$$w(S) = \frac{1}{4} \sum_{1 \leq i < j \leq n} w_{ij} \left(1 + \hat{x}_0 \hat{x}_i + \hat{x}_0 \hat{x}_j + \hat{x}_i \hat{x}_j\right).$$

Let

$$\begin{array}{rlll} \alpha(\theta) := & \max & \alpha & \\ & \text{subject to} & 1 + \frac{2}{\pi}(\arcsin(\theta x + (1-\theta)\kappa) + \arcsin(\theta y + (1-\theta)\kappa) & \\ & & + \arcsin(\theta z + (1-\theta)\kappa^2)) \geq \alpha \cdot (1 + x + y + z), & \\ & & \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0. & \end{array}$$

Since

$$\begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0$$

implies that

$$\begin{pmatrix} 1 & \theta x + (1-\theta)\kappa & \theta y + (1-\theta)\kappa \\ \theta x + (1-\theta)\kappa & 1 & \theta z + (1-\theta)\kappa^2 \\ \theta y + (1-\theta)\kappa & \theta z + (1-\theta)\kappa^2 & 1 \end{pmatrix} =$$

$$\theta \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} + (1-\theta) \begin{pmatrix} 1 & \kappa & \kappa \\ \kappa & 1 & \kappa^2 \\ \kappa & \kappa^2 & 1 \end{pmatrix} \succeq 0,$$

we must have

$$1 + \frac{2}{\pi}(\arcsin(\theta x + (1-\theta)\kappa) + \arcsin(\theta y + (1-\theta)\kappa) + \arcsin(\theta z + (1-\theta)\kappa^2)) \geq 0.$$

Thus, $\alpha(\theta)$ can be computed from

$$\begin{array}{rll} \alpha = \alpha(\theta) := \min & \frac{1 + \frac{2}{\pi}(\arcsin(\theta x + (1-\theta)\kappa) + \arcsin(\theta y + (1-\theta)\kappa) + \arcsin(\theta z + (1-\theta)\kappa^2)}{1 + x + y + z} & \\ \text{subject to} & 1 + x + y + z > 0, & \\ & \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0. & \end{array}$$

An alternative and more explicit way to compute $\alpha$ follows that in Goemans and Williamson [10].

Let

$$\alpha := \min \ \{\alpha_1, \alpha_2\}, \tag{7}$$

where

$$\begin{aligned} \alpha_1 = \quad & \min \quad \frac{4\theta(1-\frac{2x}{\pi})}{(\theta-1)(\kappa+1)^2 + 2\cos(x)(1+\cos(x))} \\ & \text{subject to} \quad 0 \le x < \arccos\left(\frac{\sqrt{1+2(1-\theta)(\kappa+1)^2}-1}{2}\right) \end{aligned}$$

and

$$\begin{aligned} \alpha_2 = \quad & \min \quad \frac{\theta(4-\frac{6x}{\pi})}{(\theta-1)(\kappa+1)^2 + 1 + 3\cos(x)} \\ & \text{subject to} \quad 0 \le x < \arccos\left(-\frac{1+(\theta-1)(\kappa+1)^2}{3}\right). \end{aligned}$$

The proof of the following lemma closely follows that of Lemma 6.3 of [10].

**Lemma 3** *Let $\alpha$ be computed from (7). Then,*

$$\begin{aligned} & 1 + \tfrac{2}{\pi}(\arcsin(\theta x + (1-\theta)\kappa) + \arcsin(\theta y + (1-\theta)\kappa) + \arcsin(\theta z + (1-\theta)\kappa^2)) \\ & \ge \alpha \cdot (1 + x + y + z) \end{aligned} \tag{8}$$

*for any $x, y, z$ such that*

$$\begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0.$$

**Proof.** Let

$$\begin{aligned} a &= \arccos(\theta x + (1-\theta)\kappa), \quad 0 \le a \le \pi, \\ b &= \arccos(\theta y + (1-\theta)\kappa), \quad 0 \le b \le \pi, \\ c &= \arccos(\theta z + (1-\theta)\kappa^2)), \quad 0 \le c \le \pi, \end{aligned}$$

then,

$$\begin{aligned} x &= \tfrac{\cos(a)-(1-\theta)\kappa}{\theta}, \\ y &= \tfrac{\cos(b)-(1-\theta)\kappa}{\theta}, \\ z &= \tfrac{\cos(c)-(1-\theta)\kappa^2}{\theta}, \end{aligned}$$

and the inequality (8) is reduced to

$$4 - \frac{2}{\pi}(a+b+c) \ge \alpha\left(\frac{(\theta-1)(\kappa+1)^2}{\theta} + \frac{1+\cos(a)+\cos(b)+\cos(c)}{\theta}\right).$$

Note that we have

$$\begin{pmatrix} 1 & \theta x+(1-\theta)\kappa & \theta y+(1-\theta)\kappa \\ \theta x+(1-\theta)\kappa & 1 & \theta z+(1-\theta)\kappa^2 \\ \theta y+(1-\theta)\kappa & \theta z+(1-\theta)\kappa^2 & 1 \end{pmatrix} \succeq 0,$$

which is equivalent to (see [10])

$$\{0 \le a \le \pi, 0 \le b \le \pi, 0 \le c \le \pi, c \le a+b, b \le a+c, a \le b+c, a+b+c \le 2\pi\}.$$

Let $S$ be the set of $(a,b,c)$ satisfying the above conditions, and let $(a,b,c)$ be the minimizer of the problem

$$\begin{array}{ll} \min & h(a,b,c) := 4 - \frac{2}{\pi}(a+b+c) - \alpha\left(\frac{(\theta-1)(\kappa+1)^2}{\theta} + \frac{1+\cos(a)+\cos(b)+\cos(c)}{\theta}\right) \\ s.t. & (a,b,c) \in S. \end{array}$$

Then, we like to prove $h(a,b,c) \ge 0$. We consider all cases of the constraint status of $S$ below.

**Case 1**. $a+b+c = 2\pi$: we now have $4 - \frac{2}{\pi}(a+b+c) = 0$. On the other hand, $\frac{(\theta-1)(\kappa+1)^2}{\theta} \le 0$ since $0 < \theta \le 1, -1 < \kappa < 1$, and

$$\begin{array}{rl} & 1+\cos(a)+\cos(b)+\cos(c) \\ = & 1+\cos(a+b)+cos(a)+\cos(b) \\ = & 2\cos^2\left(\frac{a+b}{2}\right) + 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) \\ = & 2\cos\left(\frac{a+b}{2}\right)\left(\cos\left(\frac{a+b}{2}\right) + \cos\left(\frac{a-b}{2}\right)\right) \\ \le & 0, \end{array} \tag{9}$$

where the last inequality follows from the fact that $\frac{\pi}{2} \le \frac{a+b}{2} \le \pi - \frac{|a-b|}{2}$ and thus $\cos\left(\frac{a+b}{2}\right) \le 0$ and $\cos\left(\frac{a+b}{2}\right) + \cos\left(\frac{a-b}{2}\right) \ge 0$. Therefore, $h(a,b,c) \ge 0$.

**Case 2**. $a = b+c$ or $b = a+c$ of $c = a+b$: say $c = a+b$, then by (9), we have that

$$\begin{array}{rl} & \frac{(\theta-1)(\kappa+1)^2}{\theta} + \frac{1+\cos(a)+\cos(b)+\cos(c)}{\theta} \\ = & \frac{(\theta-1)(\kappa+1)^2}{\theta} + \frac{2}{\theta}\cos\left(\frac{a+b}{2}\right)\left(\cos\left(\frac{a+b}{2}\right) + \cos\left(\frac{a-b}{2}\right)\right) \\ \le & \frac{(\theta-1)(\kappa+1)^2}{\theta} + \frac{2}{\theta}\cos\left(\frac{a+b}{2}\right)\left(1 + \cos\left(\frac{a+b}{2}\right)\right). \end{array}$$

If

$$\frac{c}{2} = \frac{a+b}{2} \ge \arccos\left(\frac{\sqrt{1+2(1-\theta)(\kappa+1)^2}-1}{2}\right),$$

then $\frac{(\theta-1)(\kappa+1)^2}{\theta} + \frac{1+\cos(a)+\cos(b)+\cos(c)}{\theta} \le 0$, thus $h(a,b,c) \ge 0$ since $1 - \frac{1}{2\pi}(a+b+c) \ge 0$ and $\alpha \ge 0$.

If

$$0 \leq \frac{c}{2} < \arccos\left(\frac{\sqrt{1+2(1-\theta)(\kappa+1)^2}-1}{2}\right),$$

then by the definition of $\alpha$ and $\alpha_1$, we have $h(a,b,c) \geq 0$.

**Case 3**. $a = 0$ or $b = 0$ or $c = 0$: say $a = 0$. Then, we have $b = c$ and then $b = c + a$ and it is reduced to Case 2.

**Case 4**. $a = \pi$ or $b = \pi$ or $c = \pi$: say $a = \pi$. Then, we have $b + c = \pi = a$ and it is also reduced to Case 2.

**Case 5**. $(a, b, c)$ belongs to the interior of $S$: we must have the gradient vector of $h(a,b,c)$, $\nabla h(a,b,c) = 0$, i.e., $\sin(a) = \sin(b) = \sin(c) = \frac{2\theta}{\alpha\pi}$; and the Hessian matrix of $h(a,b,c)$ must be positive semidefinite, i.e., $\frac{\alpha\cos(a)}{\theta} \geq 0$, $\frac{\alpha\cos(b)}{\theta} \geq 0$, and $\frac{\alpha\cos(c)}{\theta} \geq 0$. Therefore $a = b = c$, and

$$h(a,b,c) = h(a,a,a) = 4 - \frac{6a}{\pi} - \alpha\left(\frac{(\theta-1)(\kappa+1)^2}{\theta} + \frac{1+3\cos(a)}{\theta}\right).$$

If $a \geq \arccos\left(-\frac{1+(\theta-1)(\kappa+1)^2}{3}\right)$, then $h(a,a,a) \geq 0$ since $4 - \frac{6a}{\pi} \geq 0, \alpha \geq 0$ and

$$\frac{(\theta-1)(\kappa+1)^2}{\theta} + \frac{1+3\cos(a)}{\theta} \leq 0.$$

If $0 \leq a < \arccos\left(-\frac{1+(\theta-1)(\kappa+1)^2}{3}\right)$, then $h(a,a,a) \geq 0$ by the definition of $\alpha$ and $\alpha_2$. ■

(For our selection of $\theta$ and range of $\kappa$, we have found that $\alpha_2 \leq \alpha_1$ in our calculation.)

**Lemma 4** *Let $\alpha$ be computed from (7). Then,*

$$\mathrm{E}[w(S)] \geq \alpha \cdot w^{SDP} \geq \alpha \cdot w^*.$$

**Proof.**

$$\begin{aligned}
\mathrm{E}[w(S)] &= \frac{1}{4}\sum_{1\leq i<j\leq n} w_{ij}\left(1 + \mathrm{E}[\hat{x}_0\hat{x}_i] + \mathrm{E}[\hat{x}_0\hat{x}_j] + \mathrm{E}[\hat{x}_i\hat{x}_j]\right) \\
&= \frac{1}{4}\sum_{i<j} w_{ij}\left(1 + \frac{2}{\pi}(\arcsin(\theta\bar{X}_{0j} + (1-\theta)\kappa)\right. \\
&\quad \left. + \arcsin(\theta\bar{X}_{0i} + (1-\theta)\kappa) + \arcsin(\theta\bar{X}_{ij} + (1-\theta)\kappa^2))\right) \\
&\geq \frac{\alpha}{4}\sum_{i<j} w_{ij}\left(1 + \bar{X}_{0j} + \bar{X}_{0i} + \bar{X}_{ij}\right) \\
&= \alpha w^{SDP} \geq \alpha w^*.
\end{aligned}$$

■

We can now go ahead to compute $R(\eta,\gamma,\theta)$ and $R_k^*$ from (4) and (5) for any given $k$.

In fact, $\gamma$ in (5) can be explicit given as follows. Restrict $\eta \in [-1, 2\sigma-1]$ and let

$$\gamma = \begin{cases} \dfrac{\alpha}{\sigma^2 - (\eta\beta_1+\beta_2)}, & \text{if } 2\sigma^2-1 \le \eta \le 2\sigma-1 \\ \dfrac{\alpha(1-\sigma^2)}{\sigma^2((\eta\beta_1+\beta_2)-\eta)+(1+\eta)^2/4-(\eta\beta_1+\beta_2)}, & \text{if } -1 \le \eta < 2\sigma^2-1. \end{cases} \tag{10}$$

Then, we have

**Proposition 2** *If the random variable $z(\gamma,\eta)$ meets its expectation, i.e., $z(\gamma,\eta) \ge \mu_{\eta,\gamma} := \alpha + \gamma(\eta\beta_1+\beta_2)$, then*

$$w(\tilde{S}) \ge R(\eta,\theta)\cdot w^*,$$

*and*

$$R(\eta,\theta) = \begin{cases} \dfrac{\alpha(4\sigma^2-(1+\eta)^2)}{4(\sigma^2-(\eta\beta_1+\beta_2))}, & \text{if} \quad 2\sigma^2-1 \le \eta \le 2\sigma-1 \\ \dfrac{\alpha\sigma^2((1+\eta)^2/4-\eta)}{\sigma^2((\eta\beta_1+\beta_2)-\eta)+(1+\eta)^2/4-(\eta\beta_1+\beta_2)}, & \text{if} \quad -1 \le \eta < 2\sigma^2-1. \end{cases}$$

**Proof.** If $1 > \delta > \sigma$, it follows from Lemma 2 that

$$\begin{aligned} w(\tilde{S}) &\ge \frac{\sigma^2}{\delta^2}\cdot w(S) \\ &\ge \frac{\sigma^2(\mu_{\eta,\gamma}-\eta\gamma\delta-\gamma\delta(1-\delta))}{\delta^2}\cdot w^*. \end{aligned}$$

Let

$$f(\delta) = \frac{\sigma^2(\mu_{\eta,\gamma}-\eta\gamma\delta-\gamma\delta(1-\delta))}{\delta^2}.$$

It is easy to calculate its stationary point

$$\delta_0 = \frac{2\mu_{\eta,\gamma}}{(1+\eta)\gamma}\ . \tag{11}$$

Note that $f(\delta)$ is decreasing in $(0, \delta_0)$. If $2\sigma^2 - 1 \leq \eta < 2\sigma - 1$, $\mu_{\eta,\gamma} = \gamma\sigma^2$ from the definition of $\mu_{\eta,\gamma}$ and the choice of $\gamma$ in (10). It follows from (11) that

$$\delta_0 = \frac{2\sigma^2}{1+\eta} \ .$$

Since $\sigma < \delta_0 \leq 1$, it yields the minimal value for $f(\delta)$ and

$$f(\delta) \geq f(\delta_0) = \frac{\sigma^2(4\mu_{\eta,\gamma} - \gamma(1+\eta)^2)\gamma}{4\mu_{\eta,\gamma}}.$$

If $-1 \leq \eta < 2\sigma^2 - 1$, from the definition of $\mu_{\eta,\gamma}$ and the choice of $\gamma$ in (10)

$$\mu_{\eta,\gamma} = \frac{(1+\eta)^2\gamma/4 - \sigma^2\gamma\eta}{1-\sigma^2}.$$

It follows from (11) that

$$\delta_0 = \frac{(1+\eta)^2 - 4\sigma^2\eta}{2(1-\sigma^2)(1+\eta)} > 1,$$

and $f(\delta)$ achieves its minimal value at 1, that is,

$$f(\delta) \geq f(1) = \sigma^2(\mu_{\eta,\gamma} - \eta\gamma).$$

If $\delta \leq \sigma$, the number of nodes in $S$ is no more than $k$ and no vertex have to be removed from $S$. Then,

$$\begin{aligned} w(\tilde{S}) &= w(S) \\ &\geq (\mu_{\eta,\gamma} - \eta\gamma\delta - \gamma\delta(1-\delta)) \cdot w^* \\ &\geq (\mu_{\eta,\gamma} - \frac{\gamma(1+\eta)^2}{4}) \cdot w^* \end{aligned}$$

The third inequality follows from the simple fact that

$$\delta = \frac{1+\eta}{2} \tag{12}$$

yields the maximal value of

$$\eta\delta + \delta(1-\delta).$$

Using (10) we have, for $2\sigma^2 - 1 \leq \eta \leq 2\sigma - 1$,

$$\frac{\sigma^2(4\mu_{\eta,\gamma} - \gamma(1+\eta)^2)\gamma}{4\mu_{\eta,\gamma}} = \mu_{\eta,\gamma} - \frac{\gamma(1+\eta)^2}{4} = \frac{\alpha\,(4\sigma^2 - (1+\eta)^2)}{4(\sigma^2 - (\eta\beta_1 + \beta_2))}.$$

and, for $-1 \leq \eta < 2\sigma^2 - 1$,

$$\begin{aligned} \sigma^2(\mu_{\eta,\gamma} - \eta\gamma) &= \mu_{\eta,\gamma} - \gamma(1+\eta)^2/4 \\ &= \frac{\alpha\sigma^2((1+\eta)^2/4 - \eta)}{\sigma^2((\eta\beta_1 + \beta_2) - \eta) + (1+\eta)^2/4 - (\eta\beta_1 + \beta_2)}. \end{aligned}$$

This completes the proof. ■

We have calculated Table 1, which shows $R(\eta, \gamma, \theta)$ for $0.2n \leq k \leq 0.9995n$. Note that the true performance ratio $R_k^*$ for DSP should be greater than $R(\eta, \gamma, \theta)$ shown in the table.

# 5 Approximation of MC

Because the symmetry of this problem, without losing generality, we assume that the required size $k \geq n/2$ or $\sigma \geq 1/2$. The detailed greedy algorithm used in the adjusting procedure for MC is:

**3. Size Adjusting for MC** Let $\tilde{S} = S$, where $S$ is the subset determined in Step 2.

• If $|S| > k$, then for each $i \in \tilde{S}$, let $\zeta(i) = \sum_{j \in V \setminus \tilde{S}} w_{ij}$ and $\tilde{S} = \{i_1, i_2, ..., i_{|\tilde{S}|}\}$, where $\zeta(i_1) \geq \zeta(i_2) \geq ... \geq \zeta(i_{|\tilde{S}|})$. Then, remove node $i_{|\tilde{S}|}$ from $\tilde{S}$ and reassign $\tilde{S} := \{i_1, i_2, ..., i_{|\tilde{S}|-1}\}$. Repeat this swapping process till $|\tilde{S}| = k$.

• If $n/2 \leq |S| < k$, then for each $i \in V \setminus \tilde{S}$, let $\zeta(i) = \sum_{j \in \tilde{S}} w_{ij}$ and $V \setminus \tilde{S} = \{i_1, i_2, ..., i_{|V \setminus \tilde{S}|}\}$, where $\zeta(i_1) \geq \zeta(i_2) \geq ... \geq \zeta(i_{|V \setminus \tilde{S}|})$. Then, add node $i_{|V \setminus \tilde{S}|}$ to $\tilde{S}$ and reassign $V \setminus \tilde{S} := \{i_1, i_2, ..., i_{|V \setminus \tilde{S}|-1}\}$. Repeat this swapping process till $|\tilde{S}| = k$.

• If $n - k \leq |S| < n/2$, then for each $i \in \tilde{S}$, let $\zeta(i) = \sum_{j \in V \setminus \tilde{S}} w_{ij}$ and $\tilde{S} = \{i_1, i_2, ..., i_{|\tilde{S}|}\}$, where $\zeta(i_1) \geq \zeta(i_2) \geq ... \geq \zeta(i_{|\tilde{S}|})$. Then, remove node $i_{|\tilde{S}|}$ from $\tilde{S}$ and reassign $\tilde{S} := \{i_1, i_2, ..., i_{|\tilde{S}|-1}\}$. Repeat this swapping process till $|\tilde{S}| = n - k$.

• If $|S| < n - k$, then for each $i \in V \setminus \tilde{S}$, let $\zeta(i) = \sum_{j \in \tilde{S}} w_{ij}$ and $V \setminus \tilde{S} = \{i_1, i_2, ..., i_{|V \setminus \tilde{S}|}\}$, where $\zeta(i_1) \geq \zeta(i_2) \geq ... \geq \zeta(i_{|V \setminus \tilde{S}|})$. Then, add node $i_{|V \setminus \tilde{S}|}$ to $\tilde{S}$ and reassign $V \setminus \tilde{S} := \{i_1, i_2, ..., i_{|V \setminus \tilde{S}|-1}\}$. Repeat this swapping process till $|\tilde{S}| = n - k$.

■

The following lemma is straightforward:

| $\sigma = \frac{k}{n}$ | $\theta$ | $\alpha$ | $\eta$ | $\gamma$ | $R(\eta, \gamma, \theta)$ |
|---|---|---|---|---|---|
| 0.20 | 0.94 | 0.7509 | -0.8417 | 5.9521 | 0.2008 |
| 0.22 | 0.94 | 0.7514 | -0.8080 | 5.9211 | 0.2320 |
| 0.24 | 0.94 | 0.7520 | -0.7719 | 5.8990 | 0.2631 |
| 0.26 | 0.92 | 0.7381 | -0.7284 | 5.9836 | 0.2942 |
| 0.28 | 0.92 | 0.7390 | -0.6877 | 6.0083 | 0.3245 |
| 0.30 | 0.92 | 0.7399 | -0.6455 | 6.0454 | 0.3541 |
| 0.32 | 0.92 | 0.7410 | -0.6020 | 6.0947 | 0.3827 |
| 0.34 | 0.90 | 0.7284 | -0.5499 | 6.3204 | 0.4105 |
| 0.36 | 0.90 | 0.7298 | -0.5039 | 6.4235 | 0.4372 |
| 0.38 | 0.90 | 0.7314 | -0.4573 | 6.5374 | 0.4626 |
| 0.40 | 0.90 | 0.7330 | -0.4104 | 6.6593 | 0.4867 |
| 0.42 | 0.88 | 0.7220 | -0.3544 | 7.0553 | 0.5095 |
| 0.44 | 0.88 | 0.7241 | -0.3072 | 7.2155 | 0.5310 |
| 0.46 | 0.88 | 0.7264 | -0.2604 | 7.3644 | 0.5511 |
| 0.48 | 0.88 | 0.7287 | -0.2142 | 7.4910 | 0.5697 |
| 0.50 | 0.88 | 0.7311 | -0.1690 | 7.5830 | 0.5866 |
| 0.52 | 0.88 | 0.7337 | -0.1249 | 7.6282 | 0.6022 |
| 0.54 | 0.88 | 0.7364 | -0.0820 | 7.6162 | 0.6161 |
| 0.56 | 0.88 | 0.7391 | -0.0403 | 7.5412 | 0.6287 |
| 0.58 | 0.90 | 0.7517 | -0.0071 | 7.1198 | 0.6402 |
| 0.60 | 0.90 | 0.7542 | 0.0 | 5.8985 | 0.6488 |
| 0.62 | 0.92 | 0.7652 | 0.0 | 4.8650 | 0.6539 |
| 0.64 | 0.92 | 0.7674 | 0.0 | 4.1124 | 0.6563 |
| 0.65 | 0.92 | 0.7685 | 0.0124 | 3.9524 | 0.6571 |
| 0.66 | 0.92 | 0.7697 | 0.0248 | 3.8006 | 0.6577 |
| 0.68 | 0.92 | 0.7720 | 0.0490 | 3.5173 | 0.6588 |
| 0.70 | 0.90 | 0.7683 | 0.0822 | 3.3455 | 0.6597 |
| 0.72 | 0.90 | 0.7714 | 0.1045 | 3.0955 | 0.6607 |
| 0.74 | 0.90 | 0.7746 | 0.1261 | 2.8686 | 0.6615 |
| 0.76 | 0.90 | 0.7780 | 0.1470 | 2.6624 | 0.6621 |
| 0.78 | 0.88 | 0.7775 | 0.1730 | 2.5074 | 0.6630 |
| 0.80 | 0.88 | 0.7817 | 0.2273 | 2.5010 | 0.6636 |
| 0.82 | 0.88 | 0.7861 | 0.2108 | 2.0923 | 0.6687 |
| 0.84 | 0.88 | 0.7906 | 0.0 | 1.1280 | 0.6759 |
| 0.86 | 0.86 | 0.7940 | 0.0 | 0.9660 | 0.6859 |
| 0.88 | 0.82 | 0.7978 | 0.0 | 0.8142 | 0.6987 |
| 0.90 | 0.78 | 0.8042 | 0.0 | 0.6708 | 0.7149 |
| 0.92 | 0.74 | 0.8135 | 0.0 | 0.5340 | 0.7356 |
| 0.94 | 0.66 | 0.8262 | 0.0 | 0.4017 | 0.7624 |
| 0.96 | 0.58 | 0.8460 | 0.0 | 0.2717 | 0.7985 |
| 0.98 | 0.44 | 0.8790 | 0.0 | 0.1404 | 0.8515 |
| 0.99 | 0.32 | 0.9078 | 0.0 | 0.0725 | 0.8926 |
| 0.995 | 0.22 | 0.9314 | 0.0 | 0.0372 | 0.9232 |
| 0.9995 | 0.08 | 0.9764 | 0.0 | 0.0039 | 0.9755 |

Table 1: Worst-case performance ratios of the SDP-DSP algorithm

**Lemma 5** *The total weights of crossing edges between $\tilde{S}$ and $V \setminus \tilde{S}$ resulted from the approximation algorithm satisfy*

$$\textit{either } |\tilde{S}| = k \textit{ or } |V \setminus \tilde{S}| = k$$

*and*

$$w(\tilde{S}) \geq \begin{cases} \frac{k}{|S|} w(S) & \textit{if} \quad |S| \geq k \\ \frac{n-k}{n-|S|} w(S) & \textit{if} \quad n/2 \leq |S| < k \\ \frac{n-k}{|S|} w(S) & \textit{if} \quad n-k \leq |S| < n/2 \\ \frac{k}{n-|S|} w(S) & \textit{if} \quad |S| < n-k \end{cases} . \tag{13}$$

■

Similarly,

**Corollary 2** $\frac{w^*}{w(S)} \geq \frac{w^*}{w^{tol}} \geq \frac{n-k}{n}$, *where $w^{tol}$ is the total weight of the edges of $E$.*

**Proof.** The proof is similar to that of Corollary 1. ■

Then, based on Lemma 5, the deterioration function $r(\delta)$ of (2) for MC is

$$r(\delta) = \begin{cases} \frac{\sigma}{\delta} & \text{if} \quad \delta \geq \sigma \\ \frac{1-\sigma}{1-\delta} & \text{if} \quad 1/2 \leq \delta < \sigma \\ \frac{1-\sigma}{\delta} & \text{if} \quad 1-\sigma \leq \delta < 1/2 \\ \frac{\sigma}{1-\delta} & \text{if} \quad \delta < 1-\sigma \end{cases} .$$

Now we analyze $\mathrm{E}[w(S)]/w^*$ for MC. Note

$$w(S) = \frac{1}{2} \sum_{1 \leq i < j \leq n} w_{ij} \left(1 - \hat{x}_i \hat{x}_j\right).$$

Let

$$\begin{aligned} \alpha = \alpha(\theta) := \min \quad & \tfrac{1 - \frac{2}{\pi} \arcsin(\theta z + (1-\theta)\kappa^2)}{1-z} \\ \text{subject to} \quad & -1 \leq z < 1. \end{aligned} \tag{14}$$

Then, we have the following inequality whose proof resembles that for Lemma 4:

$$\mathrm{E}[w(S)] \geq \alpha \cdot w^{SDP} \geq \alpha \cdot w^*.$$

Now we can go ahead compute $R(\eta, \gamma, \theta)$ and $R_k^*$ from (4) and (5). In fact, $R(\eta, \gamma, \theta)$ can be constructed from the following four cases.

Let

$$\lambda_{\eta,\gamma}(\delta) = \mu_{\eta,\gamma} - \gamma(\eta\delta + \delta(1-\delta)) = \alpha + \gamma(\eta\beta_1 + \beta_2) - \gamma(\eta\delta + \delta(1-\delta));$$

**Case 1.** $\sigma \le \delta \le 1$:

$$R_1(\eta,\gamma,\theta) = \begin{cases} \sigma\left(2\sqrt{\gamma\mu_{\eta,\gamma}} - \eta\gamma - \gamma\right) & \text{if } \sigma \le \sqrt{\frac{\mu_{\eta,\gamma}}{\gamma}} \le 1, \\ \min\{\lambda_{\eta,\gamma}(\sigma), \sigma\lambda_{\eta,\gamma}(1)\} & \text{otherwise;} \end{cases}$$

**Case 2.** $\frac{1}{2} \le \delta \le \sigma$:

$$R_2(\eta,\gamma,\theta) = \begin{cases} (1-\sigma)\left(2\sqrt{\gamma(\mu_{\eta,\gamma} - \eta\gamma)} + \eta\gamma - \gamma\right) & \text{if } 1-\sigma \le \sqrt{\frac{\mu_{\eta,\gamma}-\eta\gamma}{\gamma}} \le \frac{1}{2}, \\ \min\{\lambda_{\eta,\gamma}(\sigma), 2(1-\sigma)\lambda_{\eta,\gamma}(1/2)\} & \text{otherwise;} \end{cases}$$

**Case 3.** $1-\sigma \le \delta \le \frac{1}{2}$:

$$R_3(\eta,\gamma,\theta) = \begin{cases} (1-\sigma)\left(2\sqrt{\gamma\mu_{\eta,\gamma}} - \eta\gamma - \gamma\right) & \text{if } 1-\sigma \le \sqrt{\frac{\mu_{\eta,\gamma}}{\gamma}} \le \frac{1}{2}, \\ \min\{\lambda_{\eta,\gamma}(1-\sigma), 2(1-\sigma)\lambda_{\eta,\gamma}(1/2)\} & \text{otherwise;} \end{cases}$$

**Case 4.** $0 \le \delta \le 1-\sigma$:

$$R_4(\eta,\gamma,\theta) = \begin{cases} \sigma\left(2\sqrt{\gamma(\mu_{\eta,\gamma} - \eta\gamma)} + \eta\gamma - \gamma\right) & \text{if } \sigma \le \sqrt{\frac{\mu_{\eta,\gamma}-\eta\gamma}{\gamma}} \le 1, \\ \min\{\lambda_{\eta,\gamma}(1-\sigma), \sigma\lambda_{\eta,\gamma}(0)\} & \text{otherwise.} \end{cases}$$

Then, we have

$$R(\eta,\gamma,\theta) = \min\{R_1(\eta,\gamma,\theta), R_2(\eta,\gamma,\theta), R_3(\eta,\gamma,\theta), R_4(\eta,\gamma,\theta)\}.$$

We have generated Table 2, which shows $R(\eta,\gamma,\theta)$ for $0.5n \le k \le 0.98n$. Note that the true performance ratio $R_k^*$ for MC should be greater than $R(\eta,\gamma,\theta)$ of the table, and $R(\eta,\gamma,\theta)$ for $0.5n \ge k \ge 0.02n$ are identical to those shown here.

# 6 Approximation of MNC

Because again the symmetry of this problem, without losing generality, we assume that the required size $\sigma \ge 1/2$. The detailed greedy algorithm used in the adjusting procedure for MNC is:

**3. Size Adjusting for MNC**

| $\sigma = \frac{k}{n}$ | $\theta$ | $\alpha$ | $\eta$ | $\gamma$ | $R(\eta, \gamma, \theta)$ |
|---|---|---|---|---|---|
| 0.50 | 0.88 | 0.831 | 0.00 | 7.41 | 0.699 |
| 0.52 | 0.90 | 0.839 | 0.00 | 6.86 | 0.725 |
| 0.54 | 0.94 | 0.855 | 0.00 | 3.91 | 0.734 |
| 0.56 | 0.96 | 0.863 | 0.00 | 2.36 | 0.721 |
| 0.58 | 0.98 | 0.871 | 0.00 | 1.61 | 0.698 |
| 0.60 | 0.98 | 0.870 | 0.00 | 1.16 | 0.671 |
| 0.62 | 0.98 | 0.870 | 0.00 | 0.76 | 0.642 |
| 0.64 | 1.00 | 0.878 | 0.00 | 0.46 | 0.616 |
| 0.66 | 1.00 | 0.878 | 0.00 | 0.11 | 0.593 |
| 0.68 | 1.00 | 0.878 | 0.00 | 0.00 | 0.562 |
| 0.70 | 1.00 | 0.878 | 0.00 | 0.00 | 0.527 |
| 0.72 | 1.00 | 0.878 | 0.00 | 0.00 | 0.492 |
| 0.74 | 1.00 | 0.878 | 0.00 | 0.00 | 0.456 |
| 0.76 | 1.00 | 0.878 | 0.00 | 0.00 | 0.421 |
| 0.78 | 1.00 | 0.878 | 0.00 | 0.00 | 0.386 |
| 0.80 | 1.00 | 0.878 | 0.00 | 0.00 | 0.351 |
| 0.82 | 1.00 | 0.878 | 0.00 | 0.00 | 0.316 |
| 0.84 | 1.00 | 0.878 | 0.00 | 0.00 | 0.281 |
| 0.86 | 1.00 | 0.878 | 0.00 | 0.00 | 0.246 |
| 0.88 | 1.00 | 0.878 | 0.00 | 0.00 | 0.210 |
| 0.90 | 1.00 | 0.878 | 0.00 | 0.00 | 0.175 |
| 0.92 | 1.00 | 0.878 | 0.00 | 0.00 | 0.140 |
| 0.94 | 1.00 | 0.878 | 0.00 | 0.00 | 0.105 |
| 0.96 | 1.00 | 0.878 | 0.00 | 0.00 | 0.070 |
| 0.98 | 1.00 | 0.878 | 0.00 | 0.00 | 0.035 |

Table 2: Worst-case performance ratios of the SDP-MC algorithm

Let $\tilde{S} = S$ where S is the subset determined in Step 2.

• If $|S| > k$, then for each $i \in \tilde{S}$, let $\zeta(i) = \sum_{j\tilde{S}} w_{ij}$ and $\tilde{S} = \{i_1, i_2, ..., i_{|\tilde{S}|}\}$, where $\zeta(i_1) \geq \zeta(i_2) \geq ... \geq \zeta(i_{|\tilde{S}|})$. Then, remove node $i_{|\tilde{S}|}$ from $\tilde{S}$ and reassign $\tilde{S} := \{i_1, i_2, ..., i_{|\tilde{S}|-1}\}$. Repeat this swapping process till $|\tilde{S}| = k$.

• If $n/2 \leq |S| < k$, then for each $i \in V \setminus \tilde{S}$, let $\zeta(i) = \sum_{j\in V\setminus\tilde{S}} w_{ij}$ and $V \setminus \tilde{S} = \{i_1, i_2, ..., i_{|V\setminus\tilde{S}|}\}$, where $\zeta(i_1) \geq \zeta(i_2) \geq ... \geq \zeta(i_{|V\setminus\tilde{S}|})$. Then, add node $i_{|V\setminus\tilde{S}|}$ to $\tilde{S}$ and reassign $V \setminus \tilde{S} := \{i_1, i_2, ..., i_{|V\setminus\tilde{S}|-1}\}$. Repeat this swapping process till $|\tilde{S}| = k$.

• If $n-k \leq |S| < n/2$, then for each $i \in \tilde{S}$, let $\zeta(i) = \sum_{j\in\tilde{S}} w_{ij}$ and $\tilde{S} = \{i_1, i_2, ..., i_{|\tilde{S}|}\}$, where $\zeta(i_1) \geq \zeta(i_2) \geq ... \geq \zeta(i_{|\tilde{S}|})$. Then, remove node $i_{|\tilde{S}|}$ from $\tilde{S}$ and reassign $\tilde{S} := \{i_1, i_2, ..., i_{|\tilde{S}|-1}\}$. Repeat this swapping process till $|\tilde{S}| = n - k$.

• If $|S| < n - k$, then for each $i \in V \setminus \tilde{S}$, let $\zeta(i) = \sum_{j\in V\setminus\tilde{S}} w_{ij}$ and $V \setminus \tilde{S} = \{i_1, i_2, ..., i_{|V\setminus\tilde{S}|}\}$, where $\zeta(i_1) \geq \zeta(i_2) \geq ... \geq \zeta(i_{|V\setminus\tilde{S}|})$. Then, add node $i_{|V\setminus\tilde{S}|}$ to $\tilde{S}$ and reassign $V \setminus \tilde{S} := \{i_1, i_2, ..., i_{|V\setminus\tilde{S}|-1}\}$. Repeat this swapping process till $|\tilde{S}| = n - k$.

■

The following lemma is also straightforward:

**Lemma 6** *The total weights of non-crossing edges between $\tilde{S}$ and $V \setminus \tilde{S}$ resulted from the approximation algorithm satisfy*

$$either\ |\tilde{S}| = k\ or\ |V \setminus \tilde{S}| = k$$

*and*

$$w(\tilde{S}) \geq \begin{cases} \frac{k(k-1)}{|S|(|S|-1)} w(S) & if \quad |S| \geq k; \\ \frac{(n-k)(n-k-1)}{(n-|S|)(n-|S|-1)} w(S) & if \quad n/2 \leq |S| < k; \\ \frac{(n-k)(n-k-1)}{|S|(|S|-1)} w(S) & if \quad n-k \leq |S| < n/2; \\ \frac{k(k-1)}{(n-|S|)(n-|S|-1)} w(S) & if \quad |S| < n-k. \end{cases} \tag{15}$$

■

Similarly,

**Corollary 3** $\frac{w^*}{w(S)} \geq \frac{w^*}{w^{tol}} \geq \frac{(n-k)(n-k-1)}{n(n-1)}$, *where $w^{tol}$ is the total weight of the edges of $E$.*

**Proof.** The proof is similar to that of Corollary 1. ■

Then, based on Lemma 6, the deterioration function $r(\delta)$ of (2) for MNC is

$$r(\delta) = \begin{cases} \frac{\sigma^2}{\delta^2} & \text{if} \quad \delta \geq \sigma; \\ \frac{(1-\sigma)^2}{(1-\delta)^2} & \text{if} \quad 1/2 \leq \delta < \sigma; \\ \frac{(1-\sigma)^2}{\delta^2} & \text{if} \quad 1-\sigma \leq \delta < 1/2; \\ \frac{\sigma^2}{(1-\delta)^2} & \text{if} \quad \delta < 1-\sigma. \end{cases}$$

To analyze $\mathrm{E}[w(S)]/w^*$ for MNC, first note

$$w(S) = \frac{1}{2} \sum_{1 \leq i < j \leq n} w_{ij} \left(1 + \hat{x}_i \hat{x}_j\right).$$

Let

$$\begin{aligned} \alpha = \alpha(\theta) := \min \quad & \tfrac{1+\frac{2}{\pi}\arcsin(\theta z + (1-\theta)\kappa^2)}{1+z} \\ \text{subject to} \quad & -1 < z \leq 1. \end{aligned} \tag{16}$$

Then, we have the following result resembling Lemma 4:

$$\mathrm{E}[w(S)] \geq \alpha \cdot w^{SDP} \geq \alpha \cdot w^*.$$

Now we can go ahead compute $R(\eta, \gamma, \theta)$ and $R_k^*$ from (4) and (5). Again, $R(\eta, \gamma, \theta)$ can be constructed from the four cases similar to MC:

**Case 1.** $\sigma \leq \delta \leq 1$:

$$R_1(\eta, \gamma, \theta) = \begin{cases} \sigma^2 \left(-\frac{(\eta\gamma+\gamma)^2}{4\mu_{\eta,\gamma}} + \gamma\right) & \text{if } \sigma \leq \frac{2\mu_{\eta,\gamma}}{\gamma+\eta\gamma} \leq 1; \\ \min\{\lambda_{\eta,\gamma}(\sigma), \sigma^2 \lambda_{\eta,\gamma}(1)\} & \text{otherwise;} \end{cases}$$

**Case 2.** $\frac{1}{2} \leq \delta \leq \sigma$:

$$R_2(\eta, \gamma, \theta) = \begin{cases} (1-\sigma)^2 \left(-\frac{(-\eta\gamma+\gamma)^2}{4(\mu_{\eta,\gamma}-\eta\gamma)} + \gamma\right) & \text{if } 1-\sigma \leq \frac{2(\mu_{\eta,\gamma}-\eta\gamma)}{\gamma-\eta\gamma} \leq \frac{1}{2}, \\ \min\{\lambda_{\eta,\gamma}(\sigma), 4(1-\sigma)^2 \lambda_{\eta,\gamma}(1/2)\} & \text{otherwise;} \end{cases}$$

**Case 3.** $1-\sigma \leq \delta \leq \frac{1}{2}$:

$$R_3(\eta, \gamma, \theta) = \begin{cases} (1-\sigma)^2 \left(-\frac{(\eta\gamma+\gamma)^2}{4\mu_{\eta,\gamma}} + \gamma\right) & \text{if } 1-\sigma \leq \frac{2\mu_{\eta,\gamma}}{\gamma+\eta\gamma} \leq \frac{1}{2}, \\ \min\{\lambda_{\eta,\gamma}(1-\sigma), 4(1-\sigma)^2 \lambda_{\eta,\gamma}(1/2)\} & \text{otherwise;} \end{cases}$$

**Case 4.** $0 \leq \delta \leq 1-\sigma$:

$$R_4(\eta,\gamma,\theta) = \begin{cases} \sigma^2\left(-\frac{(-\eta\gamma+\gamma)^2}{4(\mu_{\eta,\gamma}-\eta\gamma)}+\gamma\right) & \text{if } \sigma \leq \frac{2(\mu_{\eta,\gamma}-\eta\gamma)}{\gamma-\eta\gamma} \leq 1, \\ \min\{\lambda_{\eta,\gamma}(1-\sigma), \sigma^2\lambda_{\eta,\gamma}(0)\} & \text{otherwise.} \end{cases}$$

Again,

$$R(\eta,\gamma,\theta) = \min\{R_1(\eta,\gamma,\theta), R_2(\eta,\gamma,\theta), R_3(\eta,\gamma,\theta), R_4(\eta,\gamma,\theta)\}.$$

We have generated Table 3, which shows $R(\eta,\gamma,\theta)$ for $0.5n \leq k \leq 0.98n$. Note that the true performance ratio $R_k^*$ for MC should be greater than $R(\eta,\gamma,\theta)$ of the table, and $R(\eta,\gamma,\theta)$ for $0.5n \geq k \geq 0.02n$ are identical to those displayed here.

| $\sigma = \frac{k}{n}$ | $\theta$ | $\alpha$ | $\eta$ | $\gamma$ | $R(\eta,\gamma,\theta)$ |
|---|---|---|---|---|---|
| 0.50 | 0.75 | 0.7699 | 0 | 19.91 | 0.6026 |
| 0.52 | 0.79 | 0.7899 | 0 | 16.11 | 0.6451 |
| 0.54 | 0.87 | 0.8274 | 0 | 8.01 | 0.6353 |
| 0.56 | 0.91 | 0.8446 | 0 | 5.01 | 0.6013 |
| 0.58 | 0.95 | 0.8606 | 0 | 3.51 | 0.5587 |
| 0.60 | 0.77 | 0.7844 | 0.24 | 13.21 | 0.5258 |
| 0.62 | 0.69 | 0.7503 | 0.26 | 18.51 | 0.5049 |
| 0.64 | 0.71 | 0.7618 | 0.30 | 17.71 | 0.4828 |
| 0.66 | 0.67 | 0.7485 | 0.32 | 19.81 | 0.4586 |
| 0.68 | 0.71 | 0.7688 | 0.38 | 16.11 | 0.4320 |
| 0.70 | 0.73 | 0.7813 | 0.42 | 15.41 | 0.4030 |
| 0.72 | 0.75 | 0.7943 | 0.48 | 13.41 | 0.3724 |
| 0.74 | 0.75 | 0.7992 | 0.52 | 13.01 | 0.3397 |
| 0.76 | 0.75 | 0.8046 | 0.56 | 12.61 | 0.3053 |
| 0.78 | 0.03 | 0.6085 | 0.50 | 18.61 | 0.2746 |
| 0.80 | 0.02 | 0.6216 | 0.51 | 21.20 | 0.2547 |
| 0.82 | 0.02 | 0.6385 | 0.55 | 21.00 | 0.2305 |
| 0.84 | 0.02 | 0.6569 | 0.59 | 21.20 | 0.2057 |
| 0.86 | 0 | 0.6735 | 0.63 | 21.60 | 0.1789 |
| 0.88 | 0 | 0.6960 | 0.69 | 20.80 | 0.1498 |
| 0.90 | 0 | 0.7211 | 0.71 | 23.90 | 0.1235 |
| 0.92 | 0 | 0.7493 | 0.71 | 30.80 | 0.0986 |
| 0.94 | 0 | 0.7820 | 0.71 | 42.50 | 0.0738 |
| 0.96 | 0 | 0.8213 | 0.70 | 71.60 | 0.0509 |
| 0.98 | 0 | 0.8731 | 0.76 | 88.80 | 0.0191 |

Table 3: Worst-case performance ratios of the SDP-MNC algorithm

# 7 Approximation of MVC

For MVC, the SDP relaxation (SDP-GP) can be strengthened by adding the so-called triangle inequalities:

$$-X_{0i} - X_{0j} + X_{ij} \geq -1 \text{ for } 1 \leq i < j \leq n,$$

which can be used to improve the bound $\alpha$ for MVC. (For DSP, MC and MNC, we are not able not improve the approximation bound by adding these constraints.)

The detailed greedy algorithm used in the adjusting procedure for MVC is:

**3. Size Adjusting for MVC**

Let $\tilde{S} = S$, where $S$ is the subset determined in Step 2.

• If $|\tilde{S}| > k$, then for each $i \in \tilde{S}$, let $\zeta(i) = \sum_{j \notin \tilde{S}} w_{ij}$ and $\tilde{S} = \{i_1, i_2, ..., i_{|\tilde{S}|}\}$, where $\zeta(i_1) \geq \zeta(i_2) \geq ... \geq \zeta(i_{|\tilde{S}|})$. Then, remove node $i_{|\tilde{S}|}$ from $\tilde{S}$ and reassign $\tilde{S} = \{i_1, i_2, ..., i_{|\tilde{S}|-1}\}$. Repeat this swapping procedure till $|\tilde{S}| = k$.

• If $|\tilde{S}| \leq k$, arbitrarily add $k - |S|$ nodes to $\tilde{S}$. ■

Now, we have the following lemma

**Lemma 7** *The total weights of edges covered by $\tilde{S}$ resulted from the SDP-GP algorithm satisfies*

$$|\tilde{S}| = k$$

*and*

$$w(\tilde{S}) \geq \frac{k}{\max\{|S|, k\}} \cdot w(S).$$

Then, based on Lemma 7, the deterioration function $r(\delta)$ of (2) for MVC is

$$r(\delta) = \begin{cases} \frac{\sigma}{\delta} & \text{if} \quad \delta \geq \sigma \\ 1 & \text{otherwise} \end{cases}.$$

■

Similarly,

**Corollary 4** $\frac{w^*}{w(S)} \geq \frac{w^*}{w^{tol}} \geq \frac{k}{n}$, *where $w^{tol}$ is the total weight of the edges of $E$.*

**Proof.** The proof is similar to that of Corollary 1. ■

For MVC, recall that

$$w(S) = \frac{1}{4} \sum_{1 \le i < j \le n} w_{ij} \left(3 + \hat{x}_0 \hat{x}_i + \hat{x}_0 \hat{x}_j - \hat{x}_i \hat{x}_j\right).$$

Let

$$\begin{aligned} \alpha = \alpha(\theta) := \min \quad & \frac{3+\frac{2}{\pi}(\arcsin(\theta x+(1-\theta)\kappa)+\arcsin(\theta y+(1-\theta)\kappa)-\arcsin(\theta z+(1-\theta)\kappa^2))}{3+x+y-z} \\ \text{subject to} \quad & -x-y+z \ge -1, \\ & \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0. \end{aligned} \tag{17}$$

Again, we have the following inequality whose proof resembles that for Lemma 4:

$$\mathrm{E}[w(S)] \ge \alpha \cdot w^{SDP} \ge \alpha \cdot w^*.$$

Now we can go ahead compute $R(\eta, \gamma, \theta)$ and $R_k^*$ from (4) and (5). We choose

$$\gamma = \begin{cases} \dfrac{\alpha}{(2\sigma - (1+\eta)/2)^2 - (\eta\beta_1 + \beta_2)}, & \text{if} \quad 4\sigma - 3 \le \eta \le 2\sigma - 1 \\ \dfrac{\alpha(1-\sigma)}{\sigma(\eta\beta_1 + \beta_2 - \eta) + (1+\eta)^2/4 - (\eta\beta_1 + \beta_2)}, & \text{if} \quad -1 \le \eta < 4\sigma - 3. \end{cases} \tag{18}$$

The proof of the following theorem is an analog to that of Proposition 2, which can be found in [11].

**Proposition 3** *If the random variable $z(\gamma, \eta)$ meets its expectation, i.e., $z(\gamma, \eta) \ge \mu_{\eta,\gamma} := \alpha + \gamma(\eta\beta_1 + \beta_2)$, then*

$$w(\tilde{S}) \ge R(\theta, \eta) \cdot w^*,$$

*where*

$$R(\theta, \eta) = \begin{cases} \dfrac{\alpha\sigma(4\sigma - 2(1+\eta))}{(2\sigma - (1+\eta)/2)^2 - (\eta\beta_1 + \beta_2)}, & \textit{if} \quad 4\sigma - 3 \le \eta \le 2\sigma - 1 \\ \dfrac{\alpha\sigma((1+\eta)^2/4 - \eta)}{\sigma(\eta\beta_1 + \beta_2 - \eta) + (1+\eta)^2/4 - (\eta\beta_1 + \beta_2)}, & \textit{if} \quad -1 \le \eta < 4\sigma - 3. \end{cases}$$

■

We have generated Table 4, which shows $R(\eta, \gamma, \theta)$ for $0.4n \leq k \leq 0.98n$. Again note that the true performance ratio $R_k^*$ for MVC should be greater than $R(\eta, \gamma, \theta)$ of the table.

# 8 Final Remarks

If the size requirement $k$ is close to $n$, we may use $\kappa = \sin(\frac{\pi}{2} \cdot (2\frac{k}{n} - 1))$ in matrix $P$ of the balance rounding procedure, which results in a small improvement for the worst-case performance ratio.

In practice, after the semidefinite solution $\bar{X}$ is computed, one can do one dimensional search of $\theta$ in the rounding algorithm to yield the best possible $\tilde{S}$. Thus, the selection of $\theta$, presented in this paper, is merely used to prove the worst-case approximation ratio.

**Acknowledgment** We'd like to thank the referees for their comments that improved the presentation of the paper.

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| $\sigma = \frac{k}{n}$ | $\theta$ | $\alpha$ | $\eta$ | $\gamma$ | $R(\eta, \gamma, \theta)$ |
|---|---|---|---|---|---|
| 0.40 | 0.82 | 0.8691 | -0.4157 | 4.2531 | 0.7338 |
| 0.42 | 0.80 | 0.8675 | -0.3520 | 4.6605 | 0.7518 |
| 0.44 | 0.78 | 0.8664 | -0.2901 | 5.1341 | 0.7687 |
| 0.46 | 0.76 | 0.8657 | -0.2304 | 5.6700 | 0.7844 |
| 0.48 | 0.74 | 0.8655 | -0.1733 | 6.2403 | 0.7987 |
| 0.50 | 0.72 | 0.8657 | -0.1198 | 6.7714 | 0.8113 |
| 0.52 | 0.70 | 0.8663 | -0.0708 | 7.1330 | 0.8220 |
| 0.54 | 0.70 | 0.8700 | -0.0322 | 6.8569 | 0.8307 |
| 0.56 | 0.72 | 0.8761 | -0.0016 | 6.1515 | 0.8377 |
| 0.58 | 0.76 | 0.8840 | 0.0 | 4.5398 | 0.8425 |
| 0.60 | 0.78 | 0.8887 | 0.0 | 3.5221 | 0.8453 |
| 0.62 | 0.78 | 0.8910 | 0.0226 | 4.4361 | 0.8471 |
| 0.64 | 0.76 | 0.8914 | 0.0500 | 4.1206 | 0.8485 |
| 0.66 | 0.76 | 0.8921 | 0.0748 | 3.7722 | 0.8495 |
| 0.68 | 0.74 | 0.8944 | 0.0931 | 3.4349 | 0.8502 |
| 0.70 | 0.72 | 0.8953 | 0.1143 | 3.1240 | 0.8507 |
| 0.72 | 0.70 | 0.8965 | 0.1336 | 2.8414 | 0.8511 |
| 0.74 | 0.68 | 0.8977 | 0.1513 | 2.5910 | 0.8512 |
| 0.76 | 0.64 | 0.8983 | 0.1694 | 2.3500 | 0.8512 |
| 0.78 | 0.60 | 0.8992 | 0.1854 | 2.1365 | 0.8512 |
| 0.80 | 0.02 | 0.8964 | 0.2794 | 1.6791 | 0.8612 |
| 0.82 | 0.00 | 0.9039 | 0.3048 | 1.5784 | 0.8676 |
| 0.84 | 0.00 | 0.9115 | 0.3245 | 1.4596 | 0.8742 |
| 0.86 | 0.00 | 0.9191 | 0.3375 | 1.3074 | 0.8814 |
| 0.88 | 0.00 | 0.9268 | 0.3499 | 1.1479 | 0.8895 |
| 0.90 | 0.00 | 0.9346 | 0.3618 | 0.9807 | 0.8989 |
| 0.92 | 0.00 | 0.9428 | 0.3732 | 0.8053 | 0.9098 |
| 0.94 | 0.00 | 0.9515 | 0.3841 | 0.6210 | 0.9227 |
| 0.96 | 0.00 | 0.9612 | 0.3946 | 0.4268 | 0.9386 |
| 0.98 | 0.00 | 0.9731 | 0.4046 | 0.2209 | 0.9595 |

Table 4: Worst-case performance ratios of the SDP-MVC algorithm for $k \geq 0.40n$

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