A Note on Quadratic Convergence of the Homogeneous and Self-Dual Linear Programming Algorithm

Fang WU*, Shiquan WU* and Yinyu YE†

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Abstract: In this note we show that Ye-Todd-Mizuno's $O(\sqrt{n}L)$-iteration homogeneous and self-dual linear programming (LP) algorithm possesses quadratic convergence of the duality gap to zero. In the case of infeasibility, this result shows that a homogenizing variable quadratically converges to zero and implies that the iterates of the (original) LP variable quadratically diverge. Thus, we have established a complete asymptotic convergence result for interior-point algorithms without any assumption on the LP problem.

Key words: Linear Programming, interior point algorithms, homogeneity, self-dual, quadratic convergence.

*The Institute of Applied Mathematics, Academia Sinica, Beijing, China.

†Department of Management Sciences, The University of Iowa, Iowa City, Iowa 52242, USA. This research was supported in part by NSF grant DDM-9207347 and the K.C. WONG Education Foundation, Hong Kong, through Academia Sinica, Beijing, China.

1 Introduction

Consider linear programs in the following standard form:

$$\begin{array}{lll} (LP) & \text{minimize} & c^T x \\ & \text{subject to} & Ax = b,\ x \geq 0, \end{array}$$

where $c \in R^n$, $A \in R^{m\times n}$ and $b \in R^m$ are given, $x \in R^n$, and ^{T} denotes transpose. (LP) is said to be feasible if and only if its constraints are consistent; it is called unbounded if there is a sequence $\{x^k\}$ such that x^k is feasible for all k but $c^T x^k \to -\infty$. (LP) has a solution if and only if it is feasible and bounded. The dual to (LP) can be written as

$$\begin{array}{lll} (LD) & \text{maximize} & b^T y \\ & \text{subject to} & A^T y + s = c,\ s \geq 0, \end{array}$$

where $y \in R^m$.

We call a feasible point (x, s, y) strictly feasible if it is feasible and (x, s) is positive. A feasible point (x^*, s^*, y^*) is an optimal (complementary) solution if and only if $(x^*)^T s^* = 0$. It is well known that if (LP) has a solution, then it has a strictly complementary solution, i.e., an optimal solution satisfying

$$x_j^* + s_j^* > 0 \quad \text{for} \quad j = 1, 2, ..., n.$$

Consider a sequence of strictly feasible points $\{(x^k, s^k)\}$ such that the (complementarity or duality) gap $(x^k)^T s^k \to 0$. Then we say that this gap sequence converges quadratically to zero if

$$\limsup_{k\to\infty} \frac{(x^{k+1})^T s^{k+1}}{((x^k)^T s^k)^2} < +\infty.$$

Recently several new results have been developed for the asymptotic (local) convergence of polynomial–time interior–point algorithms. It has been shown that some interior-point algorithms for (LP) exhibit quadratic convergence of the primal-dual gap to zero under some combination of the following assumptions:

(A0) The feasible region for both the primal and dual is nonempty, i.e., (LP) has a solution.

(A1) The feasible region for both the primal and dual has a nonempty relative interior, i.e., (LP) and (LD) have strictly feasible points or a bounded optimal solution set.

(A2) The LP problem has the nondegenerate optimal solution.

(A3) The iteration sequence $\{(x^k, s^k)\}$ generated by the interior-point algorithm converges, and it converges to a strict complementarity solution.

The issue of asymptotic convergence of interior-point algorithms was extensively studied by many researchers (see references in [10]). Quadratic convergence under only (A1) was first established by Mehrotra [4], Tsuchiya [7], and Ye, Güler, Tapia and Zhang [11], The result was non-trivially extended for convex LCP by Ye and Anstreicher [10]. Quadratic convergence under only (A0) was first established by Potra [6] and Wright [8] for an infeasible-interior-point algorithm. Their algorithm also possesses $O(nL)$-iteration complexity if the initial point is suitably chosen, where L is the LP data length. As we can see that (A0), the feasibility of the LP problem, is still the minimal requirement up to date for all asymptotic convergence analyses.

Recently, Ye-Todd-Mizuno proposed a homogeneous and self-dual LP algorithms. The algorithm possesses several ideal features: it achieves $O(\sqrt{n}L)$-iteration complexity; it solves (LP) problem without any regularity assumption concerning the existence of a feasible point; it does not introduce any big M parameter; each iteration solves a system of linear equations whose dimension is almost the same as that solved in the standard interior-point algorithms.

In this note we show that Ye-Todd-Mizuno's $O(\sqrt{n}L)$-iteration homogeneous and self-dual LP algorithm also possesses quadratic convergence of the duality gap to zero for general LP. In the case of infeasibility, this shows that a homogenizing variable quadratically converges to zero, which proves that at least one of the primal and dual LP problems is infeasible and implies that the iterates of the original LP variable quadratically diverge.

2 The homogeneous and self-dual LP algorithm

We briefly review the homogeneous and self-dual LP algorithm of Ye-Todd-Mizuno. Given any $x^0 > 0$, $s^0 > 0$, $\tau^0 > 0$, $\kappa^0 > 0$, and y^0, they formulate

$$\begin{array}{lcccccl}
(HSLP) & \min & & & & ((x^0)^T s^0 + \tau^0\kappa^0)\theta & \\
 & \text{s.t.} & & Ax & -b\tau & +\bar{b}\theta & = 0, \\
 & & -A^T y & & +c\tau & -\bar{c}\theta & \geq 0, \\
 & & b^T y & -c^T x & & +\bar{z}\theta & \geq 0, \\
 & & -\bar{b}^T y & +\bar{c}^T x & -\bar{z}\tau & & = -(x^0)^T s^0 - \tau^0\kappa^0, \\
 & & y \text{ free}, & x \geq 0, & \tau \geq 0, & \theta \text{ free}, &
\end{array} \tag{1}$$

where

$$\bar{b} = b\tau^0 - Ax^0, \quad \bar{c} = c\tau^0 - A^T y^0 - s^0, \quad \bar{z} = c^T x^0 - b^T y^0 + \kappa^0. \tag{2}$$

Here $\bar{b}$, $\bar{c}$, and $\bar{z}$ represent the "infeasibility" of the initial primal point, dual point, and primal-dual "gap", respectively.

Denote by s the slack vector for the second (inequality) constraint and by κ the slack scaler for the third (inequality) constraint in (1). They observed that i.) the dual of (HSLP) has the identical form as (HSLP), ii) (HSLP) has a strictly feasible point

$$y = y^0, \quad x = x^0 > 0, \quad \tau = \tau^0 > 0, \quad \theta = 1, \quad s = s^0 > 0, \quad \kappa = \kappa^0 > 0.$$

iii) (HSLP) has an optimal solution and its optimal solution set is bounded, and iv) There is an optimal solution $(y^*, x^*, \tau^*, \theta^* = 0, s^*, \kappa^*)$ such that

$$\begin{pmatrix} x^* + s^* \\ \tau^* + \kappa^* \end{pmatrix} > 0,$$

which is called a strictly self-complementary solution. One simple choice for the initial point would be

$$y^0 = 0, \quad x^0 = e, \quad \tau^0 = 1, \quad s^0 = e, \quad \text{and} \quad \kappa^0 = 1. \tag{3}$$

For simplicity, in the rest of this paper we assume that (x^0, s^0) is chosen by (3).

Then, they apply the Mizuno-Todd-Ye predictor-corrector algorithm [5] to solving (HSLP), coupled with a termination technique. The result is the following theorem.

Theorem 1 .

The predictor-corrector algorithm, coupled with the termination technique, generates a strictly self-complementary solution, $(y^*, x^*, \tau^*, \theta^* = 0, s^*, \kappa^*)$, for (HSLP) in $O(\sqrt{n}L)$ iterations.

i. *(LP) has a solution (feasible and bounded) if and only if $\tau^* > 0$. In this case, x^*/τ^* is an optimal solution for (LP) and $(y^*/\tau^*, s^*/\tau^*)$ is an optimal solution for (LD).*

ii. *If $\tau^* = 0$, then $\kappa^* > 0$, which implies that $c^T x^* - b^T y^* < 0$, i.e., at least one of (LP) and (LD) is infeasible.*

□

3 An equivalent LCP problem and the predictor-corrector algorithm

For simplicity, let

$$x := \begin{pmatrix} x \\ \tau \end{pmatrix}, \quad s := \begin{pmatrix} s \\ \kappa \end{pmatrix}, \quad y := \begin{pmatrix} y \\ \theta \end{pmatrix},$$

and

$$M := \begin{pmatrix} 0 & c & -A^T & -\bar{c} \\ -c^T & 0 & b^T & \bar{z} \\ A & b & 0 & \bar{b} \\ \bar{c}^T & -\bar{z} & -\bar{b}^T & 0 \end{pmatrix}, \quad q := \begin{pmatrix} 0 \\ 0 \\ 0 \\ -(n+1) \end{pmatrix}.$$

Then, the HSLP algorithm equivalently solves the following linear complementarity problem (LCP):

$$x^T s = 0, \quad (x, s) \geq 0$$

and

$$\begin{pmatrix} s \\ 0 \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} + q$$

where M is skew-symmetric. Note that for any (d_x, d_s, d_y) that satisfies

$$\begin{pmatrix} d_s \\ 0 \end{pmatrix} = M \begin{pmatrix} d_x \\ d_y \end{pmatrix},$$

we have

$$(d_x)^T d_s = 0. \tag{4}$$

Also, $x^0 = e$ and $s^0 = e$ is a feasible starting point for this LCP problem. Moreover, it has a strictly complementary solution (x^*, s^*, y^*) with $x^* + s^* > 0$. Let us call this problem homogeneous linear complementarity problem (HLCP).

We now analyze the predictor-corrector algorithm for the HLCP. Also for simplicity, we assume that $x \in R^n$, $s \in R^n$ and $y \in R^m$. Let $\mathcal{F}$ be the feasible region of the HLCP and $\mathcal{F}^0$ be its "interior" (i.e., $(x, s) > 0$). We define a central path as

$$\mathcal{C}_s = \left\{ (x, s, y) \in \mathcal{F}^0 : Xs = \frac{x^T s}{n} e \right\},$$

and define the neighborhood of the path as

$$\mathcal{N}(\beta) = \{(x, s, y) \in \mathcal{F}^0 : \|Xs - \mu e\| \le \beta\mu \quad \text{where} \quad \mu = \frac{x^T s}{n}\}$$

for some $\beta \in (0, 1)$.

Given an interior feasible point $(x^k, s^k, y^k) \in \mathcal{F}^0$, consider solving the following system of linear equations for (d_x, d_s, d_y) (Kojima et al. [3]):

$$\begin{array}{rcl} X^k d_s + S^k d_x & = & \gamma \mu^k e - X^k s^k \\ \begin{pmatrix} d_s \\ 0 \end{pmatrix} & = & M \begin{pmatrix} d_x \\ d_y \end{pmatrix}, \end{array} \tag{5}$$

where $\mu^k = (x^k)^T s^k / n$, $X^k = \text{Diag}(x^k)$, $S^k = \text{Diag}(s^k)$, etc.

In the predictor step the predictor-corrector algorithm, we have $(x^k, s^k, y^k) \in \mathcal{N}(\beta)$ with $\beta = 1/4$. We solve the linear system (5) with $\gamma = 0$. Then let

$$x(\alpha) := x^k + \alpha d_x, \quad s(\alpha) := s^k + \alpha d_s, \quad y(\alpha) := y^k + \alpha d_y.$$

We determine the step size using

$$\alpha^k := \max \{\alpha : (x(\alpha), s(\alpha), y(\alpha)) \in \mathcal{N}(2\beta)\}. \tag{6}$$

Then compute the next point by

$$x^{k+1} = x(\alpha^k), \quad s^{k+1} = s(\alpha^k), \quad y^{k+1} = y(\alpha^k),$$

We can see from (4) that

$$(x^{k+1})^T s^{k+1} \le (1-\alpha^k)(x^k)^T s^k.$$

Immediately after the predictor step, we use a corrector step to generate the $(k+2)$th iterate such that

$$(x^{k+2})^T s^{k+2} = (x^{k+1})^T s^{k+1} \quad \text{and} \quad (x^{k+2}, s^{k+2}, y^{k+2}) \in \mathcal{N}(\beta).$$

Let

$$\delta^k = D_x d_s / \mu^k. \tag{7}$$

Ye et al. ([11]) and Ji et al. ([2]) essentially proved that the step-size α^k in the predictor step satisfies

$$\alpha^k \ge 2\beta / (\sqrt{\beta^2 + 4\beta \|\delta^k\|} + \beta).$$

Thus, we can derive

$$(x^{k+1})^T s^{k+1} \le \frac{\|\delta^k\|}{\beta} (x^k)^T s^k. \tag{8}$$

If we can show that

$$\|d_x\| = O(\mu^k) \quad \text{and} \quad \|d_s\| = O(\mu^k), \tag{9}$$

then from (7) and (8) the gap $(x^{k+2})^T s^{k+2} = (x^{k+1})^T s^{k+1} = O([(x^k)^T s^k]^2)$, and it converges to zero quadratically.

4 A short analysis of convergence

Let (B, N) be the partition of the index set $\{1, 2, ..., n\}$ such that for every strictly complementary solution (x^*, s^*, y^*) in (HLCP)

$$x_B^* > 0 \quad \text{and} \quad s_N^* > 0.$$

From Güler and Ye [1] , we have

$$x_j^k \ge \xi \quad \text{for} \quad j \in B \quad \text{and} \quad s_j^k \ge \xi \quad \text{for} \quad j \in N, \tag{10}$$

where ξ is a fixed positive number that is independent of k.

The following result was essentially proved in Ye and Anstreicher [10]: if d_x and d_s are obtained from the linear system (5) with $\gamma = 0$, then

$$\|(d_x)_N\| = O(\mu^k) \quad \text{and} \quad \|(d_s)_B\| = O(\mu^k). \tag{11}$$

The proofs of $\|(d_x)_B\| = O(\mu^k)$ and $\|(d_s)_N\| = O(\mu^k)$ are not trivial. However, if we decompose the HLCP as

$$\begin{pmatrix} s_B \\ s_N \\ 0 \end{pmatrix} = \begin{pmatrix} M_{BB} & M_{BN} & M_{BY} \\ M_{NB} & M_{NN} & M_{NY} \\ M_{YB} & M_{YN} & M_{YY} \end{pmatrix} \begin{pmatrix} x_B \\ x_N \\ y \end{pmatrix} + \begin{pmatrix} q_B \\ q_N \\ q_Y \end{pmatrix}.$$

Then, we have the following result resembling a lemma in [10].

Lemma 2 . *If d_x and d_s are obtained from the linear system (5) with $\gamma = 0$, then $u = (d_x)_B$, $v = (d_s)_N$ and $w = d_y$ is the solutions to the (weighted) least-squares problem*

$$\begin{array}{llllll} \min_{u,v,w} & (1/2)\|D_B^{-1}u\|^2 & +(1/2)\|D_N v\|^2 & & & \\ s.t. & M_{BB}u & & +M_{BY}w & = -M_{BN}(d_x)_N + (d_s)_B, & \\ & M_{NB}u & -v & +M_{NY}w & = -M_{NN}(d_x)_N, & \\ & M_{YB}u & & +M_{YY}w & = -M_{YN}(d_x)_N, & \end{array} \tag{12}$$

□

(We omit the proof of this lemma, which can be found in our working paper [9].)

Based on this lemma, we can establish $\|(d_x)_B\| = O(\mu^k)$ and $\|(d_s)_N\| = O(\mu^k)$, which together with (11) indicate that d_x and d_s of the predictor step satisfy relation (9). Thus, we have

Theorem 3 . *Let $\{(x^k, s^k)\}$ be the sequence generated by the HLCP Algorithm. Then, with constants $\beta = 1/4$ for the algorithm, $(x^k)^T s^k$ ($k = 0, 2, 4, ...$) converges to zero quadratically.*

Now consider solving (HSLP). Theorem 1 and relation (10) indicate that κ^k will be bounded away from zero in the infeasible case, or τ^k will be bounded away from zero in the feasible case of the LP problem. Also note that the iterates of the original LP variable are given by x^k/τ^k and s^k/τ^k in (HSLP). Therefore, Theorem 3 gives a complete quadratic convergence result:

Corollary 4 . *If the LP problem is feasible, then the solution generated by the HSLP algorithm converges to the optimal solution set quadratically; if the LP problem is infeasible, the homogenizing variable τ^k quadratically converges to zero, which proves infeasibility, and implies that the iterates of the original LP variable quadratically diverge.*

□

References

[1] O. Güler and Y. Ye, "Convergence behavior of some interior-point algorithms," to appear in *Mathematical Programming* 60 (1993).

[2] J. Ji, F. A. Potra, and S. Huang, "A predictor-corrector method for linear complementarity problems with polynomial complexity and superlinear convergence," Technical Report, Department of Mathematics, The University of Iowa (Iowa City, IA), 1991.

[3] M. Kojima, S. Mizuno and A. Yoshise, "A polynomial-time algorithm for a class of linear complementarity problems," *Mathematical Programming* 44 (1989) 1-26.

[4] S. Mehrotra, "Quadratic convergence in a primal-dual method," Technical Report 91-15, Department of IE and MS, Northwestern University (Evanston, IL.), 1991, to appear in *Mathematics of Operations Research.*

[5] S. Mizuno, M. J. Todd, and Y. Ye, "On adaptive-step primal-dual interior-point algorithms for linear programming," Technical Report No. 944, School of Operations Research and Industrial Engineering, Cornell University, (Ithaca, New York), 1990, to appear in *Mathematics of Operations Research.*

[6] F. A. Potra, "A quadratically convergent infeasible interior-point algorithm for linear programming," Report No. 28, Department of Mathematics, The University of Iowa (Iowa City, IA), 1992.

[7] T. Tsuchiya, "Quadratic convergence of Iri and Imai's algorithm for degenerate linear programming problems," manuscript, The Institute of Statistical Mathematics (Tokyo, Japan), 1991.

[8] S. J. Wright, "A path-following infeasible-interior-point algorithm for LCP," MCS Report, Mathematics and Computer Science Division, Argonne National Lab. (Argonne, IL), 1992.

[9] F. Wu, S. Wu and Y. Ye, "On quadratic convergence of the $O(\sqrt{n}L)$-iteration homogeneous and self-dual linear programming algorithm," Working Paper, The College of Business Administration, The University of Iowa (Iowa City, IA.), 1992.

[10] Y. Ye and K. M. Anstreicher, "On quadratic and $O(\sqrt{n}L)$ convergence of a predictor- corrector algorithm for LCP," Working Paper No 91-20, The College of Business Administration, The University of Iowa (Iowa City, IA.), 1991, to appear in *Mathematical Programming.*

[11] Y. Ye, O. Güler, R. A. Tapia and Y. Zhang, "A quadratically convergent $O(\sqrt{n}L)$-iteration algorithm for linear programming," *Mathematical Programming* 59 (1993) 151-162.

[12] Y. Ye, M. J. Todd, and S. Mizuno, "An $O(\sqrt{n}L)$-iteration homogeneous and self-dual linear programming algorithm," Technical Report No. 1007, School of Operations Research and Industrial Engineering, Cornell University, and Working Papers No 92-11, College of Business Administration, The University of Iowa (Iowa City, IA), 1992, to appear in *Mathematics of Operations Research.*