BRIAN WHITE - MEAN CURVATURE FLOW (MATH 258) LECTURE NOTES

NOTES BY OTIS CHODOSH

CONTENTS

These are notes from Brian White's course on mean curvature flow, taught at Stanford University in Spring 2015. Thanks to Brian for a very enjoyable class. It is likely that I have introduced errors in the compilation process. If you find any mistakes, please let me know at ochodosh@stanford.edu. Thanks in particular to David Hoffman and Shuli Chen for pointing out many typos.

1. Overview

Mean curvature flow (MCF) is a way to let submanifolds in a manifold evolve.

1.1. **Curve shortening flow.** For curves, mean curvature flow is often called "curve shortening flow." Starting with an embedded curve $\Gamma \subset \mathbb{R}^2$, we evolve it with velocity $\vec{k}$. This has several nice
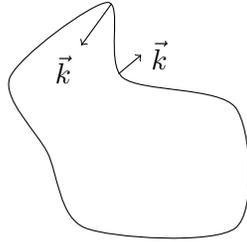

FIGURE 1. An embedded curve $\Gamma \subset \mathbb{R}^2$ and its curvature vector $\vec{k}$.

properties. First of all, the flow exhibits (short time) smoothing of the curve. More formally, we
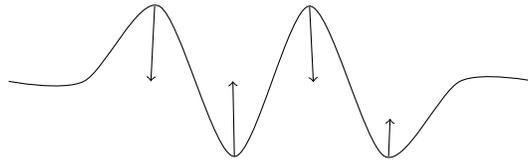

FIGURE 2. This illustrates the short time smoothing property of curve shortening flow. The curvature vector points in a direction which serves to smooth the curve out.

can write curve shortening flow as

$$\frac{\partial X}{\partial t} = \frac{\partial^2 X}{\partial s^2},$$

where s is the arc-length parameter. Because s changes with time, this is not the ordinary heat equation, but rather a non-linear heat equation. However, it still has the nice smoothing properties—if, for example, Γ is initially C^2, then for $t > 0$ small, Γ_t becomes real analytic.

Unlike the linear heat equation, singularities can occur! For example, a round circle of initial radius r_0 can be seen to evolve to a round circle of radius $r(t) = \sqrt{r_0^2 - 2t}$. At $t = \frac{r_0^2}{2}$, the circle disappears!

The arc-length decreases along curve shortening flow. In particular

$$\frac{d}{dt}\text{length} = -\int \vec{v} \cdot \vec{k}\, ds = -\int |\vec{k}|^2 < 0.$$

Note that the first equality holds for any flow. This shows that in some sense, curve shortening flow is the gradient flow of arc-length.

The curve shortening flow satisfies an avoidance principle. In particular, disjoint curves remain disjoint. This is illustrated in Figure 3.

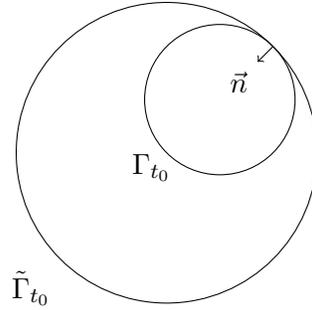Similarly, one may show that embeddedness is preserved.

FIGURE 3. If $\tilde{\Gamma}$ encloses Γ and the curves are originally disjoint, then if $\tilde{\Gamma}_t$ were to touch $\tilde{\Gamma}_t$ for some $t > 0$, then if t_0 is the time of first contact, the point of contact will look like this diagram. In particular $\vec{k}_{\tilde{\Gamma}_{t_0}} \cdot \vec{n} \leq \vec{k}_{\Gamma_{t_0}} \cdot \vec{n}$ at this point of contact. If the inequality was strict, this would immediately yield a contradiction, as it would imply that they must have crossed for $t = t_0 - \epsilon$, contradicting the choice of t_0. In general, we must reason using the maximum principle to rule this situation out.

The avoidance principle and the behavior of the shrinking circle combine to show that any closed curve flowing by curve shortening flow has a finite lifetime. This may be seen by enclosing the curve by a circle. The curve must disappear before the circle does.

FIGURE 4. The curve Γ must disappear before time $\frac{r_0^2}{2}$, which is when the circle dissapears.

Theorem 1.1. *If Γ_t is an embedded curve flowing by curve shortening flow, then if $A(t)$ is the area enclosed by Γ_t, then $A'(t) = -2\pi$.*

Proof. We compute
$$A'(t) = \int_{\Gamma_t} \vec{v} \cdot \vec{n}_{out} = \int_{\Gamma_t} \vec{k} \cdot \vec{n}_{out} = -2\pi.$$
The first equality holds for any flow, while the final equality is a consequence of Gauss–Bonnet. □

Corollary 1.2. *The lifetime of such a curve is at most $\frac{A(0)}{2\pi}$.*

The first real result proven about curve shortening flow is:

Theorem 1.3 (Gage–Hamilton [GH86])**.** *A convex curve $\Gamma \subset \mathbb{R}^2$ collapses to a "round point."*

This might seem "obvious," but we remark that it fails for other reasonable flows, e.g.

$$\vec{v} = \frac{\vec{k}}{|\vec{k}|^{\frac{2}{3}}}.$$

Under this flow, ellipses remain elliptical with the same eccentricity! One can show that a convex curve collapses to an elliptical point. More surprisingly, for the flow

$$\vec{v} = \frac{\vec{k}}{|\vec{k}|^r},$$

for $r > \frac{2}{3}$, a convex curve shrinks to a point, but rescaling it to contain unit area, the length could become unbounded (see [And03]).

Amazingly, the Gauge–Hamilton theorem remains true for embedded curves which need not be convex.

Theorem 1.4 (Grayson [Gra87]). *An embedded closed curve $\Gamma \subset \mathbb{R}^2$ eventually becomes convex under curve shortening flow.*
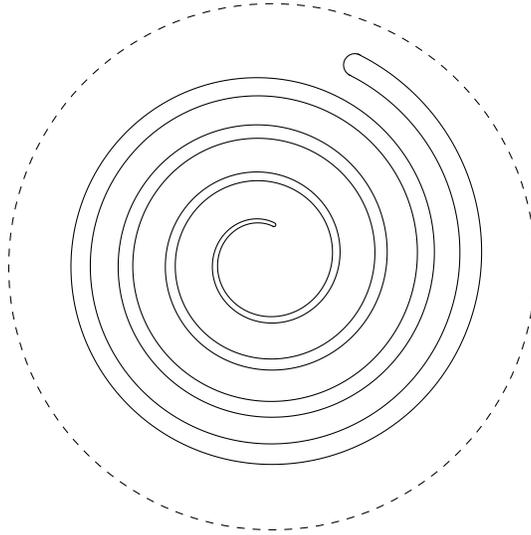
FIGURE 5. Grayson's theorem implies that this complicated spiraling curve will shrink to a round point. Amazingly, it will do so quite quickly, because it must do so before the dashed circle shrinks away, by the avoidance principle.

Corollary 1.5. *The lifetime of an embedded curve is precisely $\frac{A(0)}{2\pi}$.*

Grayson's theorem also extends to non-flat ambient geometry.

Theorem 1.6 (Grayson [Gra89]). *For simple closed curve in a compact surface $\Gamma \subset (M^2, g)$, the mean curvature flow Γ_t either tends to a round point, or smoothly tends to a closed geodesic.*

As a consequence of this, it is possible to give a relatively simple proof of the following classical result of Lusternik and Schnirelman

Corollary 1.7. *Any metric on the 2-sphere $(\mathbb{S}^2, g)$ admits at least three simple closed geodesics.*

This is sharp (as can be seen by a nearly round ellipsoid with slightly different axis). To illustrate the proof using Grayson's theorem, we will prove that there is at least one simple closed geodesic. If we smoothly foliate $\mathbb{S}^2 \setminus \{N, S\}$ by circles enclosing N, S, in the usual manner, then the small circles near N will flow to a round point, and similarly for the small circles near S. However, one can see that the orientation of the limiting points will be opposite in both cases. Hence, there must be some curve in the middle which flows to a geodesic, by continuous dependence of the flow on the initial conditions.

1.2. **Flow of hypersurfaces.** For $M^n \subset \mathbb{R}^{n+1}$, we write $\kappa_1, \ldots, \kappa_n$ for the principle curvatures with respect to $\vec{n}$. Then we define the *scalar mean curvature* $h = \sum \kappa_i$ and the *vector mean curvature* $\vec{H} = h\vec{n}$. Mean curvature flow is the flow with velocity $\vec{H}$.

Many of the properties of curve shortening flow are really properties of mean curvature flow. Short time smoothing, decreasing area, the avoidance principle, preservation of embeddedness, and finite lifetime of compact surfaces all hold. The analogue of Gage–Hamilton was proven by Huisken

Theorem 1.8 (Huisken [Hui84]). *For $n \geq 2$, if $M^n \subset \mathbb{R}^{n+1}$ is closed, convex hypersurface then it shrinks to a round point.*

However, Grayson's theorem does not extend to higher dimensions!
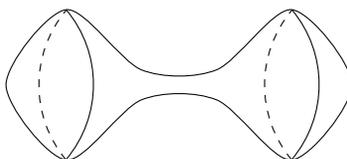
FIGURE 6. The (non-convex) dumbbell will have a "neck-pinch" before the entire surface shrinks away
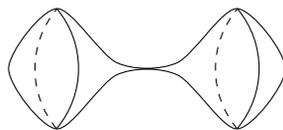
FIGURE 7. At the time of the neck-pinch, most of the surface will remain non-singular, but the diameter of the neck will have shrunk away
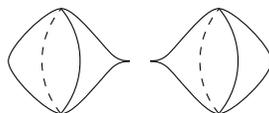
FIGURE 8. It is possible to "continue the flow through the singularity." It will become smooth immediately after, and then both components will shrink away.

A natural question that arises is: how do singularities affect the evolution? In particular, it is necessary to give a weak notion of flow after the first singular time. There are at least three possibilities: the level set flow, the Brakke flow, and what we will call the Brakke flow+. Similarly, we can ask: how large can the singular set be, and what do the singularities look like?

For the size of the singular set, the following result due to Ilmanen is the best answer for general initial surfaces:

Theorem 1.9 (Ilmanen, [Ilm94]). *For a generic smooth closed hypersurface $M_0^n \subset \mathbb{R}^{n+1}$, for a.e. $t > -0$, the mean curvature flow M_t is smooth a.e.*

Here, generic means that for a fixed smooth closed hypersurface, if we foliate a tubular neighborhood by surfaces, then a generic member of the foliation will have this property.

1.3. **Mean convex surfaces.** It turns out that the class of mean convex flows is much better understood than general mean curvature flows. See [Whi00, Whi03]. For $M^n \subset \mathbb{R}^{n+1}$ a closed embedded hypersurface, if we write $M = \partial\Omega$ for Ω a compact set, then we say that M is *mean convex* if $\vec{H}$ points into Ω.

In particular, $M_t \subset \Omega$ for $0 < t < \epsilon$. One can show that mean convexity is preserved under the flow, even through singularities. However, mean convexity still allows for interesting singularities. For example, the dumbbell in Figure 6 can be made to be mean convex. An important feature of mean convex mean curvature flows is that the evolution is unique, even without a genericity assumption. This is in contrast with the general case.

The following theorem demonstrates our improved understanding of mean convex flows. An analogous result holds in higher dimensions.

Theorem 1.10. *For $M_0^2 \subset \mathbb{R}^3$, if M_0 is mean convex, then M_t is smooth for a.e. times $t > 0$. Moreover, the singular set has parabolic Hausdorff dimension at most 1.*

In some sense, this is sharp with respect to the size of the singular set. However, much more

FIGURE 9. A rotationally symmetric mean convex torus which results in a ring of singularities.

(should) be true. Could it be possible that there is a finite number of singular times? A more ambitious conjecture is that for a generic initial surface, the singular set consists of finitely many points in spacetime.

For a mean convex flow, we can think of it as a singular foliation of Ω.
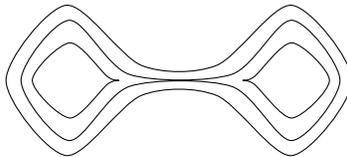
FIGURE 10. Mean convex mean curvature flow (in $\mathbb{R}^3$) as a singular foliation.

Theorem 1.11. *For $M_0^2 \subset \mathbb{R}^3$ mean convex, and $p \in \Omega$ if we dilate by λ_i around p with $\lambda_i \to \infty$, then after passing to a subsequence we get convergence to one of*

    (1) parallel planes,
    (2) concentric spheres, or
    (3) co-axial cylinders.

This point of view does miss interesting behavior. For example, if we consider the dumbbell immediately after the neck-pinch, there will be regions of high curvature, as in Figure 11. To capture this behavior, we have the following result.

Theorem 1.12. *For $M_0^2 \subset \mathbb{R}^3$ mean convex, if $p_i \in \Omega$, $p_i \to p \in \Omega$, translating by $-p_i$ and dilating by $\lambda_i \to \infty$ yields subsequential convergence to one of the following*

    (1) nested compact convex sets (e.g., spheres),
    (2) co-axial cylinders,
    (3) parallel planes,
    (4) or entire graphs of strictly convex functions.
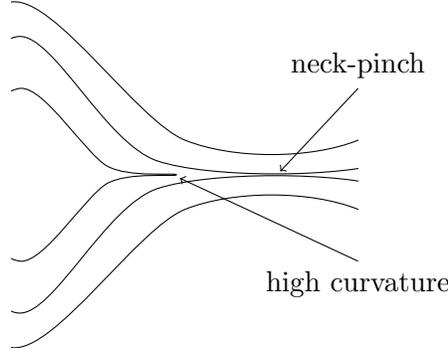
FIGURE 11. Regions of high curvature immediately after the neck-pinch singularity.

2. THE MAXIMUM PRINCIPLE

Theorem 2.1. *Suppose that M is compact and $f : M \times [0, T] \to \mathbb{R}$. Let $\varphi(t) = \min f(\cdot, t)$. Assume that for each x with $\varphi(t) = f(x, t)$, we have $\frac{\partial f}{\partial t}(x, t) \geq 0$. Then, φ is an increasing function.*

Proof. First, suppose that $\varphi(t) = f(x, t)$ implies that $\frac{\partial f}{\partial t}(x, t) > 0$. Let $\min_{M \times [0,T]} f$ be attained at (x, t). Then, $\varphi(t) = f(x, t)$. Hence

$$\frac{\partial f}{\partial t}(x, t) > 0,$$

by the hypothesis. This contradicts the choice of (x, t) unless $t = 0$. More generally, the same argument shows that $a \leq b$ implies that $\varphi(a) \leq \varphi(b)$.

In general, we can apply the above argument to

$$f(x, t) + \epsilon t$$

for $\epsilon > 0$. Then $a \leq b$ implies that

$$\varphi(a) + \epsilon a \leq \varphi(b) + \epsilon b.$$

Letting $\epsilon \searrow 0$ concludes the proof. □

Note that we did not need to assume much regularity of f, just that it is continuous everywhere and that it is differentiable at the points where it attains it minimum.

Theorem 2.2. *Suppose that $\Gamma_1(t), \Gamma_2(t)$ are compact, disjoint curves in $\mathbb{R}^2$ moving by curvature flow. Then $d(\Gamma_1(t), \Gamma_2(t))$ is increasing.*

The same theorem and proof works for hypersurfaces in $\mathbb{R}^n$.

Proof. Parametrizing Γ_i as $F_i : S^1 \times [0, T] \to \mathbb{R}^2$, we let

$$f : S^1 \times S^1 \times [0, T] \to \mathbb{R}, f(\theta_1, \theta_2, t) = |F_1(\theta_1, t) - F_2(\theta_2, t)|^2.$$

For $\varphi(t) = \min_{\theta_1, \theta_2} f(\theta_1, \theta_2, t)$, if the minimum is attained at (θ_1, θ_2), then

$$\frac{\partial f}{\partial t} = 2(F_1 - F_2) \cdot \left(\frac{\partial F_1}{\partial t} - \frac{\partial F_2}{\partial t} \right) = 2(F_1 - F_2) \cdot \left(\vec{k}_1 - \vec{k}_2 \right).$$

If we translate $\Gamma_1(t)$ and $\Gamma_2(t)$ in the direction of $F_1 - F_2$ until the surfaces touch, we conclude that at (θ_1, θ_2) it must be that $\frac{\partial f}{\partial t} \geq 0$. □

Theorem 2.3. *Let Ω be an open half plane in $\mathbb{R}^2$ and Γ_t be a closed curve moving by curvature flow. Then the number of components of $\Gamma_t \cap \Omega$ is non-increasing with time.*

Proof. We assume that the half plane is $\{x_2 > 0\}$.

Consider the set $U = \{(x,t) : x \in \Gamma_t \cap \Omega\} \subset S^1 \times [0,T]$. We claim that it cannot happen that two disjoint connected components of $W \cap \{t = T\}$ are connected by curves in U to a single connected component of $W \cap \{t = 0\}$. If this were to happen, then these curves would bound a region $W \subset S^1 \times [0,T]$ so that on the parabolic boundary $\partial W \cap \{t < T\}$, $F \cdot e_2 \geq a > 0$. Now, if an interior point of W were to attain the maximal value of the height, $F \cdot e_2$, then the maximum principle would yield a contradiction (at the maximal value of height, the curvature vector must point downward). This implies that $F \cdot e_2 \geq a$ on all of W, a contradiction.   □

This "roughly" shows that the number of intersections with Γ_t and a line is non-increasing with time. The subtlety is that we have not argued that there is an entire line of intersection for some positive time! We can argue that Γ_t is real analytic in the spatial variables for $t > 0$, and thus this is impossible. However, if the ambient space is not real analytic, things could get complicated.

For example, understanding the following question could have applications to mean curvature flow of curves in higher codimension:

Question 2.4. Consider a curvature flow with a boundary point on $\{0\} \times \mathbb{R}$ and a boundary point on $\{1\} \times \mathbb{R}$, where the boundary points are prescribed for each time t. For fixed points and an embedded initial curve, as $t \to \infty$, we can show that the flow converges to a straight line between the two points. However, at no time $t < \infty$ is the curve exactly equal to a straight line.

The question is whether or not it is possible to wiggle the boundary points so that the flow converges to a straight line in finite time.

3. Unparameterized mean curvature flow

Suppose that we have a time dependent family of curves $t \mapsto \Gamma(t)$. The velocity does not make sense until we choose a smooth parametrization $F : S^1 \times [0,T[\to \mathbb{R}^2$. However, we can check that the normal velocity $\left(\frac{\partial F}{\partial t}\right)^{\perp}$ is independent of the parametrization. In other words, unparameterized mean curvature flow takes

$$\left(\frac{\partial F}{\partial t}\right)^{\perp} = \vec{H}.$$

For example, for convex curves in $\mathbb{R}^2$, we can parametrize the curve by the angle the tangent vector makes with a fixed vector. This is not a normal parametrization, but we can still make sense of mean curvature flow in this way.

3.1. Graphs. An important special case of this is where, for $\Omega \subset \mathbb{R}^n$, the graph of $u : \Omega \times [0,T] \to \mathbb{R}$ is a hypersurface in $\mathbb{R}^{n+1}$ moving by mean curvature flow. We let $\vec{n}$ denote the upwards pointing normal vector. One may compute that

$$H = \vec{H} \cdot \vec{n} = D_i\left(\frac{D_i u}{\sqrt{1 + |Du|^2}}\right).$$

Moreover, the velocity vector is given by $\left(0, \frac{\partial u}{\partial t}\right)$. Hence, the normal component of the velocity is given by

$$\left(0, \frac{\partial u}{\partial t}\right) \cdot \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}} = \frac{1}{\sqrt{1 + |Du|^2}} \frac{\partial u}{\partial t}.$$

Thus, the non-parametric mean curvature flow of a graph can be written as

$$\boxed{\frac{\partial u}{\partial t} = \sqrt{1 + |Du|^2} D_i\left(\frac{D_i u}{\sqrt{1 + |Du|^2}}\right) = \Delta u - \frac{D^2 u(Du, Du)}{1 + |Du|^2}}$$

Let $u_1, u_2 : \bar{\Omega} \times [0, T] \to \mathbb{R}$ be solutions to mean curvature flow with $u_1 > u_2$. If

$$\min_{\bar{\Omega}}(u_1(\cdot, t) - u_2(\cdot, t)) \leq \min_{\partial \Omega}(u_1(\cdot, t) - u_2(\cdot, t)),$$

then

$$\min_{\bar{\Omega}}(u_1(\cdot, t) - u_2(\cdot, t))$$

is nondecreasing. In particular, if $u_1(\cdot, 0) - u_2(\cdot, 0) \geq a$ initially, and $u_1(x, t) - u_2(x, t) \geq a$ for $x \in \partial \Omega$, then

$$u_1 - u_2 \geq a$$

for all points in $\Omega \times [0, T]$.

In a related vein, we have:

Theorem 3.1. *For $\Gamma_1(t)$, $\Gamma_2(t)$ simple closed curves moving by curvature flow, the number of points in $\Gamma_1(t) \cap \Gamma_2(t)$ is non-increasing.*

We can prove this by localizing the statement to become a statement about graphical flows. Choose some interval $I \subset \mathbb{R}^2$ so that the curves $\Gamma_i(t)$ are graphical over I for time $[0, \epsilon]$. By shrinking ϵ, we assume that points lying over ∂I do not cross. To show this, the same proof as above shows that the number of components of $\{x \in I : u_1(x, t) - u_2(x, t) > 0\}$ is non-increasing.
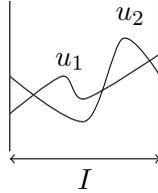
FIGURE 12. Applying the maximum principle to the number of crossings of a graphical flow of curves.

4. SHORT-TIME EXISTENCE AND SMOOTHING

For M^m a closed manifold and $\varphi : M \to \mathbb{R}^n$ a C^1 immersion (more generally, we could consider an immersion into a smooth Riemannian manifold). Then, there is a solution of mean curvature on some time interval $F_\varphi : M \times [0, \epsilon] \to \mathbb{R}^n$, with $F_\varphi(\cdot, 0) = \varphi(\cdot)$, which is an element of $C^1(M \times [0, \epsilon])$ and $C^\infty(M \times (0, \epsilon])$. Furthermore, if $\varphi_i \to \varphi$ in C^1, then ϵ can be chosen uniformly, and $F_{\varphi_i} \to F_\varphi$ in $C^1(M \times [0, \epsilon]) \cap C^\infty(M \times (0, \epsilon])$.

5. LONG TERM BEHAVIOR OF MEAN CURVATURE FLOW

Theorem 5.1. *Suppose that $t \in [0, \infty) \to \Gamma(t) \subset N$ is a curve shortening flow in a compact Riemannian manifold. Let $t_i \to \infty$ and set $\Gamma_i(t) = \Gamma(t + t_i)$. Passing to a subsequence, the Γ_i's converge smoothly to a closed geodesic.*

To see that the curve shortening flow can have a non-unique limit, one may consider $\mathbb{R}^2 \times S^1$ with the warped product metric, chosen so that the length of S^1 is f, where f is as in Figure 5. One may check that if $p(t) \times S^1$ solves curve shortening flow, then $p(t)$ flows along the gradient flow of f. This shows that we may have non-unique asymptotic limits.

Proof. We have that

$$L'(\Gamma(t)) = -\int_{\Gamma(t)} k^2 < 0,$$
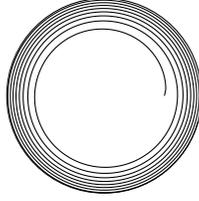
FIGURE 13. It is possible to construct a function $f : \mathbb{R}^2 \to \mathbb{R}$ so that the gradient flow of f spirals into an entire circle (and in particular does not have a unique limit as $t \to \infty$).

so $L(\Gamma(t))$ is decreasing and thus has a limit. For $a < b$, we have

$$L(\Gamma(t_i + b)) - L(\Gamma(t_i + a)) = -\int_{t_i+a}^{t_i+b} \int_{\Gamma(t)} k^2 ds dt$$

$$= -\int_a^b \int_{\Gamma_i(t)} k^2 ds dt.$$

This tends to zero as $i \to \infty$, because $L(\Gamma(t))$ has a limit. Hence, passing to a subsequence, we may arrange that

$$\sum_{i=1}^{\infty} \int_a^b \int_{\Gamma_i(t)} k^2 ds dt < \infty,$$

so

$$\int_a^b \sum_{i=1}^{\infty} \left(\int_{\Gamma_i(t)} k^2 ds \right) dt < \infty.$$

Thus, for a.e. $t \in [a, b]$, we see that

$$\sum_{i=1}^{\infty} \int_{\Gamma_i(t)} k^2 < \infty.$$

Hence,

$$\int_{\Gamma_i(t)} k^2 \to 0$$

as $i \to \infty$ for a.e. $t \in [a, b]$. Choosing $T \in [-2, -1]$ so that

$$\int_{\Gamma_i(T)} k^2 ds \to 0.$$

Parametrizing by arc-length, $\Gamma_i(T)$ is uniformly bounded in $W^{2,2}$. Hence $\Gamma_i(T) \rightharpoonup C$ weakly in $W^{2,2}$. The Sobolev inequality shows that $\Gamma_i(T) \to C$ in $C^{1,\alpha}$ for $\alpha < \frac{1}{2}$. From this, it is a simple exercise to show that C is a geodesic.

Now, for $T' > T$, we consider $F_i : S^1 \times [T, T'] \to N$. By smooth dependence on initial conditions, we see that $F_i \to C$ in $C^1(S^1 \times [T, T']; N)$ and the convergence is smooth on $(T, T') \times S^1$. This completes the proof. $\square$

Naturally, we would like to ask about higher dimensions. Suppose that $t \in [0, \infty) \to M(t)$ is an m dimensional surface. Pick $t_i \to \infty$ and set $M_i(t) := M(t + t_i)$. An identical argument shows that for a.e. t,

$$\int_{M_i(t)} H^2 dA \to 0.$$

However, the Sobolev inequality can't help us anymore!

This leads to the need for a new class of limiting surfaces. We can make sense of $M(t) \to \tilde{M}(\cdot)$ in the sense of "integral varifolds," and can show that the limit surface has zero (weak) mean curvature, and is thus a "stationary integral varifold."

Question 5.2. Even if $\tilde{M}$ is smooth, can we show that the convergence is smooth? (We only know that this is true in the case of a multiplicity 1 limit).

6. Renormalized mean curvature flow

First, think of $M^2 \subset \mathbb{R}^N$, a minimal variety. At a singularity $0 \in M$, pick $\lambda_i \to \infty$ and consider the dilated surfaces $\lambda_i M$. The limit can be seen to be a minimal cone, which is the motivation for their study. Alternatively, we could consider polar coordinates

$$\mathbb{R}^N \ni x \mapsto \left(\frac{x}{|x|}, |x|\right) \in \mathbb{S}^{N-1} \times (0, \infty),$$

but it is even better to use the conformal map

$$\mathbb{R}^N \ni x \mapsto \left(\frac{x}{|x|}, -\log|x|\right) \in \mathbb{S}^{N-1} \times \mathbb{R},$$

where we can easily see that dilation turns into translation.

Now, let us consider mean curvature flow. We want to do this for the spacetime, i.e., for $t \mapsto \Gamma(t)$, we think of the set

$$\mathcal{M} := \bigcup_t (\Gamma(t) \times \{t\}) \subset \mathbb{R}^n_x \times \mathbb{R}_t.$$

The natural dilation is $\mathscr{D}_\lambda(x, t) = (\lambda x, \lambda^2 t)$. What happens to the limit of $\mathscr{D}_\lambda \mathcal{M}$? Again, it is natural to change coordinates

$$\mathbb{R}^n \times (-\infty, 0) \ni (x, t) \mapsto \left(\frac{x}{\sqrt{2|t|}}, -\frac{1}{2}\log|t|\right) \in \mathbb{R}^n \times \mathbb{R}.$$

As before, parabolic dilation by λ corresponds to translation in the second factor. If $X \mapsto X(t) \in \mathbb{R}^n$ is a solution to mean curvature flow, i.e.,

$$\frac{\partial X}{\partial t} = H,$$

then we set

$$\hat{X} = \frac{1}{\sqrt{-2t}} X, \tau = -\frac{1}{2}\log(-t),$$

so

$$\hat{X} = \frac{1}{\sqrt{2}} e^\tau X.$$

Hence,

$$\frac{\partial \hat{X}}{\partial \tau} = \hat{X} + H(\hat{X}).$$

In other words, we see that renormalized mean curvature flow has velocity $\vec{H} + \vec{x}$. To turn this into a normal flow, we must take the velocity $\vec{H} + x^\perp$.

The flow has a "superavoidance" property: disjoint flows move apart exponentially fast.

Example 6.1. For a m-sphere of $r(t)$ centered at the origin flowing by renormalized mean curvature flow, $r(t)$ satisfies

$$\frac{dr}{dt} = -\frac{m}{r} + r.$$

So, a static sphere solves $r = \sqrt{m}$.

Suppose that $\phi : \mathbb{R}^n \to \mathbb{R}$ is smooth. Then

$$\frac{d}{dt} \int_{M(t)} \phi dA = \int \left((\nabla \phi)^\perp \cdot \vec{v} - \phi \vec{H} \cdot \vec{v} \right) dA$$

$$= \int (-\vec{H} \phi + (\nabla \phi)^\perp) \cdot \vec{v} dA$$

$$= \int (-\vec{H} \phi + (\nabla \phi)^\perp) \cdot (\vec{H} + x^\perp) dA$$

$$= \int \phi \left(-\vec{H} + \left(\frac{\nabla \phi}{\phi} \right)^\perp \right) \cdot (\vec{H} + x^\perp) dA.$$

Hence, if we choose ϕ with $\frac{\nabla \phi}{\phi} = -x$, i.e. $\phi = K e^{-\frac{1}{2}|x|^2}$ for K some normalizing constant,[1] then

$$\frac{d}{dt} \int_{M(t)} \phi \, dA = - \int \phi |\vec{H} + x^\perp|^2 dA.$$

This is known as "Huisken's monotonicity formula," [Hui90]. It is the analogue of monotoniciity in minimal surface theory.

Theorem 6.2. *For renormalized mean curvature flow, $v = \vec{H} + x^\perp$, this is the same flow as $v = e^{-\frac{|x|^2}{2m}} \tilde{H}$ where $\tilde{H}$ is the mean curvature with respect to*

$$\widetilde{ds} = e^{-\frac{|x|^2}{2m}} ds.$$

In other words, it is almost (but not quite) mean curvature flow for a weighted metric.

Now, we ask what happens for curve shortening flow, i.e., what happens if if we take a curvature flow and renormalize it around a point in the spacetime track $\mathcal{M}$? We obtain a renormalized curve shortening flow $[0, \infty) \ni t \to \Gamma(t)$.

Claim 6.3. *We have that $\Gamma(t) \cap \partial B(0,1) \neq \emptyset$.*

To see this, note that if $\Gamma(t)$ is entirely inside of the sphere, then it shrinks away in finite time, but it if it is entirely outside, it must flow off exponentially fast, contradicting the fact that it comes from a spacetime point.

Now, because the density $\int_{\Gamma(t)} \phi \, dA$ is non-increasing, we see that

$$\lim_{t \to \infty} \int_{\Gamma(t)} \phi \, dA = \Theta$$

exists. Choose $t_i \to \infty$ and set $\Gamma_i(t) = \Gamma(t + t_i)$. For $a < b$, we have that

$$\int_{\Gamma_i(b)} \phi \, dA - \int_{\Gamma_i(a)} \phi \, dA = \int_{t_i+a}^{t_i+b} \int_{\Gamma(t)} |H + x^\perp|^2 \phi \, ds dt \to 0.$$

Passing to a subsequence,

$$\sum_{i=1}^{\infty} \int_b^a \int_{\Gamma_i(t)} |H + x^\perp|^2 \phi ds dt < \infty.$$

Hence

$$\int_{\Gamma_i(t)} |H + x^\perp|^2 \phi ds \to 0,$$

----

[1] It is nice to choose $K = (2\pi)^{-\frac{n}{2}}$ so that $\int_{\mathbb{R}^m \times \{0\}} \phi dA = 1$.

for a.e., t. Because $\Gamma_i(t)$ is thus locally bounded in $W^{2,2}$, hence, $\Gamma_i(t) \rightharpoonup C(t)$ locally weakly in $W^{2,2}$ and locally in $C^{1,\alpha}$ for $\alpha < \frac{1}{2}$. The limit $C(t)$ satisfies $H = -x^\perp$. In other words, C is a geodesic for

$$g_{ij} = e^{-\frac{|x|^2}{2}}\delta_{ij}.$$

What are these geodesics? If C passes through the origin, then it is a straight line. A circle of the correct radius is also a geodesic. U. Abresch and J. Langer [AL86] have classified the geodesics. The only closed embedded geodesic is the circle, but there are many immersed examples.
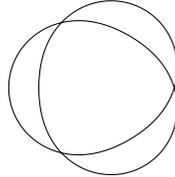

FIGURE 14. One of the Abresch–Langer immersed shrinkers.

Note that an embedded curve cannot converge to one of the examples that cross themselves. But we could obtain a line with multiplicity. In general, we know that the $C(t)$ are one of the Abresch–Langer curves. If $C(t)$ is compact, then it is unique, by existence/uniqueness of parabolic PDE's and smooth dependence on initial conditions. Note that in this case, we may conclude that the singularity was Type I, i.e. $\sqrt{t}|A|$ was bounded in the un-renormalized flow.

This argument breaks down in the non-compact setting. Suppose that $c_i : [0,T] \to \mathbb{R}^2$ is a sequence of curvature flows so that $c_i(0)$ converges in C^∞_{loc} to a line L of multiplicity m. This does not imply C^∞_{loc} convergence to a multiplicity m line at any later times.

Example 6.4. The grim reaper of speed 1, described by $y = t - \log(\cos x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ yields such an example. Dilating in spacetime by ϵ yields the equation

$$y = \frac{t}{\epsilon} - \epsilon \log\left(\cos\frac{x}{\epsilon}\right),$$

which is moving upwards with speed $1/\epsilon$. As $\epsilon \to 0$, this smoothly converges to: a double density line, for $t < 0$, a double density ray, for $t = 0$, or the empty set, for $t > 0$.

7. THE LEVEL SET APPROACH TO WEAK LIMITS

We use the Hausdorff convergence of closed sets in spacetime. For $K_i, K \subset \mathbb{R}^N$ closed, we say that $K_i \to K$ in the Hausdorff distance if every $x \in K$ is the limit of points $x_i \in K_i$ and if given any $x_i \in K_i$, any subsequential limit of $\{x_i\}$ must be in K. Equivalently, we can require that $\mathrm{dist}(\cdot, K_i) \to \mathrm{dist}(\cdot, K)$ uniformly on compact sets. We leave it as an exercise to show that for K_i a sequence of closed sets in $\mathbb{R}^N$ then there is a convergent subsequence (one may use Arzelá–Ascoli applied to the distance functions).

Theorem 7.1. *For Γ_i closed subsets of $\mathbb{R}^2 \times [0,T]$ traced out by a curvature flow on $[0,T]$. Suppose that $\Gamma_i(0) \to L$ (to a line L or a subset of L) in the Hausdorff sense, then any subsequential limit of Γ_i is contained in $L \times [0,T]$.*

Proof. Pick a disk $D_i \cap \Gamma_i(0) = \emptyset$, $D_1 \subset D_2 \subset \ldots$ an exhaustion of either side of $\mathbb{R}^2 \setminus L$ by open sets with smooth boundary curves. Let each ∂D_i move by curvature flow, and trace out a paraboloid ∂P_i in spacetime, for P_i a closed set in spacetime. By the maximum principle, $\Gamma_i \cap P_i = \emptyset$. Hence Γ does not intersect the union of the regions P_i. Applying the same argument to the other side finishes the proof. $\square$

Now, let us examine the consequence for a blowup sequence to obtain the renormalized curvature flow from a regular curvature flow. Suppose that $\Gamma_i(t)$ converges to a line L with multiplicity m. Then, the same thing (with the same line) is true for $t \geq T$ as well! Up until now, we didn't know that the line could not rotate. What about the multiplicity?

We have seen that

$$\int_{\Gamma(t)} \phi \, ds \to \Theta$$

as $t \to \infty$. Then, for each i,

$$\int_{\Gamma_i(t)} \phi \, ds \to \Theta.$$

Hence, if $\Gamma_i(t) \to C(t)$ at $t = T$, then

$$\int_{C(t)} \phi \, ds \leq \lim_{i \to \infty} \int_{\Gamma_i(t)} \phi \, ds = \Theta.$$

It would be possible to argue that we have equality here by showing that nothing is lost in the limit (this would also work in higher dimensions). However, (at least for embedded curves) we may use a different argument. We will give a different argument:

Consider $\Gamma(t)$ and $\partial B(0,1)$, which are solutions to renormalized curvature flow. We know that the number of intersections in $\Gamma(t) \cap \partial B(0,1)$ is non-increasing in time. This implies that the number of components of $\Gamma(t) \cap B(0,1)$ is non-increasing in time. Thus, it has some limit, must be the multiplicity m. This shows that m is independent of the sequence!

Now, we carefully consider the case of $m = 1$.

Theorem 7.2. *Suppose that $\Gamma_i(t)$, $0 \leq t \leq T$ are curvature flows in $B(0, R_i)$, with $R_i \to \infty$. Assume that $\Gamma_i(0)$ converges in C^1 to a multiplicity 1 line L. Then, Γ_i converges to L with multiplicity 1.*

Proof. We assume that the line L is $\mathbb{R} \times \{0\}$.

Step 1: Let $a < \infty$. We claim that there is $I < \infty$ so that for $i \geq I$ and $t \in [0, T]$, $\Gamma_i(t) \cap [-a, a]^2$ is a graph. To see this, let $0 < \epsilon < a$. By Theorem 7.1, there is I sufficiently large so that $\Gamma_i(t) \cap [-a, a]^2$ is contained in an ϵ-neighborhood of L. Let $S = \{x\} \times [-a, a]$ for some $|x| \leq a$. Then, the number of elements in $\Gamma_i(t) \cap S$ is non-increasing with time. However, there is only one element when $t = 0$. Hence, $\#(\Gamma_i(t) \cap S) \leq 1$. The intermediate value theorem then shows that $\#(\Gamma_i(t) \cap S) = 1$ for all t.

Step 2: We claim that the slope of the graph tends to zero uniformly on $B(0, R) \times [0, T]$. To see this, we may rotate the picture by any angle so that L is not vertical. We may still consider the same lines $\{x\} \times \mathbb{R}$; for i large, $\Gamma_i(0)$ will intersect this line exactly once, by the C^1 convergence to L. The same argument that we have just used proves the claim. $\qquad\square$

Theorem 7.3. *Under the same hypothesis, the curves Γ_i converges to L uniformly on $B(0, R) \cap [\delta, T]$ for all $\delta > 0$ and $R > 0$.*

First proof. This follows from parabolic Schauder estimates. $\qquad\square$

Second proof. Suppose not. Then, there is a sequence of points $p_i \in \Gamma_i(t_i)$ with $\delta \leq t_i < T$ with $|p_i|$ bounded so that

$$\limsup_{i \to \infty} \kappa_i(p_i, t_i) > 0,$$

where $\kappa_i(p_i, t_i)$ is the curvature of $\Gamma_i(t_i)$ at p_i. We may pass to a subsequence so that $p_i \to p$, $t_i \to t$ and $\kappa_i(p_i, t_i) > \frac{1}{r} > 0$. Let C_i be a circle of radius r tangent to $\Gamma_i(t_i)$ at p_i. Because the curvature of $\Gamma_i(t_i)$ at p_i is strictly greater than $\frac{1}{r}$, we have that $\#(\Gamma_i(t_i) \cap C_i) \geq 3$. Let $\tilde{C}_i$ be the circle at $t = 0$ which flows to C_i at time t_i. We have that $\#(\tilde{C}_i \cap \Gamma_i(0)) \geq 3$. However, letting $i \to \infty$,
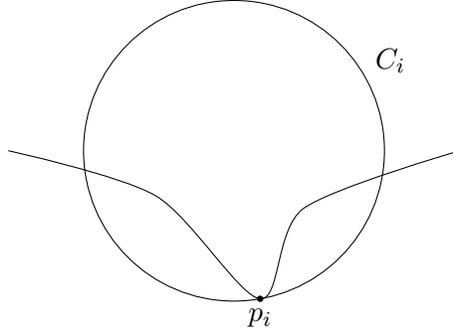
FIGURE 15. The circle C_i must intersect $\Gamma_i(t)$ in at least 3 points.

$\Gamma_i(0)$ converges to a line, and $\tilde{C}_i$ converges to some circle which intersects L but is not tangent to it. This is a contradiction. $\qquad\square$

So, we have seen that rescalings of embedded curvature flow converge to a multiplicity one circle (which is what we would like always to happen), a multiplicity one line, or a line of multiplicity at least 2. Note that the multiplicity one line always occurs at a regular point in spacetime. The converse is also true! Indeed, this follows from what we've just proven.

In particular, for $\Gamma(t)$ a curvature flow in $\mathbb{R}^2$ for $a \leq t \leq 0$. Let $\lambda_i \to \infty$ and consider resclaing by $\mathcal{D}_{\lambda_i}(x,t) = (\lambda_i x, \lambda_i^2 t)$. Assume that for almost every $t < 0$, $\Gamma_i(t)$ converges in C^1 to a multiplicity one line. We may assume this happens for $t = -1$. We've seen that the curvature tends to zero uniformly on $B(0,R) \times [-1,0]$.

To sum up, in order to prove Grayson's theorem, we must rule out multiplicity ≥ 2 lines for tangent flows to embedded curves. The first step in this direction is

Lemma 7.4. *For $\Gamma_i(t)$, $0 \leq t \leq T$ curvature flows in $B(0, R_i)$ with $R_i \to \infty$. Suppose that $\Gamma_i(t)$ converges to some line L with multiplicity m, for $t = 0$ and $t = T$. Then, for $a < \infty$, for all i sufficiently large, $\Gamma_i(t) \cap [-a,a]^2$ is the union of m graphs for all $t \in [0,T]$.*

We emphasize that $\Gamma_i(T)$ converging to L with multiplicity m is *crucial*, as can be seen by a sequence of grim reapers, which converge to a multiplicity two line at $t = 0$, but to the empty set at e.g., $t = 1$.

Proof. Assume that $L = \mathbb{R} \times \{y\}$. As before, we consider the intersection with segments $\{x\} \times \mathbb{R}$. Because $\#(\Gamma_i(t) \cap S)$ is m at $t = 0$ and T, for i large, it is thus constant. $\qquad\square$

This allows us to separate $\Gamma_i(t) \cap B(0,R)$ into curves converging to L, each with multiplicity one!

We now assume that the multiplicity of the limit line is $m = 2$ (the general case follows similarly). For t close to 0, choose $p_1(t), p_2(t)$ in each curve which are closest to the origin. Let

$$\delta(t) = \frac{|p_1(t) - p_2(t)|}{\sqrt{|t|}}.$$

By hypothesis, $\delta(t) \to 0$ as $t \nearrow 0$ (this is because the renormalized flow is converging to multiplicity two line). Hence, we may choose $t_m \nearrow 0$ with

$$\delta(t_m) = \sup_{t \geq t_m} \delta(t).$$

Let $\lambda_n = \frac{1}{\sqrt{|t_n|}}$. We have seen that $\mathcal{D}_{\lambda_m}\mathcal{M}$ converges to $L \times (-\infty, 0)$ with multiplicity 2. Hence, there is $\Omega_1 \subset \Omega_2 \subset \cdots \subset L \times (-\infty, 0)$ with $\cup_i \Omega_i = L \times (-\infty, 0)$ and $I_1 \subset I_2 \subset \cdots \subset \mathbb{R}$ with $\cup_i I_i = \mathbb{R}$ so that

$$\mathcal{M} \cap (\Omega_i \times I_i)$$

is given by the graph of two functions $u_i, v_i : \Omega_i \to I_i$ with $u_i > v_i$. Note that u_i, v_i solve the non-parametric curvature flow equation, e.g.

$$u_t = \sqrt{1 + |Du|^2} D_i \left(\frac{D_i u}{\sqrt{1 + |Du|^2}} \right) = \Delta u - \frac{D^2 u (Du, Du)}{1 + |Du|^2}.$$

Note that in the present one-dimensional setting, this actually takes the form

$$u_t = k - \frac{|Du|^2}{1 + |Du|^2} k.$$

Subtracting the equations for u_i, v_i, we may check that

$$(u_i - v_i)_t = (\delta_{kl} + a_{kl}^{(i)}) D_{ij} D_{kl}(u_i - v_i) + b_k^{(i)} D_k(u_i - v_i) + c^{(i)}(u - v)$$

where the coefficients $a_{kl}^{(i)}, b_k^{(i)}, c^{(i)}$ are all converging smoothly to zero as $i \to \infty$.

We may normalize the graphs by setting

$$w_i(x, t) := \frac{u_i(x, t) - v_i(x, t)}{u_i(0, -1) - v_i(0, -1)}.$$

Note that $w_i(0, t) \sim |p_i(t) - p_2(t)|$ (the reason that these two are not equal is that the $p_i(t)$ might not lie exactly on $\{x = 0\}$. Thus,

$$\frac{w_i(0, t)}{\sqrt{|t|}} \leq (1 + \epsilon_i) w_i(0, -1)$$

for $-1 \leq t < 0$ and some $\epsilon_i \to 0$. This, along with the parabolic Harnack inequality (which roughly says that if the solution to a parabolic equation is small at a point in space-time, then it cannot be too big at an earlier point in space-time) implies that w_i is uniformly bounded on compact subsets of $\mathbb{R} \times (-\infty, 0)$. Thus, $w_i \to w$ smoothly on compact sets and w satisfies the heat equation

$$w_t = \Delta w.$$

Moreover, we have arranged that

$$\frac{w(0, t)}{\sqrt{|t|}} \leq w(0, -1) = 1$$

for $-1 \leq t < 0$. However, this violates the strong maximum principle. For example, we can choose $\epsilon > 0$ so that

$$\epsilon e^{-1} \cos x \leq w(x, -1)$$

for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Because $\epsilon e^{-t} \cos x$ solves the heat equation, this violates the maximum principle.

Remark 7.5. As a historical remark, we note that the arguments presented here are not Grayson's original proof [Gra87]. The proof we have given here is probably simpler than Grayson's. There are now very short alternate proofs [Hui98, AB11a, AB11b]. However, the argument here generalizes to mean convex mean curvature flow in higher dimensions.

We also note that the arguments here also work for immersed curves, showing that a singularity in spacetime which is not near any points of self-intersection must be a shrinking circle.

8. WEAK COMPACTNESS OF SUBMANIFOLDS

If we want to repeat these arguments in higher dimensions, we are naturally led to trying to take the limit of submanifolds under some weak curvature bounds. To be concrete, suppose that M_i is a sequence of m-manifolds in $\mathcal{U} \subset \mathbb{R}^N$, an open set. Assume that the areas of M_i are locally uniformly bounded. We may later assume that

$$\int_{M_i} |H|^2$$

is also uniformly bounded but this will not be important for now. We ask if it is possible to understand a weak limit of the M_i.

The simplest possibility is as follows: note that any m-submanifold of $\mathcal{U}$, M, determines a Radon measure μ_M by

$$\mu_M(S) = \mathcal{H}^m(M \cap S).$$

Equivalently, for any compactly supported continuous function f, we set

$$\int f d\mu_M = \int_M f d\mathcal{H}^m.$$

Hence, for the M_i as before, we may pass to a subsequence so that $\mu_i \rightharpoonup \mu$ weakly.

This is quite a coarse procedure (as we will see later), and we would like a more refined definition. An important observation is that M actually defines a Radon measure on $\mathcal{U} \times G(m, N)$, where $G(m, N)$ is the Grassmannian of m-dimensional subspaces in $\mathbb{R}^N$. We define the measure V_M:

$$\int f dV_M = \int_M f(x, \mathrm{Tan}(M, x)) d\mathcal{H}^m$$

for $f : \mathcal{U} \times G(m, N) \to \mathbb{R}$ a continuous function of compact support. Alternatively, we have

$$V_M(S) = \mathcal{H}^m(\{x \in M : (x, \mathrm{Tan}(M, x)) \in S\}).$$

We can take a subsequence so that $V_{M_i} \rightharpoonup V$. Note that for $\pi : \mathcal{U} \times G(m, N) \to \mathcal{U}$ the projection map, we can check that $\pi_* V_M = \mu_M$.

Definition 8.1. An m-dimensional *varifold* in $\mathcal{U} \subset \mathbb{R}^N$, an open set, is a Radon measure on $\mathcal{U} \times G(m, N)$.

8.1. **Examples of varifold convergence.** We give several examples below.

$1/2n$

FIGURE 16. Denote M_n the union of every-other interval of length $\frac{1}{2n}$.

For M_n as in Figure 16, we see that $\mu_{M_n} \rightharpoonup \frac{1}{2}\mu_{[0,1]}$, and $V_{M_n} \rightharpoonup \frac{1}{2}V_{[0,1]}$.

FIGURE 17. Denote M_n the union of n intervals of height $1/n$.

The M_n in Figure 17 satisfy $\mu_{M_n} \rightharpoonup \mu_{[0,1]}$. But we may see that $V_{M_n} \not\rightharpoonup V_{[0,1]}$.

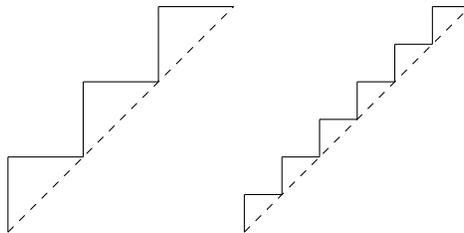
FIGURE 18. Denote M_n the zig-zag curve which is approaching the diagonal.

The M_n in Figure 18 have $\mu_{M_n} \rightharpoonup \sqrt{2}\mu_{\mathrm{diag}}$ but there is no α so that $V_{M_n} \rightharpoonup \alpha V_{\mathrm{diag}}$.

8.2. **The pushforward of a varifold.** We remark here that the pushforward by a C^1 compactly supported diffeomorphism $f : \mathcal{U} \to \mathcal{U}$ does not respect the measures μ_M: i.e., it may be that $f_{\#}\mu_M \neq \mu_{f(M)}$. For example, if M is a circle, and f shrinks the circle to a smaller radius, $f_{\#}\mu_M$ will have the same total mass as μ_M, but $\mu_{f(M)}$ will not. However, we can easily define the pushforward of a varifold $f_{\#}V$ and check that for $f_{\#}V_M = V_{f(M)}$ (this is essentially just the change of variables formula with the Jacobian).

8.3. **Integral varifolds.** The class of general varifolds will be way to general and will include numerous pathological examples. We thus would like to define a smaller class.

Lemma 8.2. *Suppose that M, M' are m-dimensional C^1 submanifolds of $\mathcal{U}$. Let*

$$Z = \{x \in M \cap M' : \mathrm{Tan}(M, x) \neq \mathrm{Tan}(M', x)\}.$$

Then, $\mathcal{H}^m(Z) = 0$.

Proof. If M, M' are hypersurfaces, then Z is an $(m-1)$-dimensional C^1-submanifold, by transversality. In higher co-dimension, one may show that $M \cap M'$ is contained in a $(m-1)$-dimensional C^1 submanifold, by projecting onto a lower dimensional space. $\qquad\square$

Corollary 8.3. *Suppose that $S \subset \cup_i M_i$ and $S \subset \cup_i M_i'$ for M_i, M_i', m-dimensional C^1 submanifolds of $\mathcal{U}$. Define T on S by*

$$T(x) = \mathrm{Tan}(M_i, x),$$

where i is the first i so that $x \in M_i$, i.e., $x \in M_i \setminus \cup_{j<i} M_j$. Define T' similarly. Then

$$T(x) = T'(x)$$

for a.e. $x \in S$.

Thus, for S a Borel subset of a C^1 submanifold $M \subset \mathcal{U}$, we define V_S to be the varifold given by

$$\int f dV_S = \int_S f(x, T(M, x)) d\mathcal{H}^m.$$

This does not depend on the choice of M, by the previous corollary. This allows us to give the following two equivalent definitions

Definition 8.4. An *integral m-varifold* is one that can be written as

$$V = \sum_{i=1}^{\infty} V_{S_i}$$

Definition 8.5. Suppose that $\theta \in \mathscr{L}^1_{\mathrm{loc}}(\mathcal{U}; \mathbb{Z}^+; \mathcal{H}^m)$ and $S = \{\theta > 0\} \subset Z \cup (\cup_i M_i)$ where Z has $\mathcal{H}^m(Z) = 0$ and the M_i are m-dimensional C^1 submanifolds of $\mathcal{U}$. This data defines an *integral m-varifold V_θ* by

$$\int f dV_\theta = \int f(x, T(x))\theta(x) d\mathcal{H}^m(x) = \sum_i \int_{M_i \setminus \cup_{j<i} M_j} f(x, \mathrm{Tan}(M_i, x))\theta(x) d\mathcal{H}^m.$$

Note that to relate the two definitions, we can easily see that $\theta = \sum \mathbf{1}_{S_i}$.

8.4. **First variation of a varifold.** First, we recall

Theorem 8.6 (Divergence theorem). *Suppose that M is a C^2-manifold in $\mathcal{U}$ and X a compactly supported C^1 vector field in $\mathcal{U}$. We set*

$$\mathrm{div}_M X = \sum_i \nabla_{e_i} X \cdot e_i,$$

for e_i an orthonormal basis for $\mathrm{Tan}(M, x)$. Then,

$$\int_M \mathrm{div}_M X = \int_M \mathrm{div}_M X^\perp + \int_M \mathrm{div}_M^T$$

$$= -\int_M X \cdot H d\mathcal{H}^m + \int_{\partial M} X \cdot \nu d\mathcal{H}^{m-1}.$$

Note that the right hand side makes sense if X is just in C^1. If there is a distributional vector field H making this true, then we say that H is the *weak mean curvature*.

Now, for V an m-varifold, we define the *first variation of V* by

$$\delta V(X) = \int \mathrm{div}_T X dV(x, T)$$

where

$$\mathrm{div}_T X = \sum_i \nabla_{e_i} X \cdot e_i$$

for e_i an orthonormal basis for T. If V is an integral varifold, this can be written as

$$\delta V(X) = \int \mathrm{div}_{T(V,x)} X d\mu_V.$$

Trivially, if $V_i \rightharpoonup V$, then $\delta V_i(X) \to \delta V(X)$. Suppose that we have local bounds on the first variation in the form

$$|\delta V(X)| \le C_K \|X\|_0$$

for $\mathrm{supp}\, X \subset K \Subset \mathcal{U}$. Then the Riesz representation theorem implies that there is a Radon measure λ on $\mathcal{U}$ and a λ-measurable unit vector Λ such that

$$\delta V(X) = \int X \cdot \Lambda d\lambda.$$

Decomposing λ with respect to μ_V, there is $\lambda_{ac} \ll \mu_V$ and λ_{sing} so that

$$\delta V(X) = \int X \cdot \Lambda \, d\lambda_{ac} + \int X \cdot \Lambda \, d\lambda_{sing}$$

$$= \int X \cdot \Lambda \underbrace{\frac{d\lambda_{ac}}{d\mu_V}}_{:=-H} d\mu_V + \int X \cdot \underbrace{\Lambda}_{:=\nu} d\lambda_{sing}.$$

Thus, in the case that V has locally bounded first variation, we have

$$\delta V(X) = -\int X \cdot H d\mu_V + \int X \cdot \nu d\lambda_{sing}.$$

The following deep theorem due to Allard [All72] is the reason that the class of integral varifolds is a reasonable one to study.

Theorem 8.7 (Allard's compactness theorem)**.** *Suppose that $V_i \rightharpoonup V$ is a sequence of integral varifolds converging weakly to a varifold V. If the V_i have locally uniformly bounded first variation, i.e., there is C_K independent of i so that for $\mathrm{supp}\, X \subset K \Subset \mathcal{U}$, we have*

$$|\delta V_i(X)| \le C_K \|X\|_0,$$

then V is also an integral varifold.

Note that in the theorem, we trivially obtain the bounds

$$|\delta V(X)| \le C_K \|X\|_0.$$

For example, a sequence of hypersurfaces satisfy the hypothesis of Allard's theorem if and only if

$$\int_K |H_i| d\mu_{M_i} + \int_K d\sigma_i \leq K$$

where $d\sigma_i$ is the boundary measure for M_i.

Example 8.8. Note that the quantities "$|H|$" and "$d\sigma$" can get "mixed up" in the limit. For example consider a sequence of ellipses converging to a line with multiplicity two Note that $\int |H_i| =$
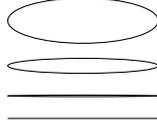

FIGURE 19. Ellipses converging to a line of multiplicity two.

2π and $\sigma_i = 0$ but in the limit, $H = 0$ but $\sigma \neq 0$. Conversely, a sequence of polygons converging to a circle has $H = 0$ but nontrivial boundary measure (at the vertices), but the circle only has mean curvature and no boundary.

Theorem 8.9. *Suppose that for V_i integral varifolds, we have $V_i \rightharpoonup V$ and that V_i has locally bounded first variation and no generalized boundary. Equivalently, we are assuming that*

$$\delta V_i(X) = -\int X \cdot H_i d\mu_{V_i}.$$

Assume that

$$\int_K |H_i|^2 d\mu_i \leq C_K < \infty$$

for $K \Subset \mathcal{U}$. Then

(1) V is an integral varifold
(2) We have

$$\int H_i \cdot X d\mu_i \to \int H \cdot X d\mu$$

Furthermore, if X is a continuous vector field with compact support in $\mathcal{U} \times G(m, N)$, then

$$\int H_i \cdot X(x, T_{V_i}(x)) d\mu_i(x) \to \int H \cdot X(x, T_V(x)) d\mu_V.$$

We note that any L^p for $p > 1$ could replace L^2 here.

Proof. Local bounds for H in L^2 imply local bounds in L^1. Thus, V is an integral varifold, by Allard's theorem. Note that

$$\delta V_i(X) = -\int_K H_i \cdot X \leq \left(\int_K |H_i|^2\right)^{\frac{1}{2}} \left(\int_K |X|^2\right)^{\frac{1}{2}} \leq C_K^{\frac{1}{2}} \left(\int_K |X|^2\right)^{\frac{1}{2}}.$$

Thus,

$$|\delta V(X)| \leq C_K^{\frac{1}{2}} \left(\int_K |X|^2\right)^{\frac{1}{2}}.$$

From this, the rest of the claims follow easily from this, because the Riesz representation theorem implies that $H \in \mathcal{L}^2_{loc}(d\mu_V)$. □

9. Brakke flow

We now discuss Brakke's weak mean curvature flow [Bra78].

Definition 9.1. An m-dimensional integral Brakke flow in $\mathcal{U} \subset \mathbb{R}^N$ is a one-parameter family of Radon measures on $\mathcal{U}$, $[a,b] \ni t \mapsto \mu(t)$ so that

(1) for a.e. t, $\mu(t) = \mu_{V(t)}$ for an integral m-dimensional varifold $V(t)$ in $\mathcal{U}$, so that

$$\delta V_t(X) = -\int H(x) \cdot X d\mu_t$$

for some $H(t) \in \mathcal{L}^2_{loc}(\mu(t))$,
(2) If $f \in C^1_c(\mathcal{U} \times [a,b])$ has $f \geq 0$, then

$$\int f(\cdot, b) d\mu_b - \int f(\cdot, a) d\mu_a \leq \int_a^b \int \left(-|H|^2 f + H \cdot \nabla f + \frac{\partial f}{\partial t} \right) d\mu_t dt.$$

Remark 9.2. We note that if M_t is a smooth mean curvature flow, then

$$\frac{d}{dt} \int_{M_t} f dA = \int_{M_t} \left((-H \cdot v)f + \nabla^\perp f \cdot v + \frac{\partial f}{\partial t} \right) dA$$

$$= \int_{M_t} \left(-|H|^2 f + H \cdot \nabla f + \frac{\partial f}{\partial t} \right),$$

where the first equality holds for any smooth flow with velocity v. An obvious question is why we require the inequality, rather than equality in the definition of Brakke flow. The reason for this is that only the inequality is possibly preserved under weak limits. For example, we have seen that the weak limit of rescaled grim reapers is a multiplicity two line for $t < 0$ and is empty for $t > 0$!

Theorem 9.3. *Suppose that μ_t is an m-dimensional integral Brakke flow. Assume that $\mathcal{U}$ contains $\{|x|^2 \leq r^2\}$. Let $\phi = (r^2 - |x|^2 - 2mt)^+$. Then*

$$\int \phi^4 d\mu_t$$

is decreasing as $t \to \infty$.

Proof. For $f = \frac{1}{4}\phi^4$, we compute

$$\nabla f = \phi^3 \nabla \phi,$$

so

$$\text{div}_M(\nabla f) = 3\phi^2 |\nabla \phi|^2 + \phi^3 \text{div}_M(\nabla \phi) \geq -2m\phi^3.$$

Moreover,

$$\frac{\partial f}{\partial t} = \phi^3 \frac{\partial \phi}{\partial t} = -2m\phi^3.$$

Thus, using f as a test function in the definition of Brakke flow yields

$$\int f d\mu_b - \int f d\mu_a \leq \int_a^b \int \left(-|H|^2 f + H \cdot \nabla f + \frac{\partial f}{\partial t} \right) d\mu_t dt$$

$$\leq \int_a^b \int \left(-\text{div}_M(\nabla f) + \frac{\partial f}{\partial t} \right) d\mu_t dt$$

$$\leq 0,$$

as desired. $\qquad\square$

Corollary 9.4. *For an integral m-dimensional Brakke flow μ_t on $\mathcal{U} \subset \mathbb{R}^N$ defined on $[a,b]$, for $K \Subset \mathcal{U}$, we have uniform mass bounds, i.e., there is c_K independent of t, so that*

$$\mu_t(K) \leq c_K < \infty$$

for $t \in [a,b]$.

Theorem 9.5. *An integral m-dimensional Brakke flow on $\mathcal{U}$ satisfies*

$$\int_a^b \int_K H^2 d\mu_t dt < \infty$$

for any $K \Subset \mathcal{U}$.

Proof. Suppose that $\phi \in C_c^2(\mathcal{U})$, $\phi \geq 0$, is time independent. Note that $\frac{|\nabla \phi|^2}{\phi}$ is bounded. Hence, because

$$\nabla \phi \cdot H \leq \frac{1}{2} \frac{|\nabla \phi|^2}{\phi} + \frac{1}{2} \phi H^2,$$

we have

$$\int \phi d\mu_a - \int \phi d\mu_b \geq \int_a^b \int (\phi H^2 - \nabla \phi \cdot H) d\mu dt$$

$$\geq \int_a^b \left(\frac{1}{2} \phi H^2 - \frac{1}{2} \frac{|\nabla \phi|^2}{\phi} \right) d\mu_t dt.$$

Hence, rearranging this, we obtain

$$\frac{1}{2} \int_a^b \int \phi H^2 d\mu_t dt \leq \int \phi d\mu_a - \int \phi d\mu_b + \frac{1}{2} \int_a^b \int \frac{|\nabla \phi|^2}{\phi} d\mu_t dt.$$

Now, we may bound

$$\int_a^b \int \frac{|\nabla \phi|^2}{\phi} d\mu_t dt \leq C(\phi) c_{\mathrm{spt}\,\phi}(b-a),$$

so putting this together, we have

$$\frac{1}{2} \int_a^b \phi H^2 d\mu_t dt \leq C(\phi) c_{\mathrm{supp}\,\phi}(1 + b - a),$$

which gives the desired bound. $\square$

Theorem 9.6. *An integral m-dimensional Brakke flow on $\mathcal{U}$ satisfies*

$$\lim_{\tau \nearrow t} \mu(\tau) \geq \mu(t) \geq \lim_{\tau \searrow t} \mu(\tau).$$

In other words, for $\phi \in C_c^0(\mathcal{U})$ with $\phi \geq 0$, we have

$$\lim_{\tau \nearrow t} \int \phi \mu_\tau \geq \int \phi d\mu_t \geq \lim_{\tau \searrow t} \int \phi d\mu_\tau.$$

Proof. First, assume that $\phi \in C_c^2(\mathcal{U})$ (the general case follows by density). Then, we have

$$\int \phi d\mu_d - \int \phi d\mu_c \leq \int_c^d \int (-\phi H^2 + H \cdot \nabla \phi) d\mu_t dt$$

$$\leq \int_c^d \frac{1}{2} \frac{|\nabla \phi|^2}{\phi} d\mu_t dt$$

$$\leq C(\phi) c_{\mathrm{supp}\,\phi}(d-c).$$

Thus,

$$f(t) := \int \phi d\mu_t - C(\phi) c_{\mathrm{supp}\,\phi} t$$

is decreasing in t. This implies that

$$f(t^-) \geq f(t) \geq f(t^+),$$

which finishes the proof, as the linear part of f is continuous. $\qquad\qquad \square$

Note that we have proven

Theorem 9.7. *For an integral m-dimensional Brakke flow on $\mathcal{U}$ and $\phi \in C_c^2(\mathcal{U})$, $\phi \geq 0$, the map*

$$t \mapsto \int \phi \, d\mu_t - C(\phi) c_{\mathrm{supp}\, \phi} t$$

is decreasing.

9.1. A compactness theorem for integral Brakke flows.

Theorem 9.8. *Suppose that $[a, b] \ni t \mapsto \mu_i(t)$ is a sequence of integral Brakke flows. Assume that the local bound on area are uniform, i.e.*

$$\sup_i \sup_{t \in [a,b]} \mu_i(t)(K) \leq c_K < \infty$$

for all $K \Subset \mathcal{U}$. Then, after passing to a subsequence

   (1) *we have weak convergence $\mu_i(t) \rightharpoonup \mu_t$ for all t,*
   (2) *$t \mapsto \mu_t$ is an integral Brakke flow*
   (3) *for a.e. t, after passing to a further subsequence which depends on t, the associated varifolds converge nicely $V_i(t) \to V(t)$.*

Proof. Choose $\phi \in C_c^2(\mathcal{U})$, $\phi \geq 0$. Recall that

$$L_i^\phi(t) = \int \phi \, d\mu_i(t) - C(\phi) c_{\mathrm{supp}\, \phi} t$$

is a decreasing function of t. Passing to a subsequence depending on ϕ, we have that $L_i^\phi(t)$ converges to a decreasing function $L(t)$. Hence,

$$\int \phi \, d\mu_i(t)$$

has a limit, for all t. By repeating this process on a countable dense subset of $C_c^2(\mathcal{U}; \mathbb{R}^+)$, we may arrange that

$$\mu_i(t) \rightharpoonup \mu(t)$$

for all t.

Now, we replace $\mathcal{U}$ by $\mathcal{U}' \Subset \mathcal{U}$, for simplicity. Thus, we may assume that $\mu_i(t)(\mathcal{U}) \leq C < \infty$ independently of i and t. Additionally, we have proven above that

$$\int_a^b H^2 \, d\mu_i(t) dt \leq D < \infty.$$

Let $[c, d] \subset [a, b]$. Then

$$\int \phi \, d\mu_i(c) - \int \phi \, d\mu_i(d) \geq \int_c^d \int \left(\phi H_i^2 - \nabla \phi \cdot H_i - \frac{\partial \phi}{\partial t} \right) d\mu_i(t) dt$$

Thus,

$$\int \phi \, d\mu_i(c) - \int \phi \, d\mu_i(d) + \epsilon D + \int_c^d \int \frac{1}{2\epsilon} |\nabla \phi|^2 d\mu_i(t) dt$$
$$\geq \int_c^d \int \left(\phi H_i^2 - \nabla \phi \cdot H_i + \epsilon H_i^2 + \frac{1}{2\epsilon} |\nabla \phi|^2 - \frac{\partial \phi}{\partial t} \right) d\mu_i(t) dt.$$

Note that

$$\nabla\phi \cdot H_i \leq \frac{1}{2\epsilon}|\nabla\phi|^2 + \frac{\epsilon}{2}H_i^2,$$

so

$$\phi H_i^2 - \nabla\phi \cdot H_i + \epsilon H_i^2 + \frac{1}{2\epsilon}|\nabla\phi|^2 \geq \frac{1}{2}\epsilon H_i^2,$$

which in particular is positive. Now, we may pass to the limit in i and use Fatou's lemma to see that

$$\int \phi d\mu(c) - \int \phi d\mu(d) + \epsilon D + \int_c^d \int \frac{1}{2\epsilon}|\nabla\phi|^2 d\mu(t)dt$$

$$\geq \int_c^d \liminf_i \int \left(\phi H_i^2 - \nabla\phi \cdot H_i + \epsilon H_i^2 + \frac{1}{2\epsilon}|\nabla\phi|^2\right) d\mu_i(t)dt - \int_c^d \int \frac{\partial\phi}{\partial t}d\mu_t dt$$

Thus, for a.e. t we have

$$C(t) := \liminf_i \int \left(\phi H_i^2 - \nabla\phi \cdot H_i + \epsilon H_i^2 + \frac{1}{2\epsilon}|\nabla\phi|^2\right) d\mu_i(t) < \infty.$$

Pass to a subsequence (depending on t!) so that this becomes a limit, rather than a lim. inf. Because the integrand is at least $\frac{\epsilon}{2}H_i^2$, we see that μ_i are integral varifolds with mean curvature uniformly in $\mathcal{L}^2(\mu_i)$. Hence, we can apply the strengthened form of Allard's compactness theorem to pass to a subsequence so that

$$V_i(t) \to V(t),$$

where $V(t)$ is an integral varifold with $H \in \mathcal{L}^2(d\mu_V)$. In particular,

$$\int H_i \cdot X d\mu_{V_i(t)} \to \int H \cdot X d\mu_{V(t)},$$

for $X \in C_c(\mathcal{U}; T\mathcal{U})$ a continuous vector field.

Note that for a.e. t, $V(t)$ is well defined independent of the subsequence depending on t. This is because an integral varifold V is uniquely determined by its associated measure μ_V. However, we emphasize that the convergence of $V_i(t)$ to V as varifolds requires extracting the subsequence depending on t, as we have done above.

Now, returning to $C(t)$, each term converges to what we expect, except for the H_i^2 terms, which might drop in general (by weak convergence). Hence, we see that

$$C(t) \geq \int \left(\phi H^2 - \nabla\phi \cdot H + \epsilon H^2 + \frac{1}{2\epsilon}|\nabla\phi|^2\right) d\mu_t.$$

This inequality goes in the right direction for us to conclude that μ_t is an integral Brakke flow. $\square$

9.2. **Self shrinkers.** By the proof of uniform mass bounds for Brakke flows, we also get the following extension result:

Corollary 9.9. *Suppose that $[0, T) \ni t \mapsto \mu_t$ is an integral m-dimensional Brakke flow on $\mathcal{U} \subset \mathbb{R}^N$. Then $\mu(T^-) := \lim_{t \nearrow T} \mu(t)$ exists.*

From this, we obtain

Corollary 9.10. *Suppose that $M \subset \mathbb{R}^N$ is a m-dimensional self-shrinker, i.e.*

$$(-\infty, 0) \ni t \mapsto \sqrt{|t|}M$$

is a mean curvature flow, or alternatively, M is minimal for the weighted area $\int e^{-\frac{|x|^2}{2}}dA$. Then
   (1) M has

$$\sup_r \frac{\text{area}_{\mathbb{R}^N}(M \cap B_r)}{r^m} < \infty,$$

(2) M is asymptotic to a cone at infinity in a weak sense.

Proof. Let $\mu(t)$ be the Brakke flow $t \mapsto \mu_{\sqrt{|t|}M}$. Then, we have just seen that

$$\lim_{t \nearrow 0} \mu(t) = \mu$$

exists. Note that $\mu(t)$ is just a rescaling of M by $\sqrt{|t|}$, so because the limit is independent of the sequence of $t \nearrow 0$, we see immediately that μ is a cone. Furthermore,

$$\limsup_{t \nearrow 0} \frac{\text{area}\left(M \cap \overline{B(0, |t|^{-\frac{1}{2}})}\right)}{|t|^{-\frac{m}{2}}} = \limsup \mu_{\sqrt{|t|}M}(\overline{B(0,1)}) \leq \mu(\overline{B(0,1)}) < \infty,$$

proving the theorem. $\qquad\square$

9.3. **Existence by elliptic regularization.** We now describe Ilmanen's construction [Ilm94] of Brakke flows by "elliptic regularization."

Theorem 9.11. *Let z denote the height function in $\mathbb{R}^{N+1}$ and $\vec{e}$ the upward pointing unit vector. Then $M \subset \mathbb{R}^{N+1}$ is a critical point for $\int_M e^{-\lambda z} dA$ if and only if*

$$t \mapsto M - \lambda t \vec{e}$$

is a mean curvature flow.

Proof. If $s \mapsto M_s$ is a variation of M with velocity X, we compute

$$\frac{d}{ds} \int_{M_s} e^{-\lambda z} dA = \int (-H + \nabla^\perp(-\lambda z)) \cdot X e^{-\lambda z} dA = \int (-H - \lambda \vec{e}^\perp) \cdot X e^{-\lambda z} dA.$$

On the other hand, note that the flow $t \mapsto M - \lambda \vec{e} t$ has velocity $-\lambda \vec{e}$ and hence normal velocity $-\lambda \vec{e}^\perp$. Comparing these two computations proves the theorem. $\qquad\square$

Now, for Σ a compact m-dimensional surface in $\mathbb{R}^N$, let $M_\lambda \subset \mathbb{R}^{N+1}$ minimize $\int e^{-\lambda z} dA$ subject to the constraint $\partial M_\lambda = \Sigma$. We can show that M_λ exists and in nice situations (e.g., for hypersurfaces of low dimension) is regular except for a small singular set. So, by the above computation $t \mapsto M_\lambda - \lambda t \vec{e}$ is a mean curvature flow.

Our goal is to send $\lambda \to \infty$. We would like to show that these converge to a limit Brakke flow $\mu(t)$ which is translation invariant, i.e. $\mu(t) = \Sigma(t) \times \mathbb{R}$ for $\Sigma(t)$ an m-dimensional Brakke flow in $\mathbb{R}^N$ with $\Sigma(0) = \Sigma$.
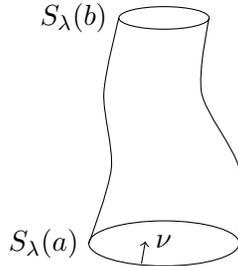
FIGURE 20. The surface $M_\lambda(a, b)$.

Set $M_\lambda(a, b) = M_\lambda \cap \{a < z < b\}$ and set $S_\lambda(z_0) = M_\lambda \cap \{z = z_0\}$ (see Figure 20). Then, we have that, for ν the upward pointing normal vector to $\partial M_\lambda(a, b)$ in $M_\lambda(a, b)$

$$0 = \int_{M_\lambda(a,b)} \text{div}_M(\vec{e}) = \int_{M_\lambda(a,b)} -H \cdot \vec{e} + \int_{S_\lambda(b)} \vec{e} \cdot \nu - \int_{S_\lambda(a)} \vec{e} \cdot \nu$$

$$= \int_{M_\lambda(a,b)} \lambda |\vec{e}^\perp|^2 + \int_{S_\lambda(b)} |\vec{e}^T| - \int_{S_\lambda(a)} |\vec{e}^T|.$$

We may rearrange this to yield

$$\int_{M_\lambda(a,b)} \lambda |\vec{e}^\perp|^2 + \int_{S_\lambda(b)} |\vec{e}^T| = \int_{S_\lambda(a)} |\vec{e}^T|.$$

In particular,

$$\int_{S_\lambda(z)} |\vec{e}^T|$$

is a decreasing function of z. Now, we have

$$\begin{aligned}
\text{area}(M_\lambda(a,b)) &= \int_{M_\lambda(a,b)} |\vec{e}^\perp|^2 + |\vec{e}^T|^2 \\
&\leq \frac{1}{\lambda} \int_{S_\lambda(0)} |\vec{e}^T| + \int_{M_\lambda(a,b)} |\vec{e}^T|^2 \\
&= \frac{1}{\lambda} \int_{S_\lambda(0)} |\vec{e}^T| + \int_{z=a}^{z=b} \int_{S_\lambda(z)} |\vec{e}^T| \\
&\leq (\lambda^{-1} + b - a) \int_{S_\lambda(0)} |\vec{e}^T| \\
&\leq (\lambda^{-1} + b - a) \,\text{area}(\Sigma).
\end{aligned}$$

Thus, the flows have uniform area bounds on compact sets in space-time. Thus, a subsequence converges to a limit Brakke flow (strictly speaking, these flows have boundary, but we could work in the upper half-space, i.e., $\{z > 0\}$, where they do not have boundary).

We thus have obtained a Brakke flow $\mu(t)$ in $\mathbb{R}^N \times \mathbb{R}^+$. We would like to show that (1) the flow is translation invariant, i.e. $\mu(t) = \Sigma(t) \times \mathbb{R}^+$ for $\Sigma(t)$ a Brakke flow and (2) the flow has initial conditions $\Sigma \times \mathbb{R}^+$.

We start by showing translation invariance. Suppose that ϕ is a nice compactly supported nonnegative function on $\mathbb{R}^N \times \mathbb{R}^+$. Define $\phi^\tau(x,z) = \phi(x, z - \tau)$ to be "upwards translation by τ." Let $t \mapsto \mu_\lambda(t)$ be the translating Brakke flow constructed above, which limits to $\mu(t)$ along some subsequence of $\lambda \to \infty$.

Note that (we will use the shorthand $\nu(f) = \int f d\nu$ for ν a Radon measure)

$$\mu_\lambda(t)(\phi) = \mu_\lambda(t + \tau/\lambda)(\phi).$$

Recall that there is a constant c_ϕ depending on ϕ but independent of λ so that

$$t \mapsto \mu_\lambda(t)(\phi) - c_\phi t$$

is decreasing in time. Hence, if $t < s$, for λ large so that

$$t < t + \tau/\lambda < s,$$

we see that

$$\mu_\lambda(t)(\phi) - c_\phi t \geq \mu_\lambda(t)(\phi^\tau) - c_\phi(t + \tau/\lambda) \geq \mu_\lambda(s)(\phi) - c_\phi s.$$

Sending $\lambda \to \infty$ along the subsequence so that $\{\mu_\lambda(t)\}$ converges to $\{\mu(t)\}$, we have that

$$\mu(t)(\phi) - c_\phi t \geq \mu(t)\phi^\tau - c_\phi t \geq \mu(s)(\phi) - c_\phi s,$$

i.e., we have

$$\mu(t)(\phi) \geq \mu(t)\phi^\tau \geq \mu(s)(\phi) - c_\phi(s - t).$$

Sending $s \searrow t$, we obtain

$$\mu(t)(\phi) \geq \mu(t)\phi^\tau \geq \mu(t^+)(\phi).$$

This holds for every t. Moreover, for all but a countably many t, $\mu(\cdot)$ is continuous at t. Thus, we see that for a.e., t, $\mu(t) = \Sigma(t) \times \mathbb{R}^+$, as desired.

Now, we would like to show that $\mu(0) = \mu_{\Sigma \times \mathbb{R}^+}$. To do so, we will use the flat norm $\mathscr{F}(\cdot)$. For A, B closed m-dimensional cycles, the flat norm $\mathscr{F}(A - B)$ is the infimum of the area of $m + 1$ chains spanning $A - B$.

Let π denote the projection onto $\mathbb{R}^N \times \{b\}$ and let $A_\lambda = \pi(M_\lambda(0, b))$. We compute

$$\text{area}(A_\lambda) \leq \int_{M_\lambda(0,b)} |\vec{e}^{\perp}|$$

$$\leq \left(\int_{M_\lambda(0,b)} |\vec{e}^{\perp}|^2 \right)^2 \text{area}(M_\lambda(a, b))^{\frac{1}{2}}.$$

The area term is uniformly bounded. Moreover, we have seen above that the divergence theorem implies that

$$\int_{M_\lambda(0,b)} |\vec{e}^{\perp}|^2 \leq \lambda^{-1} \text{area}(\Sigma).$$

Putting this together, we see that $\text{area}(A_\lambda) \to 0$. Because b is bounded, we can then use this to see that the area of the grey region in Figure 21 is tending to zero.
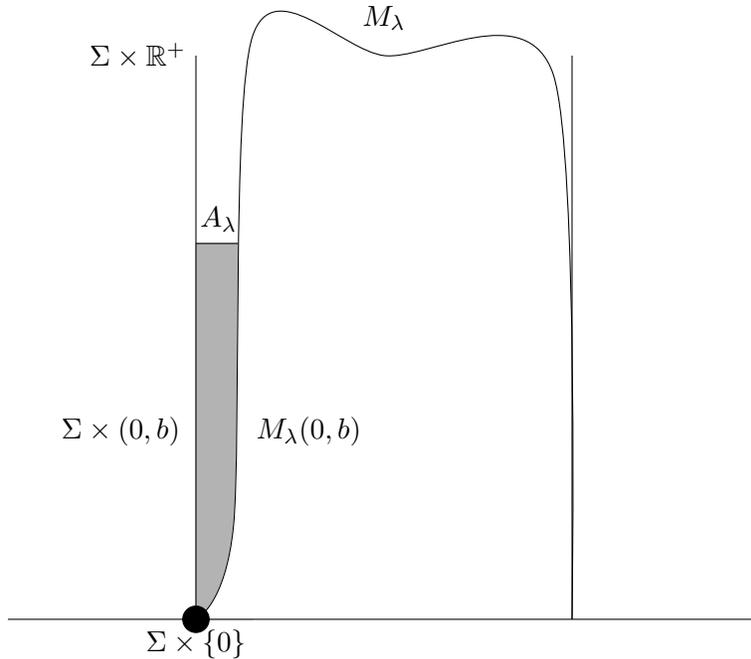
FIGURE 21. Showing that $\mu(0) = \mu_{\Sigma \times \mathbb{R}^+}$. We will show that the area of the grey region is tending to zero, which implies convergence in the flat norm.

This shows that

$$\mathscr{F}(M_\lambda(0, b) + A_\lambda - \Sigma \times (0, b)) \to 0.$$

Recall that mass is lower semicontinuous under flat convergence. In particular, we have that

$$\text{area}(\Sigma \times [0, b]) \leq \liminf_{\lambda \to \infty} \text{area}(M_\lambda(0, b) + A_\lambda)$$

$$= \liminf_{\lambda \to \infty} \text{area}(M_\lambda(0, b))$$

$$\leq \limsup_{\lambda \to \infty} \text{area}(M_\lambda(0, b))$$

$$\leq \limsup_{\lambda \to \infty} (\lambda^{-1} + b)\, \text{area}(\Sigma)$$

$$= b\, \text{area}(\Sigma)$$

$$= \text{area}(\Sigma \times (0, b)).$$

In particular, in addition to flat norm convergence, the masses converge (rather than dropping down)! We now may use the following general result

Proposition 9.12. *Suppose that $T_i \xrightarrow{\mathscr{F}} T$. Then, we have seen that the masses satisfy $\mathbb{M}(T) \leq \liminf \mathbb{M}(T_i)$. By passing to a subsequence, we may assume that the associated Radon measures μ_{T_i} converge. Then, we have that*

$$\mu_T \leq \lim_{i \to \infty} \mu_{T_i}.$$

Roughly speaking, this means that even locally mass can only drop down (we already used the global version of this fact). So, if the total mass converges, then the measures must converge (if they dropped down somewhere, then because the mass cannot jump up somewhere else, this would mean that the mass actually dropped down).

This combines to show that $\mu(0) = \Sigma \times \mathbb{R}^+$. Thus, we've completed the existence theory. Note that we've found solutions $\Sigma(t)$ with the extra convenient property that the flow $t \mapsto \Sigma(t) \times \mathbb{R}^+$ is the limit of smooth (or with a small singular set) flows.

9.4. **Why doesn't the flow disappear immediately?** A natural worry is that the flow we just constructed immediately disappears (this is a well defined Brakke flow!). We will show that the Brakke flows constructed by elliptic regularization cannot disappear, at least for a short time interval.

Definition 9.13. The support of a Brakke flow in $\mathcal{U}$, $[0, \infty) \ni t \mapsto \mu(t)$ is defined as

$$\overline{\bigcup_t \text{supp}(\mu(t)) \times \{t\}} \subset \mathcal{U} \times \mathbb{R}^+.$$

Note that without taking the closure, this is unlikely to be a closed set: for example the shrinking sphere sweeps out a paraboloid in space-time, but then disappears, so without the closure, we would be missing the point where the flow shrinks away.

We'll prove the following fact later using the monotonicity formula:

Fact 9.14. *If Brakke flows converge, then so do their supports (in the Hausdorff sense).*

Now fix $\Sigma \subset \mathbb{R}^N$ an initial closed hypersurface. Choose p, q inside and outside of Σ respectively. We can find small spheres $S(p), S(q)$ around p, q so that they are disjoint from Σ. When we minimize the $e^{-\lambda z}$-weighted area, then M_λ (the minimizer with $\partial M_\lambda = \Sigma$) will be disjoint from $S(p)_\lambda$ and $S(q)_\lambda$. Moreover, this will remain true as $\lambda \to \infty$. Note that we know explicitly what $S(p)_\lambda, S(q)_\lambda$ converge to because we understand shrinking spheres.

Now, take a straight line from p to q. Note that $M_\lambda(t)$ must intersect this line at least until it intersects one of the p, q. But the shrinking spheres show that this is only possible after a definite amount of time, say ϵ. So, for $0 \leq t \leq \epsilon$, $M_\lambda(t)$ will always intersect this line. Thus, the limit flow cannot shrink away immediately!

9.5. **Ilmanen's enhanced flow.** Ilmanen has observed that his elliptic regularization procedure can be further refined to give the space-time track a structure of an integral current (or flat chain). To do so, one can work at the level of the surfaces M_λ and show that their space-time track has the structure of a current/chain. Taking the limit (note that the current/chain could loose mass in the limit), we see that the space-time track admits a current/chain with the same support as the

support of the Brakke flow, as defiend above. This is a very convenient property, as it allows for the use of homological arguments, e.g., the discussion of (signed) intersection numbers, etc.

10. Monotonicity and entropy

For $M^m \subset \mathbb{R}^N$, we define

$$F(M) = \frac{1}{(4\pi)^{\frac{m}{2}}} \int_M e^{-\frac{|x|^2}{4}} d\mathcal{H}^m.$$

More generally, for a Radon measure μ, we set

$$F_m(\mu) = \frac{1}{(4\pi)^{\frac{m}{2}}} \int e^{-\frac{|x|^2}{4}} d\mu$$

Define the function

$$\rho(x) = \frac{1}{(4\pi)^{\frac{m}{2}}} e^{-\frac{|x|^2}{4}}.$$

Then, if M_t flows with velocity $H + \frac{1}{2}x^\perp$, then we have seen that $F(M_t)$ is decreasing with respect to t. Indeed, if $a < b$, then

$$F(M_b) - F(M_a) \leq \int_a^b \int_{M_t} \rho(x) \left| H + \frac{1}{2}x^\perp \right|^2 d\mathcal{H}^m dt.$$

Note that if M_t is a strong flow, then we have equality here, but we have kept this in the form we would have for a general Brakke flow. If $F(M_b) = F(M_a)$, then $H = -\frac{1}{2}x^\perp$ for $t \in (a,b)$, so $M_a = M_t = M_b$ for $a \leq t \leq b$.

Let $t_i \nearrow \infty$ and set $M_i(t) = M(t + t_i)$. Because F is decreasing, we can obtain local bounds on mass, so we can take the limit of the flows. In the limit, the surface will be an integral varifold self-shrinker, i.e., it will (weakly) satisfy $H = -\frac{1}{2}x^\perp$. Note that if the surfaces are non-compact, we must run this argument with a cutoff function, but the end result is the same.

How can we interpret $F(M)$? For a Radon measure μ, set $A(r) = \mu(B(0,r))$. Then, integrating by parts (assuming that $A(r)$ grows sub-exponentially), we obtain

$$F_m(\mu) = \frac{1}{(4\pi)^{\frac{m}{2}}} \int \omega_m \frac{r^{m+1}}{2} e^{-\frac{r^2}{4}} \frac{A(r)}{\omega_m r^m} dr$$

In particular, we see that $F(M)$ is bounded above and below by the area ratio:

$$c \sup_r \frac{A(r)}{\omega_m r^m} \leq F(M) \leq \sup_r \frac{A(r)}{\omega_m r^m}.$$

Colding and Minicozzi have defined [CM12] a related quantity, *entropy*, as

$$E(M) = \sup_{\lambda > 0, p \in \mathbb{R}^N} F(\lambda M + p).$$

By the above bounds, we have

$$c \sup_{B(p,r)} \frac{A(r)}{\omega_m r^m} \leq E \leq \sup_{B(p,r)} \frac{A(r)}{\omega_m r^m}.$$

10.1. **Monotonicity for mean curvature flow.** For $\mu(t)$ a Brakke flow defined for $(-T, 0)$, we have that

$$\frac{1}{(4\pi|t|)^{\frac{m}{2}}} \int e^{-\frac{|x|^2}{4\pi|t|}} d\mu(t)$$

is a decreasing function of t. More generally,

$$\frac{1}{(4\pi|t - t_0|)^{\frac{m}{2}}} \int e^{-\frac{|x - x_0|^2}{4\pi|t - t_0|}} d\mu(t)$$

is decreasing for $t < t_0$. Note that this quantity is bounded above by the entropy $E(\mu_t)$ (translate to $(0,0)$ and rescale the surface). From this, we see

Corollary 10.1. *The entropy $E(\mu(t))$ is decreasing with respect to t.*

10.2. **Gaussian density.** Define

$$\mathscr{C}_\Lambda = \{\text{Radon measures on } \mathbb{R}^N \text{ with } E_m(\mu) \le \Lambda\}$$

Note that this class is preserved by the Brakke flow. Moreover $\mathscr{C}_\Lambda$ or the set of flows in $\mathscr{C}_\Lambda$ is compact, because the entropy bound yields local area bounds.

Suppose that f is a continuous, bounded (or more generally $|f| \le c(1 + |x|)^k$) and $\mu_i \in \mathscr{C}_\Lambda$ has $\mu_i \rightharpoonup \mu \in \mathscr{C}_\Lambda$ (this is always the case, up to passing to a subsequence). It is easy to show that

$$\int f e^{-\frac{|x|^2}{4r^2}} d\mu_i \to \int f e^{-\frac{|x|^2}{4r^2}} d\mu,$$

by using a cutoff function if necessary. Set

$$\rho_{x_0, t_0}(x, t) = \frac{1}{(4\pi|t - t_0|)^{\frac{m}{2}}} e^{-\frac{|x - x_0|^2}{4|t - t_0|}}.$$

Then, if $\mu(t)$ is a Brakke flow (with spacetime track $\mathcal{M}$) and $X = (x_0, t_0)$, $r > 0$, let

$$\Theta(\mathcal{M}, X, r) = \int \frac{1}{(4\pi r^2)^{\frac{m}{2}}} e^{-\frac{|x - x_0|^2}{4r^2}} d\mu(t_0 - r^2)$$

Note that monotonicity implies that $\Theta(\mathcal{M}, X, r)$ is increasing with r. (Note that this is reminiscent of monotonicity for minimal surfaces).

Hence, as $r \searrow 0$, the limit exists, so we can set

$$\Theta(\mathcal{M}, X) := \lim_{r \searrow 0} \Theta(\mathcal{M}, X, r).$$

We call this the *Gaussian density* of $\mathcal{M}$ at X.

Proposition 10.2. *If the flows $\mathcal{M}_i \rightharpoonup \mathcal{M}$ then*

$$\Theta(\mathcal{M}, X) \ge \limsup_i \Theta(\mathcal{M}_i, X).$$

Proof. We have that

$$\Theta(\mathcal{M}_i, 0) \le \Theta(\mathcal{M}_i, 0, r) \to \Theta(\mathcal{M}, 0, r),$$

for any $r > 0$. Letting $r \searrow 0$, the proposition follows.   □

Proposition 10.3. *Suppose that $\mathcal{M}_i \rightharpoonup \mathcal{M}$, $X_i \to X$, $r_i \to 0$. Then*

$$\limsup_i \Theta(\mathcal{M}_i, X_i, r_i) \le \Theta(\mathcal{M}, X).$$

Proof. Translating by X_i, we can assume that $X_i = X = 0$. Then, for $r > 0$, for i sufficiently large, we have that $r_i < r$. Thus,

$$\limsup_i \Theta(\mathcal{M}_i, 0, r_i) \le \limsup_i \Theta(\mathcal{M}_i, 0, r) = \Theta(\mathcal{M}, 0, r).$$

This holds for all r. Letting $r \searrow 0$, the proposition follows.   □

Theorem 10.4. *If $\mathcal{M}_i \rightharpoonup \mathcal{M}$, $X_i \to X$, $\lambda_i \to \infty$, where X_i, X are always strictly after the initial time of the respective flows, then up to a subsequence*

$$\mathscr{D}_{\lambda_i}(\mathcal{M}_i - X_i)$$

converges to an eternal limit Brakke flow $\tilde{\mathcal{M}}$. Moreover $E(\tilde{\mathcal{M}}) \le \Theta(\mu, X)$.

We note that even if the flow is only defined on a subset of $\mathbb{R}^N$, $\mathcal{U}$, we can define

$$f = R^{-2}(R^2 - |x|^2 - 2mt)^+$$

and for R sufficiently small so that the support of f is contained in $\mathcal{U}$, we can show that $\mu_t(f)$ is decreasing. Hence, setting

$$g = \frac{1}{(4\pi|t|)^{\frac{m}{2}}} e^{-\frac{|x|^2}{4|t|}},$$

we can show that $\mu_t(fg)$ is decreasing in t. This allows us to define

$$\Theta(\mathcal{M}, 0) := \lim_{r \searrow 0} \mu_t(fg).$$

This agrees with the other definition of Gaussian density, if μ_t is defined on all of $\mathbb{R}^N$.

Now, suppose that $\mathcal{M}$ is an ancient flow in $\mathbb{R}^N$. Then, we may set

$$\Theta(\mathcal{M}) = \lim_{r \to \infty} \Theta(\mathcal{M}, X, r).$$

It is a simple exercise to check that this does not depend on X. Note that

$$\Theta(\mathcal{M}) = \sup_r \Theta(\mathcal{M}, X, r) = E(\mathcal{M}),$$

where $E(\mathcal{M})$ is the supremum in time of the entropy of $\mathcal{M}_t$.

Recall that if we set

$$\Theta_{euc}(M, 0, r) = \frac{A(r)}{\omega_m r^m},$$

then the Gaussian area satisfies

$$F(M) \ge \inf_{0 < r < \infty} \Theta_{euc}(M, 0, r).$$

Moreover,

$$F(M) \ge \inf_{\epsilon < r < \epsilon^{-1}} \Theta_{euc}(M, 0, r) - \delta(\epsilon),$$

where $\delta(\epsilon) \to 0$ as $\epsilon \to 0$.

Corollary 10.5. *We have that*

$$E(M) \ge \Theta_{euc}(M, 0) := \lim_{r \to 0} \Theta_{euc}(M, 0, r),$$

assuming the limit exists. Similarly,

$$E(M) \ge \Theta_{euc}(M, \infty) := \lim_{r \to \infty} \Theta_{euc}(M, 0, r),$$

assuming that the limit exists.

Suppose that we have a sequence of converging Brakke flows $\mathcal{M}_i \rightharpoonup \mathcal{M}$. We have seen that

$$\Theta(\mathcal{M}, X) \ge \limsup \Theta(\mathcal{M}_i, X_i)$$

if $X_i \to X$.

Proposition 10.6. *For any non-zero ancient flow, $\Theta(\mathcal{M}) \ge 1$.*

Proof. Let T denote the extinction time of the flow. For a.e. $t < T$, $\mu(t)$ is a non-zero integral varifold. Thus $\Theta_{euc}(\mu(t), x)$ is a nonzero integer for $\mu(t)$ a.e. x. Thus, for such an x,

$$E(\mu(t)) \geq \Theta_{euc}(\mu(t), x) \geq 1.$$

Because $\Theta(\mathcal{M})$ is the supremum of $E(\mu(t))$, the claim follows.   □

Corollary 10.7. *If $\mathcal{M}$ is a Brakke flow, then $\Theta(\mathcal{M}, X) \geq 1$ for all $X \in \operatorname{supp} \mathcal{M}$.*

Proof. We can choose $X_i = (x_i, t_i)$ converging to X so that $\Theta_{euc}(\mu(t_i), x_i) \geq 1$. We claim that

$$\Theta(\mathcal{M}, X_i) \geq \Theta_{euc}(\mu(t_i), x_i) \geq 1,$$

which finishes the proof by upper semicontinuity. To see this, we may translate so that $X_i = 0$. Then, we would like to show that for any flow, $\mathcal{M}$, we have

$$\Theta(\mathcal{M}, 0) \geq \Theta_{euc}(\mu(0), 0).$$

To see this, monotonicity implies that for some $\delta(r) \to 0$ as $r \to 0$,

$$\Theta(\mathcal{M}, (0, r^2), r + s) \geq \Theta(\mathcal{M}, (0, r^2), r) \geq \Theta_{euc}(0) - \delta(r) \geq 1 - \delta(r).$$

Sending $r \to 0$, we have

$$\Theta(\mathcal{M}, 0, s) \geq \Theta_{euc}(\mu(0), 0).$$

Now, the claim follows after letting $s \to 0$.   □

11. A version of Brakke's regularity theorem

First, we recall Allard's regularity theorem [All72] (see also [Sim83]), which says:

Theorem 11.1 (Allard's regularity theorem). *There is $\epsilon = \epsilon(m, N)$ with the following property. If M is a minimal variety (i.e., a stationary integral varifold) and $\Theta_{euc}(M, x) < 1 + \epsilon$, then x is a regular point of M.*

The corresponding regularity theorem for Brakke flows was proven by Brakke [Bra78] (see also [KT14]).

Theorem 11.2 (Brakke's regularity theorem). *There is $\epsilon = \epsilon(m, N)$ with the following property. Suppose that $\mathcal{M}$ is an integral Brakke flow and that $\Theta(\mathcal{M}, X) \leq 1 + \epsilon$, then X is a regular point for the flow.*

There is a subtle point about the second statement, because Brakke flows are allowed to suddenly vanish. So, one way to interpret the statement is that in a *backwards* parabolic neigbhorhood of X, the flow is a smooth flow of surfaces. If the flow comes from elliptic regularization, then it can be seen to be a smooth flow of surfaces both forward and backwards in time.

Here, we will discuss proofs of weaker versions of these results from [Whi05].

Theorem 11.3 (Easy Brakke 1). *Suppose that $\mathcal{M}_i$ are smooth, multiplicity one Brakke flows and $\mathcal{M}_i \rightharpoonup \mathcal{M}$. Assume that $\Theta(\mathcal{M}, X) = 1$. Then, there is an open neighborhood of X in space-time so that in this neighborhood, the convergence is smooth.*

Proof. We may assume that $X = 0$. We write $\kappa(\mathcal{M}_i, Y)$ for the norm of the second fundamental form at Y. We claim that $\kappa(\mathcal{M}_i, \cdot)$ is bounded on a spacetime neighborhood of X for sufficiently large i.

Suppose that this fails. Then, there is $X_i \to 0$ so that $\kappa(\mathcal{M}_i, X_i) \to \infty$. It is convenient to define $\|(x, t)\| = \max\{|x|, |t|^{\frac{1}{2}}\}$ (note that this norm, instead of the usual ℓ^2-rnom yields convenient space-time neighborhoods). Choose

$$R_i > 2\|X_i\|$$

so that $R_i \to 0$ sufficiently slowly, so that $R_i \kappa(\mathcal{M}_i, X_i) \to \infty$. Let $\mathcal{U}_i$ be the R_i neighborhood of 0, i.e.

$$\mathcal{U}_i = \{(x, t) : \|(x, t)\| < R_i\}.$$

Choose $Y_i \in \mathcal{M}_i \cap \mathcal{U}_i$ achieving

$$\max_{Y \in \mathcal{M}_i \cap \mathcal{U}_i} \kappa(\mathcal{M}_i, Y) \operatorname{dist}(Y, \partial \mathcal{U}_i).$$

Note that this quantity must tend to ∞, because of the way we chose R_i.

Let $\tilde{\mathcal{M}}$ be a subsequential limit of

$$\tilde{\mathcal{M}}_i = \kappa(\mathcal{M}_i, Y_i)(\mathcal{M}_i - Y_i)$$

and set

$$\tilde{\mathcal{U}}_i = \kappa(\mathcal{M}_i, Y_i)(\mathcal{U}_i - Y_i).$$

Then, we have that

$$\max_{Y \in \tilde{\mathcal{M}}_i \cap \tilde{\mathcal{U}}_i} \kappa(\tilde{\mathcal{M}}_i, Y) \operatorname{dist}(Y, \partial \tilde{\mathcal{U}}_i) = \kappa(\tilde{\mathcal{M}}_i, 0) \operatorname{dist}(0, \partial \tilde{\mathcal{U}}_i).$$

Note that $\kappa(\tilde{\mathcal{M}}_i, 0) = 1$ by normalization. Hence, for we see that

$$\kappa(\tilde{\mathcal{M}}_i, Y) \leq \frac{\operatorname{dist}(0, \partial \tilde{\mathcal{U}}_i)}{\operatorname{dist}(Y, \partial \tilde{\mathcal{U}}_i)},$$

which tends to 1 uniformly for Y in a compact set. Note that $\tilde{\mathcal{U}}_i$ converges to $\mathbb{R}^N \times \mathbb{R}$. Hence, we actually have that

$$\tilde{\mathcal{M}}_i \to \tilde{\mathcal{M}}$$

smoothly on compact subsets of space-time. Moreover, $\kappa(\tilde{\mathcal{M}}, 0) = 1$.

On the other hand, by monotonicity we see that

$$1 = \Theta(\tilde{\mathcal{M}}, 0) \leq \Theta(\tilde{\mathcal{M}}, 0, \infty) = \Theta(\tilde{\mathcal{M}}) \leq \Theta(\mathcal{M}, 0) = 1.$$

The first equality holds because 0 is a regular point of $\tilde{\mathcal{M}}$ and the last equality holds by assumption. Thus, the whole flow $\tilde{\mathcal{M}}$ is self-similar, but 0 is a regular point. This implies that $\tilde{\mathcal{M}}$ is a plane, contradicting the fact that $\kappa(\tilde{\mathcal{M}}, 0) = 1$. □

Given this version of Brakke's theorem, we can improve it to more useful statements. First, we show the following:

Theorem 11.4 (Gap Theorem). *Suppost that $\mathcal{M}$ is a self-similar Brakke flow which is not a multiplicity 1 plane. Then $\Theta(\mathcal{M}) \geq \eta > 1$, where $\eta = \eta(m, N)$.*

Proof. Suppose not. Then, there is a sequence of self-similar, non-planar, Brakke flows with $\Theta(\mathcal{M}_i) \to 1$. By compactness we may pass to a subsequential limit so that $\mathcal{M}_i \rightharpoonup \mathcal{M}$, which will necessarily be self-similar and have $\Theta(\mathcal{M}) = 1$. We have seen that the Euclidean densities of $\mathcal{M}$ at, say, $t = -1$ are bounded by $\Theta(\mathcal{M}) = 1$. Thus, Allard's theorem implies that $\mathcal{M}(-1)$ is smooth everywhere *and* $\mathcal{M}_i(-1) \to \mathcal{M}(-1)$ smoothly. Thus, for i large, we have that the $\mathcal{M}_i$ are smooth for all $t < 0$. However, as they are non-flat, their curvature must be blowing up near 0 (in space-time). Thus, there exists (x_i, t_i) with $\|(x_i, t_i)\| \to 0$ so that $\kappa(\mathcal{M}_i, (x_i, t_i)) \to \infty$.

On the other hand, we can apply the Easy Brakke 1 to $\mathcal{M}_i \cap \{t \leq t_i\}$ to see that because

$$\mathcal{M}_i \cap \{t \leq t_i\} \hookrightarrow \mathcal{M} \cap \{t \leq 0\},$$

and $\Theta(\mathcal{M}, 0) \leq \Theta(\mathcal{M}) = 1$, implying that the convergence $\mathcal{M}_i \to \mathcal{M}$ is smooth at 0. □

Theorem 11.5 (Easy Brakke 2). *Let $\mathcal{G}$ denote the class of integral Brakke flows $\mathcal{M}$ so that every point X with $\Theta(\mathcal{M}, X) = 1$ must be a regular point.*

Then, $\mathcal{G}$ is closed. Moreover, if $\mathcal{M} \in \mathcal{G}$ and if $\Theta(\mathcal{M}, X) < \eta$, for η from the Gap Theorem, then X is a regular point.

Note that Brakke's (hard) theorem says that $\mathscr{G}$ is actually the set of all integral Brakke flows.

Proof. Let $\mathcal{M}_i \in \mathscr{G}$ have $\mathcal{M}_i \rightharpoonup \mathcal{M}$. Suppose that $X \in \mathcal{M}$ and $\Theta(\mathcal{M}, X) \leq \eta - 2\epsilon$, for some $\epsilon > 0$. By semi-continuity of the density, there is $I < \infty$ and some neighborhood $\mathcal{U}$ of X so that

$$\Theta(\mathcal{M}_i, \cdot) \leq \eta - \epsilon$$

in $\mathcal{U}$ for $i \geq I$. Thus, we see that any tangent flow to $\mathcal{M}_i$ at $Y \in \mathcal{U}$ has entropy at most $\eta - \epsilon$. The gap theorem implies that it must be a multiplicity one tangent plane, so $\Theta(\mathcal{M}_i, Y) = 1$.

Therefore, Y must be a regular point of $\mathcal{M}_i$, because $\mathcal{M}_i \in \mathscr{G}$. By Easy Brakke 1 applied to $\mathcal{M}_i \cap \mathcal{U} \rightharpoonup \mathcal{M} \cap \mathcal{U}$, we see that the convergence is smooth, so X must be a regular point of $\mathcal{M}$. $\quad\square$

12. Stratification

We would like to discuss the stratification of Brakke flows. See [Alm00, Fed70, Whi97]. For simplicity, we start with the stratification of minimal surfaces.

Lemma 12.1. *For $M \subset \mathbb{R}^N$, $0 \in M$, let $V(M) := \{x \in \mathbb{R}^N : M = M + x\}$.*
  (1) *Trivially, $V(M)$ is an additive subgroup of $\mathbb{R}^N$.*
  (2) *If M is a cone, then $V(M)$ is a linear subspace and $M = C \times V(M)$, where C is the cone $C = M \cap (V(M))^\perp$.*
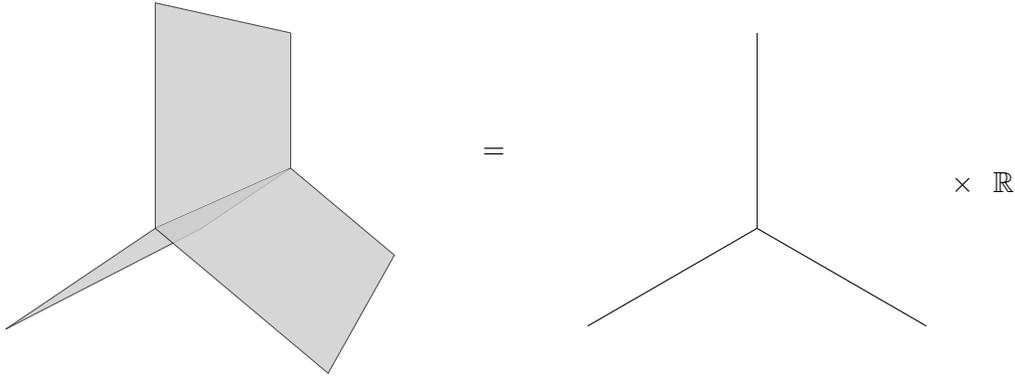

FIGURE 22. The spine (or, equivalently, the set of possible vertex points) of a minimal cone.

Theorem 12.2. *If $M \subset \mathbb{R}^N$ is a minimal cone, then*

$$\max_M \Theta(M, \cdot) = \Theta(M, 0) = \Theta(M).$$

Moreover, if

$$\mathrm{spine}(M) = \{x : \Theta(M, x) = \Theta(M)\},$$

then $\mathrm{spine}(M) = V(M)$.

Proof. First, note that $\Theta(M, x) \leq \Theta(M)$ with equality if and only if M is dilation invariant around x (this follows from the monotonicity formula).

Now, suppose that $x \in V(M)$. Because $M + x = M$, we see that $\Theta(M, x) = \Theta(M, 0)$. Thus, $x \in \mathrm{spine}(M)$. This shows that $V(M) \subset \mathrm{spine}(M)$.

On the other hand, if $a \in \mathrm{spine}(M)$, then M is invariant by dilation around a and 0. In particular, composing the maps $x \mapsto a + \lambda(x - a) = (1 - \lambda)a + \lambda x$ and $x \mapsto \frac{1}{\lambda}x$, we see that M is invariant under

$$x \mapsto \left(\frac{1 - \lambda}{\lambda}\right)a + x.$$

Setting $\lambda = \frac{1}{2}$, we see that M is invariant under $x \mapsto x + a$. This shows that $\mathrm{spine}(M) \subset V(M)$. $\quad\square$

Now, if M is a minimal variety, we set

$$\Sigma_k = \{x \in M : \text{each tangent cone at } x \text{ has a spine of dimension} \leq k\}.$$

Then, the main result concerning stratification is

Theorem 12.3. *The set Σ_k satisfies $\dim_{Haus}(\Sigma_k) \leq k$.*

Example 12.4. Let us consider M which minimizes m-dimensional area mod 2. Clearly, there can be no tangent planes of multiplicity bigger than 1. Thus, every point in $M \setminus \Sigma_{m-1}$ must be regular, so we automatically get $\dim_{Haus} \mathcal{S} \leq m - 1$, where $\mathcal{S}$ is the singular set. Next, if we consider 1-dimensional area minimizing mod 2 cones, we see that they must be the union of rays. Moreover, in order for them to have no boundary mod 2 at the origin, there must be an even number of rays. Thus, we see that there are no-nontrivial 1-dimensional cones which minimize area mod 2. From
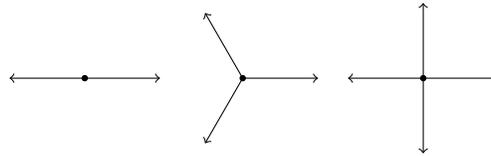

FIGURE 23. Possible 1-dimensional cones. The middle one cannot be a tangent cone to M, as it has boundary point at the origin. The other two are possible cones.
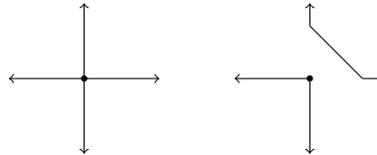

FIGURE 24. Cones with four or more rays cannot be area minimizing, as we could reduce the area of a compact piece, as shown.

this, we see that any point in $M \setminus \Sigma_{m-2}$ must be a regular point.

Putting this together with the stratification theorem, we see that the singular set satisfies $\dim_{Haus} \mathcal{S} \leq m - 2$.

For Brakke flows, the situation is similar but slightly more complicated. Let $\mathcal{M}$ be a Brakke flow. We know that

$$\mathcal{D}_\lambda(\mathcal{M} - X) \rightharpoonup \mathcal{M}'$$

subsequentially, where $\mathcal{M}'$ is an *tangent flow at* X.

Theorem 12.5. *We have that $\Theta(\mathcal{M}, 0) = \Theta(\mathcal{M}')$, so $\mathcal{M}' \cap \{t < 0\}$ is self-similar.*

Observe that we do not have any information about $t \geq 0$! Indeed, below we will see various examples of different possible behaviors for $t \geq 0$.

In general, for $\mathcal{M}$ a self-similar flow, we define the "spatial spine"

$$V(\mathcal{M}) = \{x \in \mathbb{R}^N : \Theta(\mathcal{M}, (x, 0)) = \Theta(\mathcal{M})\}$$

As before, it is not hard to show that

$$V(\mathcal{M}) = \{x \in \mathbb{R}^N : \mathcal{M} \cap \{t < 0\} \text{ is invariant under translation by } (x, 0)\}$$

and that $V(\mathcal{M})$ is a linear subspace of $\mathbb{R}^N$.

Moreover, we can show that the set

$$\{X : \Theta(\mathcal{M}, X) = \Theta(\mathcal{M})\}$$

must be one of the following:

(1) $V(\mathcal{M}) \times \{0\}$, e.g., a cylinder
(2) $V(\mathcal{M}) \times \mathbb{R}$, e.g., a minimal cone
(3) $V(\mathcal{M}) \times (-\infty, a]$, for $a \geq 0$, e.g., $\mathcal{M}$ remains a minimal cone until time a, and then flows in some other manner. We call this case an *quasi-static cone*.

An interesting example of (3) is the tangent flow to a cusp singularity from an immersed plane curve. For $t < 0$, it is a multiplicity 2 line, while for $t > 0$ it is empty.

Now, to discuss the stratification of a general Brakke flow, if $\mathcal{M}'$ is a tangent flow, then we set $d(\mathcal{M}')$ to be the dimension of the spatial spine $V(\mathcal{M}')$. Then, we set $D(\mathcal{M}') = d + 2$ if $\mathcal{M}'$ is a static cone for all time, and $D(\mathcal{M}') = d$ otherwise. Note that in case (2) above, $D = d + 2$, and otherwise $D = d$.

Theorem 12.6. *For $\mathcal{M}$ an integral Brakke flow, let*

$$\Sigma_k = \{X : D(\mathcal{M}') \leq k \text{ for every tangent flow at } X\}.$$

Then

$$\dim_{par.\ Haus} \Sigma_k \leq k$$

Here, the parabolic Hausdorff dimension is the dimension with respect to the metric on space-time given by, e.g.,

$$d((x,t),(x',t')) = |x - x'| + |t - t'|^{\frac{1}{2}}.$$

Corollary 12.7. *For a.e., t_0,*

$$\dim_{Haus}(\Sigma_k \cap \{t = t_0\}) \leq k - 2.$$

We now discuss an example of the stratification. Some time-slices of the flow are illustrated in Figure 25.
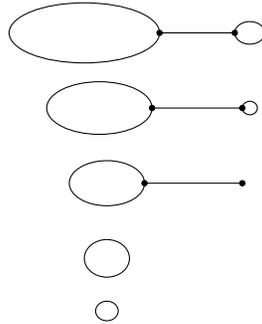

FIGURE 25. An example of a Brakke flow of curves (known as the *network flow*).

The space-time illustrated in Figure 26 has the following tangent flows:

(a) The tangent flow is a static multiplicity one plane, as this is a regular point.
(b) The tangent flow is a shrinking circle.
(c) This is a quasi-static cone. The tangent flow is a static triple junction for $t < 0$, but at $t = 0$ one arc disappears and the other two flow outwards smoothly.
(d) This is a static triple junction.
(e) This tangent flow is what one might call *quasi-regular*: it is a multiplicity one line for $t < 0$, but then disappears at $t = 0$.
(f) This is a self-similar shrinker which looks like this: ——◯ . At $t = 0$ it becomes a single arc, and dissapears immediately.
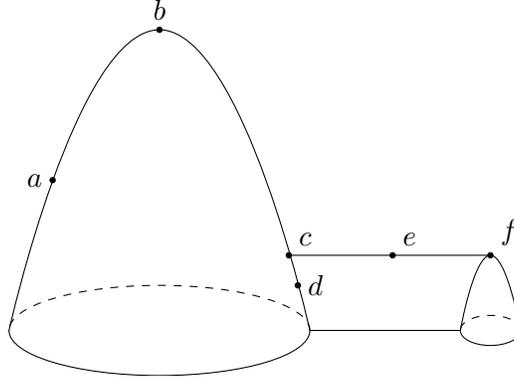
FIGURE 26. The space-time track of the network flow from Figure 25 exhibits several interesting tangent flows, as we discuss below.

12.1. **Ruling out the worst singularities.** Now, if $\mathcal{M}$ is an m-dimensional integral Brakke flow, we know that for any tangent flow, $d \leq m$, so $D \leq m+2$ (this is the trivial estimate). So, the worst thing we could get, from the point of view of stratification is a higher multiplicity plane which does not disappear. Then, $D = m+2$. The next possibility is a static cone with an $(m-1)$-dimensional spine, i.e., a union of half-planes. This has $D = m+1$. So, if we can rule these possibilities out, then we can already say that the parabolic Hausdorff dimension of the singular set is at most m.

Note that a shrinking $\mathbb{S}^1 \times \mathbb{R}^{m-1}$ shows that in general the dimension of the singular set should be at least $m-1$.

13. EASY PARITY THEOREM

We will restrict ourselves to hypersurfaces, but modified versions of these results hold in general.

Theorem 13.1. *Define $\mathscr{G}$ to be the class of Brakke flows $\mathcal{M}$ so that density 1 points are regular and so that if a closed curve C has $C \cap \operatorname{sing}(\mathcal{M}) = \emptyset$ and C intersects $\operatorname{reg}(\mathcal{M})$ (the set of regular and multiplicity one points) transversely, then $C \cap \mathcal{M}$ has an even number of elements.*

Then, $\mathscr{G}$ is closed.

Proof. Pick $\mathcal{M}$ which is a limit of such flows, and pick C such a curve. By the versions of Brakke's theorem proved above, the convergence to $\mathcal{M}$ is smooth in a neighborhood of C. Thus, the desired property passes to the limit. $\qquad\square$

Theorem 13.2. *Let $\mathcal{M}$ be an m-dimensional integral Brakke flow arising from elliptic regularization. Consider the set*

$$W = \{X : \Theta(\mathcal{M}, X) < 2\},$$

which is open by upper-semicontinuity of density. Then $\operatorname{sing}(\mathcal{M}) \cap W$ has parabolic Hausdorff dimension at most $m-1$. Moreover, away from a set of dimension at most $m-2$, $\operatorname{sing}(\mathcal{M}) \cap W$ has tangent flows which are all $C \times \mathbb{R}^{m-3}$ for C a static smooth 3-D cone, or $\mathbb{S}^1 \times \mathbb{R}^{m-1}$.

Remark 13.3. Colding–Ilmanen–Minicozzi [CIM15] have shown that if one tangent flow is $\mathbb{S}^k \times \mathbb{R}^{m-k}$, then they all are. Subsequently Colding–Minicozzi [CM15] showed that in this case, the flow is unique, i.e., there is no rotation.

To prove the above theorem, we consider the possible tangent flows at a singularity with density $\Theta < 2$. First we consider the static/quasi-static cones:

(1) The first possibility would be a static plane of multiplicity ≥ 2. This could contribute dimension $m+2$ to the singular set. But, it cannot happen by density considerations.

(2) Similarly, a quasi-static plane of multiplicity ≥ 2 could contribute dimension m, but it is also ruled out by density considerations.

(3) A static (resp. quasi-static) union of half planes (i.e., a 1-D minimal cone times $\mathbb{R}^{m-1}$) could contribute dimension $m + 1$ (resp. $m - 1$). However, $\Theta < 2$ implies that there must be exactly 3 half-planes of multiplicity 1, which is ruled out by parity, or otherwise the cone is a flat, multiplicity one cone.

(4) A static (resp. quasi-static) 2-D minimal cone crossed with $\mathbb{R}^{m-2}$ could contribute dimension m (resp. $m - 2$). Such a cone intersected with the unit sphere is a geodesic network. By $\Theta < 2$ and parity considerations, there cannot be any junctions, so such a cone cannot exist (besides a multiplicity one plane).

Thus, we see that the worst static cone that could happen is a 3-d cone times $\mathbb{R}^{m-3}$, contributing dimension at most $m - 1$. We must also consider the possible shrinkers:

(1) One possibility is a 1-D shrinker times $\mathbb{R}^{m-1}$, which could contribute dimension $m - 1$. The argument above shows that the 1-D shrinker cannot have any junctions (e.g., it cannot be the shrinking spoon ——◯. Hence, it is a smooth, embedded shrinker, and is thus a round $\mathbb{S}^1$. Thus, $\mathbb{S}^1 \times \mathbb{R}^{m-1}$ is the only possibility in this case.

(2) Continuing on, we could consider a 2-D shrinker times $\mathbb{R}^{m-2}$, contributing at most $m - 2$, and so on.

Putting this together with the stratification theorem implies the above result.

14. The maximum principle

Theorem 14.1. *Suppose that M is a space-time support of an m-dimensional integral Brakke flow in Ω, $t \mapsto \mu(t)$. Let $u : \Omega \to \mathbb{R}$ be a smooth function so that at (x_0, t_0),*

$$\frac{\partial u}{\partial t} < \operatorname{tr}_m D^2 u,$$

where $D^2 u$ is the spacial Hessian, and tr_m is the sum of the smallest m-eigenvalues. Then,

$$u|_{M \cap \{t \leq t_0\}}$$

cannot have a local maximum at (x_0, t_0).

Proof. Assume otherwise. We may clearly assume that $(x_0, t_0) = (0, 0)$, $\mathcal{M} = \mathcal{M} \cap \{t \leq 0\}$ and $u|_M$ has a strict local maximum at $(0, 0)$ (otherwise we could replace u by $u - |x|^2 - |t|^4$).

Let $Q(r) = B(r) \times (-r^2, 0]$ (where $B(r)$ is the spatial open ball centered at the origin). Choose r small enough so that $-r^2$ is past the initial time of the flow, $u|_{M \cap \overline{Q}}$ has a maximum at $(0, 0)$ and nowhere else, $\frac{\partial u}{\partial t} < \operatorname{tr}_m D^2 u$ on $\overline{Q}$ and $u|_{M \cap (\overline{Q} \setminus Q)} < 0 < u(0, 0)$.

Now, setting $u^+ = \max\{u, 0\}$, we plug $(u^+)^4$ into the definition of Brakke flow. Thus,

$$
\begin{aligned}
0 &\leq \int_B (u^+)^4 d\mu(0) \\
&= \int_B (u^+)^4 d\mu(0) - \int_B (u^+)^4 d\mu(-r^2) \\
&\leq \int_{-r^2}^0 \int \left(-|H|^2 (u^+)^4 + H \cdot \nabla (u^+)^4 + \frac{\partial}{\partial t} (u^+)^4 \right) d\mu(t) dt \\
&\leq \int_{-r^2}^0 \int \left(-\operatorname{div}_{\mathcal{M}} \nabla (u^+)^4 + \frac{\partial}{\partial t} (u^+)^4 \right) d\mu(t) dt \\
&= \int_{-r^2}^0 \int 4 \left(-3(u^+)^2 |\nabla^{\mathcal{M}} u^+|^2 - (u^+)^3 \operatorname{div}_{\mathcal{M}} \nabla (u^+) + (u^+)^3 \frac{\partial u^+}{\partial t} \right) d\mu(t) dt
\end{aligned}
$$

$$\leq \int_{-r^2}^0 \int 4(u^+)^3 \left(-\operatorname{tr}_m D^2 u^+ + \frac{\partial u^+}{\partial t} \right) d\mu(t) dt$$
$$< 0.$$

This is a contradiction, completing the proof. $\qquad\square$

As a consequence of this, we obtain

Theorem 14.2 (Weak barrier principle)**.** *Let M be the space-time support of an m-dimensional integral Brakke flow in Ω. Suppose that $t \mapsto N(t)$ is a 1-parameter family of domains in Ω so that $t \mapsto \partial N(t)$ is a smooth 1-parameter family of hypersurfaces. Assume that $M(t) = \{x : (x,t) \in M\} \subset N(t)$.*
If $p \in M(\tau) \cap \partial N(\tau)$, then $v(p,\tau) \geq h_m(p,\tau)$, where $v(p,\tau)$ is the speed of $\partial N(\tau)$ at p in the inward direction ν and h_m is the sum of the m-smallest principle eigenvalues of ∂N.

Proof. Let $f : \Omega \to \mathbb{R}$ be defined by

$$f(x,t) = \begin{cases} -\operatorname{dist}(x, \partial N(t)) : x \in N(t) \\ \operatorname{dist}(x, \partial N(t)) : x \notin N(t) \end{cases}$$

and let $e_1, \ldots, e_{n-1}$ denote the principal directions of $\partial N(t)$ at p. Then $e_1, \ldots, e_{n-1}, \nu$ is an orthonormal basis at p. We compute

$$D^2 f(p) = \begin{pmatrix} \kappa_1 & & & \\ & \ddots & & \\ & & \kappa_{n-1} & \\ & & & 0 \end{pmatrix}.$$

Set $u = e^{\alpha f}$. Then $Du = \alpha e^{\alpha f} Df$, so

$$D^2 u = \alpha^2 e^{\alpha f} Df^T Df + \alpha e^{\alpha f} D^2 f.$$

From this, we readily see that the eigenvalues of $D^2 u$ at p (note that $f(p) = 0$) are $\alpha \kappa_1, \ldots, \alpha \kappa_{n-1}, \alpha^2$. For α sufficiently large, we see that

$$\operatorname{tr}_m D^2 u|_p = \alpha h_m.$$

On the other hand,

$$\frac{\partial u}{\partial t} = \alpha e^{\alpha f} \frac{\partial f}{\partial t} = \alpha v(p,t).$$

By assumption $f|_M$ has a maximum at (p,t), so the conclusion follows from the maximum principle proven above. $\qquad\square$

Theorem 14.3 (Barrier principle for hypersurfaces)**.** *Suppose that M is the space-time support of an m-dimensional integral Brakke flow in Ω, which is an $(m+1)$-dimensional manifold. Let $M(t)$ denote the t-time slice of M. Suppose that $t \mapsto N(t)$ is a family of closed domains in Ω so that $t \mapsto \partial N(t)$ is a smooth 1-parameter family of hypersurfaces. Assume that $\partial N(t)$ is compact and connected and $v_{\partial N,in} \leq H_{\partial N,in}$ everywhere. Suppose that $M(0) \subset N(0)$ and $\partial N(0) \setminus M(0)$ is nonempty. Then, $M(t)$ is contained in the interior of $N(t)$ for $t > 0$.*

First we prove

Lemma 14.4. *Assumptions as in the barrier principle. If $M(0) \subset N(0)$, then $M(t) \subset N(t)$.*

Proof. Let $\tilde{N}_\epsilon(t)$ be the region with $\partial \tilde{N}_\epsilon(0) = \partial N(0)$ and which flows with speed $v_{\partial N,in} - \epsilon$. If ϵ is sufficiently small, this flow will be smooth on an interval comparable to that of the definition of $N(t)$. We can apply the weak maximum principle and then set $\epsilon \to 0$. $\qquad\square$

Now, to prove the barrier principle, because we are assuming that $\partial N(0)$ has some points which are disjoint from $M(0)$. Thus, we may push $\partial N(0)$ inwards slightly near these points, to find a new set $\hat{N}(0)$ with $\hat{N}(0)$ smooth, and $M(0) \subset \hat{N}(0) \subsetneq N(0)$. Flow $\partial \hat{N}(0)$ by mean curvature flow; it will remain smooth at least for a short time. The classical maximum principle shows that $\partial N(t)$ and $\partial \hat{N}(t)$ immediately become disjoint. Applying the above lemma to $\hat{N}(t)$ yields the desired result.

15. Mean convex flows

The following analysis of mean convex flows was proven in [Whi00, Whi03]. Suppose that $M(0)$ is mean convex. Then, the cylinder $M(0) \times \mathbb{R}$ is a good barrier for the elliptic regularization minimization procedure. Indeed, we can find (local) minimizers of $\int e^{-\lambda z}$ which are *smooth* graphs over $M(0)$, even in high dimensions. In particular, the elliptic regularization surfaces can be assumed to be smooth in this case.

15.1. **Curvature estimates along the flow.** We can check that mean convexity is preserved by smooth mean curvature flow. Moreover, the following theorem follows readily from Hamilton's maximum principle for tensors [Ham82].

Theorem 15.1. *For smooth, compact, mean convex mean curvature flow, if κ_1 is the smallest principle curvature and h is the mean curvature,*

$$\min \frac{\kappa_1}{h}$$

is non-decreasing in time. If it is negative it is strictly increasing unless the minimum is attained at the boundary.

In particular, for the translating solitons minimizing $\int e^{-\lambda z}$, we see that either $\min \frac{\kappa_1}{h} \geq 0$ (i.e., we are in the convex case) or the minimum is attained at the boundary.

Proposition 15.2. *As $\lambda \to \infty$, the quantity $\min \frac{\kappa_1}{h}$ on the translator tends to $\min \frac{\kappa_1}{h}$ on $M(0) \times \mathbb{R} = \lambda > -\infty$.*

Thus, for mean convex flows, we may always assume that $\frac{\kappa_1}{h} \geq \lambda > -\infty$.

15.2. **One-sided minimization.** Consider $[0, \infty) \ni t \mapsto K(t)$, closed regions so that $\partial K(t)$ is flowing by mean convex mean curvature flow. Of course, it is clear that

$$\text{area}(\partial K(t)) \leq \text{area}(\partial K(0)).$$

However, the flows have the following much stronger property: suppose that $K(t) \subset \hat{K} \subset K(0)$. Then,

$$\text{area}(\partial K(t)) \leq \text{area}(\partial \hat{K}).$$

To prove this, let Σ denote the least area surface containing $K(t)$ inside of $K(0)$. By the maximum principle, it cannot touch $\partial K(0)$. Evolve $\partial K(0)$ as $\partial K(t)$ until it touches Σ. Because Σ is minimal at points not touching $K(t)$, this can never happen. Thus, $\Sigma = \partial K(t)$. Any regularity issues here could be dealt with at the level of the translators.

An immediate consequence of the one-sided minimization is

Theorem 15.3. *If $B \subset K(0)$, then $\text{area}(\partial K(t) \cap B) \leq \text{area}(\partial B)$.*

This immediately gives uniform local area bounds for mean convex flows. Moreover, we have the following more refined result.

Theorem 15.4. *Assume that $B(x, r) \subset K(0)$ and $\partial K(t) \cap B(x, r) \subset \text{slab}(\epsilon r)$, where $\text{slab}(\epsilon r)$ is an ϵr neighborhood of some hyperplane containing x. Then one of the following occurs:*

(1) $(K(t) \cap B) \setminus \mathrm{slab}(\epsilon r)$ *is* 1 *component of* $B \setminus \mathrm{slab}(\epsilon r)$. *The* 1*-sided minimization property lets us conclude that*

$$\mathrm{area}(\partial K(t) \cap B) \leq (1 + c\epsilon)\omega_m r^m$$

(2) $K(t) \cap B \subset \mathrm{slab}(\epsilon r)$. *Then, we have*

$$\mathrm{area}(\partial K(t) \cap B) \leq (2 + c\epsilon)\omega_m r^m.$$

(3) *Finally* $(K(t) \cap B) \setminus \mathrm{slab}(\epsilon r)$ *is both components of* $B \setminus \mathrm{slab}(\epsilon r)$. *Then,*

$$\mathrm{area}(\partial K(t) \cap B) \leq c\epsilon r^m.$$

Moreover, in this case, we can show that $B(x, r/2) \subset K(t)$.

The argument for each case is similar, so we only discuss (1). The area bound is an immediate consequence of the one-sided minimization property and the competitor surface constructed in Figure 27.
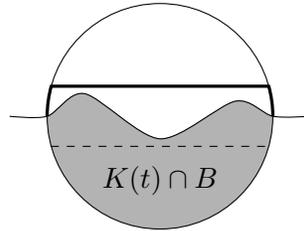

FIGURE 27. The competitor surface in case (1) above is illustrated by a bold line.

Corollary 15.5. *Suppose that* $\mathcal{M}$ *is a static plane of multiplicity* k *obtained as a tangent flow to mean convex mean curvature flow. Then* $k = 0, 1, 2$. *Moreover, we can see that for the* $k = 2$ *case, the mean curvature points "in," i.e., before the limit, there is components of* $K(t)$ *on "both sides."*

15.3. **Blow-up limits of mean convex flows.** Thus, we are lead to the question: what blow-up flows with $\Theta \leq 2$ are possible?

Step 1: Suppose that $\tilde{\mathcal{M}}$ is a (quasi-)static minimal cone. Then, it must be a plane of multiplicity one or two. We have seen that it cannot be a higher multiplicity cone. Moreover, a union of half planes must have at least 4 half-planes by parity, and this cannot be 1-sided minimizing. Finally, we can use these results along with the bound $\frac{\kappa_1}{h} \geq \lambda > -\infty$ to rule out "curved" minimal cones: the cone must be regular and multiplicity 1 away from the vertex (if not, we could find a singular cone with a spine, which we have just ruled out), so we can use this inequality to see that it must be flat.

Step 2: Suppose that $\tilde{\mathcal{M}}$ is a non-static shrinker. We have that $M(-1)$ is minimal with respect to the Gaussian metric. Moreover $\Theta < 2$ on $M(-1)$ (otherwise there would be a time-symmetry so the whole flow would have to be static/quasi-static). Thus, $M(-1)$ is completely regular! By a result of Huisken [Hui90], we have that $M(-1) = \mathbb{S}^k \times \mathbb{R}^{m-k}$.

Step 3: Suppose that $\tilde{\mathcal{M}}$ is a (quasi-)static minimal variety. Then, we claim that $\tilde{\mathcal{M}}$ is a plane (of multiplicity 1 or 2), or two parallel planes each of multiplicity one. To see this, note that if there is a point of density 2, then the surface must be (time) dilation invariant, i.e., it must be a shrinker. In this case we can reduce to Step 1. Hence, we can assume that $\Theta < 2$ everywhere on $\tilde{\mathcal{M}}$. Any tangent cone will be a blow-up limit which is a minimal cone. The density bound implies that it must be a multiplicity 1 cone. Hence, $\tilde{\mathcal{M}}$ is regular everywhere. Now, we can conclude that $\tilde{\mathcal{M}}$ is flat, using the $\frac{\kappa_1}{h} \geq \lambda > -\infty$ bound.

Lemma 15.6 (Key Lemma). *Suppose that for all large R, $\tilde{\mathcal{M}} \cap (B(0, R) \times [-R^2, R^2])$ is weakly close, in a scale-invariant sense, to a static multiplicity 2 plane. Then, $\tilde{\mathcal{M}}$ is a multiplicity 2 plane or two parallel planes, each of multiplicity one.*

Proof. Any tangent cone at infinity must be a static, multiplicity 2 plane by the assumptions and above arguments. Let $K(t)$ denote the closed region bounded by $\tilde{\mathcal{M}}(t)$. Suppose that $\lambda \to 0$. We thus have that $\lambda K(0)$ subsequentially converges to a plane through 0 (this is a consequence of the fact that we get a multiplicity two plane, and in this case, the mean curvature "points in"). Hence, if $t_i \geq 0$ and if $\lambda \to 0$, then because $K(t_i) \subset K(0)$, we have that $\lambda(K(t_i))$ converges subsequentially to some subset of a plane through 0.

The easy case is if

$$\bigcap_t K(t) \neq \emptyset.$$

Then, in this case $\partial(\cap_t K(t))$ is a minimal surface (in fact, a plane or two parallel planes). To see this, we may simply consider $\tilde{\mathcal{M}}(t)$ as $t \to \infty$. By assumption the limit is nonzero, and it must be minimal. Thus, by Step 3 above, it must be a multiplicity 1 or 2 plane. Thus, because $\cap_t K(t)$ is either a plane or a slab, we can decompose $\tilde{\mathcal{M}}(t)$ into two pieces, one on each side of $\cap_t K(t)$. Because each component must have density at infinity at most (and at least) 1, they are both static
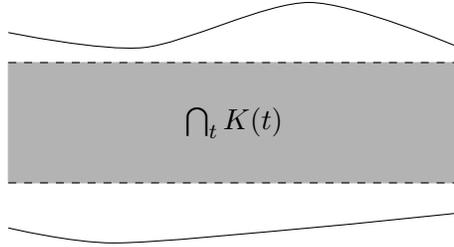

FIGURE 28. The set $\cap_t K(t)$ divides $\tilde{\mathcal{M}}(t)$ into two components

planes.

Now, we must consider the hard case, when

$$\bigcap_t K(t) = \emptyset.$$

Let $R(t) = \mathrm{dist}(0, K(t))$. By hypothesis, $R(t) \to \infty$ as $t \to \infty$ (note that $R(t)$ is increasing by the nested property). Scaling considerations show that $R(t) \leq o(\sqrt{t})$ (if not, the tangent flow at infinity will be missing a paraboloid in space-time, but we know that it is a (quasi-)static plane). Hence,

$$\frac{R(t)}{\sqrt{t}} \to 0$$

as $t \to \infty$. This might not occur monotonically, but we choose t_i so that this happens "as slowly as possible," i.e.,

$$\frac{R(t_i)}{\sqrt{t_i}} = \max_{t \geq t_i} \frac{R(t)}{\sqrt{t}}.$$

Now we have

$$\mathscr{D}_{1/R(t_i)}(\tilde{\mathcal{M}} - (0, t_i)) \rightharpoonup \hat{\mathcal{M}}.$$

Let $\hat{K}(t)$ denote the corresponding region. By the above considerations, $\hat{K}(t)$ is contained in a plane through the origin. However, for[2] $t > 0$, the choice of blow-up sequence yields $\text{dist}(0, \hat{K}(t) = 1$. This is a contradiction, as a plane with a hole must vanish instantly under the flow. □

Theorem 15.7 (Sheeting Theorem). *Suppose that $\mathcal{M}_i$ is a blowup of an smooth flow, with the limit a static multiplicity 2 plane. Then the convergence is smooth on compact sets of space-time.*
The same holds for quasi-static multiplicity planes, with the conclusion restricted to $t < 0$.

Proof. By choosing a blowup sequence correctly, we can arrange that the limit is at most ϵ away from a static multiplicity 2 plane for $r \geq 1$, but it is exactly ϵ away from a static multiplicity 2 plane at the scale $r = 1$ (note that the smooth flow is close to a multiplicity 1 plane for very small scales, so there is some intermediate scale which is bounded away from the multiplicity 2 plane). Now, the key lemma implies that the blow-up sequence is either two parallel planes each with multiplicity 1, or else it is a multiplicity two plane. The latter cannot occur, because we have bounded the scale $r = 1$ away from a multiplicity two plane.

This implies that (locally) the two sheets of the flow separate and converge with unit multiplicity, thus smoothly. □

The proof given for curve shortening flow in §7 readily generalizes to this case to give a contradiction, showing that (quasi-)static multiplicity two planes do not arise as blow-up limits for mean-convex mean curvature flows. Note that we can immediately rule out static multiplicity two planes by using the strong maximum principle in space-time.

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