The Pigeonhole Principle

The **pigeonhole principle** is the following:

If *m* objects are placed into *n* bins, where $m > n$, then some bin contains at least two objects.

(We proved this in Lecture #02)

Why This Matters

- The pigeonhole principle can be used to show a surprising number of results must be true because they are “too big to fail.”
- Given a large enough number of objects with a bounded number of properties, eventually at least two of them will share a property.
- The applications are extremely deep and thought-provoking.

Using the Pigeonhole Principle

- To use the pigeonhole principle:
  - Find the *m* objects to distribute.
  - Find the $n < m$ buckets into which to distribute them.
  - Conclude by the pigeonhole principle that there must be two objects in some bucket.
- The details of how to proceeds from there are specific to the particular proof you're doing.

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$$\left.\begin{array}{|r|}\hline 1\\\hline 11\\\hline 111\\\hline 1111\\\hline 11111\\\hline 111111\\\hline 1111111\\\hline 11111111\\\hline 111111111\\\hline 1111111111\\\hline\end{array}\right\}\text{There are 10 objects here.}$$

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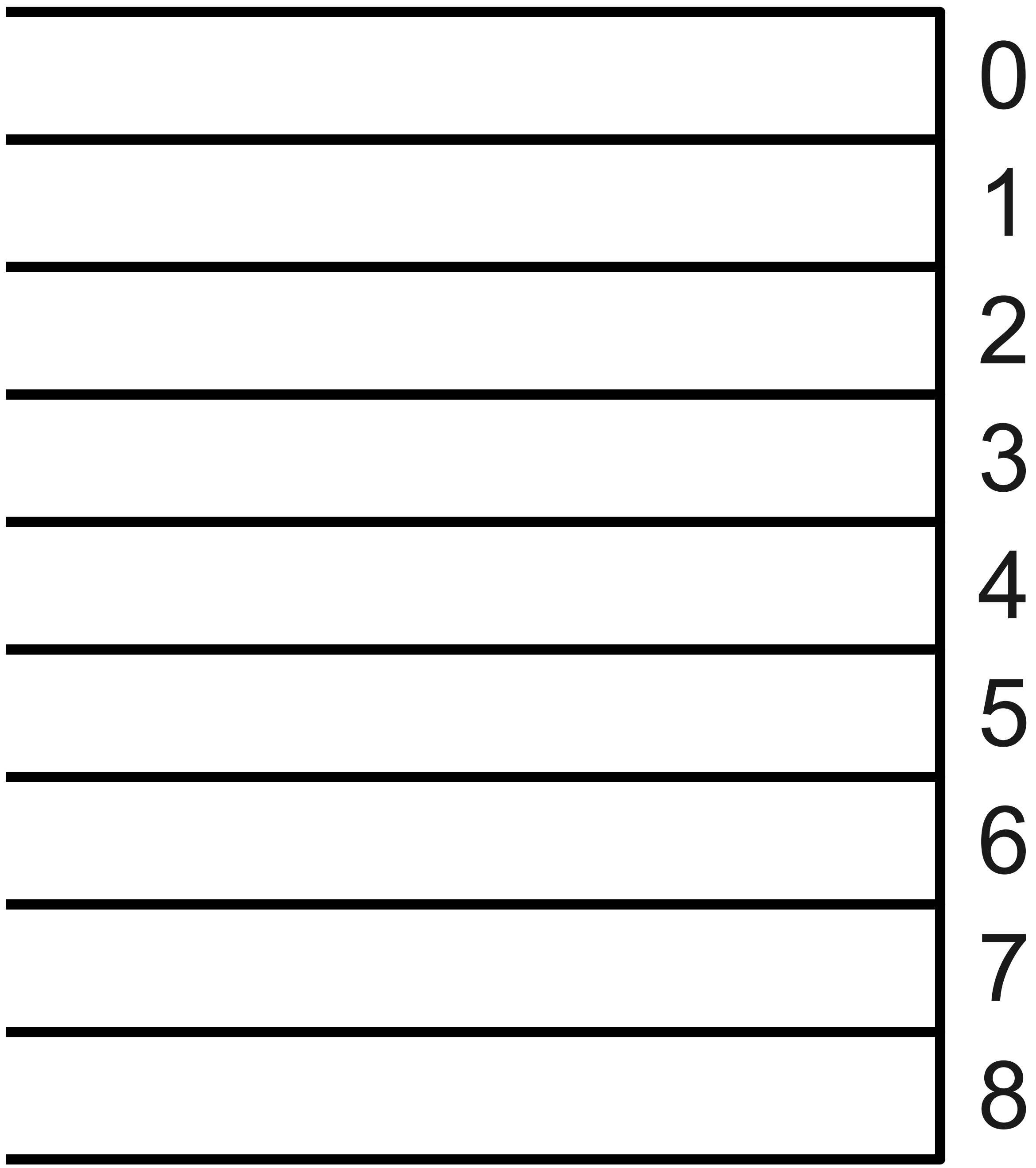
1
11
111
1111
11111
111111
1111111
11111111
111111111
1111111111



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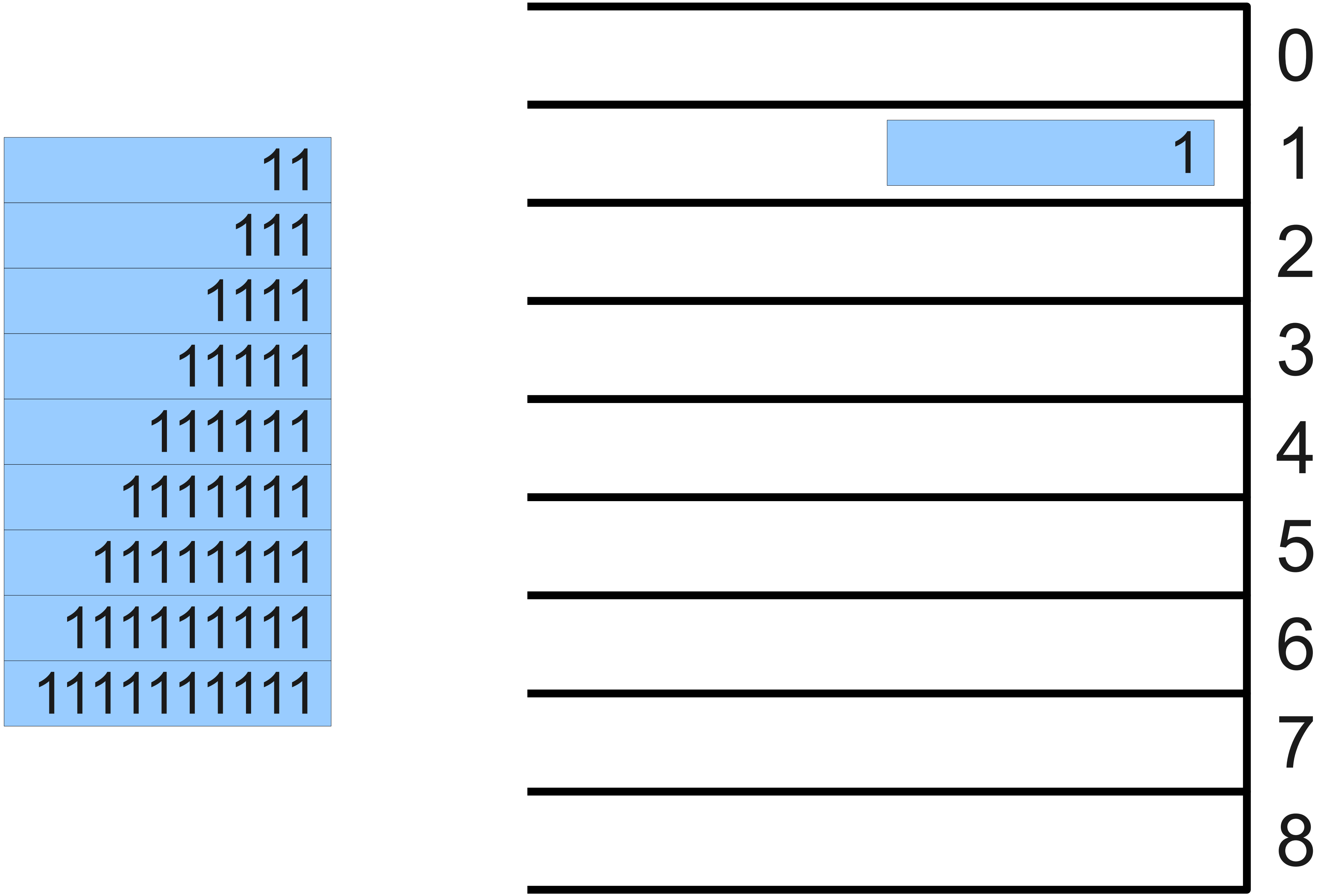

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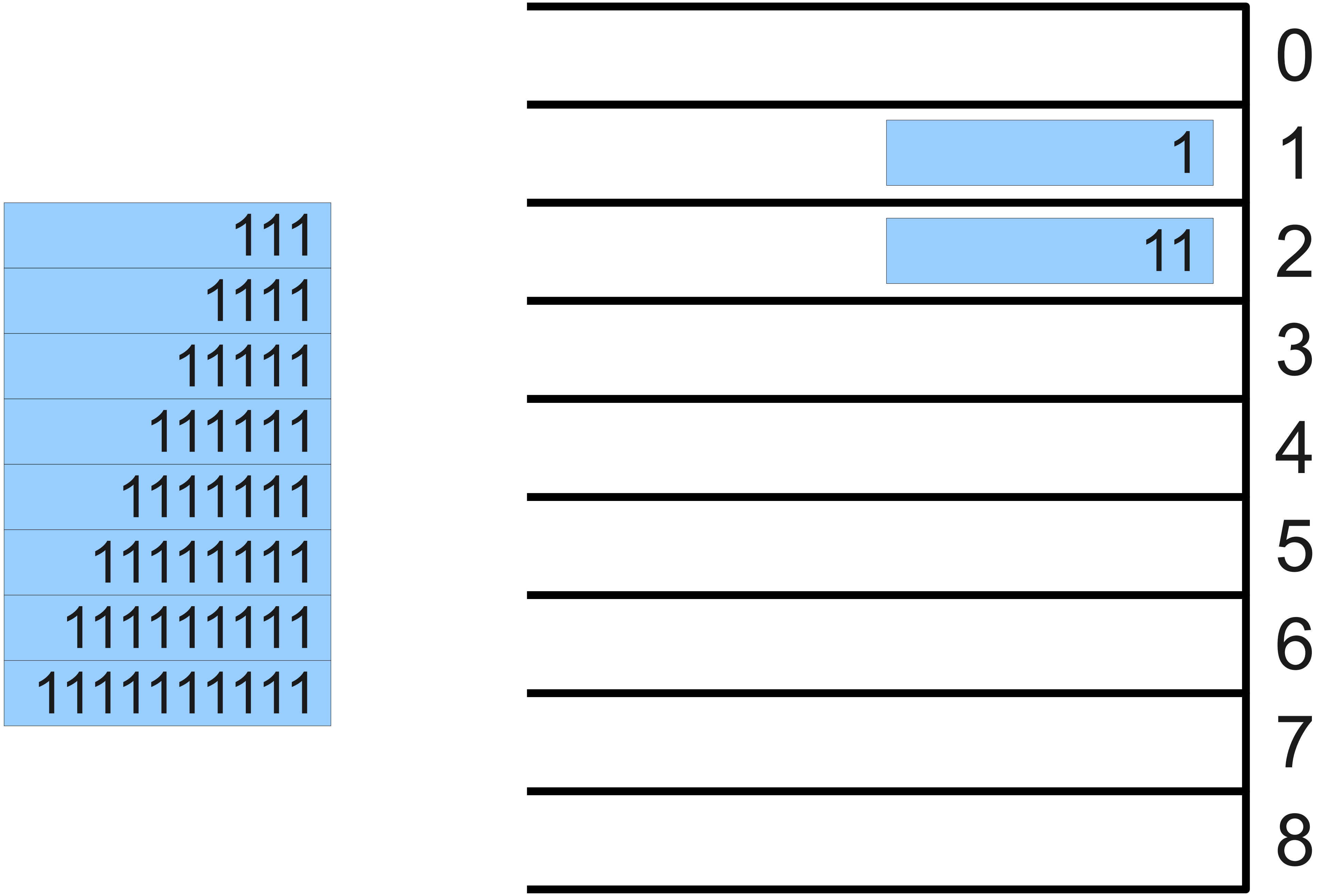

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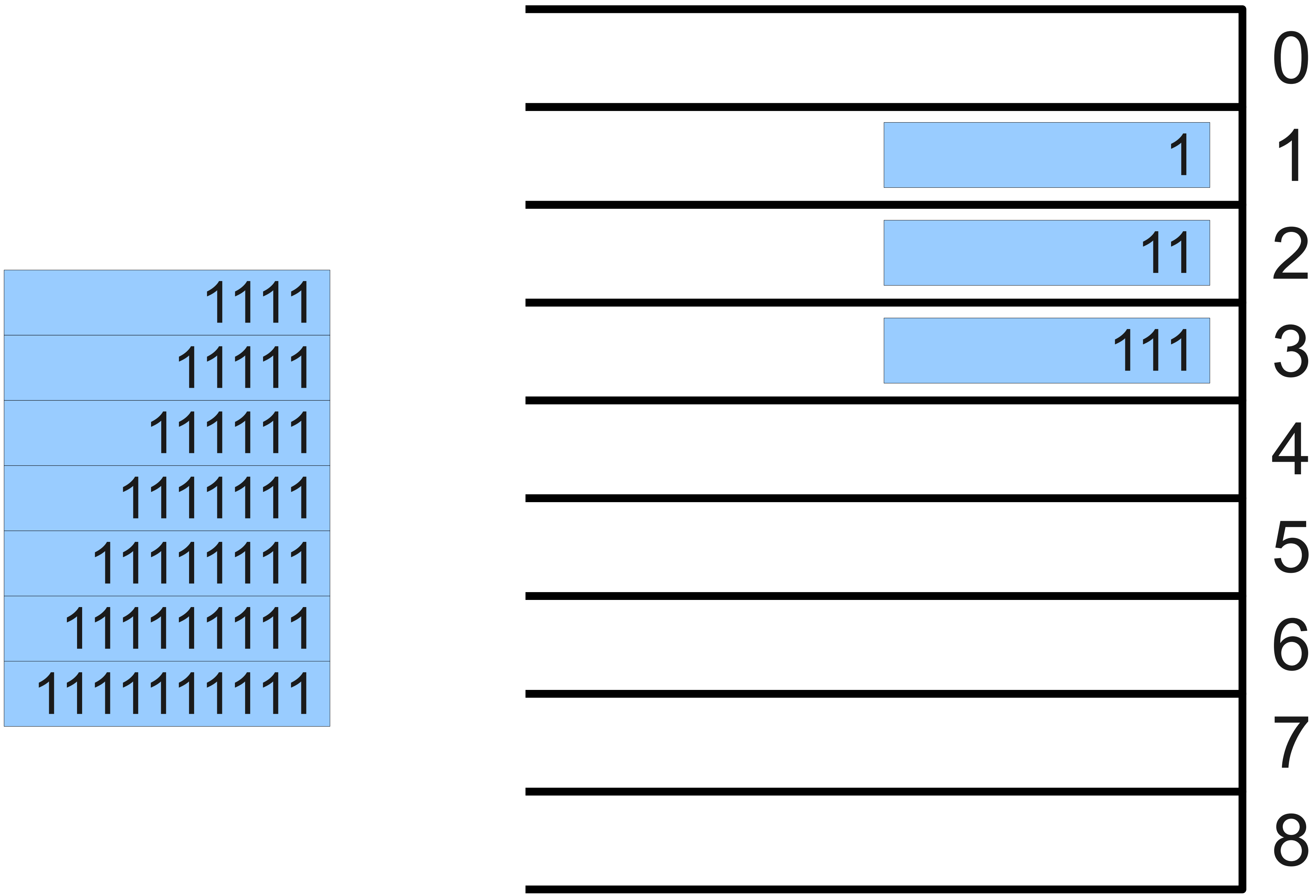

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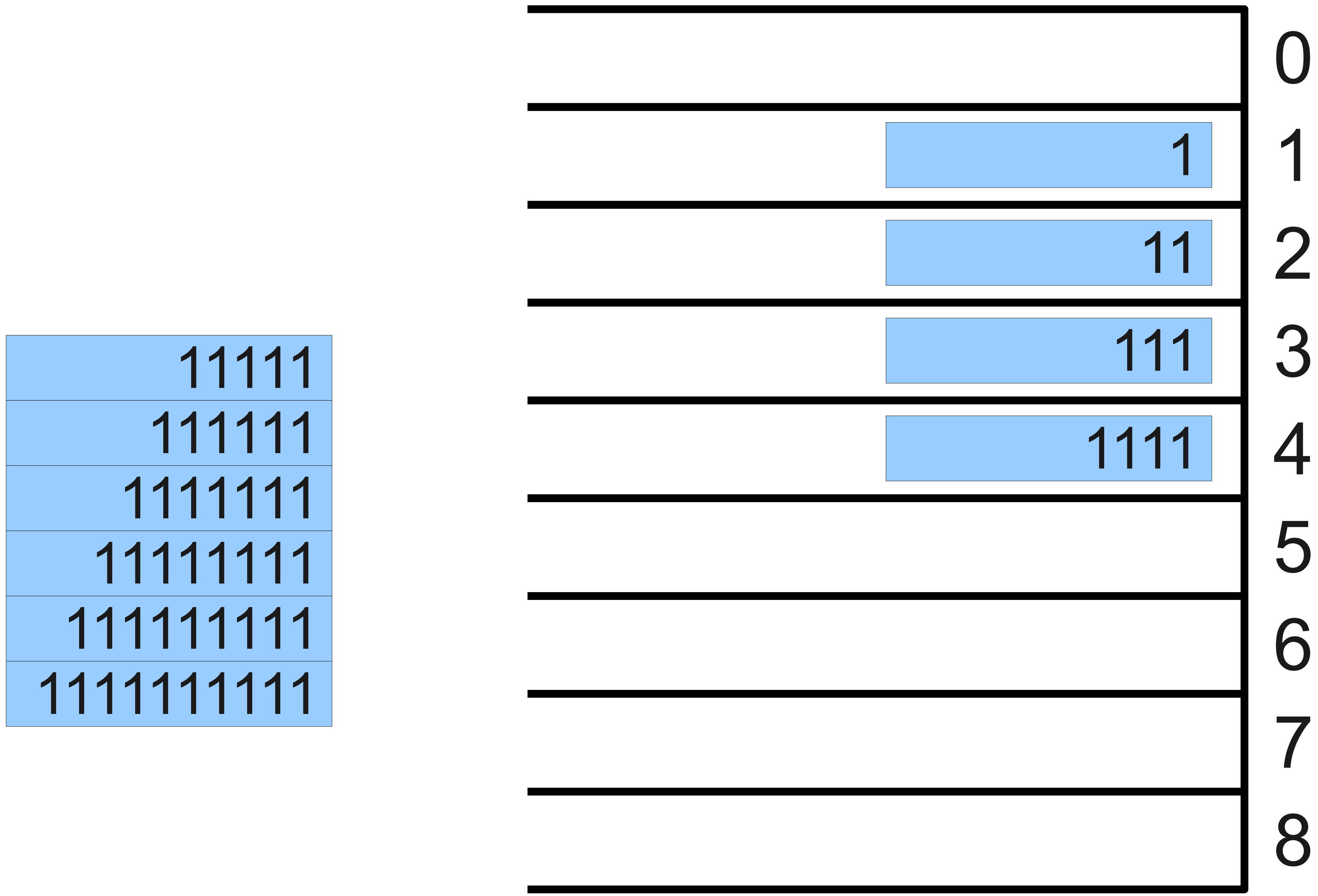

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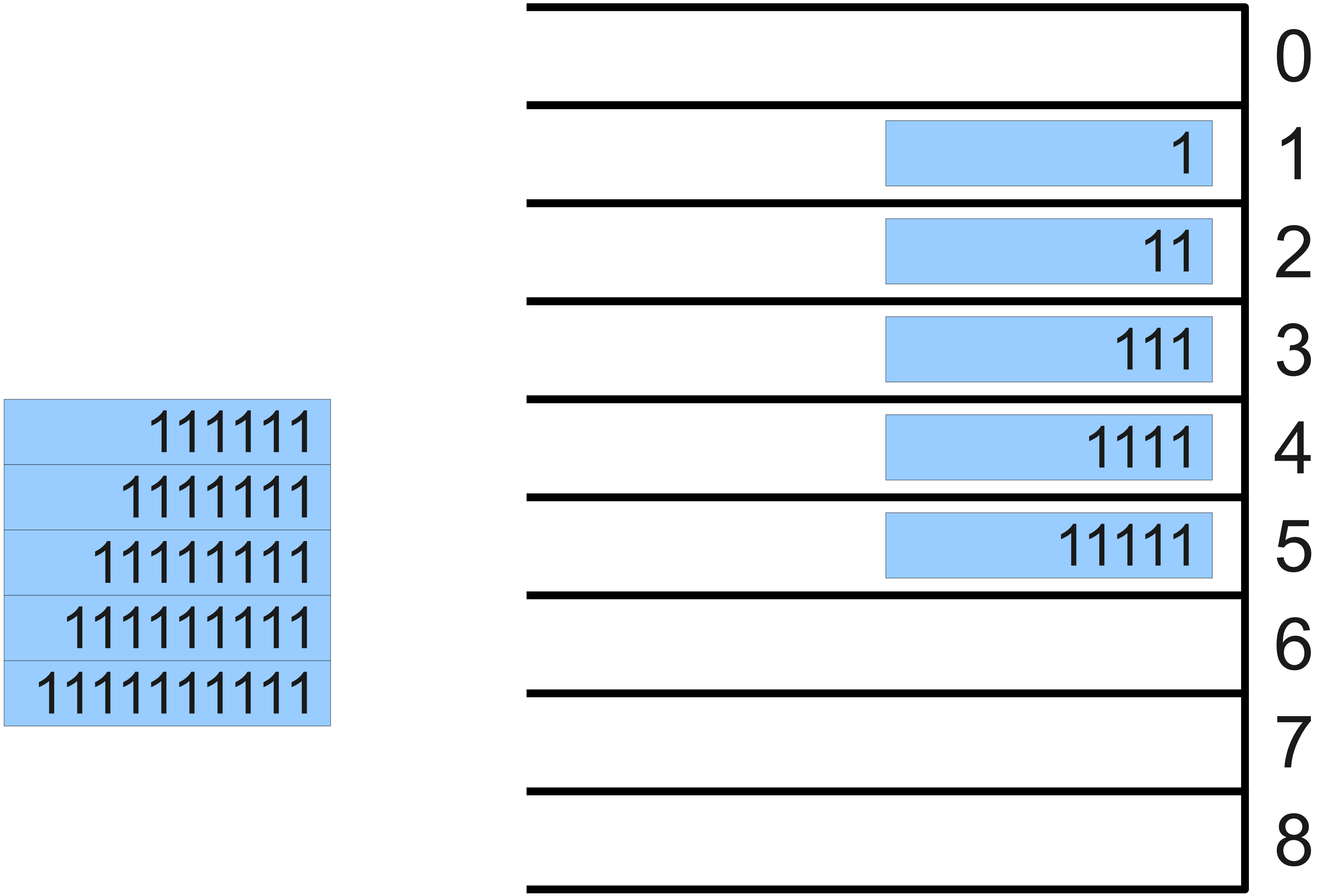

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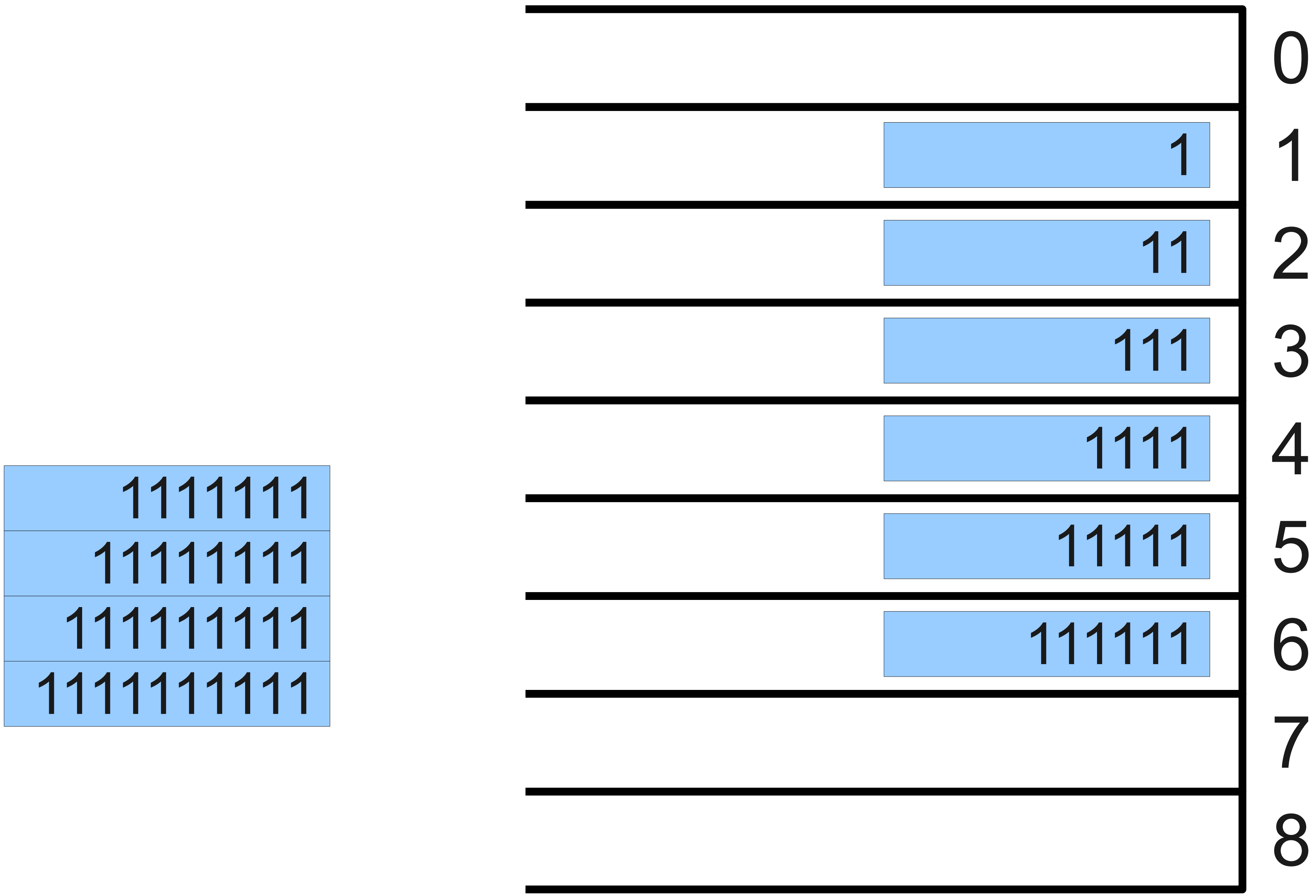

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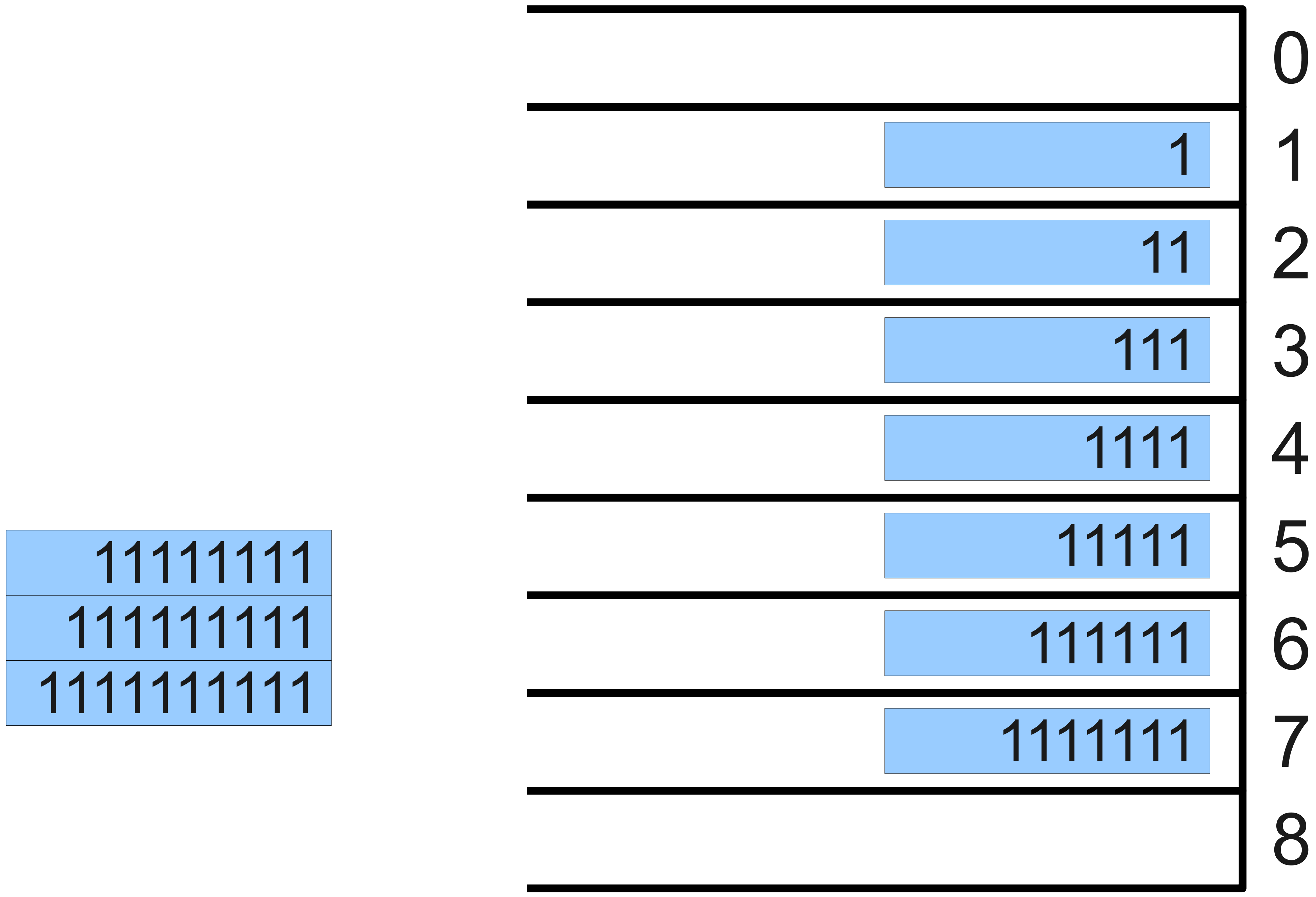

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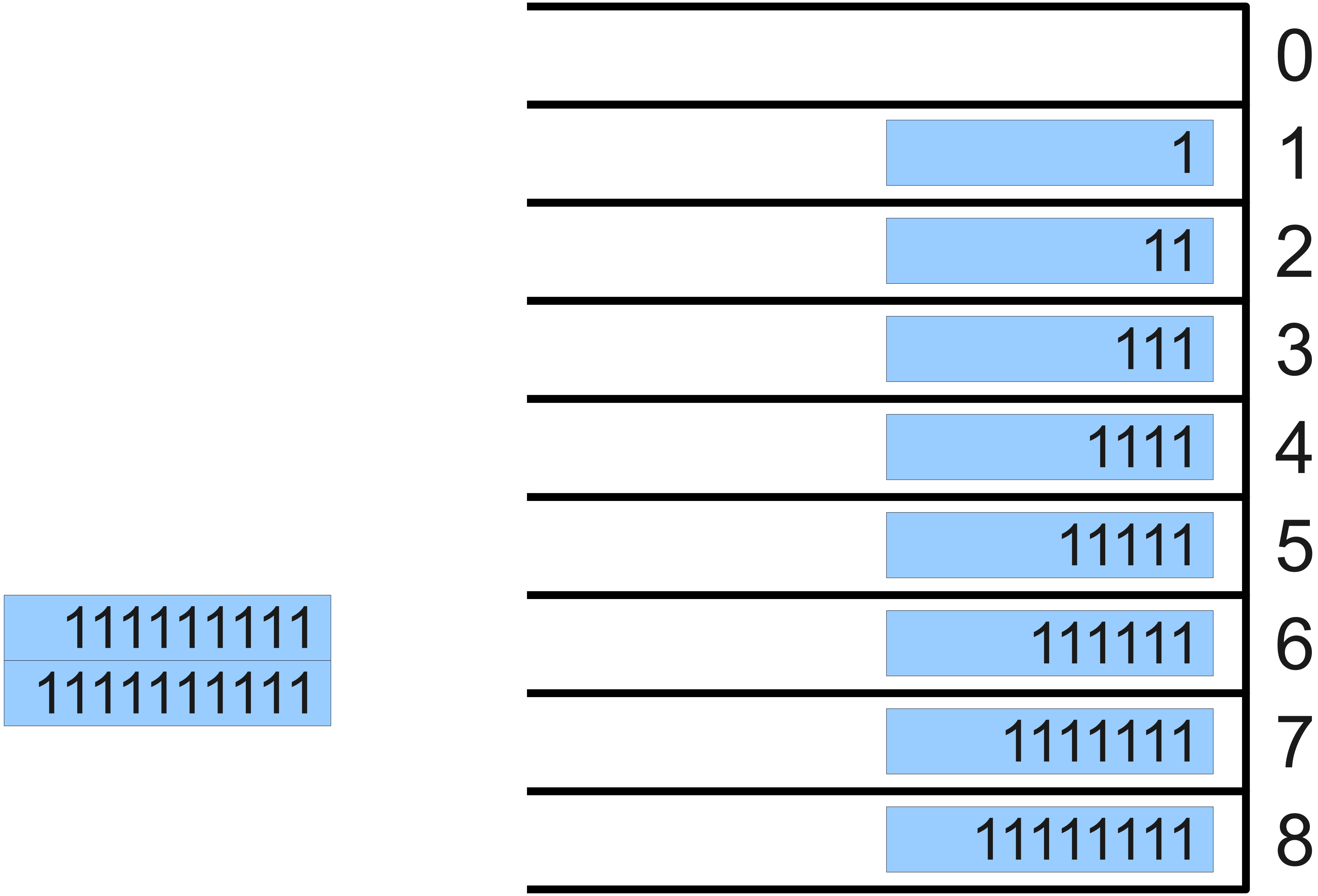

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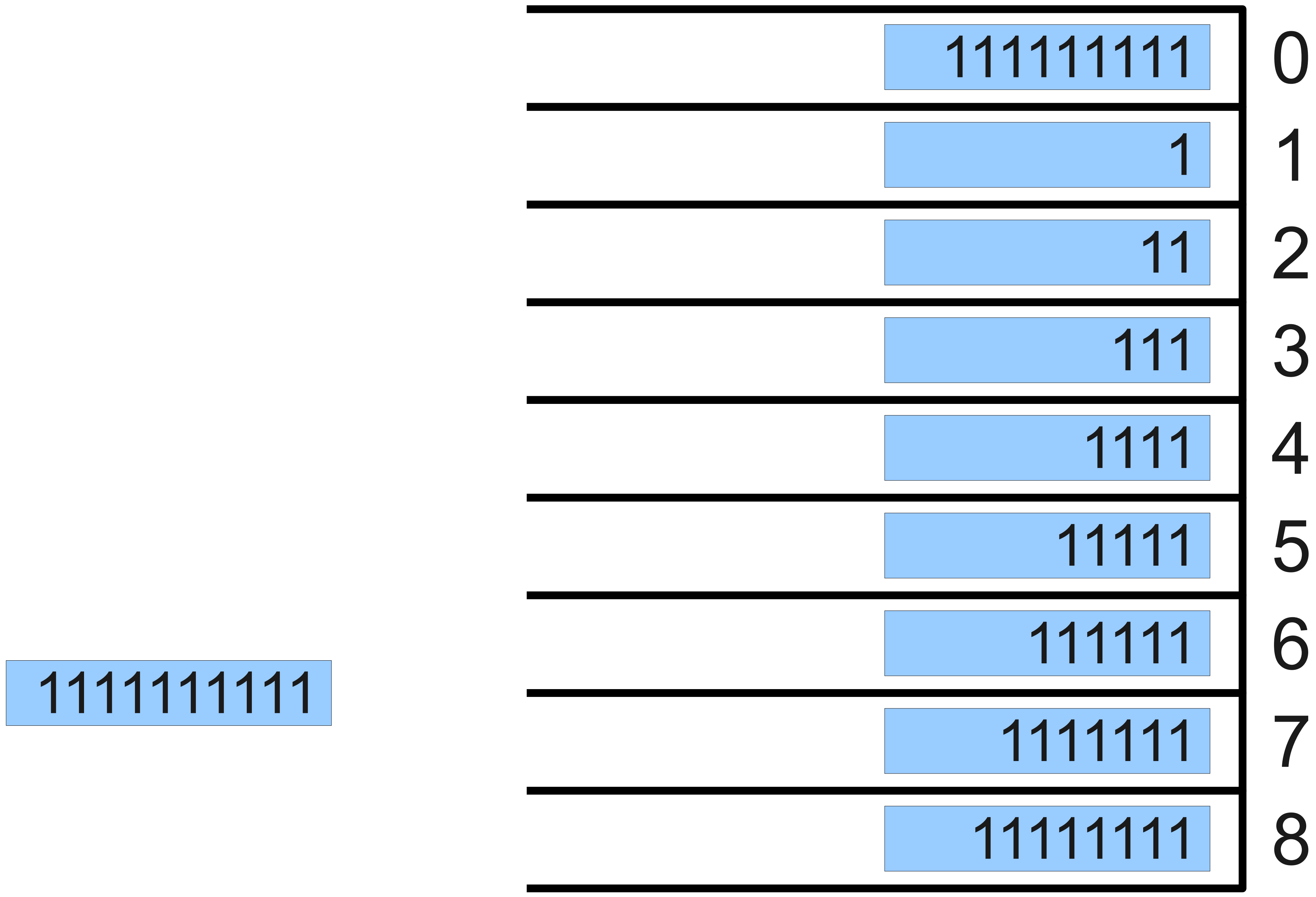

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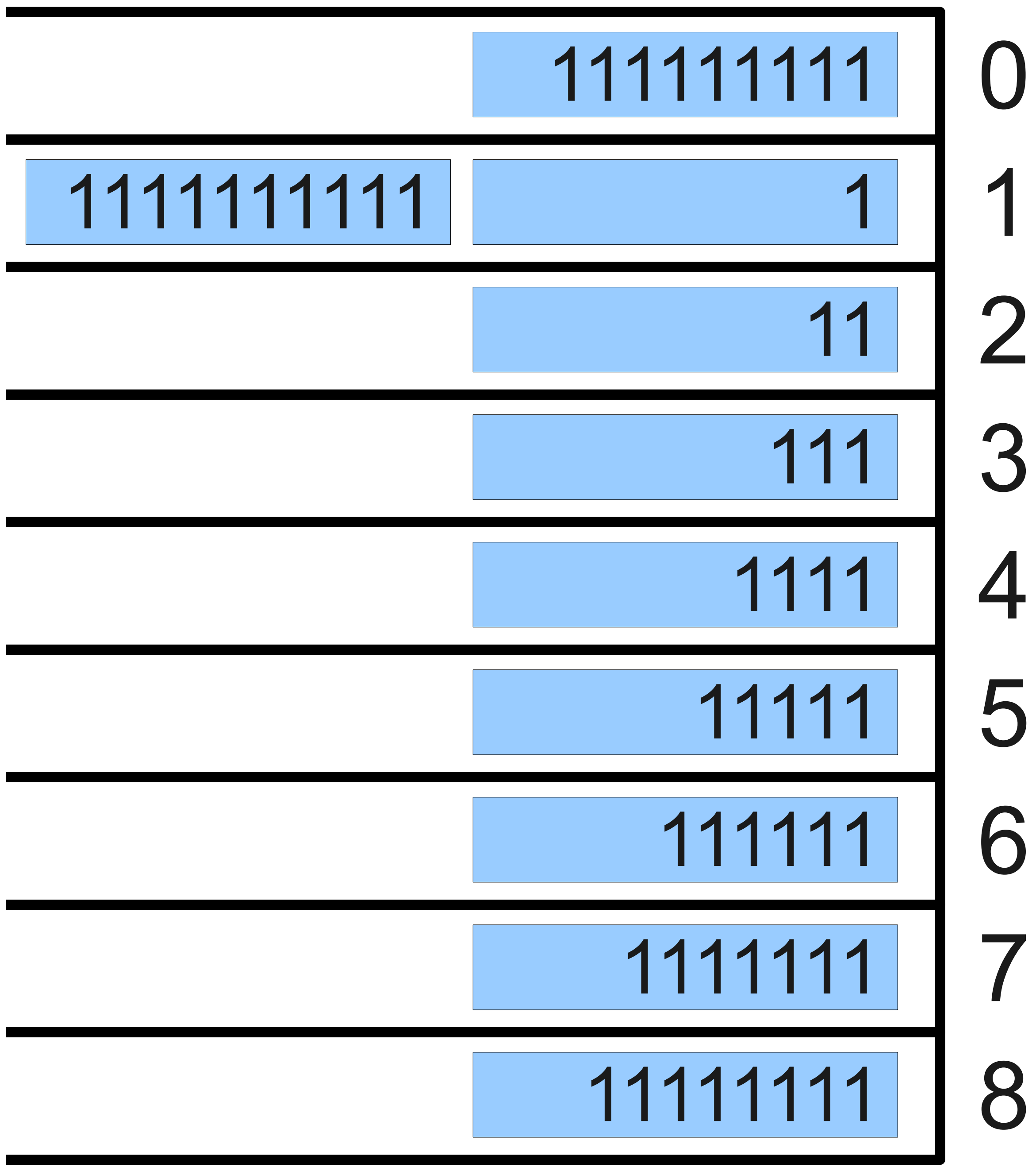

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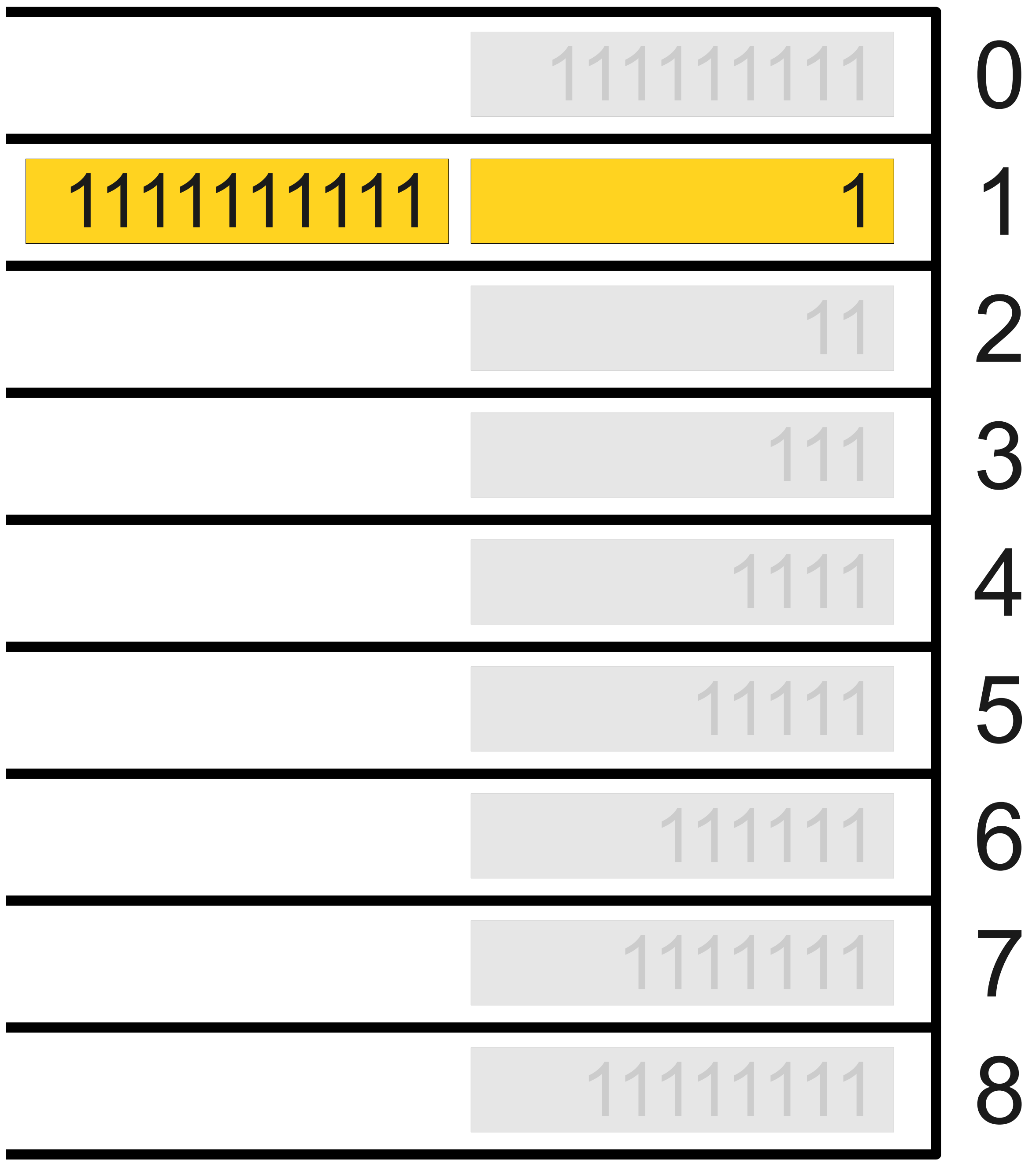

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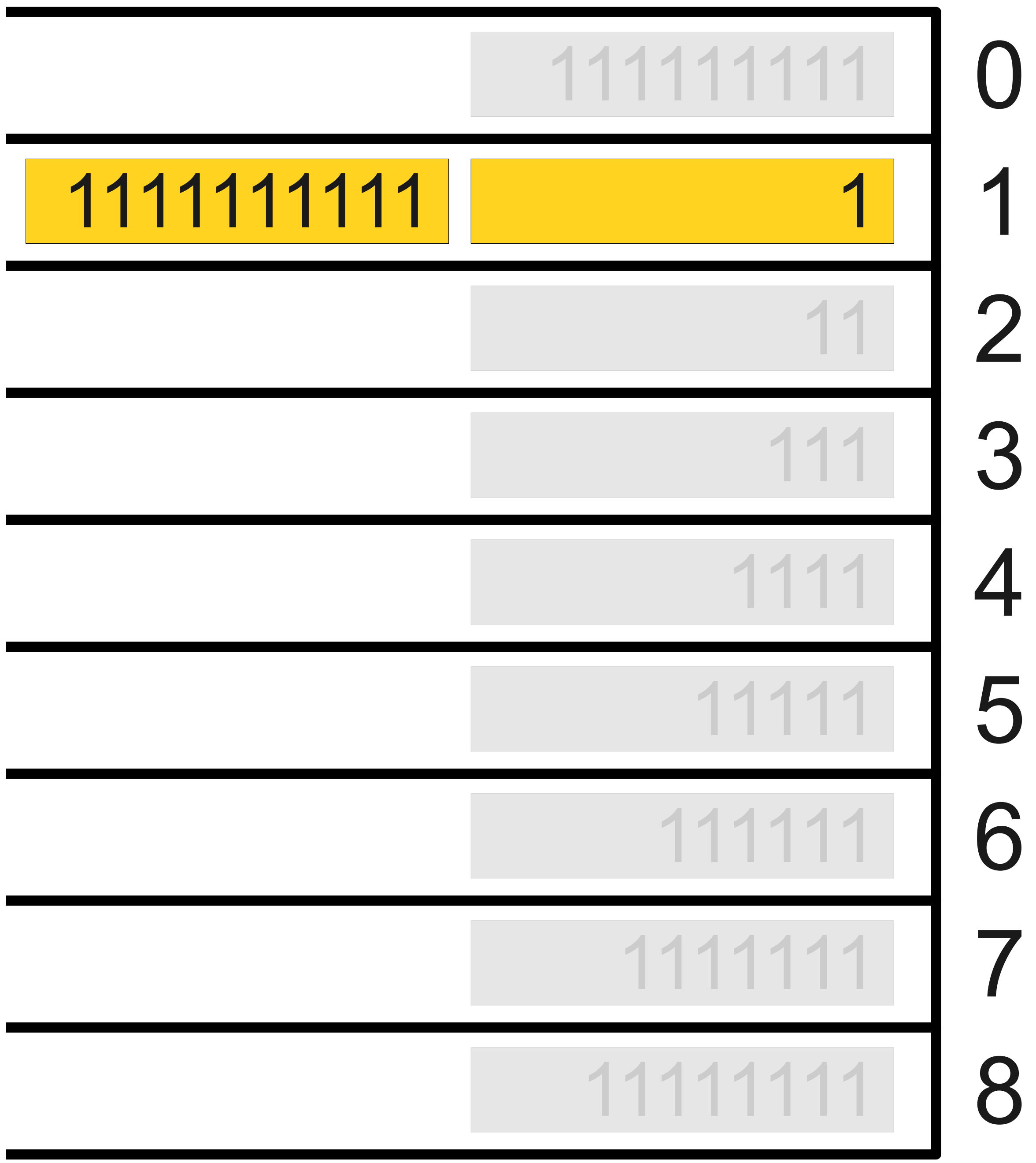

```
  1111111111
-          1
------------
  1111111110
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Proof Idea

- For any natural number $n \geq 2$ generate the numbers 1, 11, 111, … until $n + 1$ numbers are generated.
- There are n possible remainders modulo n, so two of these numbers have the same remainder.
- Their difference is a multiple of n.
- Their difference consists of just 1s and 0s.

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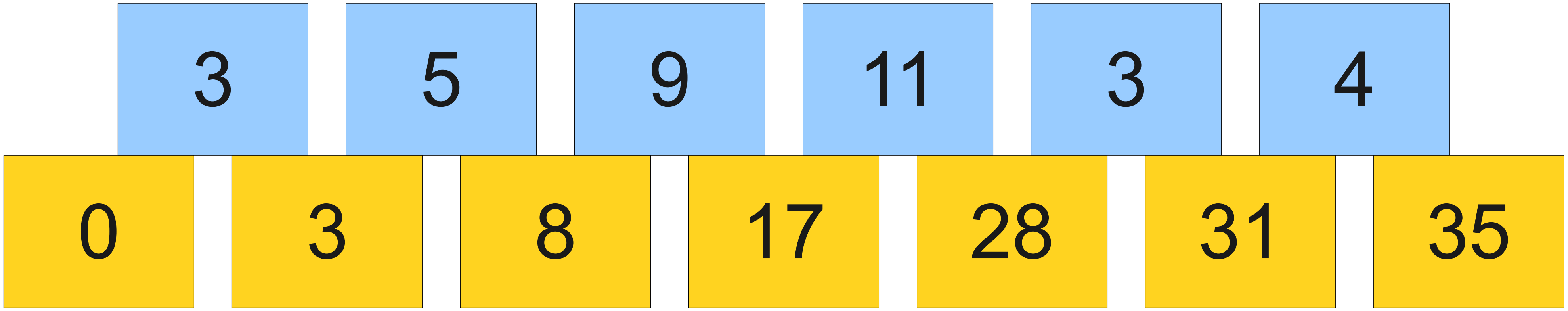

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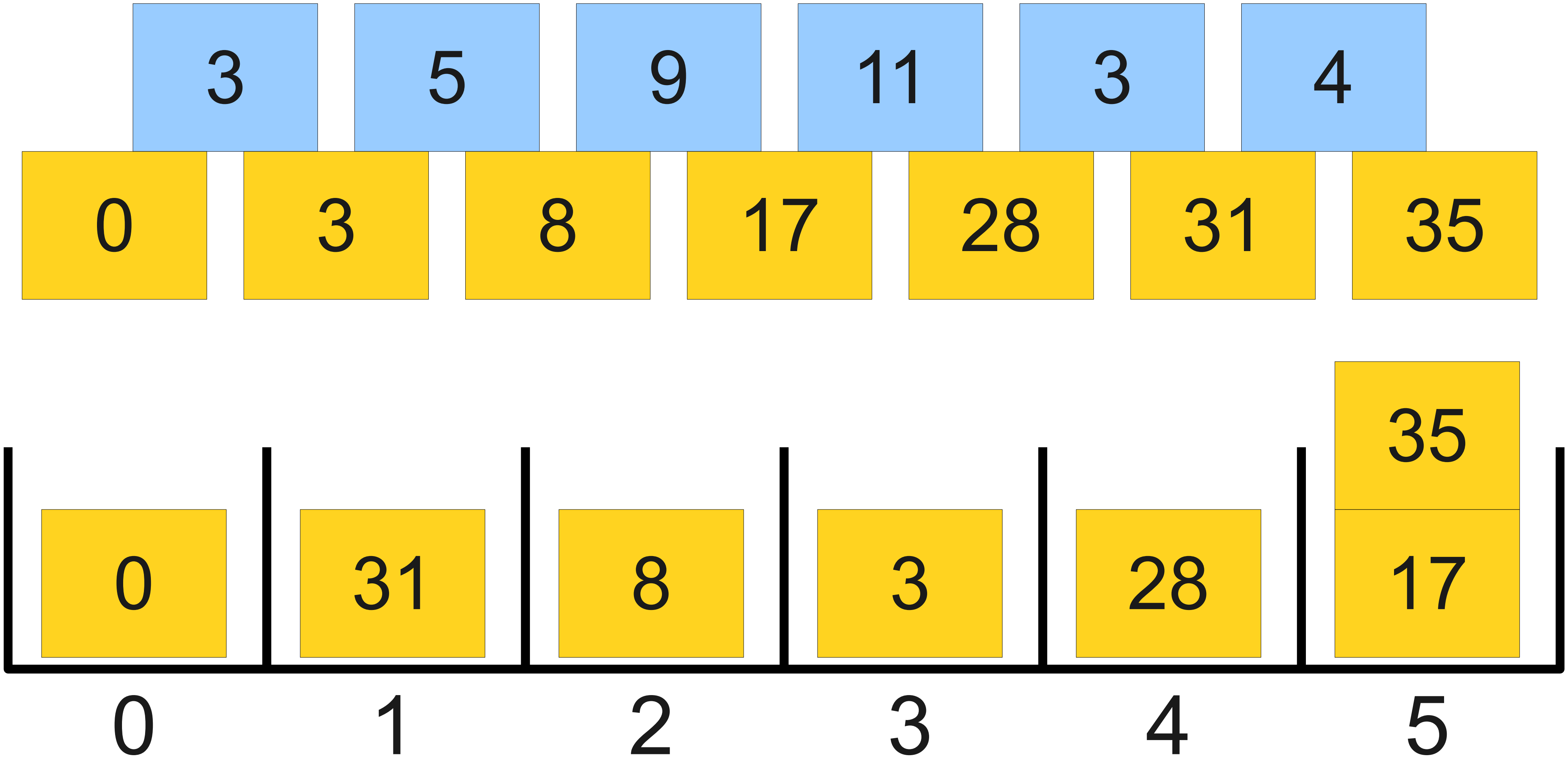

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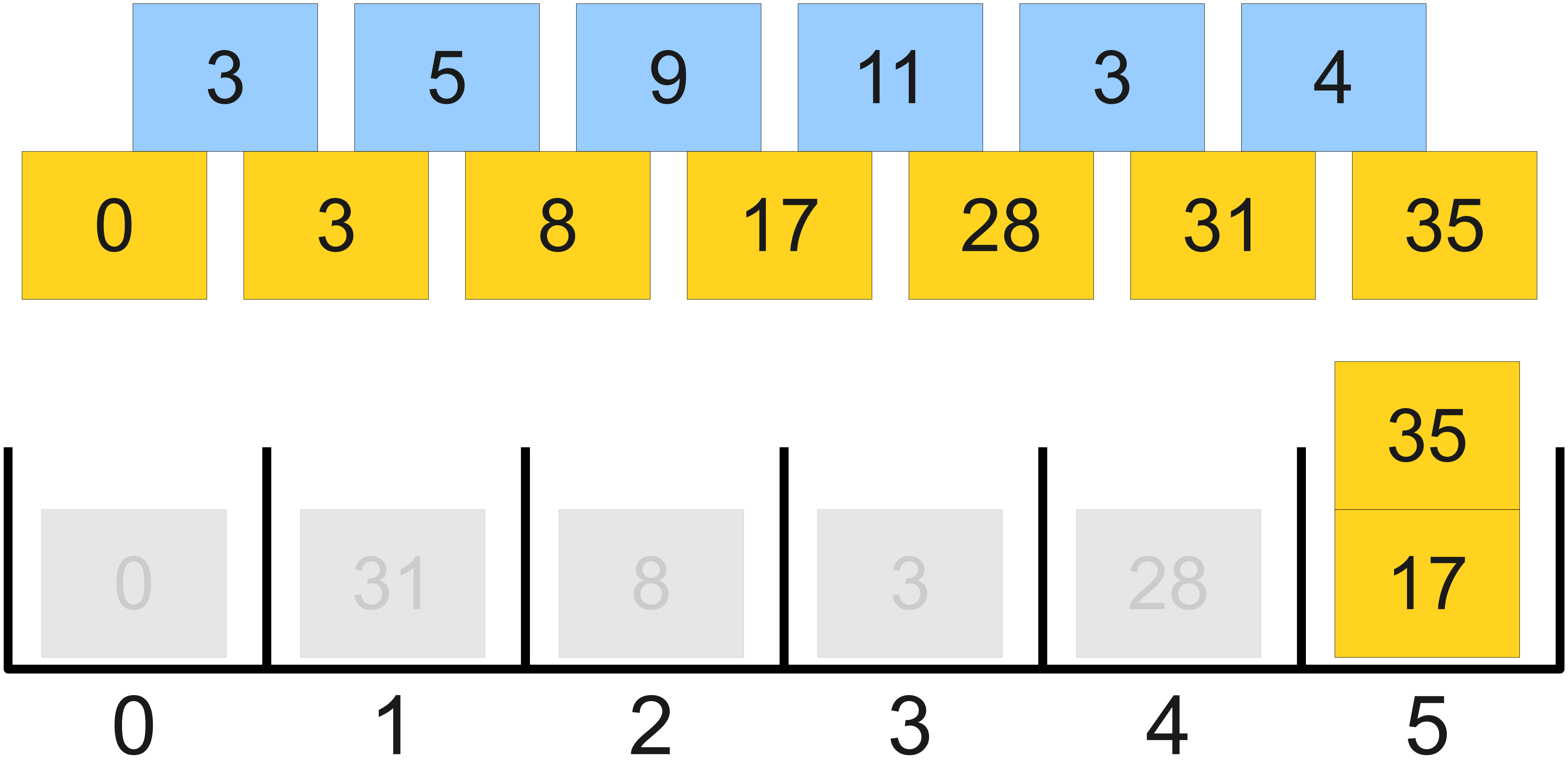

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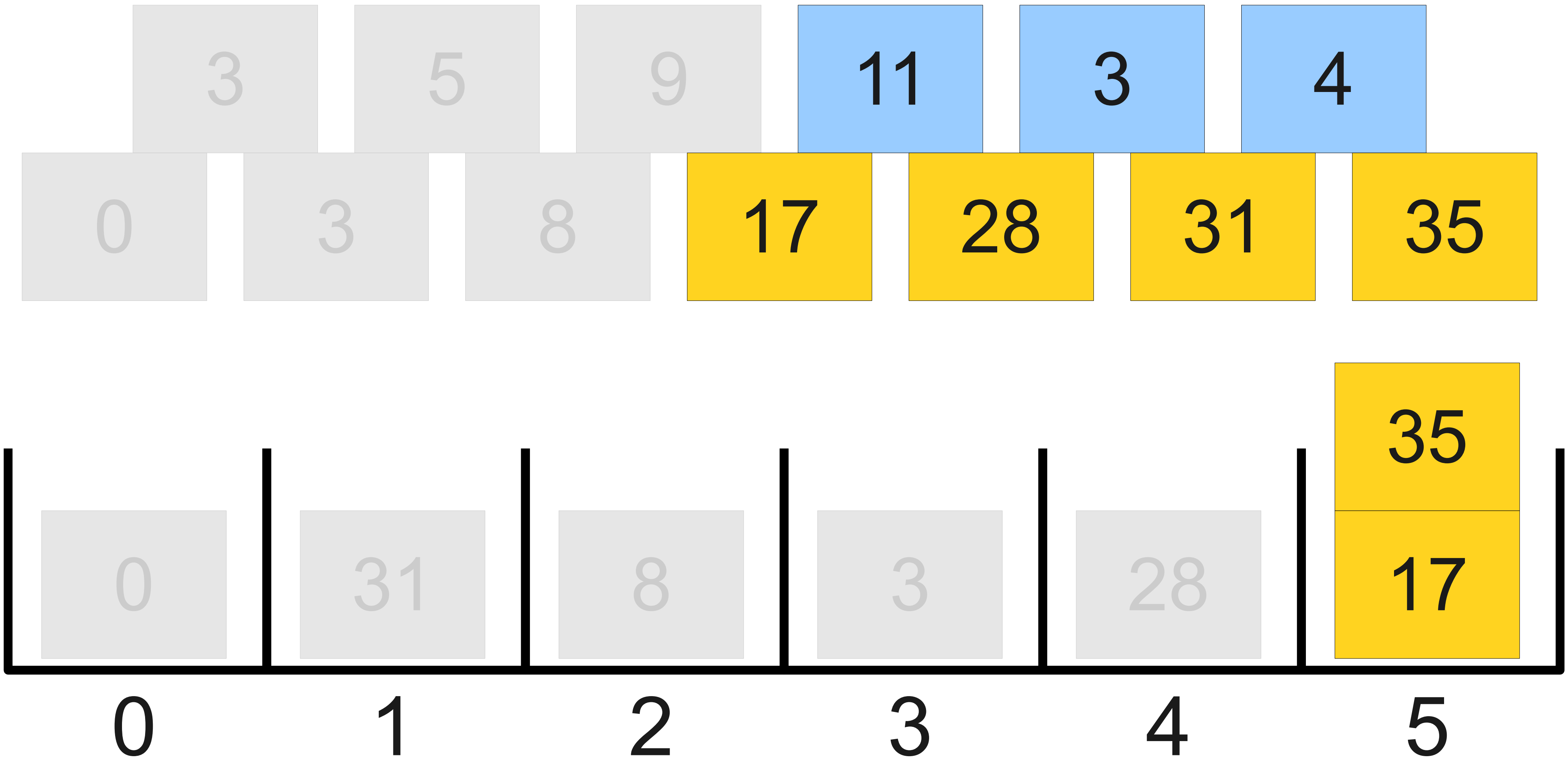

The Formalization

- Compute the sum of the first 0, first 1, first 2, …, first *n* elements.
  - This is a set of *n* + 1 integers.
- Consider the numbers of remainders modulo *n*.
  - There are *n* of them.
- Therefore, two partial sums must have the same remainder modulo *n*.
- Their difference is therefore 0 modulo *n*.
- The sum of the elements between these two points therefore is a multiple of *n*.

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Since $x > y$, we know $x \geq y + 1$.

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The Limits of Data Compression

The Pigeonhole Principle and Functions

- Consider a function $f : A \to B$ for finite sets A and B.
- If $|A| > |B|$, then by the pigeonhole principle some element of B has at least two elements of A that map to it.
- Thus f cannot be injective.

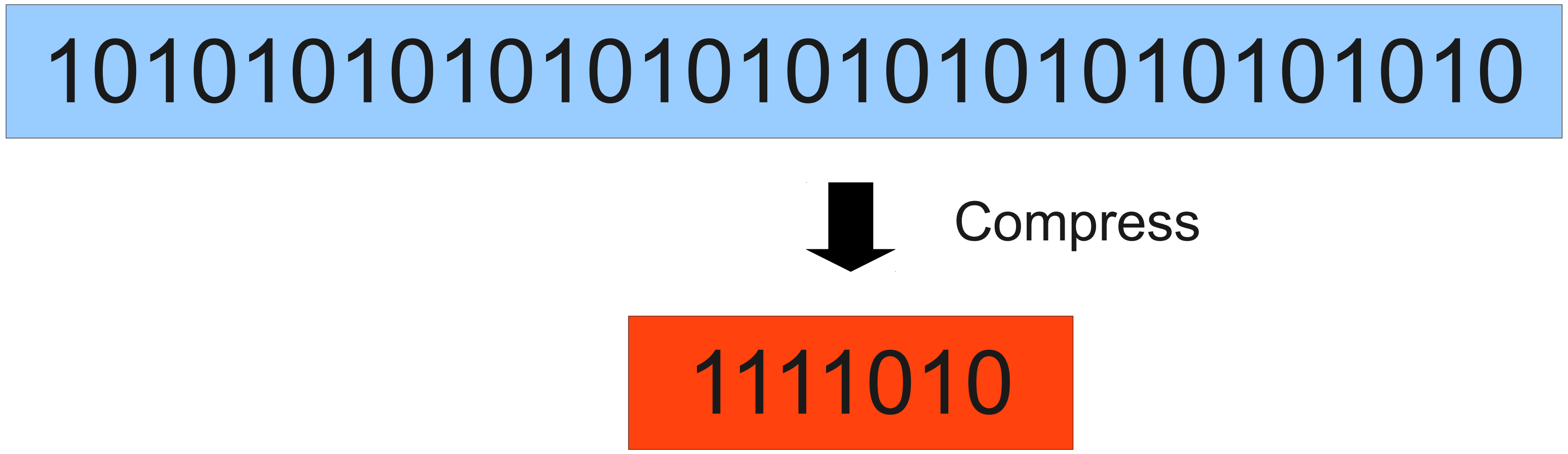
Bitstrings

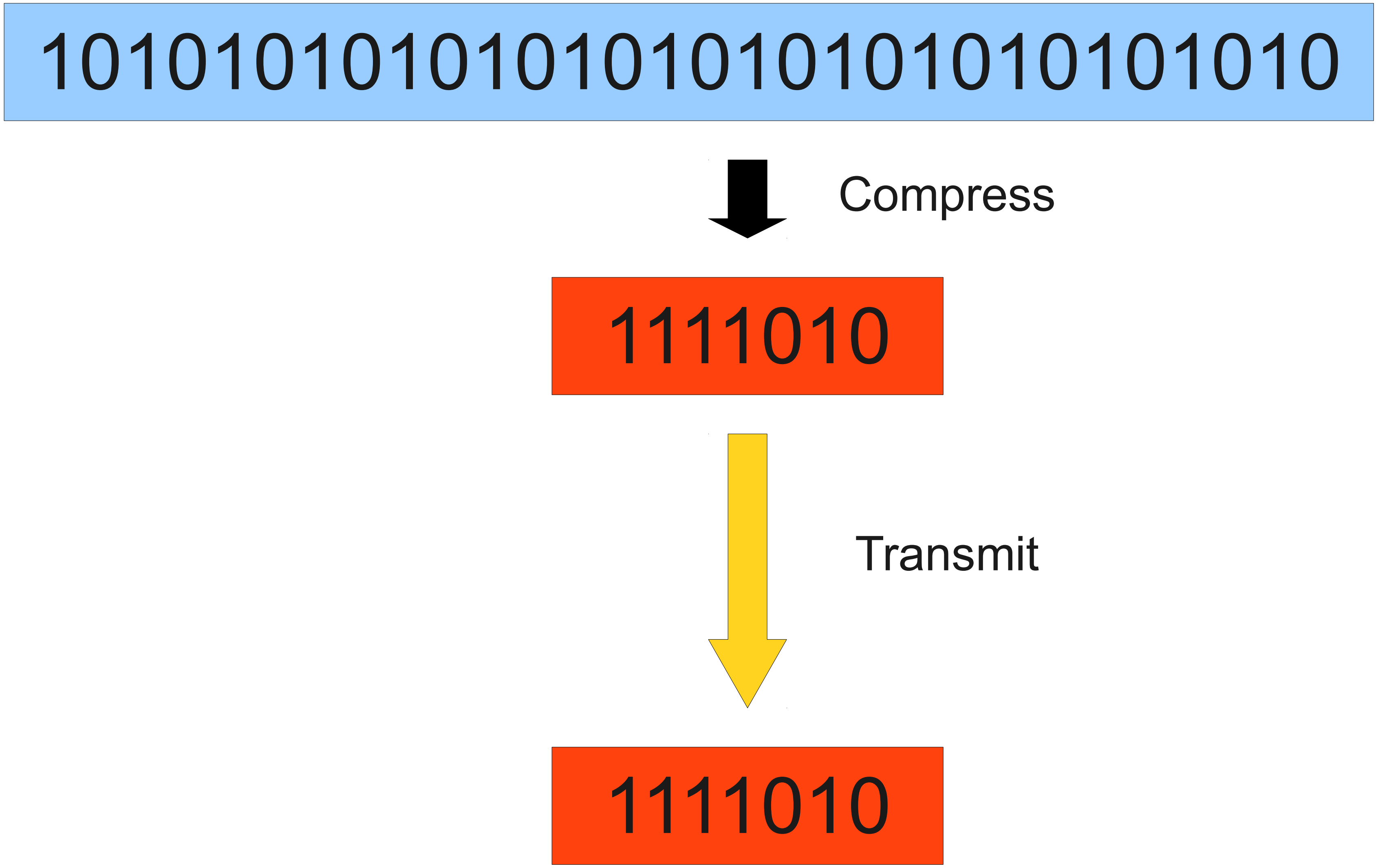
- A **bitstring** is a finite sequence of 0s and 1s.
- Examples:
  - 11011100
  - 010101010101
  - 0000
  - ε (the **empty string**)
- There are 2^n bitstrings of length exactly n.
- There are $2^0 + 2^1 + \dots + 2^{n-1} = 2^n - 1$ bitstrings of length less than n.
  - We proved this by induction earlier in the quarter.

Data Compression

- Inside a computer, all data are represented as sequences of 0s and 1s (bitstrings)
- To transfer data (across a network, on DVDs, on a flash drive, etc.), it is advantageous to try to reduce the number of 0s and 1s before transferring it.
- Most real-world data can be compressed by exploiting redundancies.
  - Text repeats common patterns ("the", "and", etc.)
  - Bitmap images use similar colors throughout the image.
- **Idea**: Replace each bitstring with a *shorter* bitstring that contains all the original information.
  - This is called **lossless data compression**.

101010101010101010101010101010


101010101010101010101010101010
Compress
1111010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Compress
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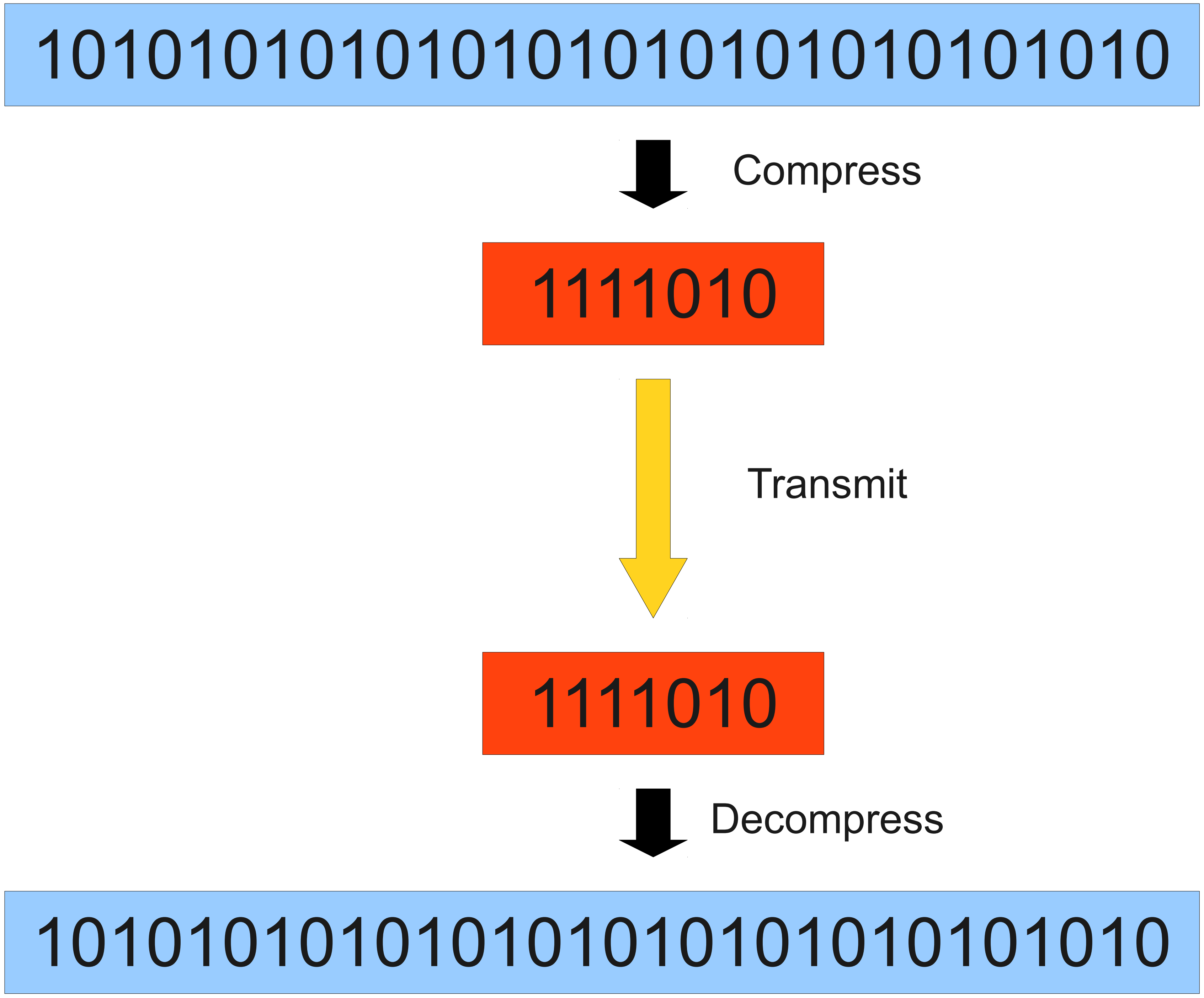

101010101010101010101010101010
Compress
1111010
Transmit
1111010
Decompress
101010101010101010101010101010

Lossless Data Compression

- In order to losslessly compress data, we need two functions:
  - A **compression function** C, and
  - A **decompression function** D.
- These functions must be inverses of one another: $D = C^{-1}$.
  - Otherwise, we can't uniquely encode or decode some bitstring.
- Since C is invertible, C must be a bijection.

A Perfect Compression Function

- Ideally, the compressed version of a bitstring would always be shorter than the original bitstring.
- **Question**: Can we find a lossless compression algorithm that always compresses a string into a shorter string?
- To handle the issue of the empty string (which can't get any shorter), let's assume we only care about strings of length at least 10.

A Counting Argument

- Let $\mathbb{B}^n$ be the set of bitstrings of length n, and $\mathbb{B}^{<n}$ be the set of bitstrings of length less than n.
- How many bitstrings of length n are there?
  - **Answer**: 2^n
- How many bitstrings of length *less than n* are there?
  - **Answer:** $2^0 + 2^1 + \ldots + 2^{n-1} = 2^n - 1$
- Using our earlier result, by the pigeonhole principle, there cannot be an injective function from $\mathbb{B}^n$ to $\mathbb{B}^{<n}$.
- Since every bijection is an injection, there is no bijection between $\mathbb{B}^n$ and $\mathbb{B}^{<n}$.
- **There is no perfect compression function!**

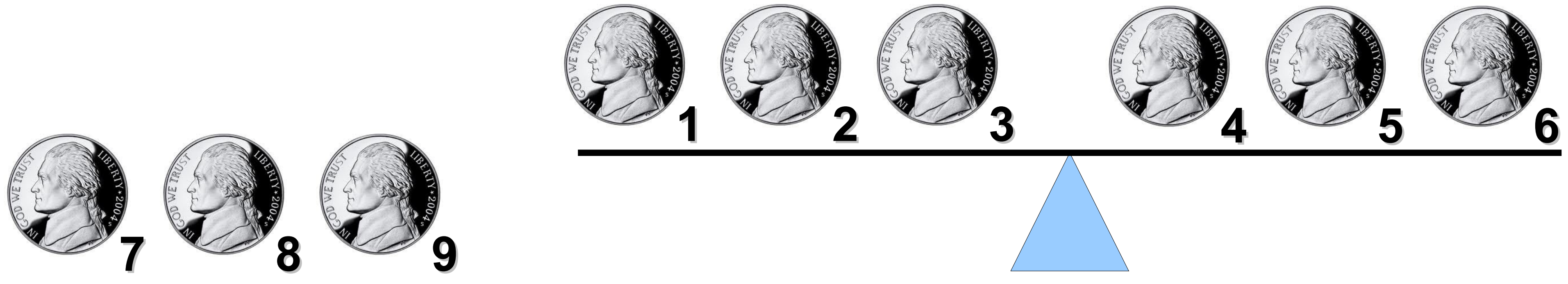
Why this Result is Interesting

- Our result says that no matter how hard we try, it is **impossible** to compress every string into a shorter string.
- No matter how clever you are, you cannot write a lossless compression algorithm that always makes strings shorter.
- In practice, only highly redundant data can be compressed.
- The fields of **information theory** and **Kolmogorov complexity** explore the limits of compression; if you're interested, go explore!

The Limits of Counterfeit Detection

The Counterfeit Coin Problem

- Given 3^n coins, one of which weighs more than the rest, find that coin with at most *n* weighings on a balance.



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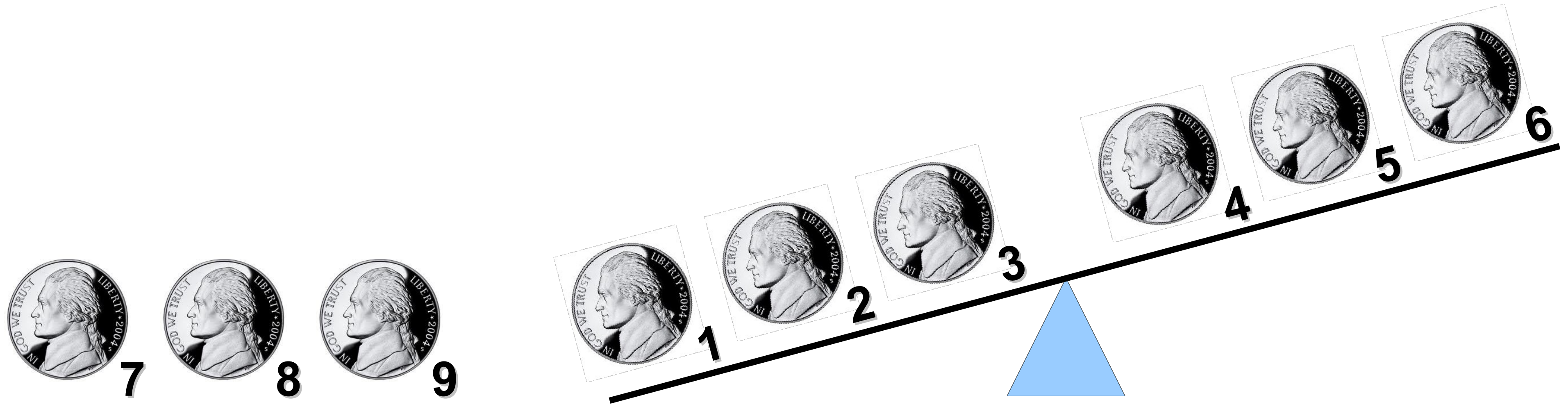

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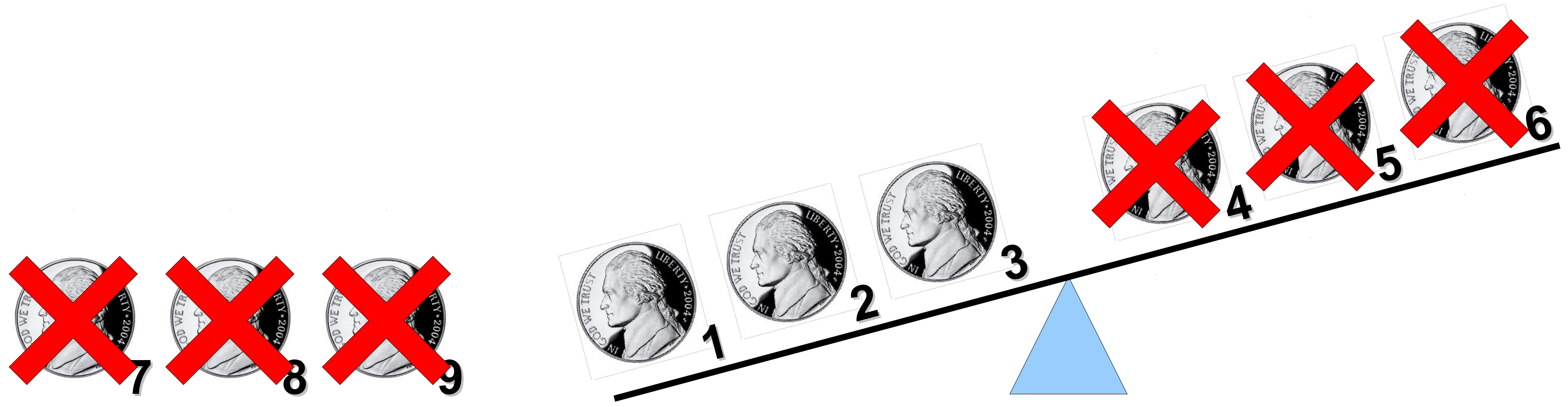

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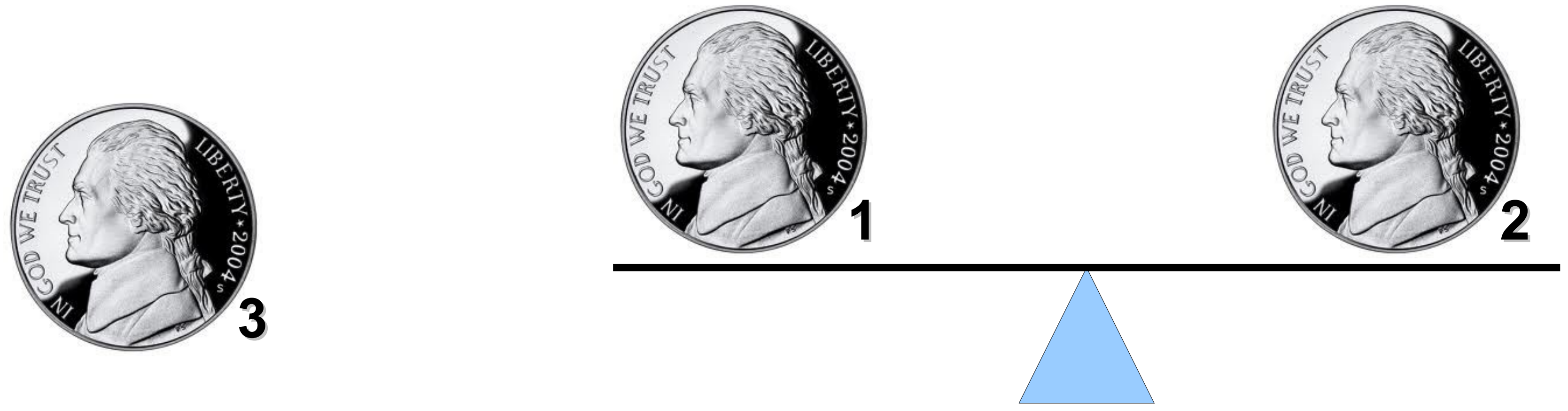
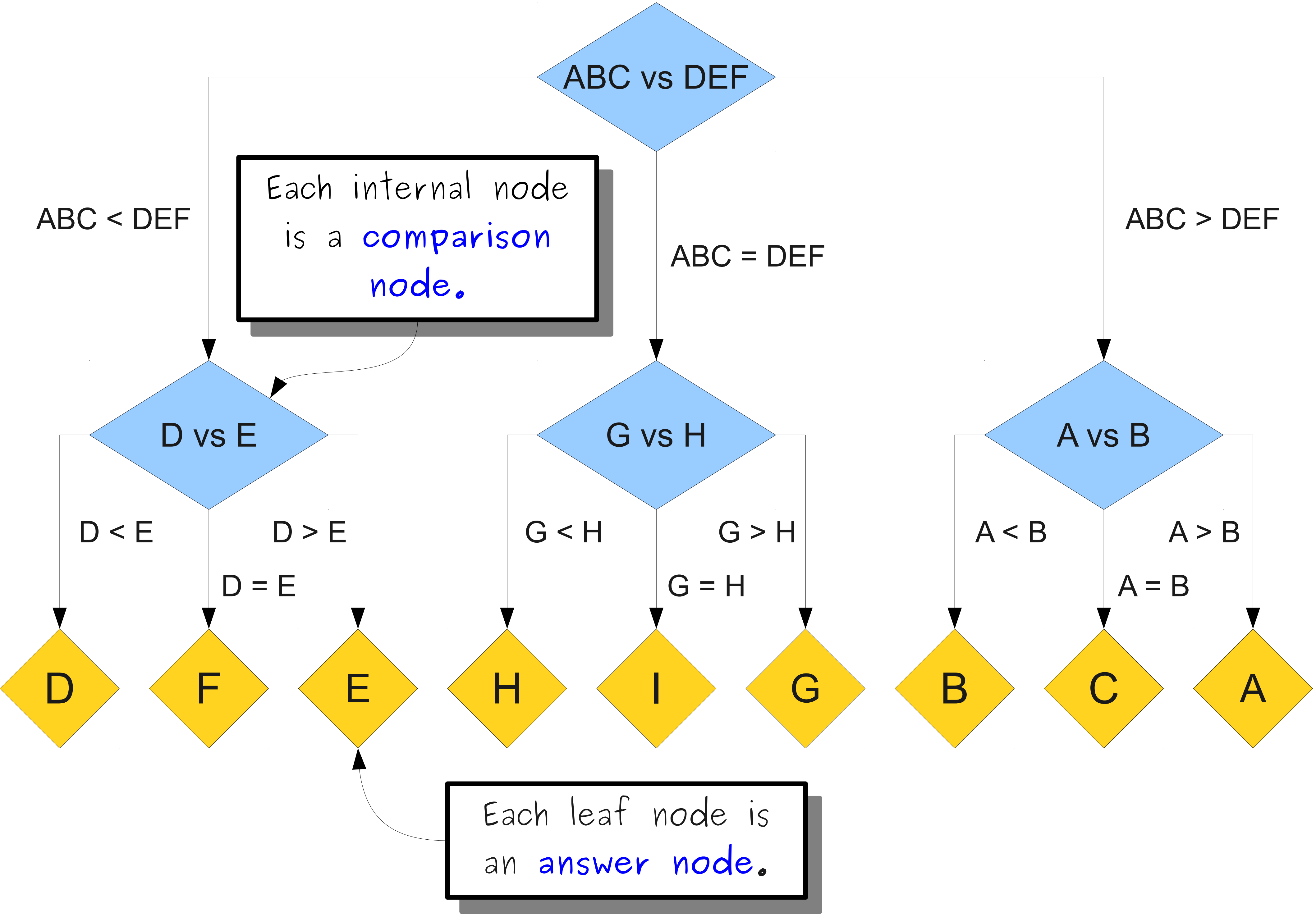


The Counterfeit Coin Problem

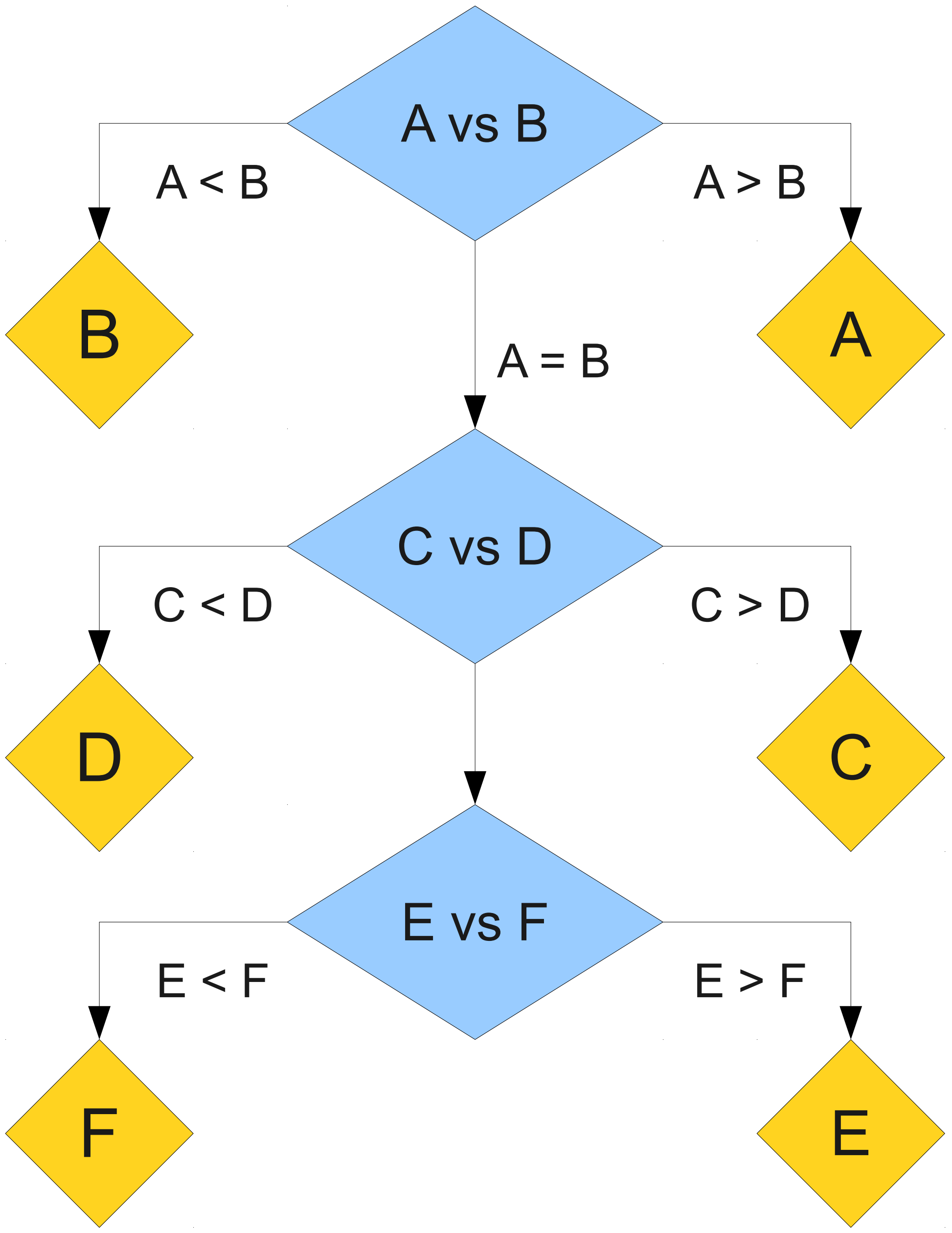
- Given 3^n coins, one of which weighs more than the rest, find that coin with at most *n* weighings on a balance.
- **Question**: Can we find the counterfeit coin out of *more* than 3^n coins using just *n* weighings?

Modeling an Algorithm

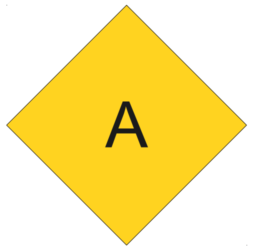
- In order to reason about the maximum number of coins, we need to find some way to reason about all possible algorithms for finding the coin.
- Main assumption: The only operation we can perform on the coins is weighing them on the scale.
    - We can't test their density, give them to the Secret Service, etc.
- We'll call such an algorithm a **comparison-based algorithm**, since the only way of distinguishing coins is through comparisons.


ABC vs DEF
ABC < DEF
ABC = DEF
ABC > DEF
Each internal node is a comparison node.
D vs E
G vs H
A vs B
D < E
D = E
D > E
G < H
G = H
G > H
A < B
A = B
A > B
D
F
E
H
I
G
B
C
A
Each leaf node is an answer node.

An Algorithm for Six Coins



An Algorithm for One Coin



Reasoning about Algorithms

- In this setup, every algorithm corresponds to a tree structure consisting of **comparisons** and **answers**.
- Each comparison node produces one of three outputs.
- Each answer node immediately ends the algorithm with the answer.
- Reasoning about these structures will tell us about the counterfeit coin problem.

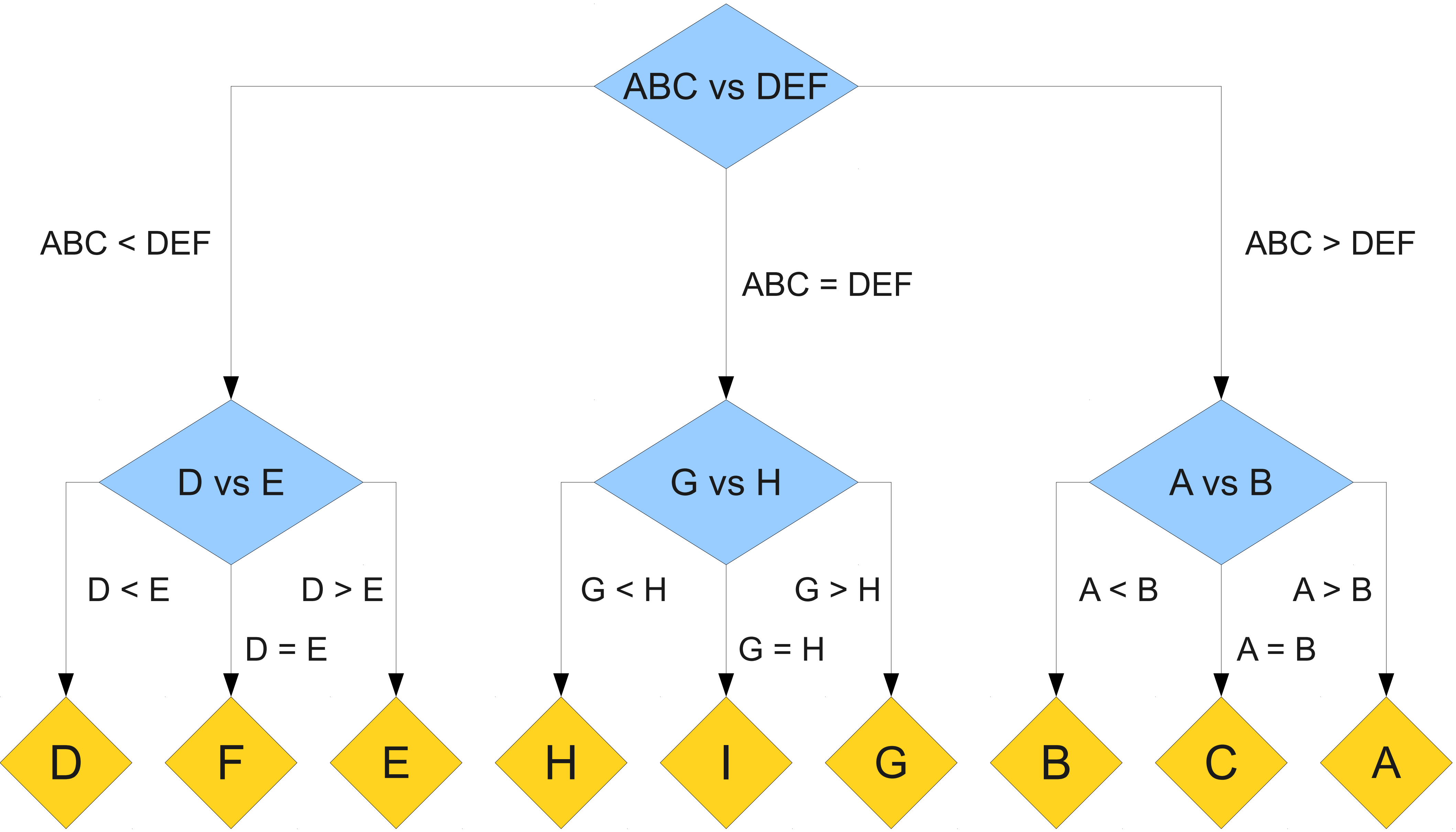
Reasoning about Inputs

- To be precise, we need to reason about the inputs to our algorithm.
- An input is a collection of *k* coins, exactly one of which is heavier than the rest.
  - It doesn't matter how much heavier it is; just that it weighs more than the rest.
- This means that there are exactly *k* possible inputs to the algorithm – one in which the first coin is counterfeit, one in which the second coin is counterfeit, etc.

A Critical Observation

- Suppose that we have an algorithm for finding which of *k* coins is counterfeit.
- There must be at least *k* answer nodes in the tree.
- Reasoning from the pigeonhole principle:
  - Run the algorithm on all *k* possible inputs.
  - Consider the set of counterfeit coins that arrive at each answer node in the tree.
  - If there are more coins than answer nodes, there must be two different coins that arrive at the same answer node.
  - The algorithm has to be wrong at least one of the two inputs.

How many answer nodes can there be in the tree?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D
F
E
H
I
G
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A

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Corollary 1: There is no comparison-based algorithm for finding which of $3^n + k$ coins is counterfeit in n weighings for any $k > 0$.

Proof: By our previous result, we need at least $\log_3 (3^n + k)$ comparisons to determine which of $3^n + k$ coins is heaviest. If $k > 0$, then $\log_3 (3^n + k) > \log_3 3^n = n$. Thus we need strictly more than n weighings to find which of the coins is counterfeit. ■

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Corollary 2: The comparison-based algorithm we developed in class is optimal.

What Just Happened?

- **This is our first theorem about the difficulty of a specific problem!**
- Our procedure was as follows:
    - Build a mathematical model of computation for finding the counterfeit coin.
    - Given the model, reason about the behavior of that model on various inputs.
    - Write a proof that formalizes our reasoning about that behavior.
- We will build many more models like this one later in the quarter.

Suppose you have a set of coins. There is a counterfeit coin among them that weighs more than the rest of the coins.

If you have *n* weighings, what is the largest number of coins for which you can solve this problem?

Suppose you have a set of coins. There **may be** a counterfeit coin among them **(though there doesn't have to be)**. If there is a counterfeit, it will weigh more than the rest of the coins.

If you have *n* weighings, what is the largest number of coins for which you can solve this problem?

The Possibly Counterfeit Coin Problem

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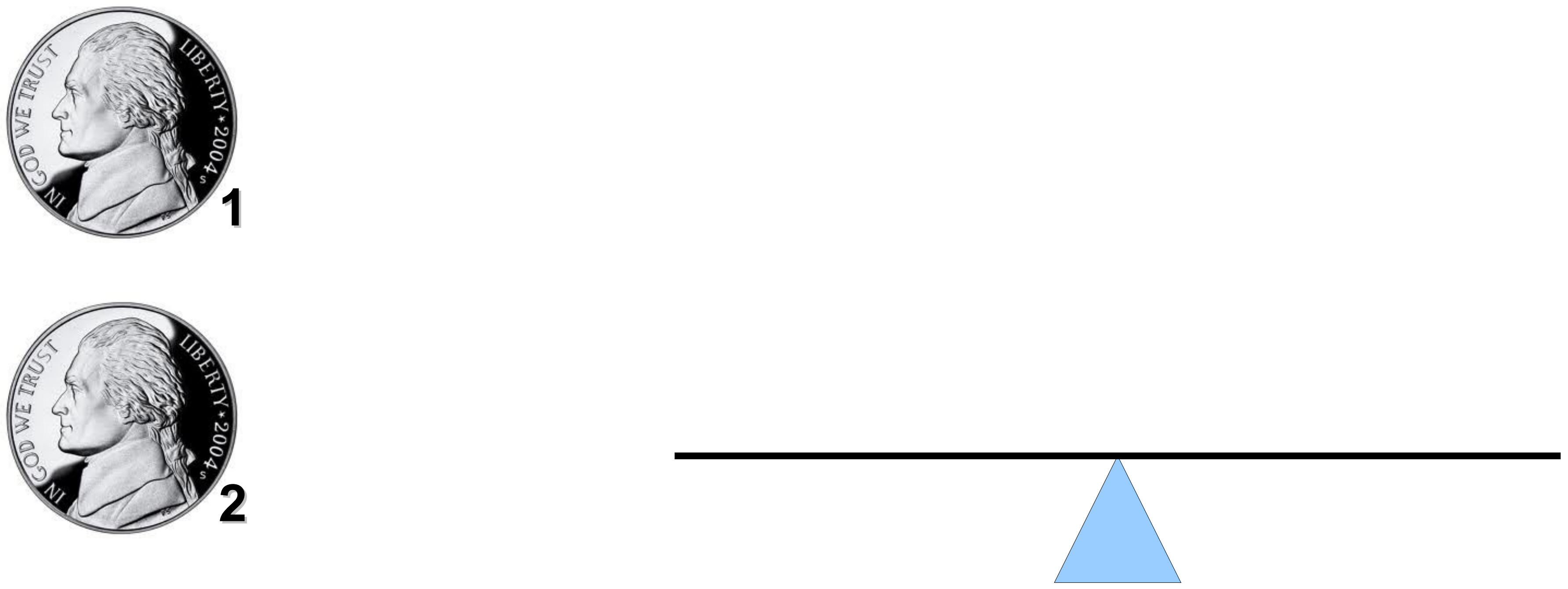

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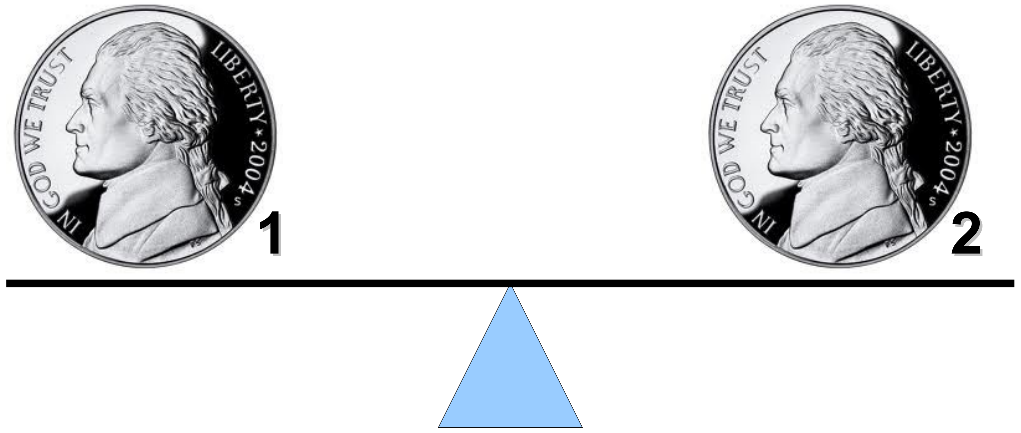

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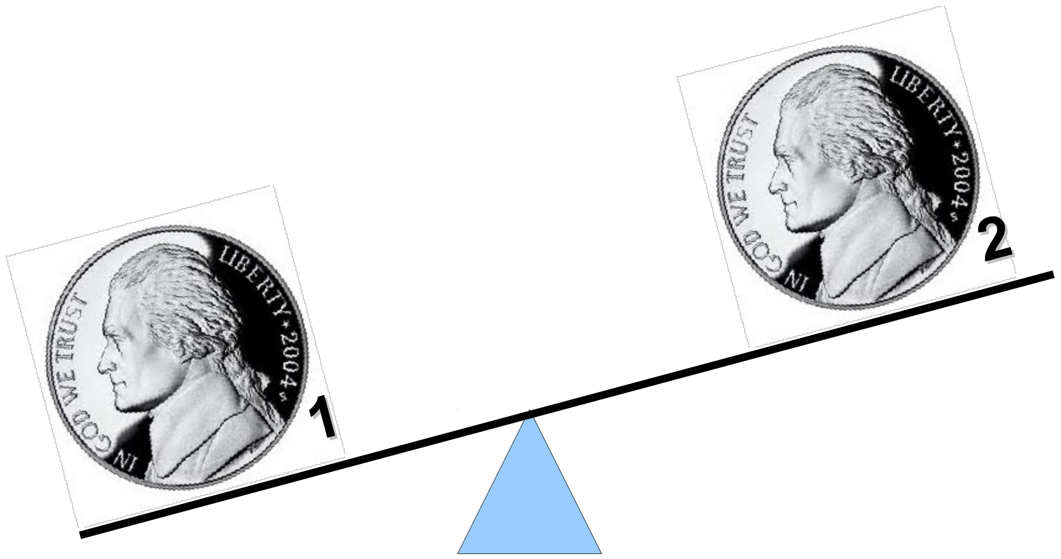

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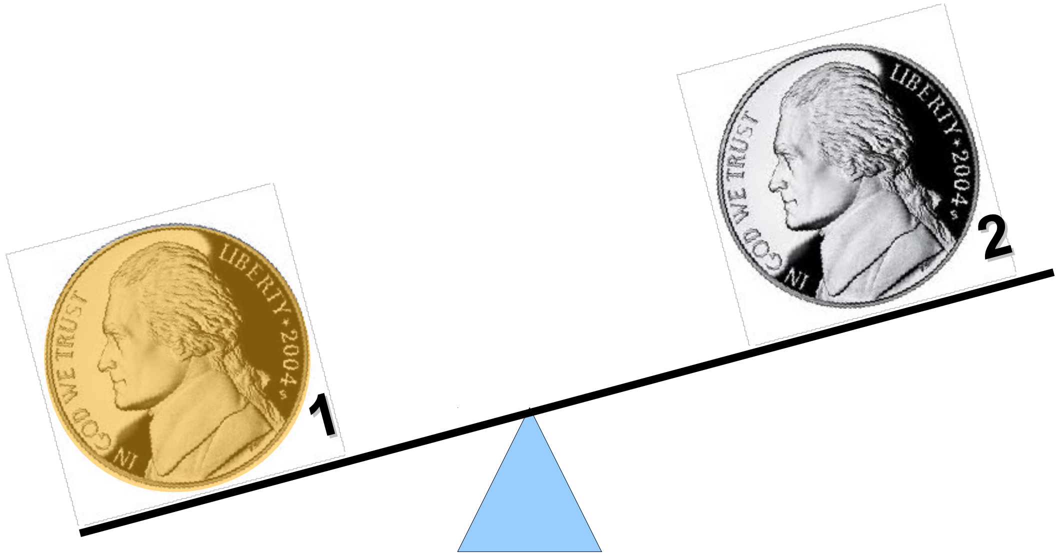

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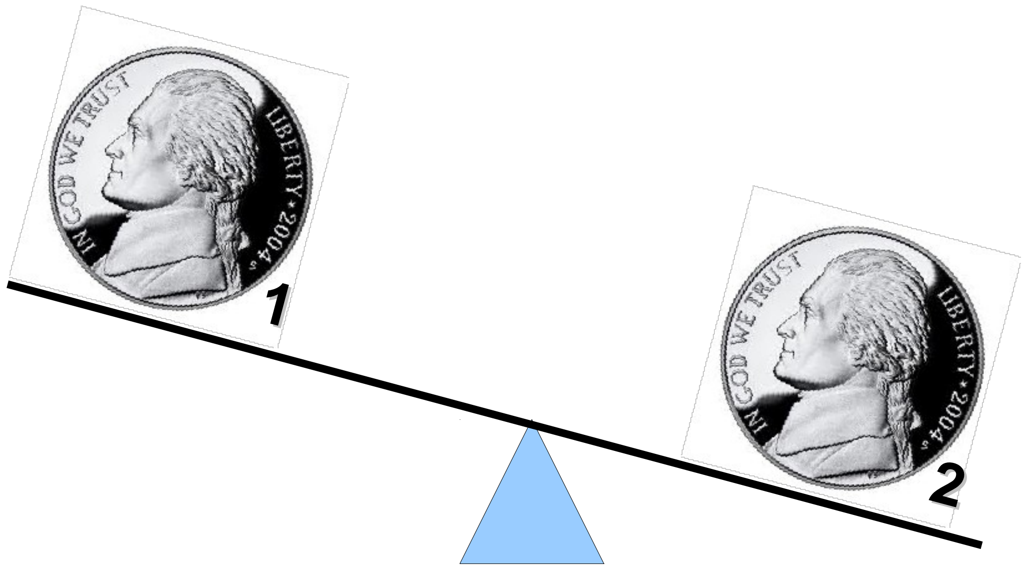

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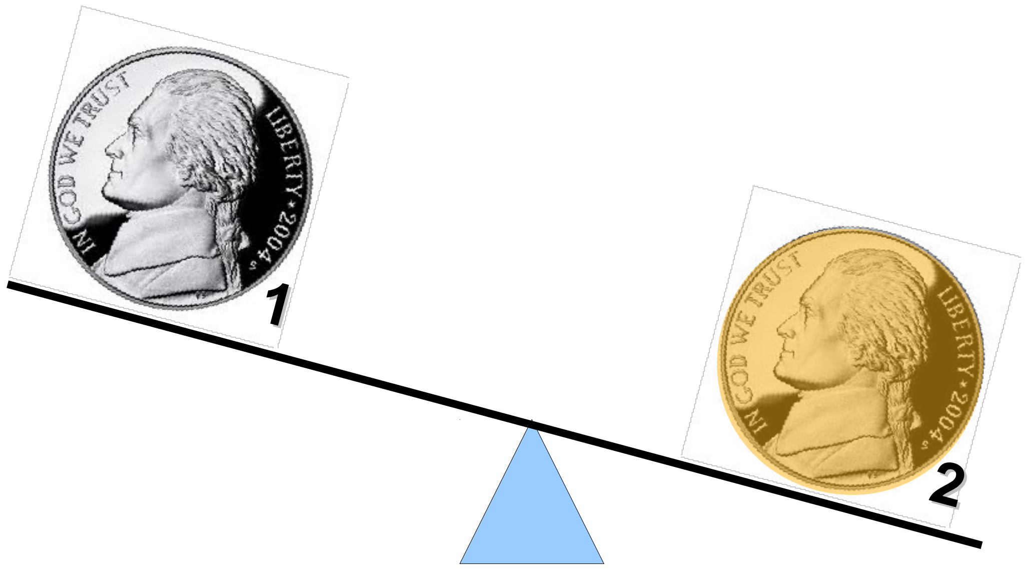

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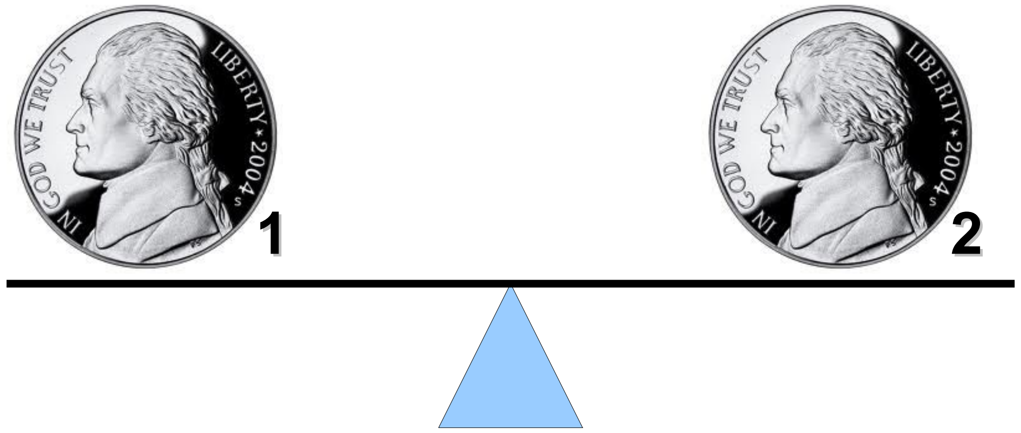

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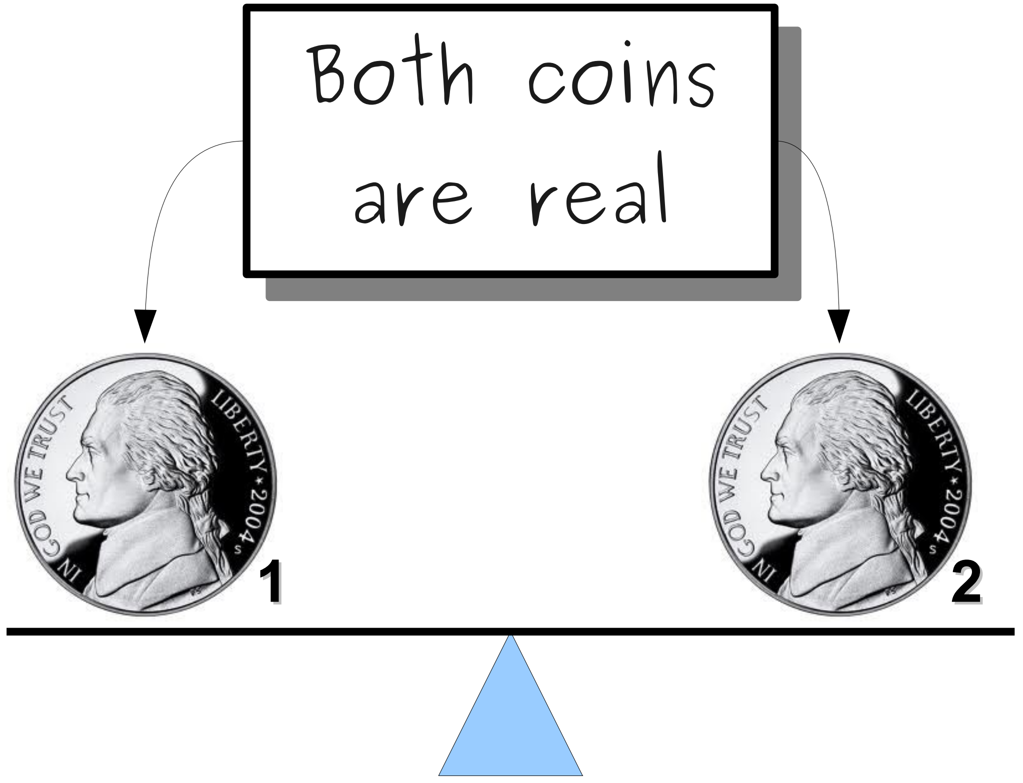

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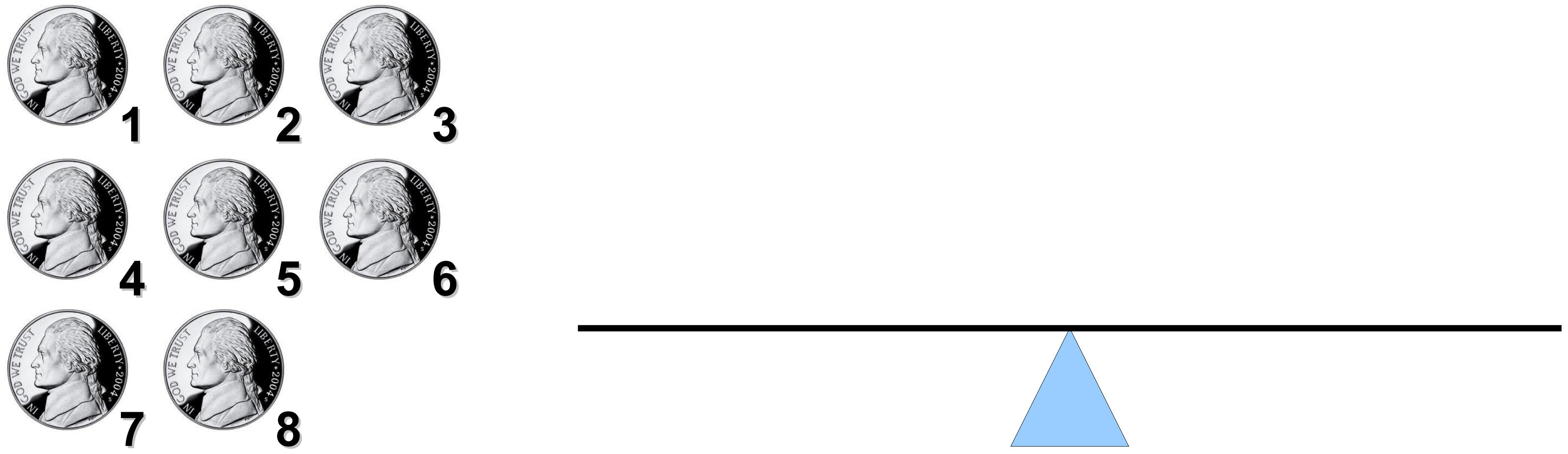

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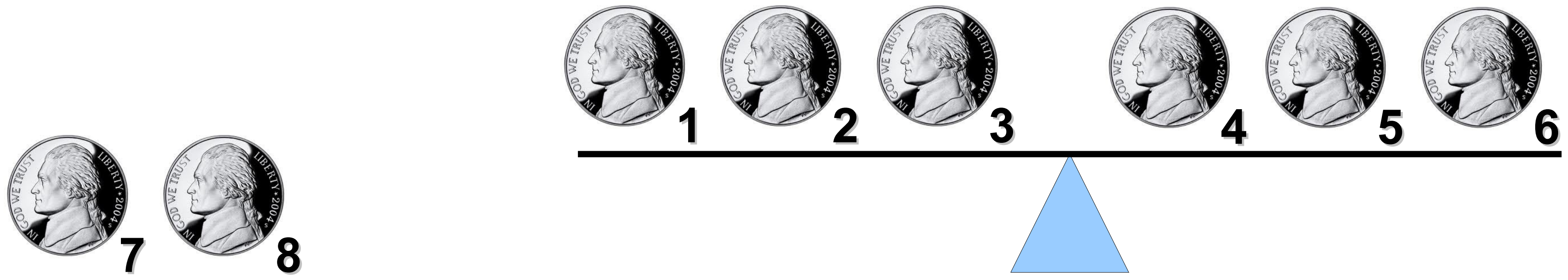

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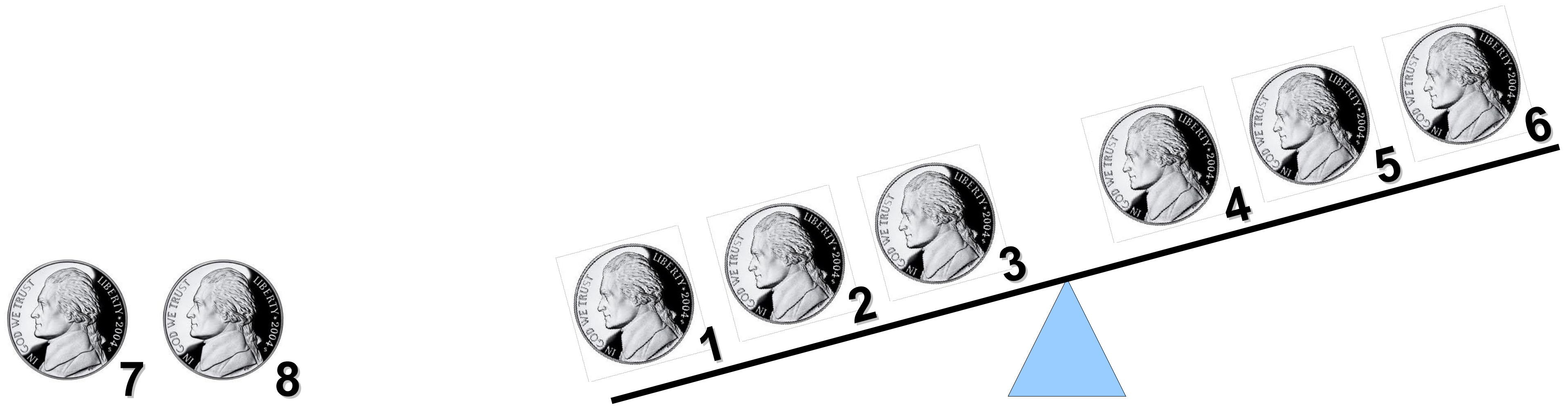

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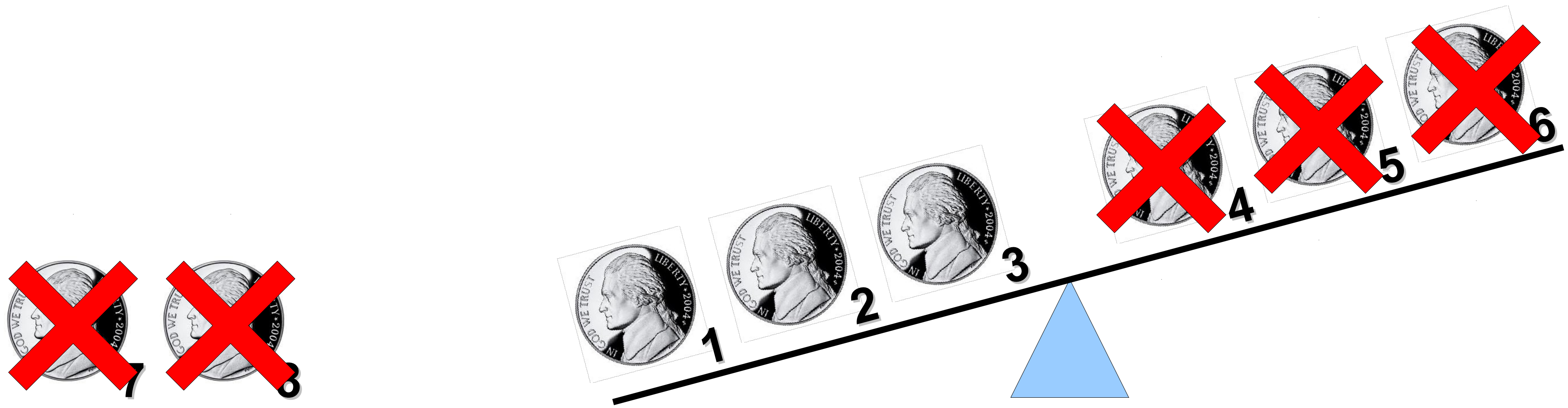

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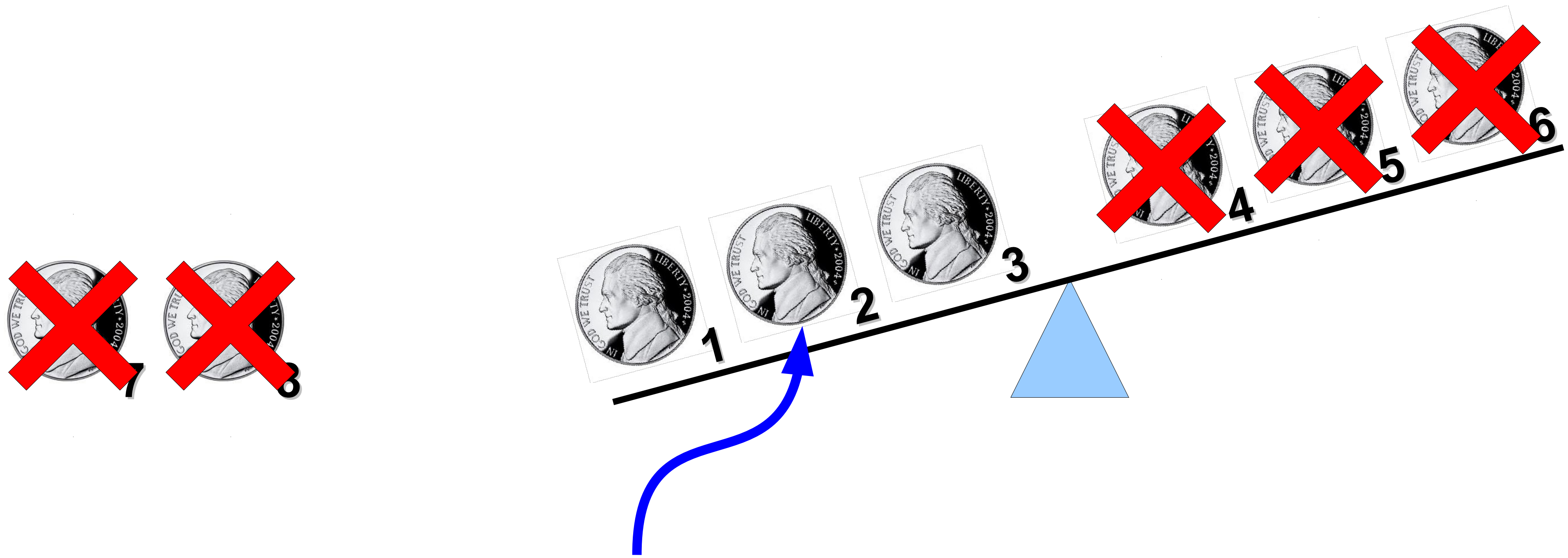

We know for a fact that one of these is heavier than the rest, so we can use our old algorithm!

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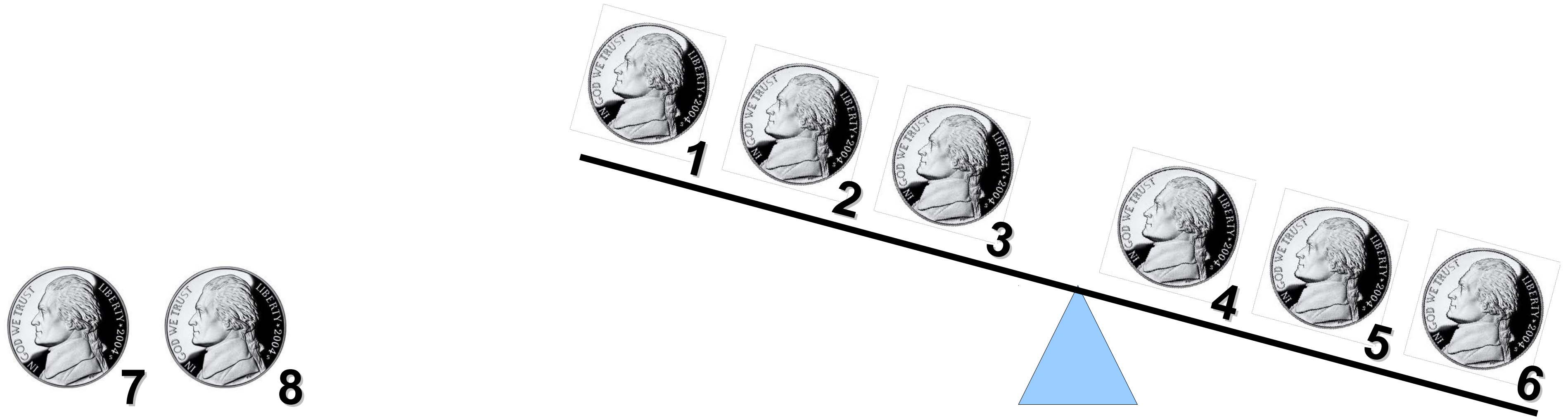

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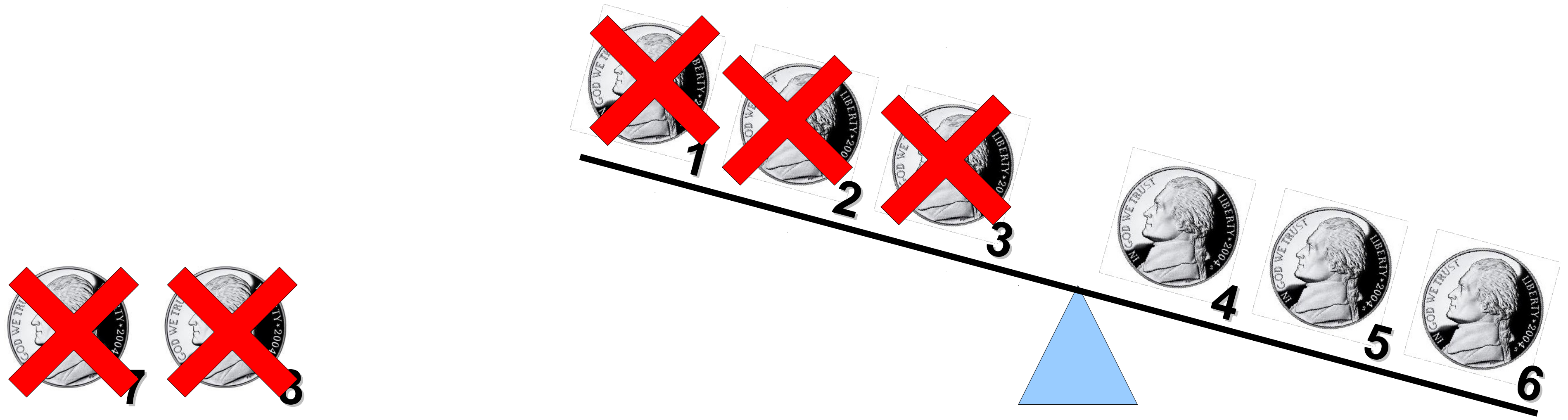

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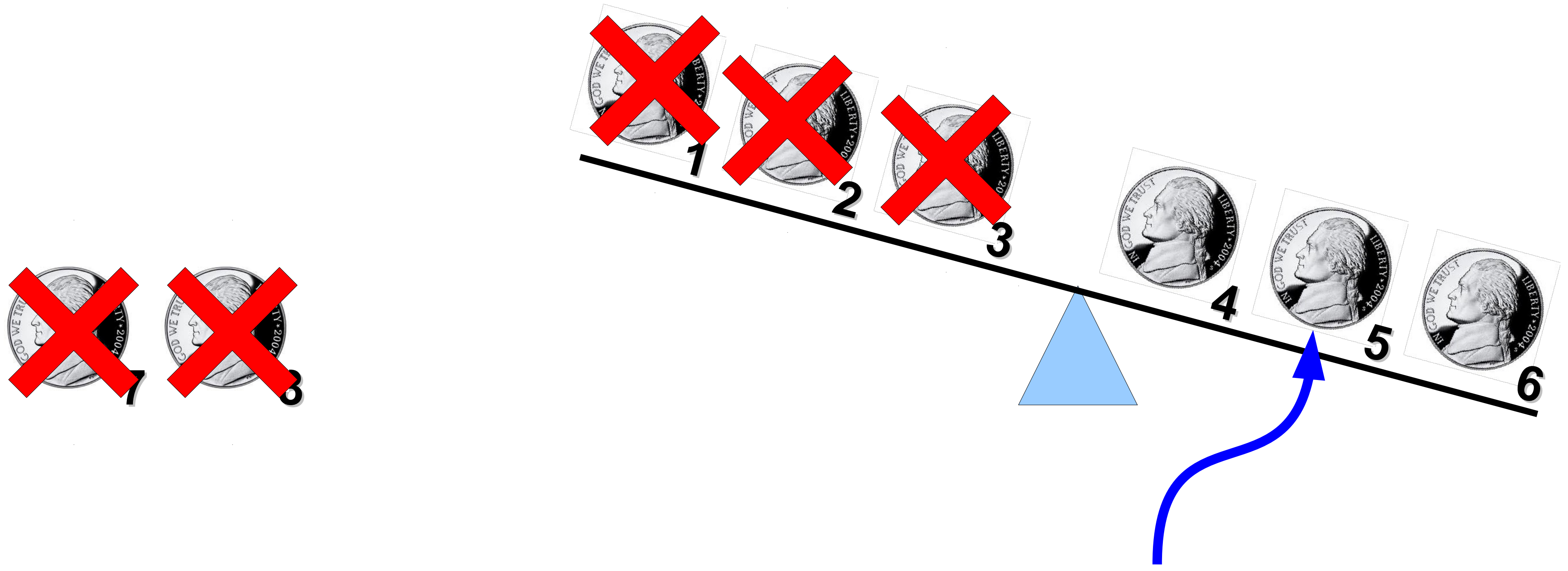

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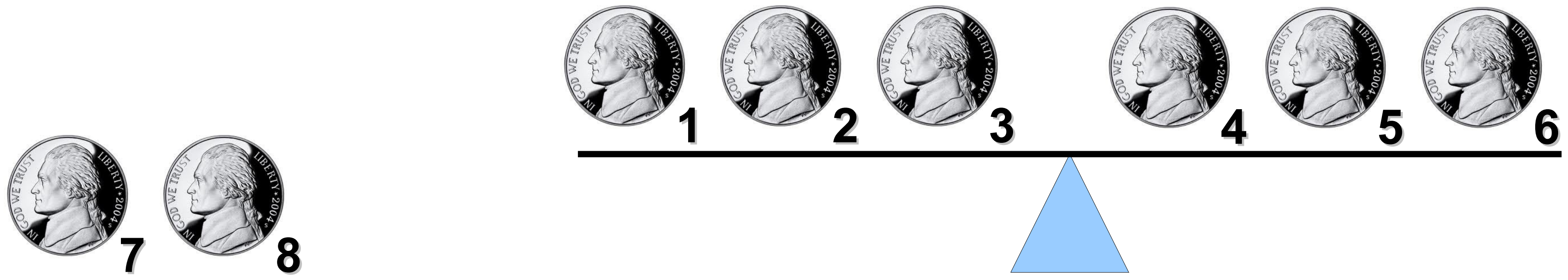

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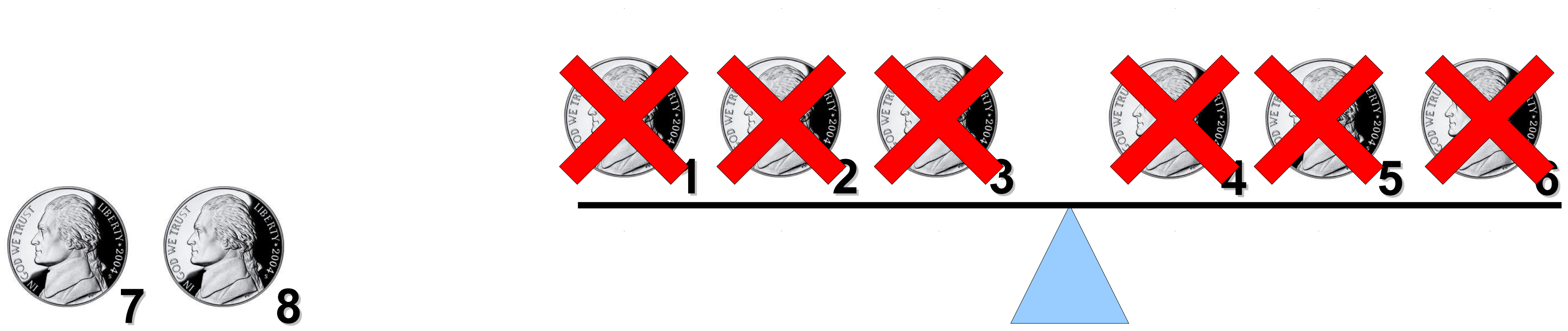

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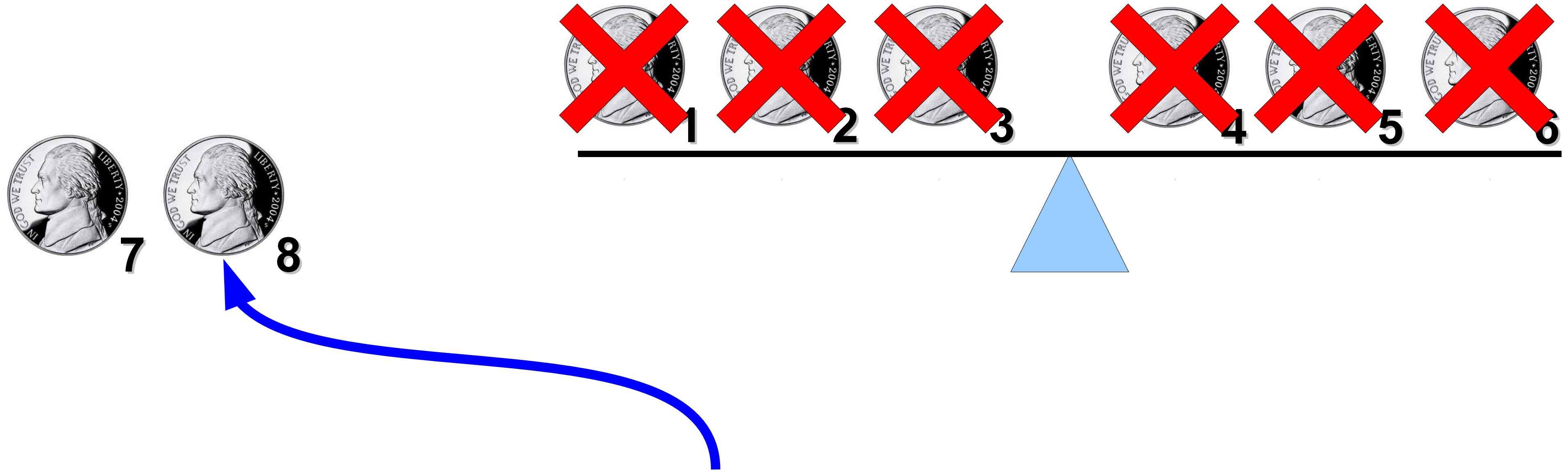


One of these two may be counterfeit, so we can use the case for just one weighing to check.

The Possibly Counterfeit Coin Problem

- With *n* weighings, we have a strategy for finding which of $3^n - 1$ coins, if any, is heavier.
- If *n* = 0, then we can check $3^0 - 1 = 0$ coins and determine they are all real.
- Otherwise:
    - Split the coins into groups of size 3^{n-1}, 3^{n-1}, and $3^{n-1} - 1$. Call them A, B, and C.
    - Weigh A vs. B.
    - If A or B is heavier than the other, one of the 3^{n-1} coins in it is counterfeit. We can find it in *n* – 1 weighings.
    - Otherwise, nothing in A or B is counterfeit. Recursively check the $3^{n-1} - 1$ coins using *n* – 1 weighings.

Is this solution optimal?

Using Our Model

- Let's use the same reasoning as before.
- How many different inputs are there if there are *k* coins?
  - **Answer:** $k + 1$: one for each coin, plus one for “no coin is counterfeit.”
- How many answer nodes are in an algorithm that makes *n* comparisons?
  - **Answer:** 3^n
- Solving $k + 1 = 3^n$, we get $k = 3^n – 1$.
- **Our algorithm has to be optimal!**

Suppose you have a set of coins. There may be a counterfeit coin among them (though there doesn't have to be). If there is a counterfeit, it is heaver than the rest of the coins.

If you have *n* weighings, what is the largest number of coins for which you can solve this problem?

Suppose you have a set of coins. There may be a counterfeit coin among them (though there doesn't have to be). If there is a counterfeit, it **can be either heavier or lighter** than the rest of the coins.

Find the counterfeit coin, if any, and tell whether it is heavier or lighter than the rest.

If you have *n* weighings, what is the largest number of coins for which you can solve this problem?

Decision Tree Reasoning

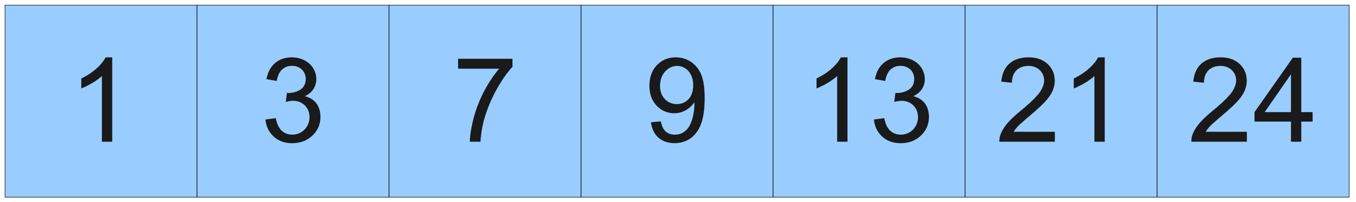
- Before we even try thinking up a solution, let's see if we can figure out a best-case scenario.
- How many different inputs are there with k coins?
  - **Answer**: $2k + 1$: Each of the k coins can be heavier or lighter, or they might all be the same.
- How many answer nodes are there in an algorithm that makes n comparisons?
  - **Answer**: 3^n, as before.
- Solving, we get that $2k + 1 = 3^n$, so $k = (3^n – 1) / 2$.

Complicated Counterfeit Coin Problem

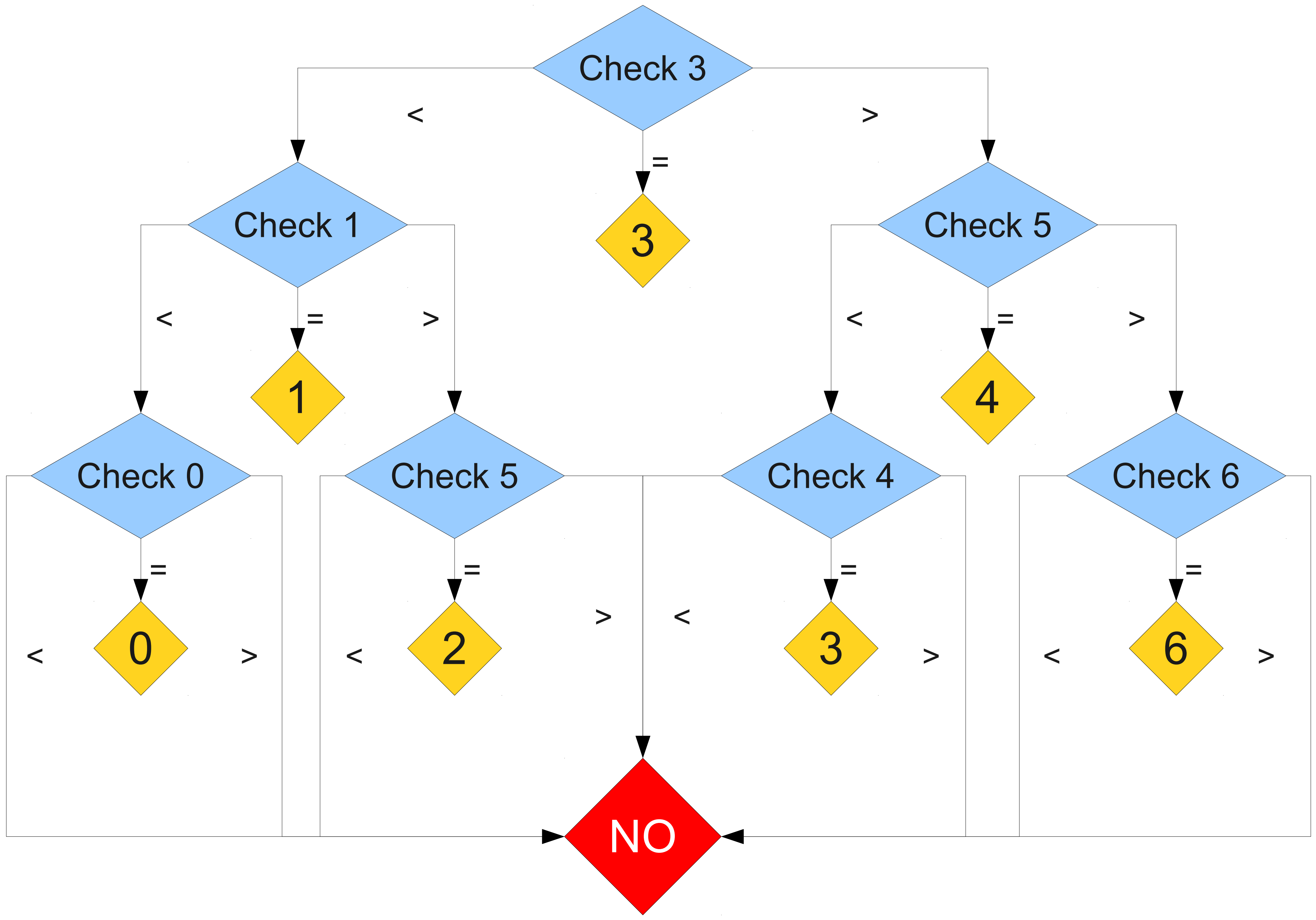
- We haven't even thought up a way to solve this problem yet, but we already have an upper bound on how many coins we can differentiate.
- It turns out that there is an algorithm that can achieve this upper bound, but it's fairly difficult to come up with.
- **Fun puzzle: Given $(3^n – 1) / 2$ coins, determine which (if any) is different, and whether it's heavier or lighter than the rest, in *n* weighings.**

More Advanced Applications

The Complexity of Searching



How efficiently can we search
a sorted array for a single value?


Check 3
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1
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Check 5
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Check 6
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0
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<
2
>
<
3
>
<
6
>
NO

A Lower Bound on Searching

- Any **comparison-based** searching algorithm on a sorted array must make at most $\log_3 (n + 1)$ comparisons.
- Why?
  - We need there to be $n + 1$ answer nodes, one for each of the n positions in the array, and one to signal "not found."
  - By our previous result, the tree must have depth $\log_3 (n + 1)$ to achieve this result.
- This does **not** mean that it's possible to build an algorithm that makes this many comparisons; instead, it shows that no algorithm can do any better.

A Lower Bound on Sorting

Sorting

21	7	24	1	9	3	13

Sorting

1	3	7	9	13	21	24

Sorting Algorithms

- Suppose *S* is a totally-ordered set.
- Let $x_1, x_2, \ldots, x_n$ be *n* distinct elements drawn from *S*, ordered arbitrarily.
- **Goal**: Reorder the elements to put the sequence into sorted order.
- Perhaps the single most-studied problem in all computer science.

Comparison Sorting

- A **comparison-based sorting algorithm** is a sorting algorithm in which the only way to gain information about the elements being sorted is to compare them.
- Similar to the counterfeit coin problem – we can get the relative weights of the elements, but not their absolute weights.
  - The **relative ordering** of the inputs can be known, not the **absolute values.**
- Question: How many comparisons are required in order to sort *n* values?

Defining Our Inputs

x_1	x_2	x_3	x_4
x_1	x_2	x_4	x_3
x_1	x_3	x_2	x_4
x_1	x_3	x_4	x_2
x_1	x_4	x_2	x_3
x_1	x_4	x_3	x_2

x_2	x_1	x_3	x_4
x_2	x_1	x_4	x_3
x_2	x_3	x_1	x_4
x_2	x_3	x_4	x_1
x_2	x_4	x_1	x_3
x_2	x_4	x_3	x_1

x_3	x_1	x_2	x_4
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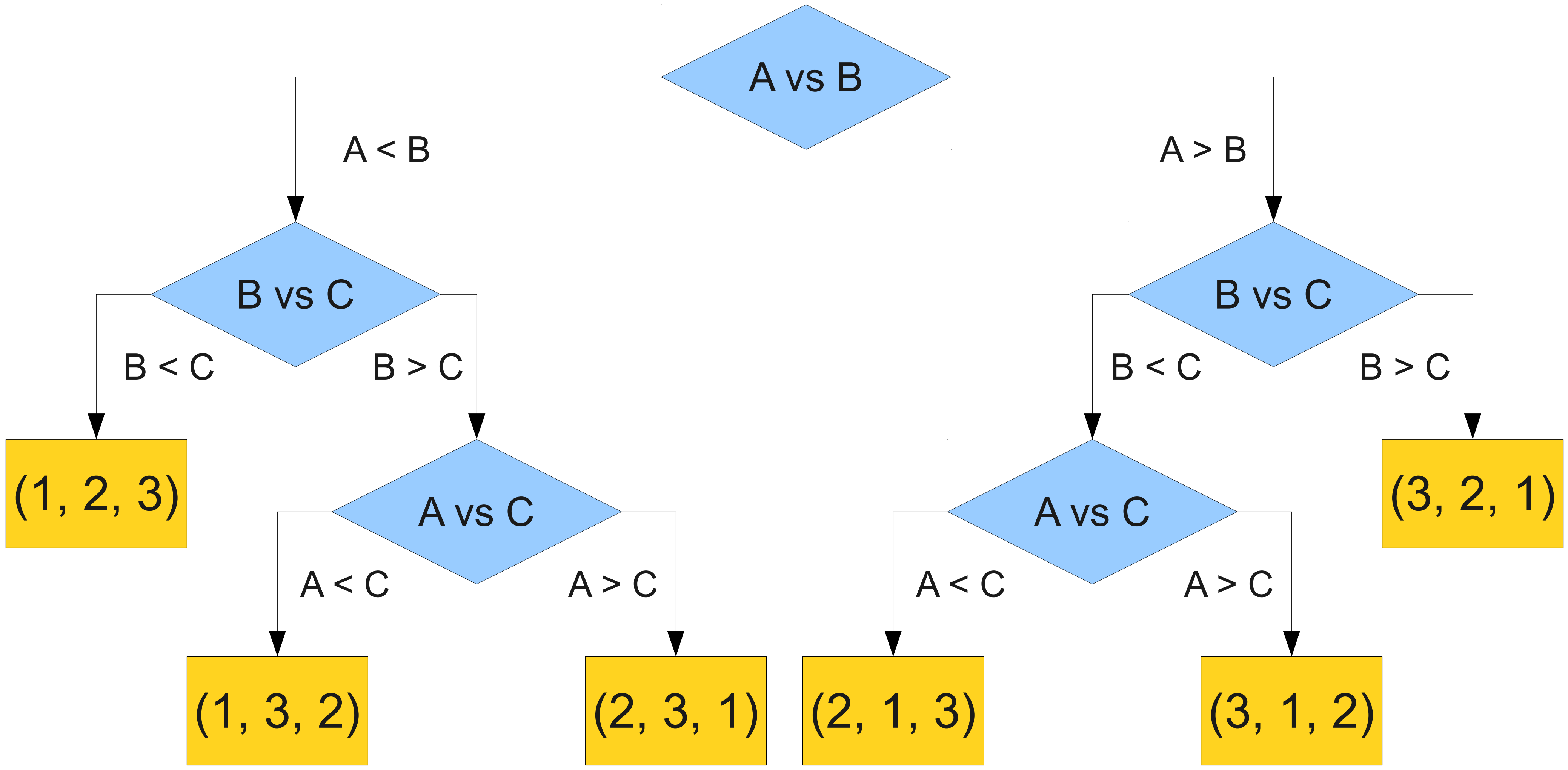
x_3	x_2	x_1	x_4
x_3	x_2	x_4	x_1
x_3	x_4	x_1	x_2
x_3	x_4	x_2	x_1

x_4	x_1	x_2	x_3
x_4	x_1	x_3	x_2
x_4	x_2	x_1	x_3
x_4	x_2	x_3	x_1
x_4	x_3	x_1	x_2
x_4	x_3	x_2	x_1

Defining Our Inputs

- Suppose we are sorting *n* distinct values.
- There are *n*! different permutations of those values.
- Consequently, there are *n*! different inputs, one for each permutation.

Defining Our Algorithmic Model



Defining Our Algorithmic Model

- Represent as a tree.
  - Internal nodes encode the comparisons being made.
  - Leaf nodes have to say which input permutation we received.
- Why does this work?
  - If we are just counting comparisons, it doesn't matter what the code looks like to do the comparison; it just matters what comparisons are made.
  - We don't need to see what rearrangements the algorithm actually does. If the algorithm can't identify what the input permutation is, then it can't be guaranteed that it it's correct.

The Analysis

- Using a similar proof to before, we can show that in a tree with k comparisons, there are at most 2^k answer nodes.
- There are $n!$ different inputs of length n, each of which must have a unique answer.
- Combined, we must have $2^k \geq n!$, or otherwise two different input permutations would reach the same answer (one gets sorted incorrectly).
- Thus $\boldsymbol{k \geq \log_2 n!}$. We have to do at least $\log_2 n!$ comparisons in the worst-case.

Stirling's Approximation

- A result called **Stirling's approximation** says that as n grows large, $\log_2 n! \approx n \log_2 n$
- Consequently, any comparison-based sorting algorithm must make (approximately) $n \log_2 n$ comparisons.
- Thus there cannot be a comparison-based sorting algorithm that does less than $\Theta(n \log_2 n)$ work making comparisons.
- Since some comparison-based sorting algorithms have this runtime (mergesort, heapsort, etc.), these algorithms are **asymptotically optimal**.

Why All This Matters

- We've spent the last few weeks exploring proof techniques and defining mathematical structures.
- These techniques make it possible to reason about fundamental questions in computing.
- In about a week, we'll begin exploring more elaborate models of computation using similar techniques.

Next Time

- **Propostional Logic**
  - How do we start formalizing our intuitions about mathematical truth?
  - How do we justify proofs by contradiction and contrapositive?