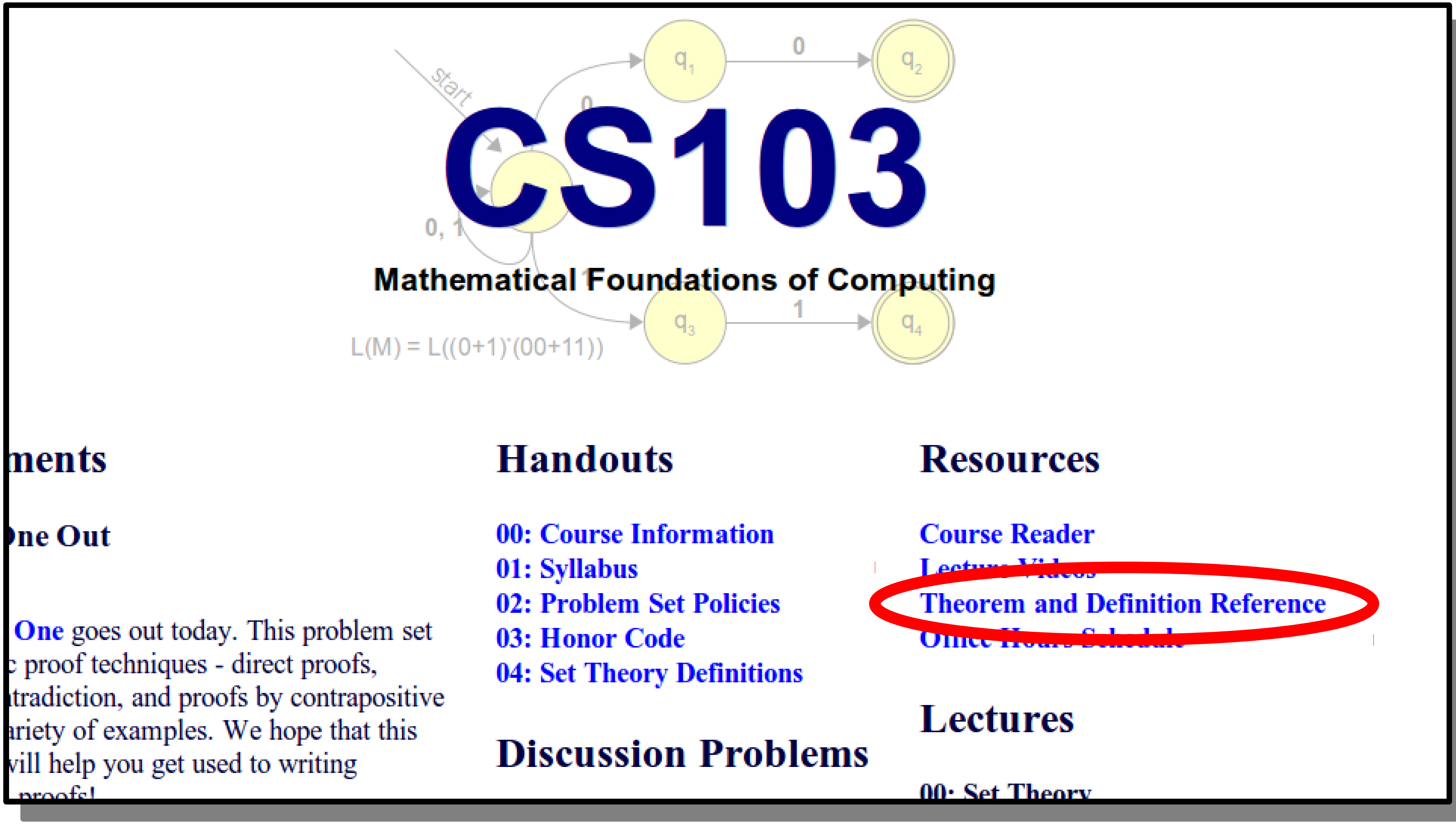
Indirect Proofs

Announcements

- Problem Set 1 out.
- **Checkpoint** due Monday, September 30.
  - Grade determined by attempt rather than accuracy. It's okay to make mistakes – we want you to give it your best effort, even if you're not completely sure what you have is correct.
  - We will get feedback back to you with comments on your proof technique and style.
  - The more an effort you put in, the more you'll get out.
- **Remaining problems** due Friday, October 4.
  - Feel free to email us with questions!

Submitting Assignments

- You can submit assignments by
  - handing them in at the start of class,
  - dropping it off in the filing cabinet near Keith's office (details on the assignment handouts), or
  - emailing the submissions mailing list at **cs103-aut1314-submissions@lists.stanford.edu** and attaching your solution as a PDF. (Please don't email the staff list directly with submissions) See Handout #02 for more details.
- Late policy:
  - Three "late periods:" extend due date by one class period.
  - Can use at most one per assignment.
  - No work accepted more than one class period after the due date.



CS103

Mathematical Foundations of Computing

ments

ne Out

One goes out today. This problem set
proof techniques - direct proofs,
tradiction, and proofs by contrapositive
riety of examples. We hope that this
vill help you get used to writing
proofs!

Handouts

00: Course Information
01: Syllabus
02: Problem Set Policies
03: Honor Code
04: Set Theory Definitions

Discussion Problems

Resources

Course Reader
Lecture Videos
Theorem and Definition Reference
Office Hours Schedule

Lectures

00: Set Theory

Office hours start tomorrow.

Schedule available on the course website.

Friday Four Square

- Good snacks!
- Good company!
- Good game!
- Good fun!
- **Today at 4:15 in front of Gates.**

Outline for Today

- **Logical Implication**
  - What does "If *P*, then *Q*" mean?
- **Proof by Contrapositive**
  - The basic method.
  - An interesting application.
- **Proof by Contradiction**
  - The basic method.
  - Contradictions and implication.
  - Contradictions and quantifiers.

Logical Implication

Implications

- An **implication** is a statement of the form

  If $\boldsymbol{P}$, then $\boldsymbol{Q}$.

- When discussing implications in the abstract, we denote that P implies Q by writing $\boldsymbol{P \to Q}$.
- When $P \to Q$, we call P the **antecedent** and Q the **consequent**.

What Implication Means

- The statement $P \to Q$ means exactly the following:

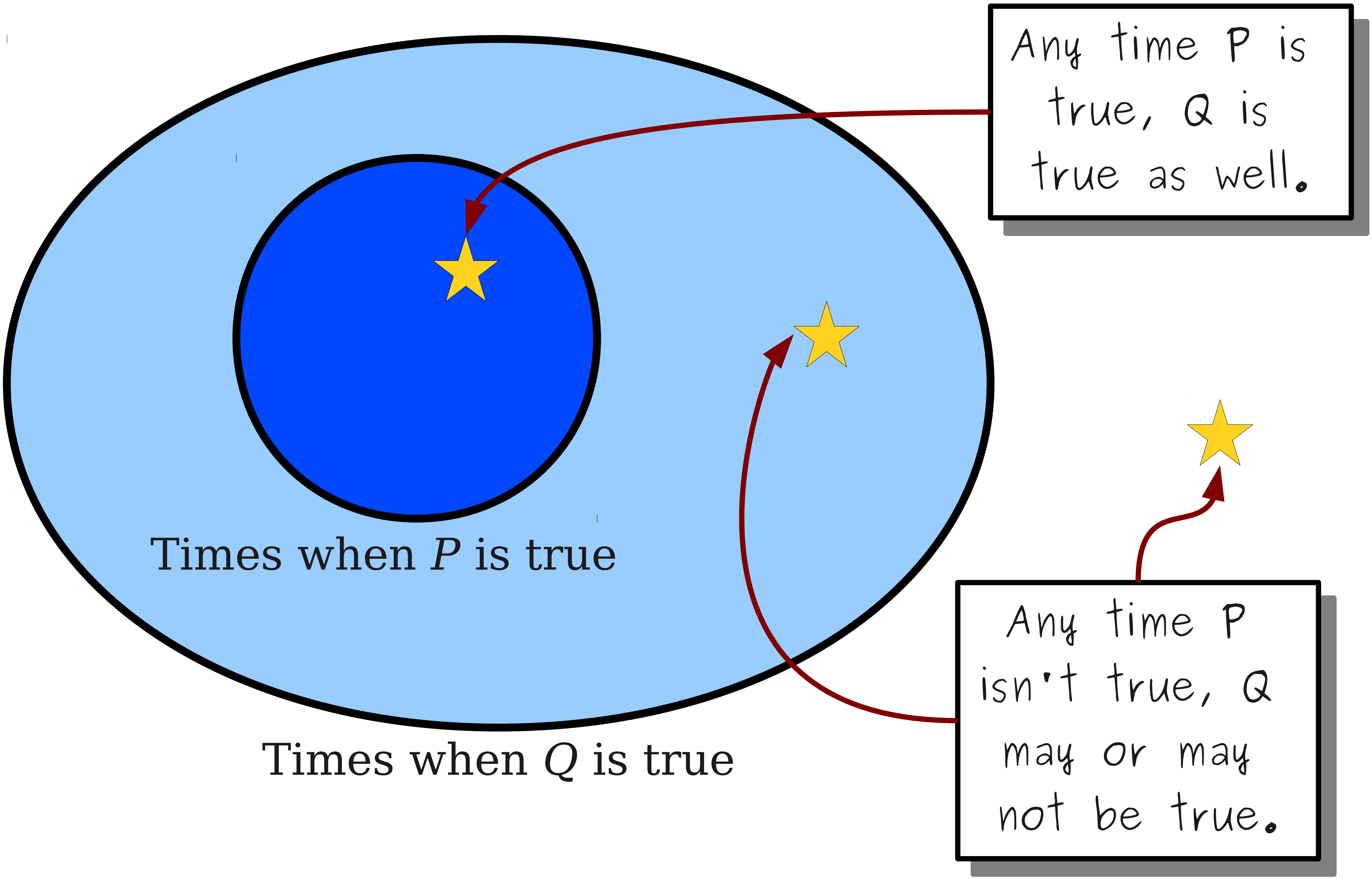
If *P* is true, then
***Q* must be true as well.**

- For example:
  - n is an even integer $\to$ n^2 is an even integer.
  - $(A \subseteq B$ and $B \subseteq A) \to A = B$

What Implication Doesn't Mean

- $P \to Q$ ***doesn't*** mean that whenever Q is true, P is true.
  - "If you die in Canada, you die in real life" doesn't mean that if you die in real life, you die in Canada.
- $P \to Q$ ***doesn't*** say anything about what happens if P is false.
  - "If an animal is a puppy, you should hug it" doesn't mean that if that animal isn't a puppy, you shouldn't hug it.
  - **Vacuous truth:** If P is never true, then $P \to Q$ is always true.
- $P \to Q$ ***doesn't*** say anything about causality.
  - "If I like math, then 2 + 2 = 4" is true because any time that I like make, 2 + 2 = 4 is true.
  - "If I don't like math, then 2 + 2 = 4" is also true, since whenever I don't like math, 2 + 2 = 4 is true.

Implication, Diagrammatically



Proof by Contrapositive

Honk **If** You Love Formal Logic

Suppose that you're driving this car and you *don't* get honked at.

What can you say about the people driving behind you?

The Contrapositive

- The **contrapositive** of "If *P*, then *Q*" is the statement "If **not** *Q*, then **not** *P*."
- Example:
  - "If I stored the cat food inside, then the raccoons wouldn't have stolen my cat food."
  - Contrapositive: "If the raccoons stole my cat food, then I didn't store it inside."
- Another example:
  - "If you liked it, then you should have put a ring on it."
  - Contrapositive: "If you shouldn't have put a ring on it, then you didn't like it."

An Important Proof Strategy

To show that $P \to Q$, you may instead show that $\neg Q \to \neg P$.

This is called a ***proof by contrapositive***.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Since n is odd, $n = 2k + 1$ for some integer k. Then

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

Therefore, there exists an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$.

Thus n^2 is odd, as required. ■

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Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Since n is odd, $n = 2k + 1$ for some integer k. Then

$n^2 =$

$n^2 =$

$n^2 =$

There … m

(nam … $2m + 1$.

Thus n^2 is odd, as required. ■

Notice the structure of the proof. We begin by announcing that it's a proof by contrapositive, then state the contrapositive, and finally prove it.

Biconditionals

- Combined with what we saw on Wednesday, we have proven that, if n is an integer:

If n is even, then n^2 is even.

If n^2 is even, then n is even.

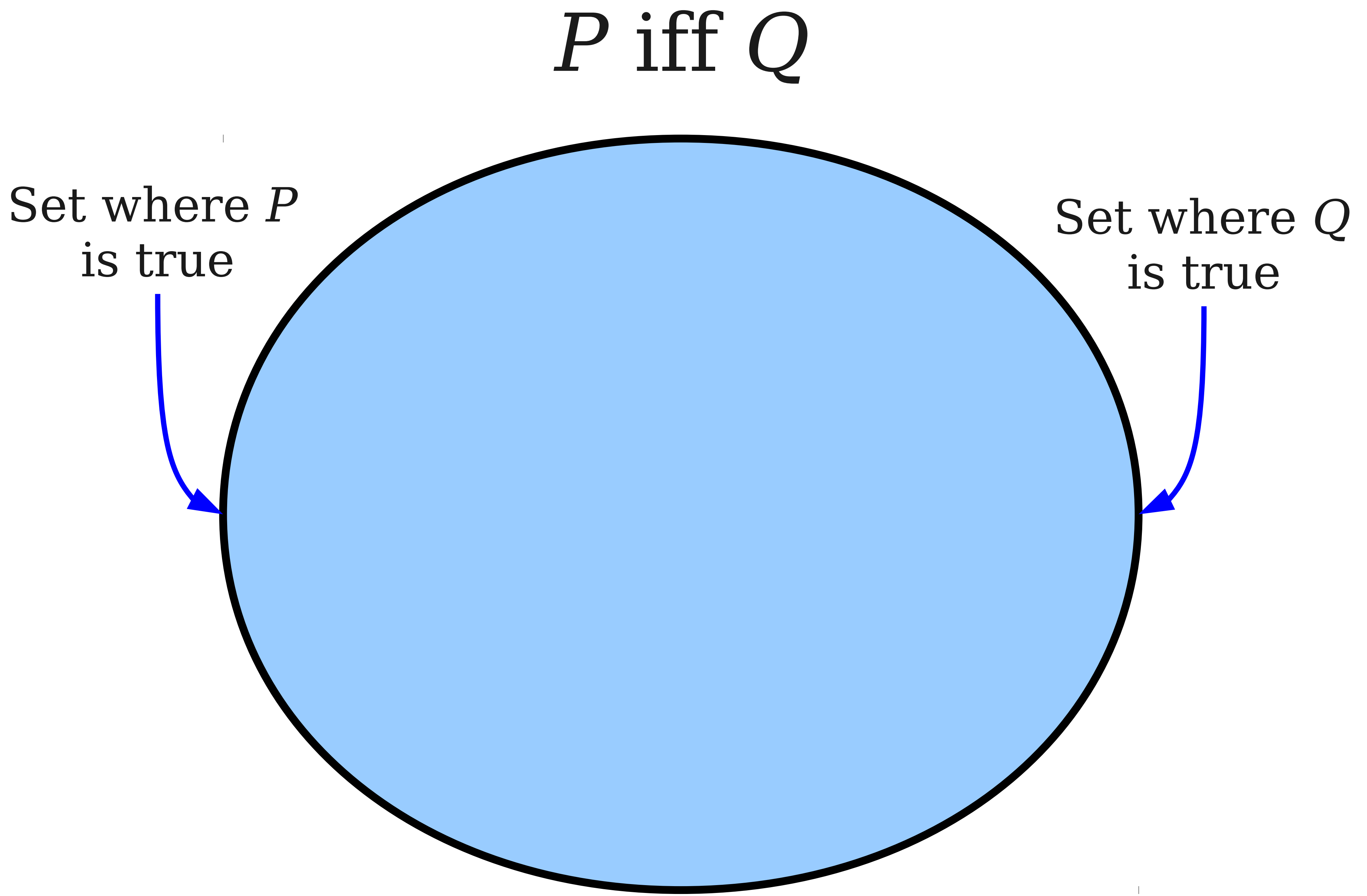
- Therefore, if n is an integer:

n is even if and only if n^2 is even.

- "If and only if" is often abbreviated **iff**:

n is even iff n^2 is even.

- This is called a **biconditional**.



P iff Q
Set where P is true
Set where Q is true

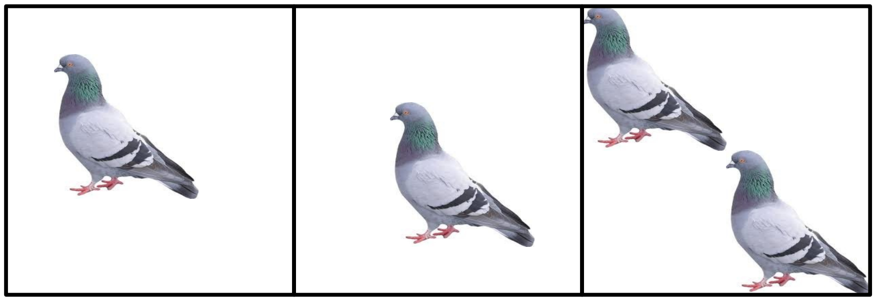
Proving Biconditionals

- To prove ***P* iff *Q***, you need to prove that *P* implies *Q* and that *Q* implies *P*.
- You can any proof techniques you'd like to show each of these statements.
  - In our case, we used a direct proof and a proof by contrapositive.

The Pigeonhole Principle

The Pigeonhole Principle

- Suppose that you have n pigeonholes.
- Suppose that you have $m > n$ pigeons.
- If you put the pigeons into the pigeonholes, some pigeonhole will have more than one pigeon in it.



Theorem: Let m objects be distributed into n bins. If $m > n$, then some bin contains at least two objects.

Proof: By contrapositive; we prove that if every bin contains at most one object, then $m \leq n$.

Let x_i be the number of objects in bin i. Since m is the number of total objects, we have that

$$m = x_1 + x_2 + \ldots + x_n$$

Since every bin has at most one object, $x_i \leq 1$ for all i. Thus

$$\begin{aligned} m &= x_1 + x_2 + \ldots + x_n \\ &\leq 1 + 1 + \ldots + 1 \qquad (n \text{ times}) \\ &= n \end{aligned}$$

So $m \leq n$, as required. ■

Using the Pigeonhole Principle

- The pigeonhole principle is an enormously useful lemma in many proofs.
  - We'll spend a full lecture on it in a few weeks.
- General structure of a pigeonhole proof:
  - Find m objects to distribute into n buckets, with $m > n$.
  - Using the pigeonhole principle, conclude that some bucket has at least two objects in it.
  - Use this conclusion to show the desired result.

Some Simple Applications

- Any group of 367 people must have a pair of people that share a birthday.
  - 366 possible birthdays (pigeonholes)
  - 367 people (pigeons)
- Two people in San Francisco have the exact same number of hairs on their head.
  - Maximum number of hairs ever found on a human head is no greater than 500,000.
  - There are over 800,000 people in San Francisco.
- Each day, two people in New York City drink the same amount of water, to the thousandth of a fluid ounce.
  - No one can drink more than 50 gallons of water each day.
  - That's 6,400 fluid ounces. This gives 6,400,000 possible numbers of thousands of fluid ounces.
  - There are about 8,000,000 people in New York City proper.

Some Words of Caution

An Incorrect Proof

Theorem: For any sets A and B,
if $x \notin A \cap B$, then $x \notin A$.

Proof: By contrapositive; we show that
if $x \in A \cap B$, then $x \in A$.

Since $x \in A \cap B$, $x \in A$ and $x \in B$.
Consequently, $x \in A$ as required. ■

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if $x \notin A \cap B$, then $x \notin A$.

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Since $x \in A \cap B$, $x \in A$ and $x \in B$.
Consequently, $x \in A$ as required. ■

Common Pitfalls

To prove $P \to Q$ by contrapositive, prove

$$\boldsymbol{\neg Q \to \neg P}$$

Be careful not to prove

$$\boldsymbol{\neg P \to \neg Q}$$

(Proving $\neg P \to \neg Q$ proves $Q \to P$, which isn't what you want!)

More Generally

- When doing a proof by contrapositive, your proof is only valid if you actually prove the contrapositive of the statement you want to prove.
- Make sure to set up the proof correctly; double- and triple-check you have taken the contrapositive correctly!
- This is true in general of most indirect proofs.

Proof by Contradiction

“When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth.”

- Sir Arthur Conan Doyle*, The Adventure of the Blanched Soldier*

Proof by Contradiction

- A **proof by contradiction** is a proof that works as follows:
  - To prove that P is true, assume that P is not true.
  - Based on the assumption that P is not true, conclude something impossible.
  - Assuming the logic is sound, the only valid explanation is that the assumption that P is not true is incorrect.
  - Conclude, therefore, that P is true.

Theorem: There is no integer that is both even and odd.

Proof: By contradiction; suppose some integer is both even and odd. Let that integer be k.

Since k is even, there is some $r \in \mathbb{Z}$ such that $k = 2r$. Since k is odd, there is some $s \in \mathbb{Z}$ such that $k = 2s + 1$.

Therefore, $2r = 2s + 1$, so $2r - 2s = 1$ and therefore $r - s = ½$. Since r and s are integers, their difference is an integer. But this is impossible, since ½ is not an integer.

We have reached a contradiction, so our assumption must have been wrong. Thus there is no integer that is both even and odd. ■

Theorem: There is no integer that is both even and odd.

Proof: By contradiction; suppose some integer is both even and odd. Let that integer be k.

The three key pieces:

1. State that the proof is by contradiction.
2. State what you are assuming is the negation of the statement to prove.
3. State you have reached a contradiction and what the contradiction entails.

In CS103, please include all these steps in your proofs!

We have reached a contradiction, so our assumption must have been wrong. Thus there is no integer that is both even and odd. ■

Rational and Irrational Numbers

Rational and Irrational Numbers

- A **rational number** is a number r that can be written as

$$r = \frac{p}{q}$$

  where p and q are integers and $q \neq 0$.
- A number that is not rational is called **irrational**.
- Useful theorem: If r is rational, r can be written as p / q where $q \neq 0$ and where p and q have no common factors other than ± 1.

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, we have $p = \sqrt{2}\, q$, so $p^2 = 2q^2$.

Since q^2 is an integer and $p^2 = 2q^2$, we have that p^2 is even. By our earlier result, since p^2 is even, we know p is even. Thus there is an integer k such that $p = 2k$.

Therefore, $2q^2 = p^2 = (2k)^2 = 4k^2$, so $q^2 = 2k^2$.

Since k^2 is an integer and $q^2 = 2k^2$, we know q^2 is even. By our earlier result, since q^2 is even, we have that q is even. But this means that both p and q have 2 as a common divisor. This contradicts our earlier assertion that their only common divisors are 1 and -1.

We have reached a contradiction, so our assumption was incorrect. Consequently, $\sqrt{2}$ is irrational. ■

A Famous and Beautiful Proof

Theorem: √2 is irrational.

Proof: By contradiction; assume √2is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{}$, and p and q have no common divisors other than 1 and -1.

The three key pieces:

1. State that the proof is by contradiction.
2. State what you are assuming is the negation of the statement to prove.
3. State you have reached a contradiction and what the contradiction entails.

In CS103, please include all these steps in your proofs!

contradicts our earlier assertion that their only common divisors are 1 and -1.

We have reached a contradiction, so our assumption was incorrect. Consequently, √2 is irrational. ■

Vi Hart on Pythagoras and
the Square Root of Two:

http://www.youtube.com/watch?v=X1E7I7_r3Cw

A Word of Warning

- To attempt a proof by contradiction, make sure that what you're assuming actually is the opposite of what you want to prove.
- Otherwise, the core logic of your proof will be incorrect.
- Also true in proofs by contrapositive, but can be a lot more subtle in proofs by contradiction.

Negations of Standard Statements

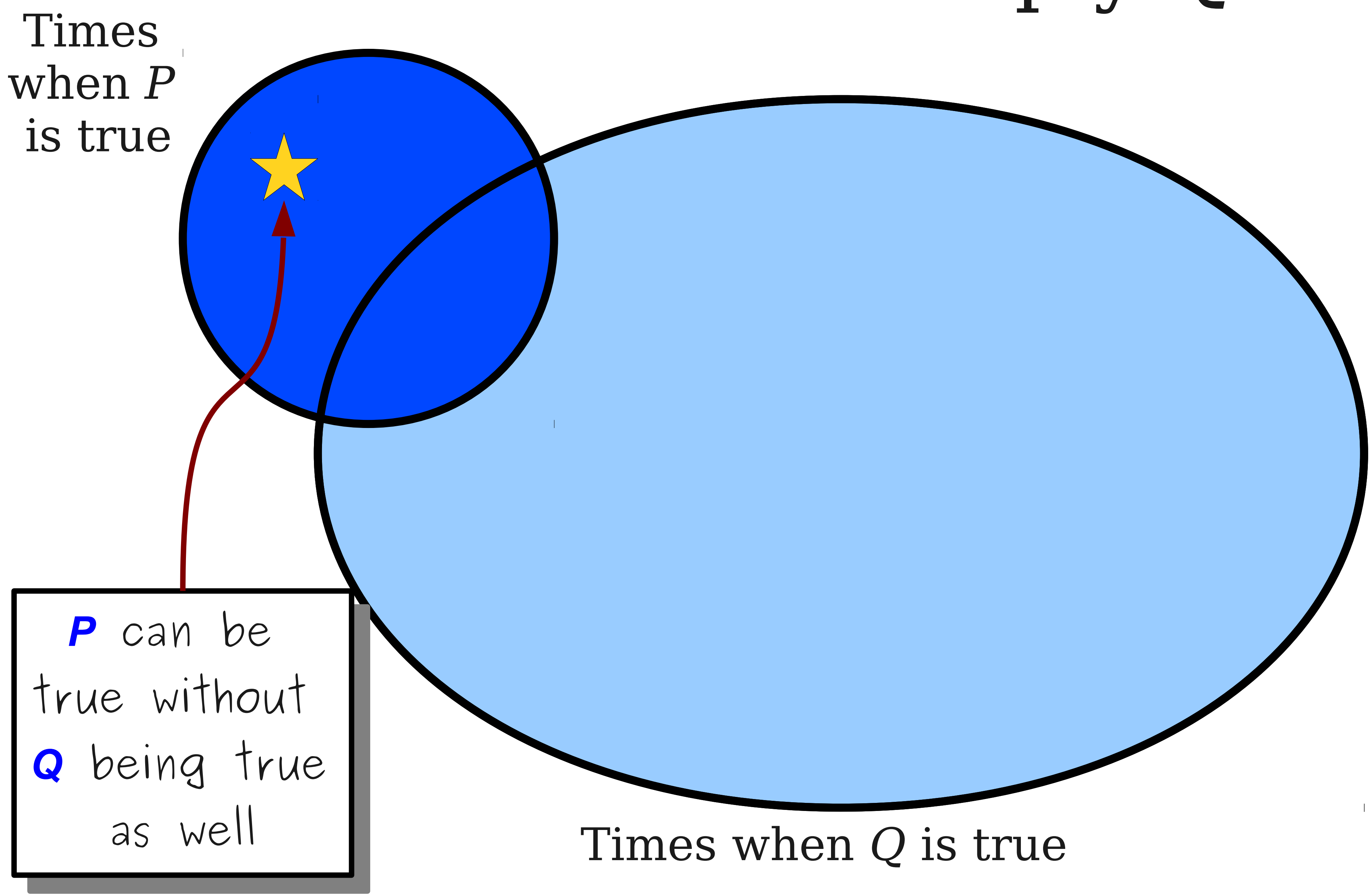
- It's good to know how to negate three general types of statements:
  - **Implications:** "If *P*, then *Q*."
  - **Universal statements:** "For all *x*, P(*x*) is true."
  - **Existential statements:** "There exists an *x* where P(*x*) is true."
- Let's quickly go over how to prove these statements by contradiction.

Negating Implications

When P Doesn't Imply Q

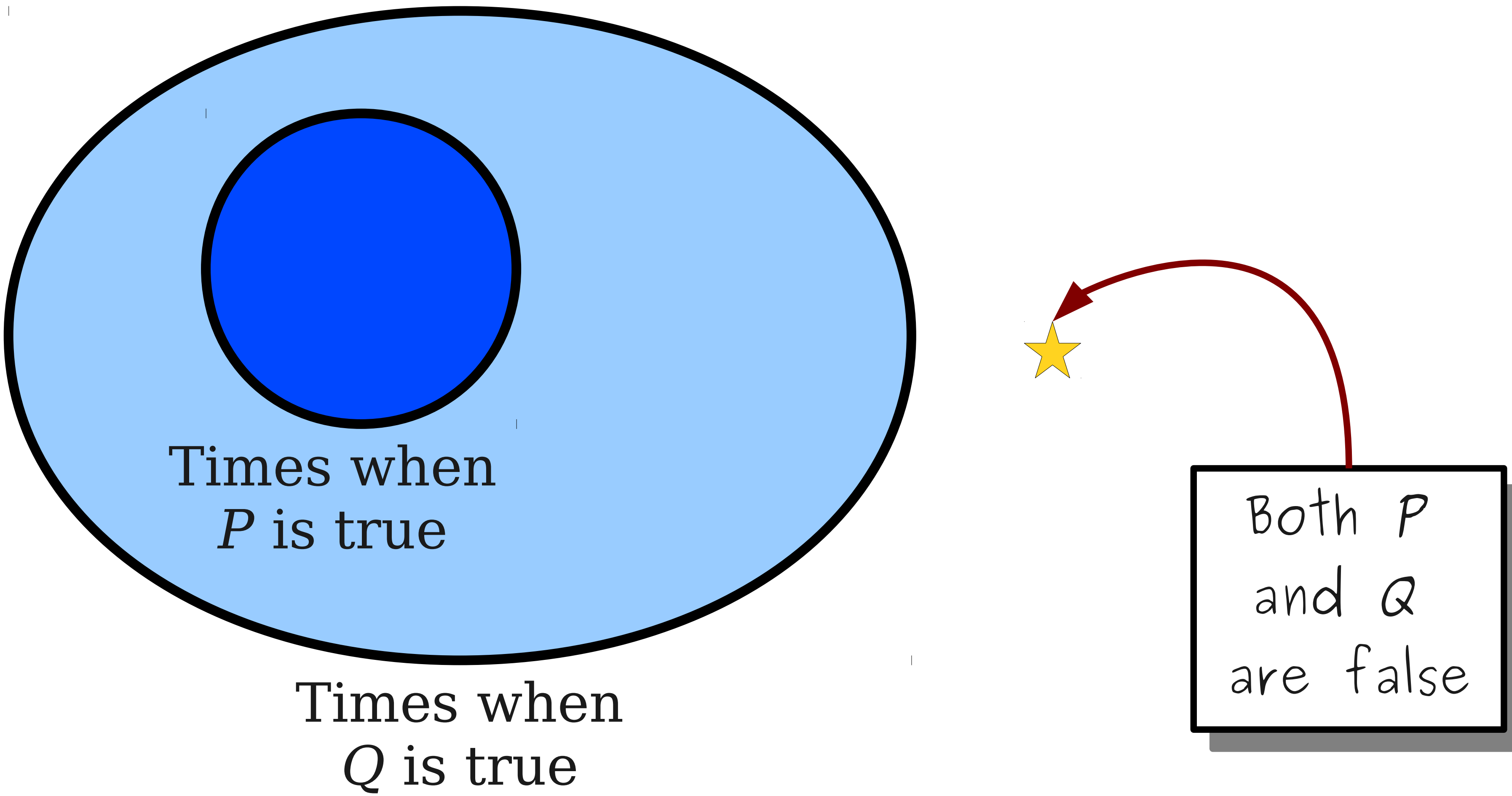
- Recall: What does "If P, then Q" mean?
  - **Answer**: If P is true, then Q is true as well.
- When will "If P, then Q" be false?
  - **Answer**: P is true, but Q is false.
- The only way to disprove that P implies Q is to show that there is some way for P to be true and Q to be false.

When P Doesn't Imply Q



A Common Mistake

- To show that $P \to Q$ is false, it is not sufficient to find a case where P is false and Q is false.



Contradictions and Implications

- Suppose we want to prove that $P \to Q$ is true by contradiction.
- The proof will look something like this:
  - Assume that P is true and Q is false.
  - Using this assumption, derive a contradiction.
  - Conclude that $P \to Q$ must be true.

A Simple Proof by Contradiction

Theorem: If n is an integer and n^2 is even, then n is even.
Proof: By contradiction; assume n is an integer and n^2 is even, but that n is odd.

Since n is odd, $n = 2k + 1$ for some integer k.

Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

Now, let $m = 2k^2 + 2k$. Then $n^2 = 2m + 1$, so by definition n^2 is odd. But this is impossible, since n^2 is even.

We have reached a contradiction, so our assumption was false. Thus if n is an integer and n^2 is even, n is even as well. ■

A Simple Proof by Contradiction

Theorem: If n is an integer and n^2 is even, then n is even.

Proof: By contradiction; assume n is an integer and n^2 is even, but that n is odd.

The three key pieces:

1. State that the proof is by contradiction.
2. State what the negation of the original statement is.
3. State you have reached a contradiction and what the contradiction entails.

In CS103, please include all these steps in your proofs!

We have reached a contradiction, so our assumption was false. Thus if n is an integer and n^2 is even, n is even as well. ■

Negating Existential and Universal Statements

An Incorrect Proof

Theorem: For any natural number *n*, the sum of all natural numbers less than *n* is not equal to *n*.

Proof: By contradiction; assume that for any natural number *n*, the sum of all smaller natural numbers is equal to *n*. But this is clearly false, because $5 \neq 1 + 2 + 3 + 4 = 10$. We have reached a contradiction, so our assumption was false and the theorem must be true. ■

An Incorrect Proof

Theorem: For any natural number *n*, the sum of all natural numbers less than *n* is not equal to *n*.

Proof: By contradiction; assume that for any natural number *n*, the sum of all smaller natural numbers is equal to *n*. But this is clearly false, because $5 \neq 1 + 2 + 3 + 4 = 10$. We have reached a contradiction, so our ass[...] the theorem must be true. ■

Is this *really* the negation of the original statement?

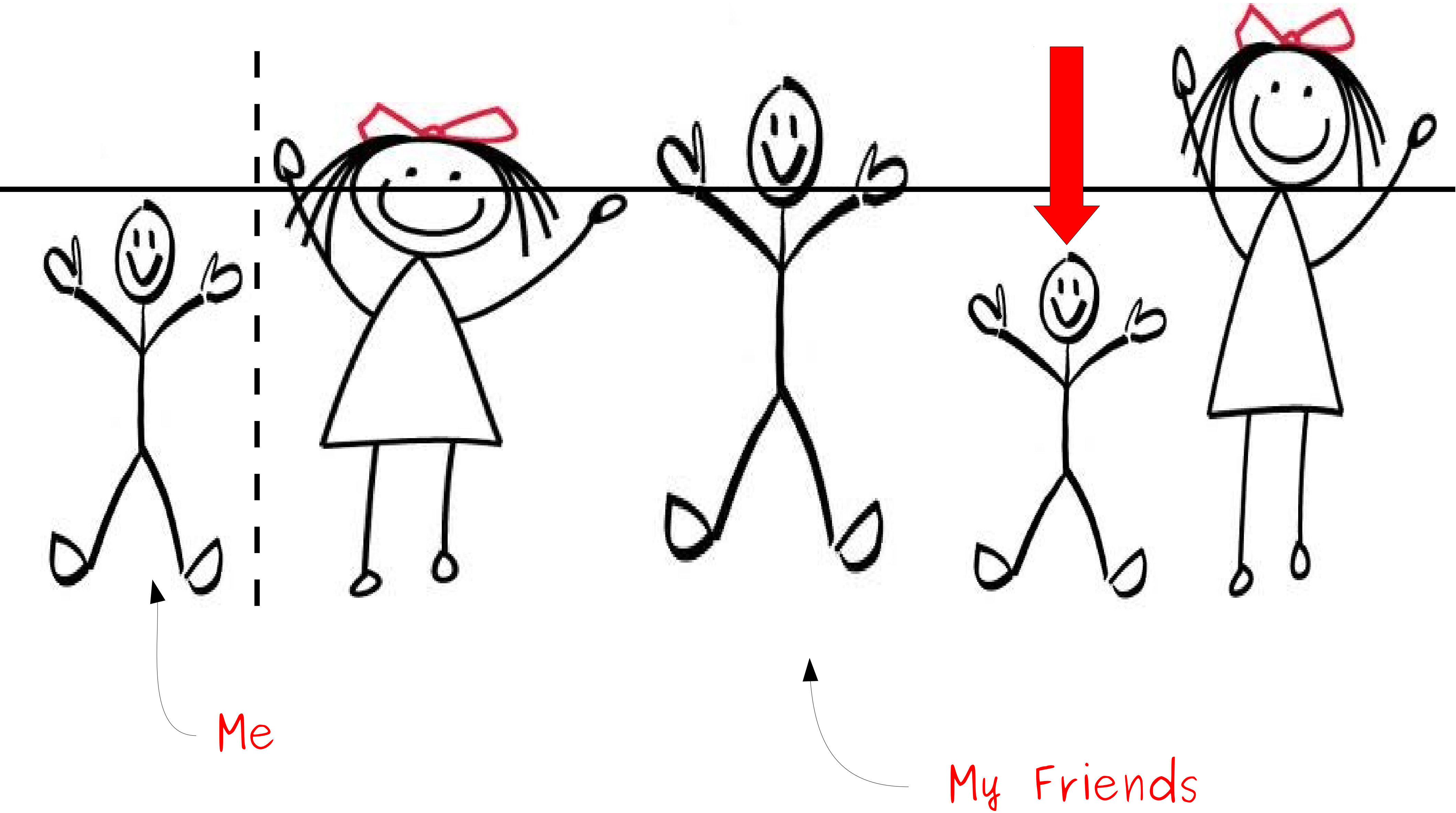
The negation of the universal statement

For all *x*, P(*x*) is true.

is **not**

For all *x*, P(*x*) is false.

“All My Friends Are Taller Than Me”



The negation of the universal statement

For all *x*, P(*x*) is true.

is the existential statement

There exists an *x* such that P(*x*) is false.

For all natural numbers *n*,
the sum of all natural numbers
smaller than *n* is not equal to *n*.

becomes

There exists a natural number *n* such that
the sum of all natural numbers
smaller than *n* ***is*** equal to *n*

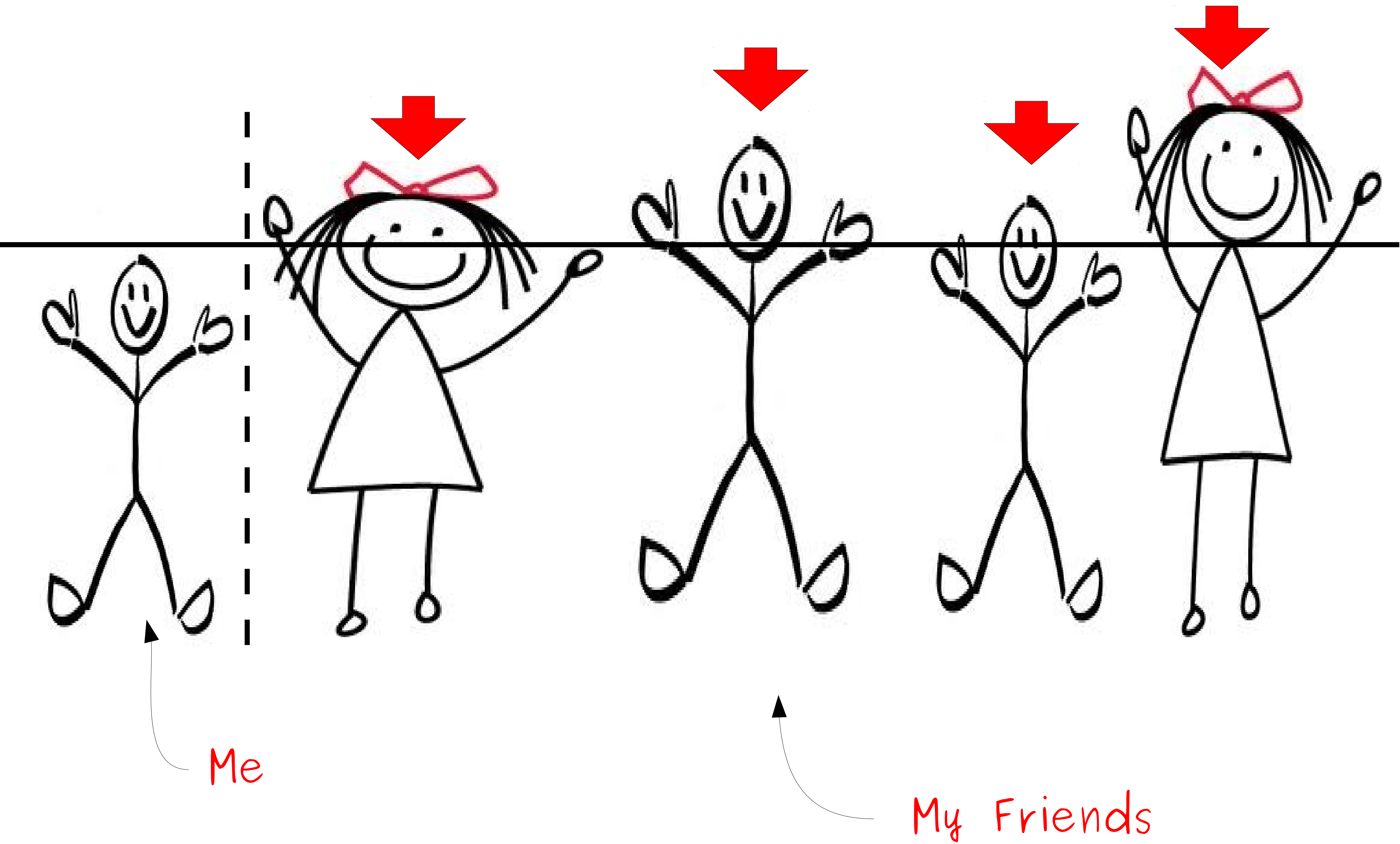
The negation of the existential statement

There exists an *x* such that P(*x*) is true.

is **not**

There exists an *x* such that P(*x*) is false.

“Some Friend Is Shorter Than Me”



The negation of the existential statement

There exists an *x* such that P(*x*) is true.

is the universal statement

For all *x*, P(*x*) is false.

Negating Implications

"If *P*, then *Q*"

becomes

"*P* but not *Q*"

Negating Universal Statements

"For all *x*, P(*x*) is true"

becomes

"There is an *x* where P(*x*) is false."

Negating Existential Statements

"There exists an *x* where P(*x*) is true"

becomes

"For all *x*, P(*x*) is false."

Next Time

- **Proof by Induction**
  - Proofs on sums, programs, algorithms, etc.