Regular Expressions

Problem Set Four is due using a late period in the box up front.

Concatenation

- The **concatenation** of two languages L_1 and L_2 over the alphabet Σ is the language

$$L_1L_2 = \{ wx \in \Sigma^* \mid w \in L_1 \land x \in L_2 \}$$

- Intuitively, the set of all strings formed by concatenating some string from L_1 and some string from L_2.
- Conceptually similar to the Cartesian product of two sets, only with strings.

Language Exponentiation

- We can define what it means to “exponentiate” a language as follows:
- $L^0 = \{ \varepsilon \}$
  - The set containing just the empty string.
  - Idea: Any string formed by concatenating zero strings together is the empty string.
- $L^{n+1} = LL^n$
  - Idea: Concatenating (n+1) strings together works by concatenating n strings, then concatenating one more.

The Kleene Closure

- An important operation on languages is the **Kleene Closure**, which is defined as

$$L^* = \bigcup_{i=0}^{\infty} L^i$$

- Mathematically:

$$\boldsymbol{w \in L^*} \quad \textbf{iff} \quad \boldsymbol{\exists n \in \mathbb{N}.\ w \in L^n}$$

- Intuitively, all possible ways of concatenating any number of copies of strings in L together.

Closure Properties

- The regular languages are closed under the following operations:
  - Complementation
  - Union
  - Intersection
  - Concatenation
  - Kleene closure

Another View of Regular Languages

Rethinking Regular Languages

- We currently have several tools for showing a language is regular.
  - Construct a DFA for it.
  - Construct an NFA for it.
  - Apply closure properties to existing languages.
- We have not spoken much of this last idea.

Constructing Regular Languages

- **Idea**: Build up all regular languages as follows:
  - Start with a small set of simple languages we already know to be regular.
  - Using closure properties, combine these simple languages together to form more elaborate languages.
- *A bottom-up approach to the regular languages.*

Regular Expressions

- **Regular expressions** are a family of descriptions that can be used to capture the regular languages.
- Often provide a compact and human-readable description of the language.
- Used as the basis for numerous software systems (Perl, **`flex`**, **`grep`**, etc.)

Atomic Regular Expressions

- The regular expressions begin with three simple building blocks.
- The symbol **Ø** is a regular expression that represents the empty language Ø.
- The symbol **ε** is a regular expression that represents the language { ε }
  - ***This is not the same as Ø!***
- For any `a` ∈ Σ, the symbol `a` is a regular expression for the language { `a` }

Compound Regular Expressions

- We can combine together existing regular expressions in four ways.
- If R_1 and R_2 are regular expressions, $\boldsymbol{R_1R_2}$ is a regular expression for the **concatenation** of the languages of R_1 and R_2.
- If R_1 and R_2 are regular expressions, $\boldsymbol{R_1 \mid R_2}$ is a regular expression for the **union** of the languages of R_1 and R_2.
- If R is a regular expression, $\boldsymbol{R^*}$ is a regular expression for the **Kleene closure** of the language of R.
- If R is a regular expression, $\boldsymbol{(R)}$ is a regular expression with the same meaning as R.

Operator Precedence

- Regular expression operator precedence:

$$(R)$$

$$R^*$$

$$R_1 R_2$$

$$R_1 \mid R_2$$

- So **`ab*c|d`** is parsed as **`((a(b*))c)|d`**

Regular Expression Examples

- The regular expression **`trick|treat`** represents the regular language { **`trick`**, **`treat`** }
- The regular expression **`booo*`** represents the regular language { **`boo`**, **`booo`**, **`boooo`**, … }
- The regular expression **`candy!(candy!)*`** represents the regular language { **`candy!`**, **`candy!candy!`**, **`candy!candy!candy!`**, … }

Regular Expressions, Formally

- The **language of a regular expression** is the language described by that regular expression.
- Formally:
  - $\mathscr{L}(\varepsilon) = \{\varepsilon\}$
  - $\mathscr{L}(\emptyset) = \emptyset$
  - $\mathscr{L}(\texttt{a}) = \{\texttt{a}\}$
  - $\mathscr{L}(R_1R_2) = \mathscr{L}(R_1)\ \mathscr{L}(R_2)$
  - $\mathscr{L}(R_1 \mid R_2) = \mathscr{L}(R_1) \cup \mathscr{L}(R_2)$
  - $\mathscr{L}(R^*) = \mathscr{L}(R)^*$
  - $\mathscr{L}((R)) = \mathscr{L}(R)$

Worthwhile activity: Apply this recursive definition to

`a(b|c)((d))`

and see what you get.

Regular Expressions are Awesome

- Let $\Sigma = \{$`0`, `1`$\}$
- Let $L = \{\ w \in \Sigma^* \mid w$ contains `00` as a substring $\}$

(0 | 1)*00(0 | 1)*

11011100101
0000
11111011110011111

Regular Expressions are Awesome

- Let $\Sigma = \{$`0`, `1`$\}$
- Let $L = \{\ w \in \Sigma^* \mid w$ contains `00` as a substring $\}$

$$\boldsymbol{\Sigma^* 00 \Sigma^*}$$

11011100101
0000
11111011110011111

Regular Expressions are Awesome

Let $\Sigma = \{0, 1\}$

Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$

The length of a string w is denoted |w|

Regular Expressions are Awesome

- Let $\Sigma = \{$`0`, `1`$\}$
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$

$$\Sigma\Sigma\Sigma\Sigma$$

0000
1010
1111
1000

Regular Expressions are Awesome

- Let $\Sigma = \{$`0`, `1`$\}$
- Let $L = \{\ w \in \Sigma^* \mid |w| = 4\ \}$

$$\boldsymbol{\Sigma^4}$$

0000
1010
1111
1000

Regular Expressions are Awesome

- Let Σ = {`0`, `1`}
- Let L = { w ∈ Σ* | w contains at most one `0` }

1*(0 | ε)1*

11110111
111111
0111
0

Regular Expressions are Awesome

- Let Σ = {`0`, `1`}
- Let L = { w ∈ Σ* | w contains at most one `0` }

1*0?1*

11110111
111111
0111
0

Regular Expressions are Awesome

- Let Σ = { **`a`**, **`.`**, **`@`** }, where **`a`** represents “some letter.”
- Regular expression for email addresses:

Regular Expressions are Awesome

- Let Σ = { `a`, `.`, `@` }, where `a` represents "some letter."
- Regular expression for email addresses:

aa*(.aa*)*@aa*.aa*(.aa*)*

cs103@cs.stanford.edu
first.middle.last@mail.site.org
barack.obama@whitehouse.gov

Regular Expressions are Awesome

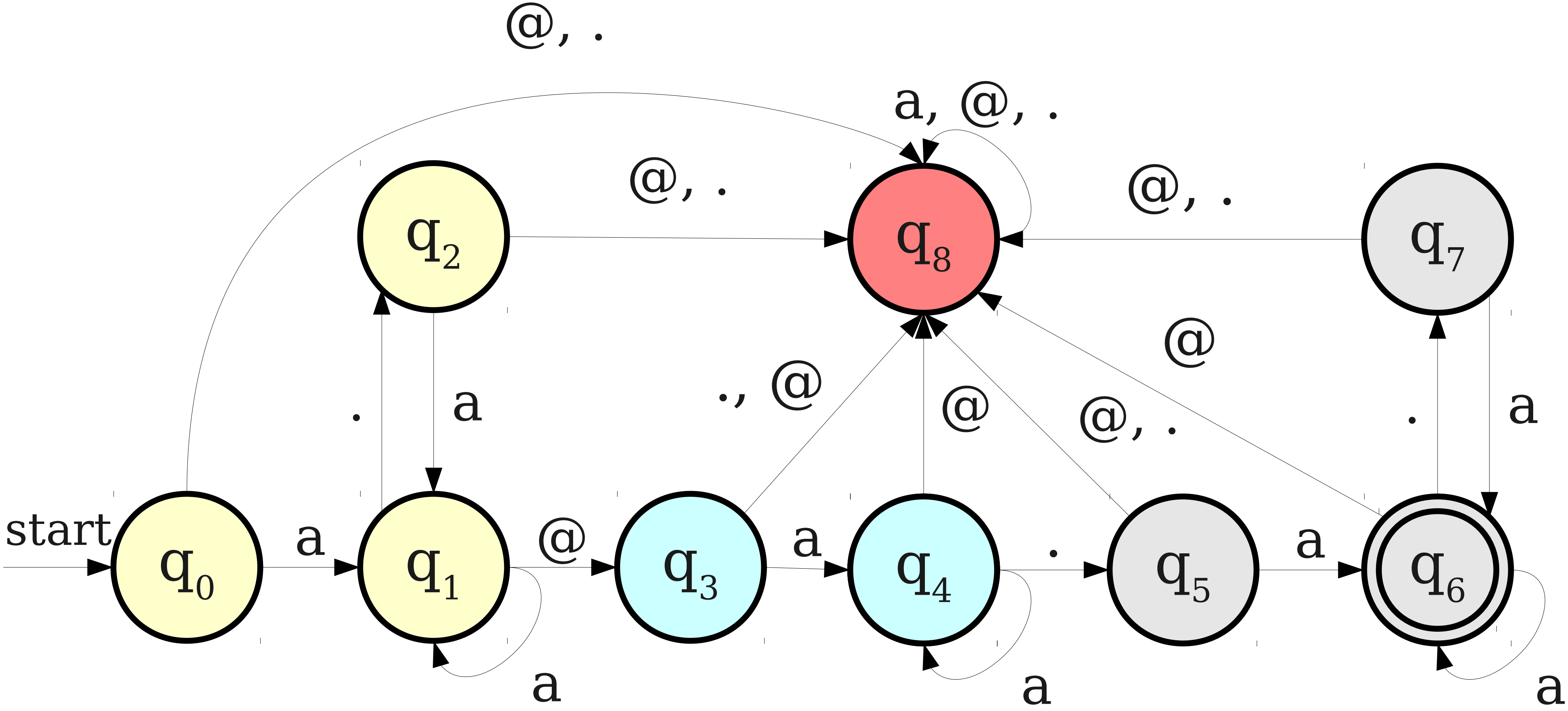
- Let Σ = { `a`, `.`, `@` }, where `a` represents "some letter."
- Regular expression for email addresses:

$$a^+(.a^+)^*@a^+(.a^+)^+$$

cs103@cs.stanford.edu
first.middle.last@mail.site.org
barack.obama@whitehouse.gov

Regular Expressions are Awesome

$$\mathbf{a^{+}(.a^{+})^{*}@a^{+}(.a^{+})^{+}}$$



Shorthand Summary

- R^n is shorthand for $RR \dots R$ (n times).
- Σ is shorthand for “any character in Σ.”
- $R?$ is shorthand for $(R \mid \varepsilon)$, meaning “zero or one copies of R.”
- R^+ is shorthand for RR^*, meaning “one or more copies of R.”

Break for Announcements!

Midterm Logistics

- Midterm is tomorrow, October 29, from 7PM - 10PM
- Room determined by last name:
  - A – G: Go to **Gates B01**
  - H – K: Go to **Gates B03**
  - L – P: Go to **200-002**
  - Q – V: Go to **420-041**
  - W – Z: Go to **Herrin T175**

Your Questions

When writing a logic statement, do you have to include the universal or existential quantifier for every variable that you state? I thought you had to, but this one from lecture doesn't:

$$Tallest(x) \to \forall y.\ (x \neq y \to IsShorterThan(y, x))$$

This example is a "sentence fragment" in first-order logic; without a definition of x, this isn't a valid statement. All variables need to be quantified.

"When writing first-order logic statements with quantifiers, which one out of the following would be correct?

$$\forall x\ P(x).\ \exists y.\ R(y)$$

or

[✓ $\forall x.\ (P(x) \rightarrow \exists y.\ R(y))$]

If you find that the function $f\colon A \to B$ is not surjective, have you proven that $|A| < |B|$? Or do you still need to do additional proof steps?

$$f : \mathbb{N} \to \mathbb{N}$$

$$f(n) = 137$$

“What is the best thing to do to prepare for the exam between now and 7PM tomorrow?”

“Is there some mathematical automaton that can determine whether or not two first-order logical statements are equivalent?”

More on that later in the quarter...

Back to Regular Expressions!

The Power of Regular Expressions

Theorem: If R is a regular expression, then $\mathscr{L}(R)$ is regular.

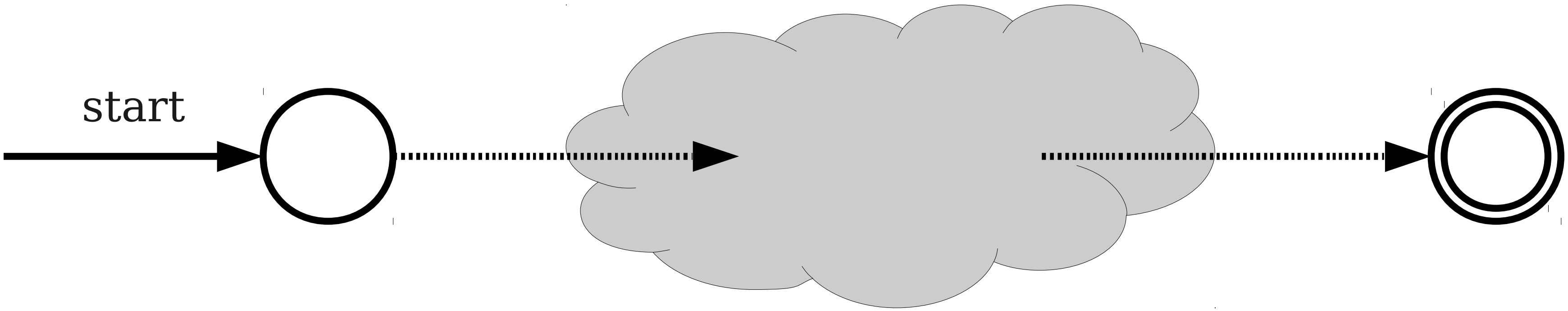
Proof idea: Show how to convert a regular expression into an NFA.

A Marvelous Construction

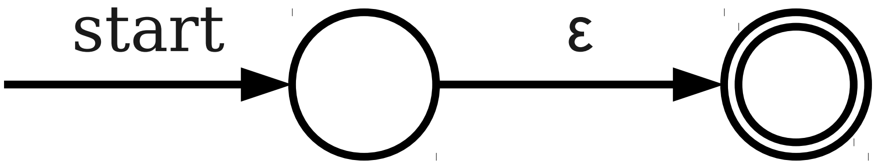
- The following theorem proves the language of any regular expression is regular:
- ***Theorem:*** For any regular expression R, there is an NFA N such that

  $$\mathscr{L}(R) = \mathscr{L}(N)$$

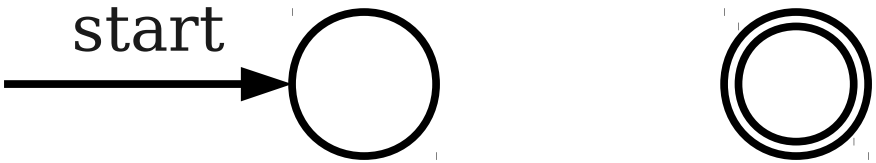
  - N has exactly one accepting state.
  - N has no transitions into its start state.
  - N has no transitions out of its accepting state.



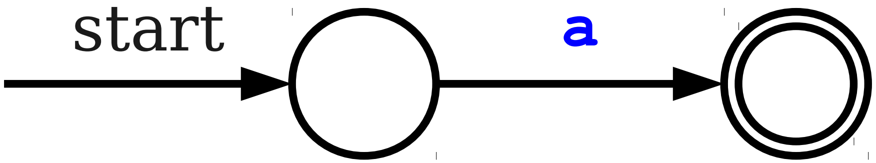
Base Cases



Automaton for ε

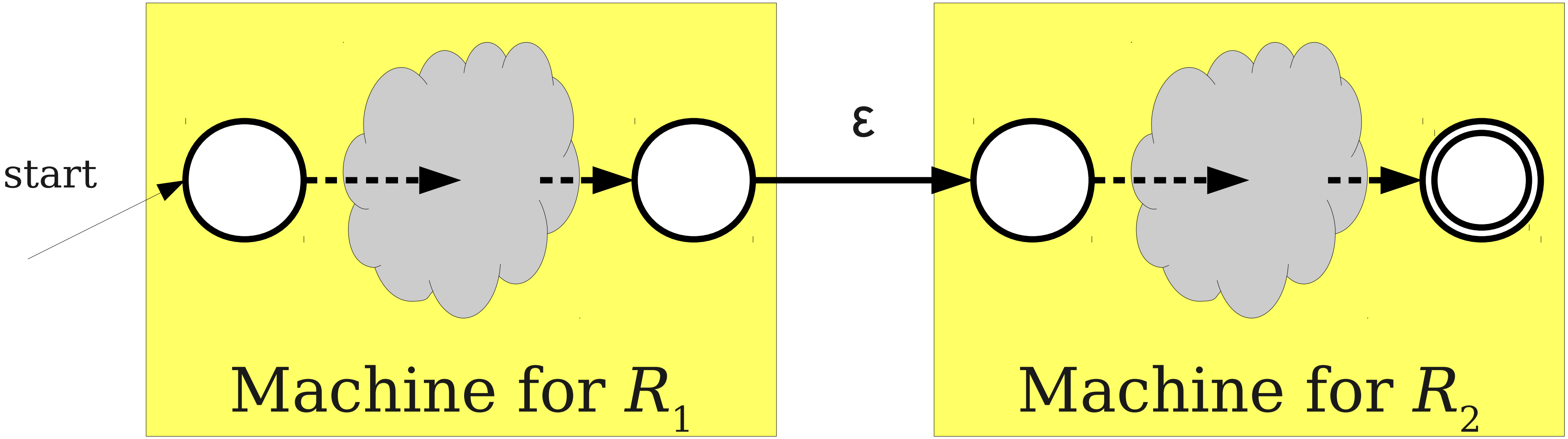


Automaton for Ø

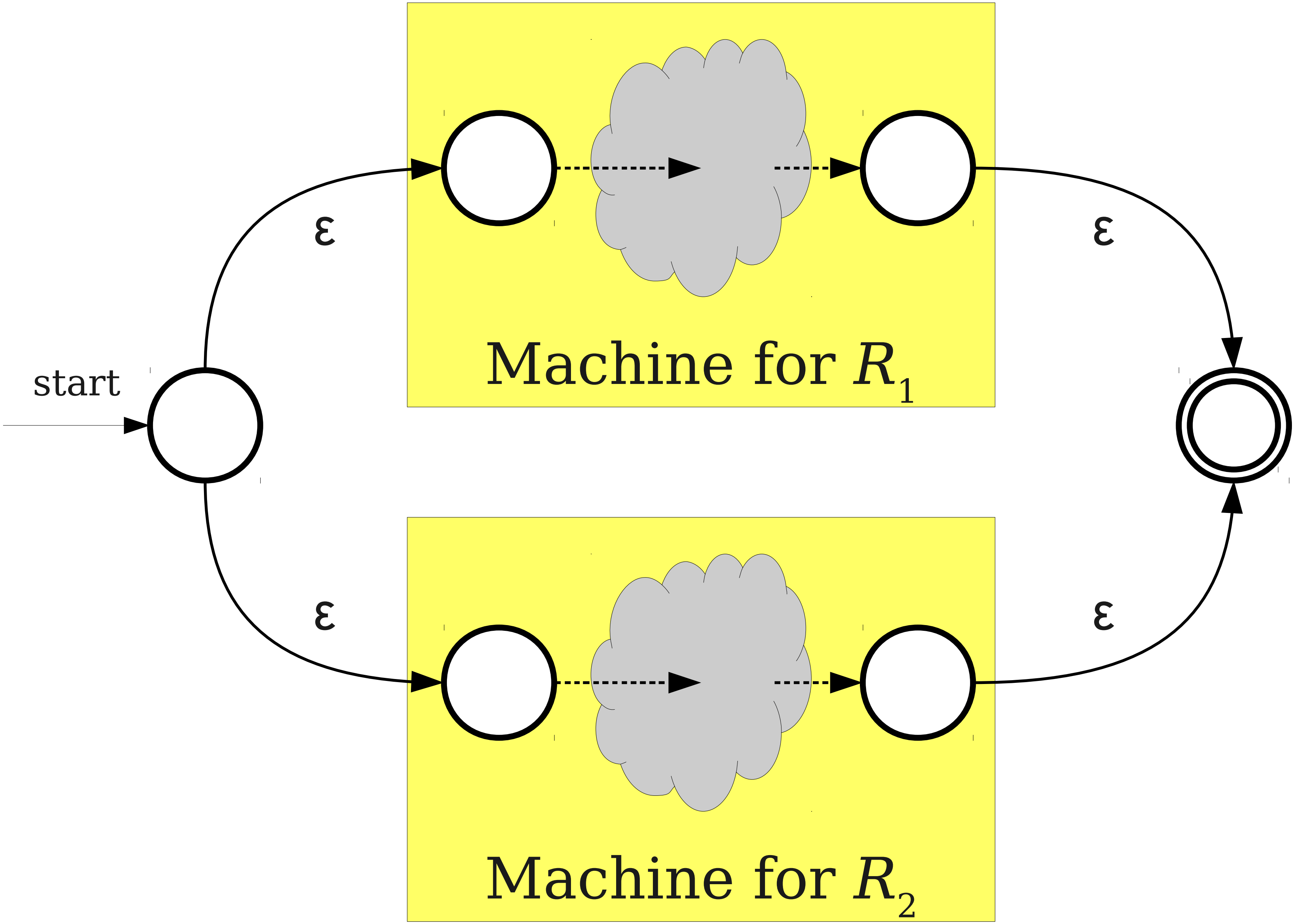


Automaton for single character **`a`**

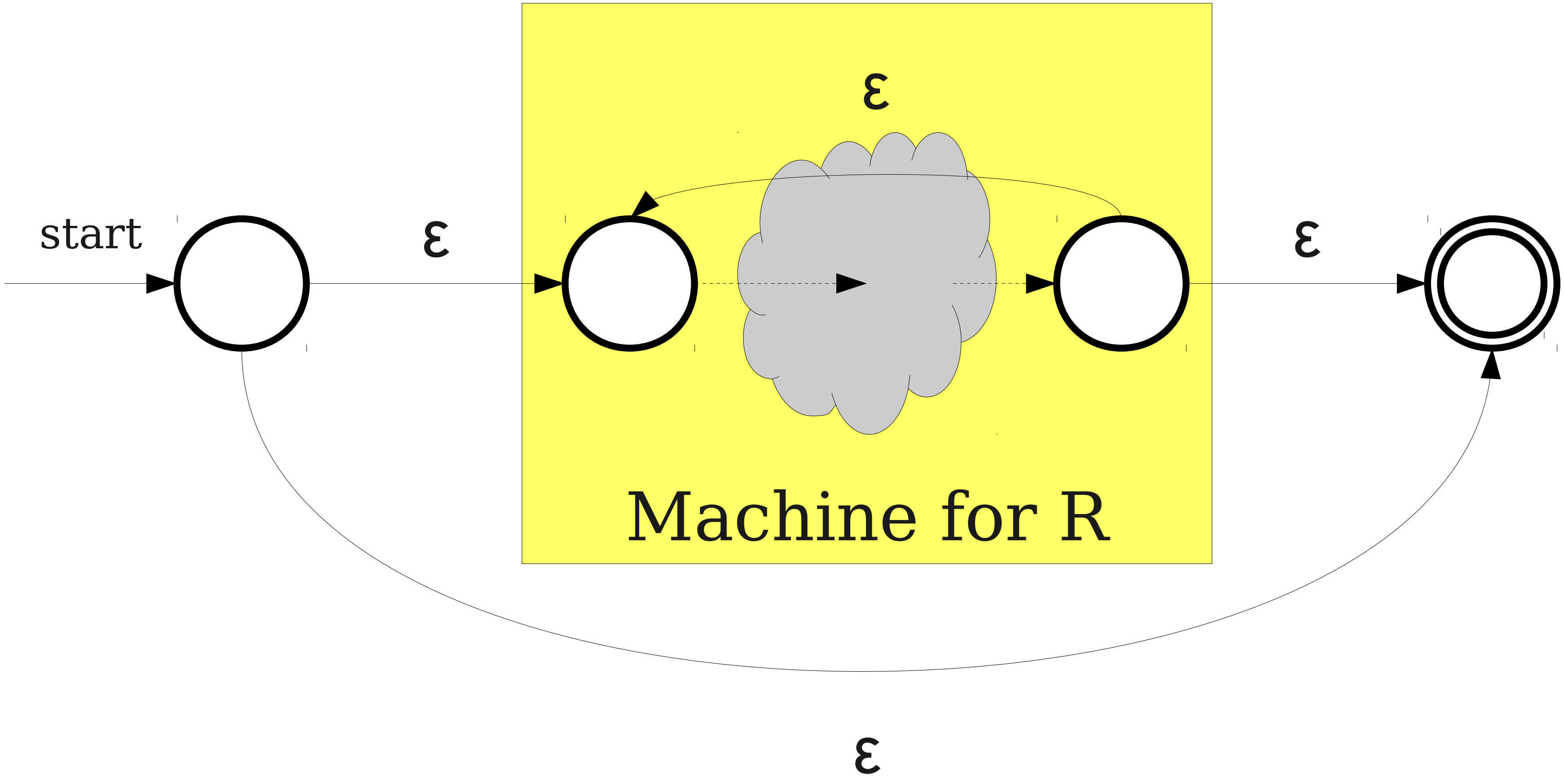
Construction for R_1R_2



Construction for $R_1 \mid R_2$



Construction for *R**



Why This Matters

- Many software tools work by matching regular expressions against text.
- One possible algorithm for doing so:
    - Convert the regular expression to an NFA.
    - (Optionally) Convert the NFA to a DFA using the subset construction.
    - Run the text through the finite automaton and look for matches.
- Runs extremely quickly!

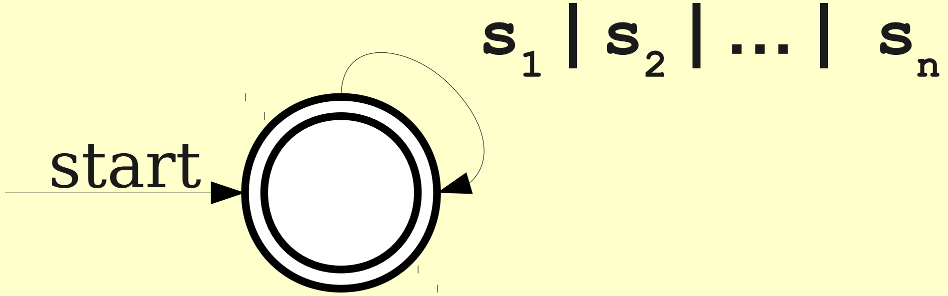
The Power of Regular Expressions

Theorem: If L is a regular language, then there is a regular expression for L.

This is not obvious!

Proof idea: Show how to convert an arbitrary NFA into a regular expression.

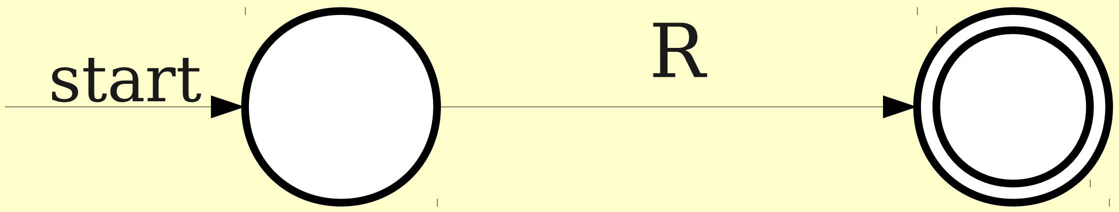
From NFAs to Regular Expressions



Regular expression: **$(s_1 \mid s_2 \mid \dots \mid s_n)^*$**

Key idea: Label transitions with arbitrary regular expressions.

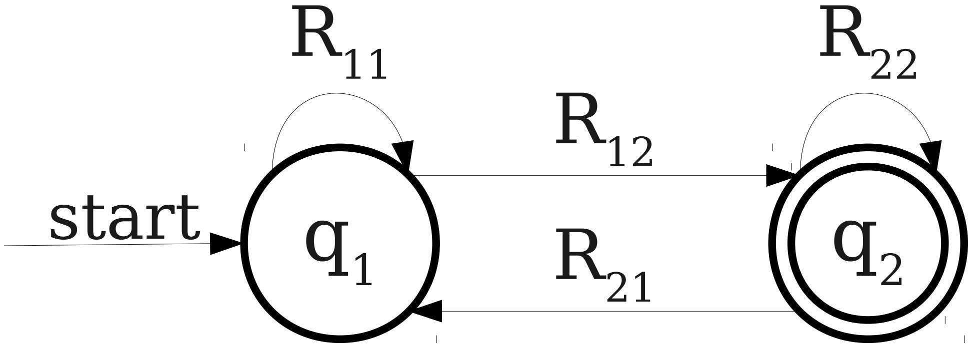
From NFAs to Regular Expressions



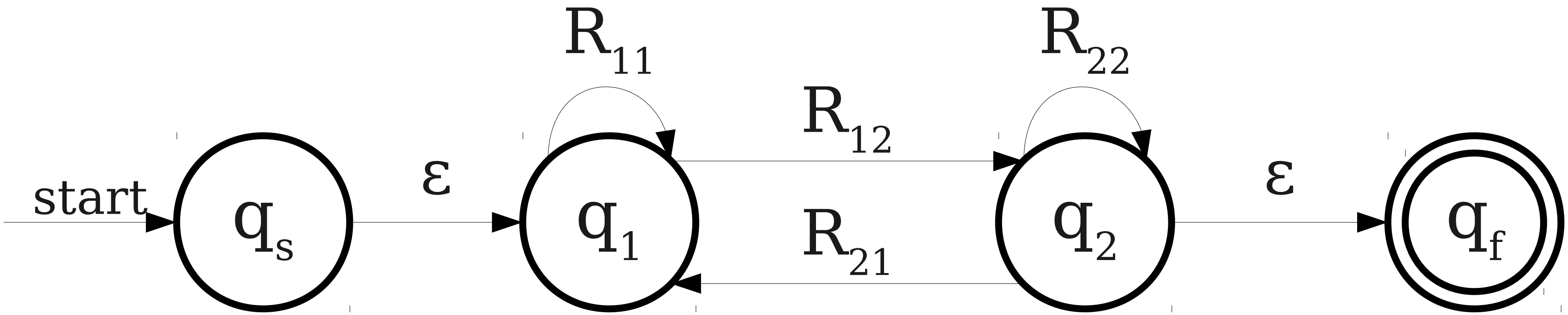
Regular expression: **R**

Key idea: If we can convert any NFA into something that looks like this, we can easily read off the regular expression.

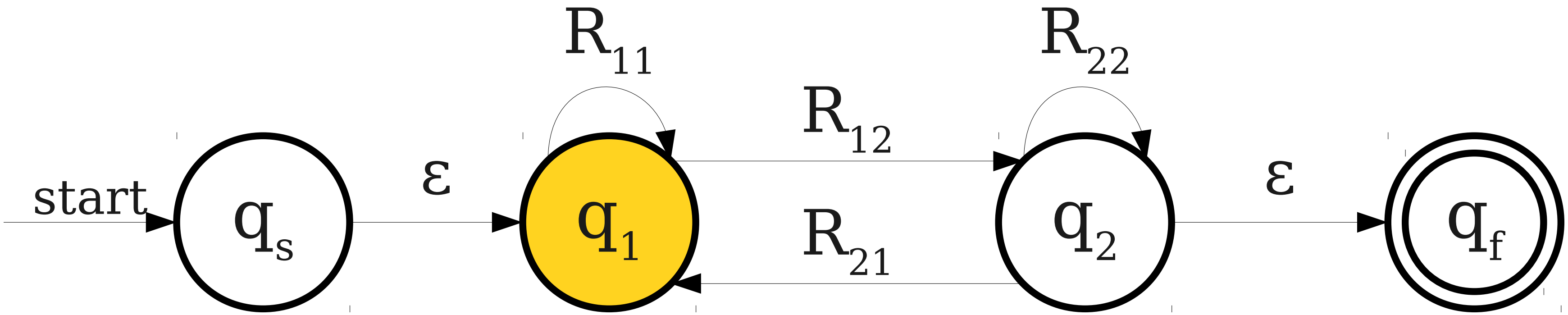
From NFAs to Regular Expressions



From NFAs to Regular Expressions

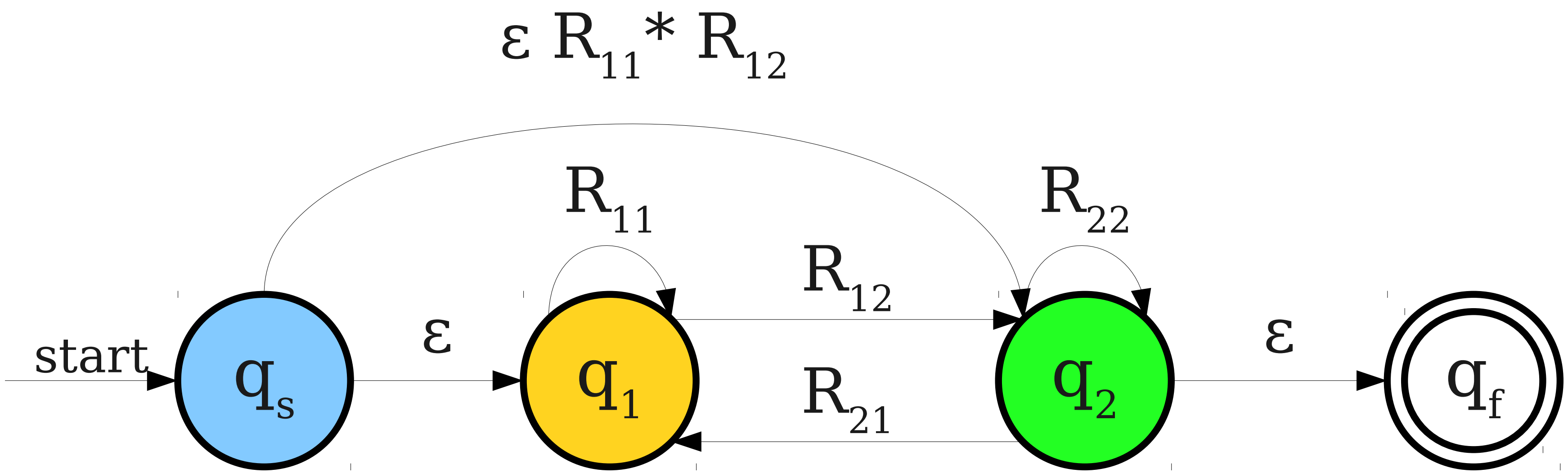


From NFAs to Regular Expressions



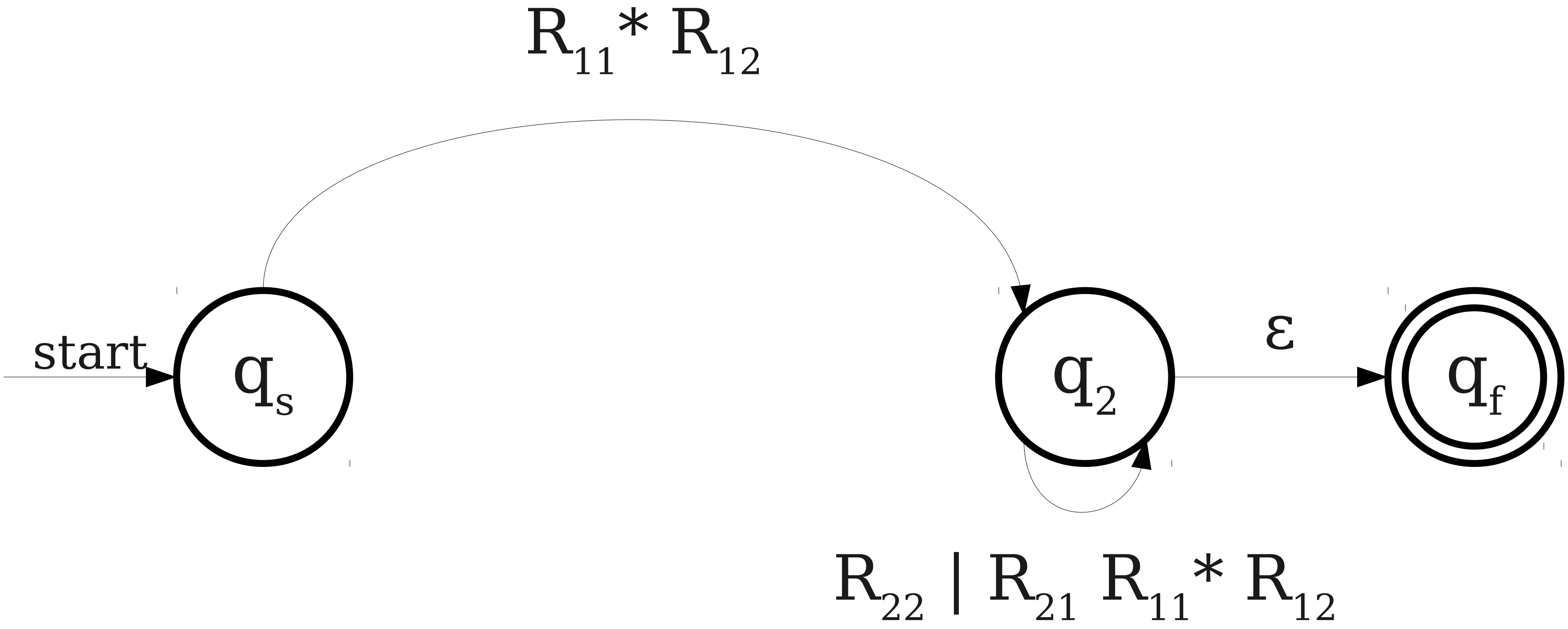
Could we eliminate this state from the NFA?

From NFAs to Regular Expressions



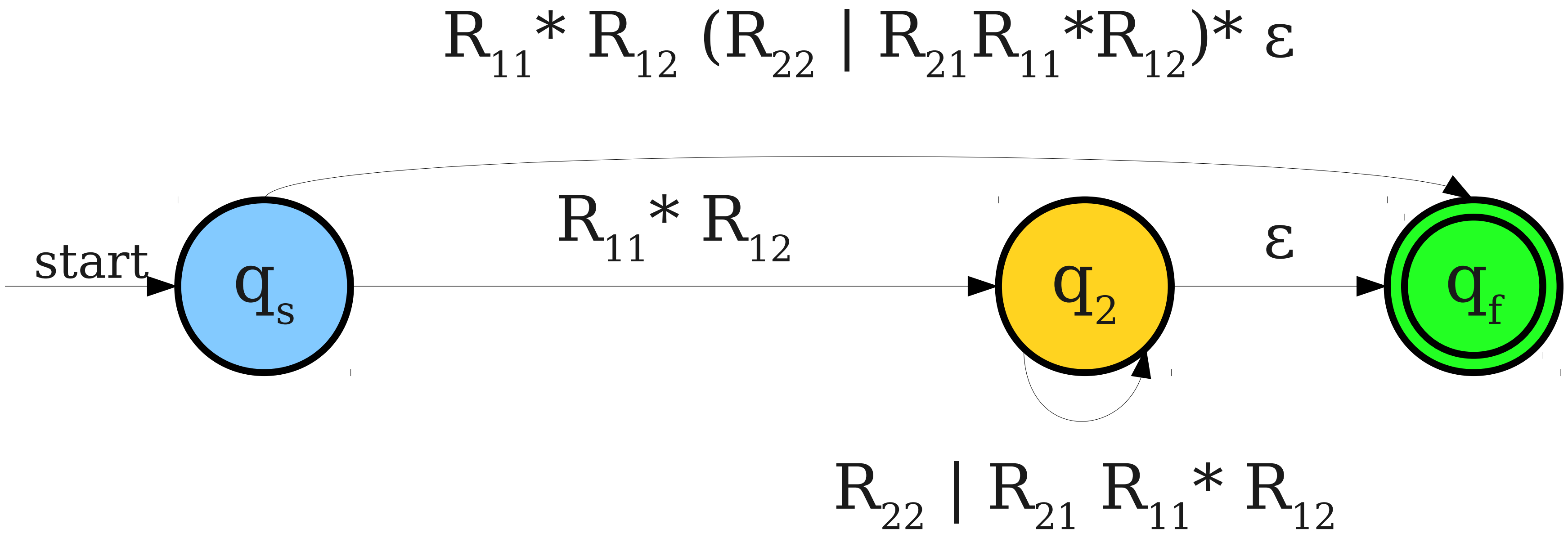
Note: We're using **concatenation** and **Kleene closure** in order to skip this state.

From NFAs to Regular Expressions



Note: We're using union to combine these transitions together.

From NFAs to Regular Expressions



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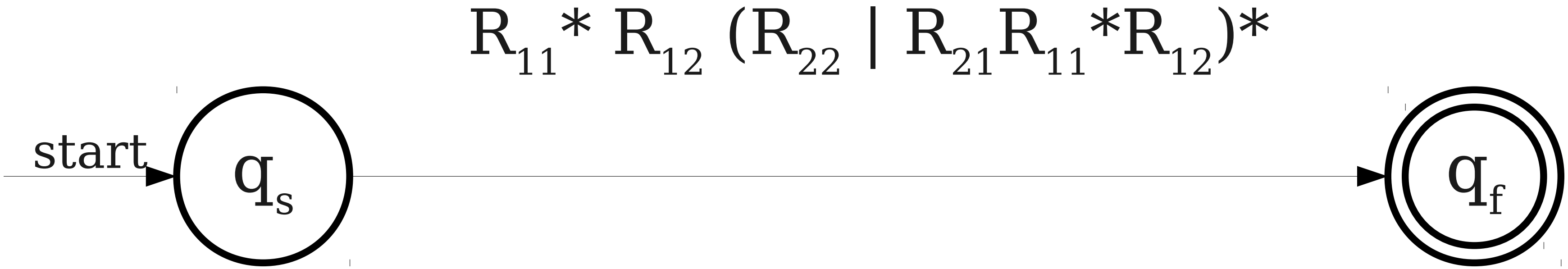
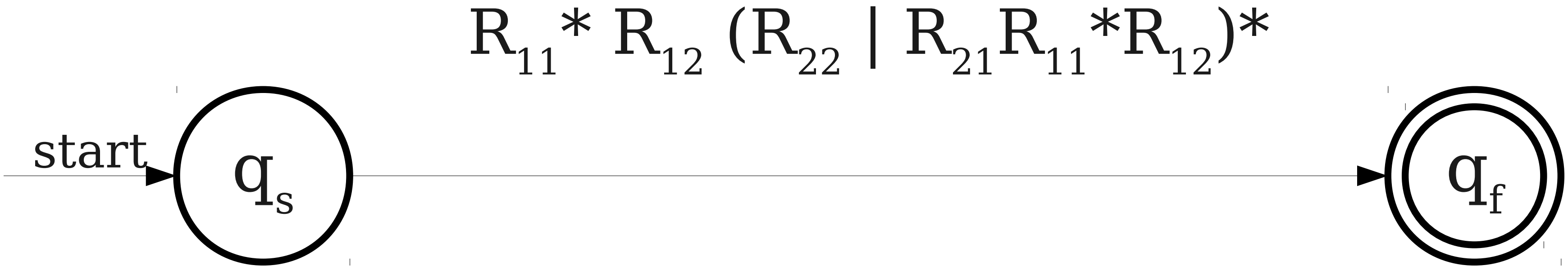
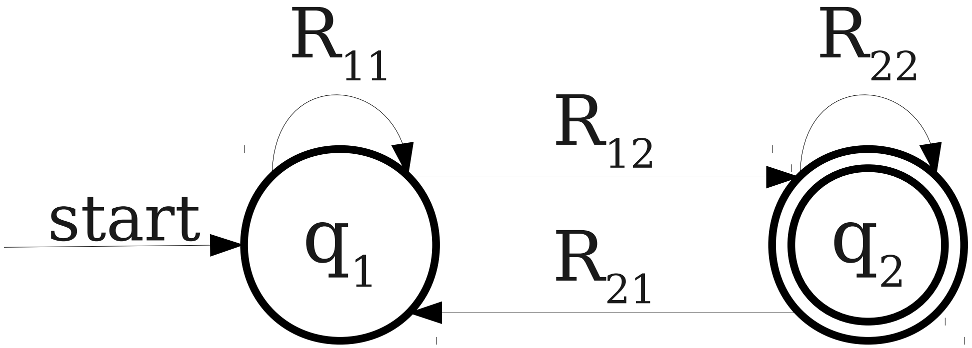


From NFAs to Regular Expressions

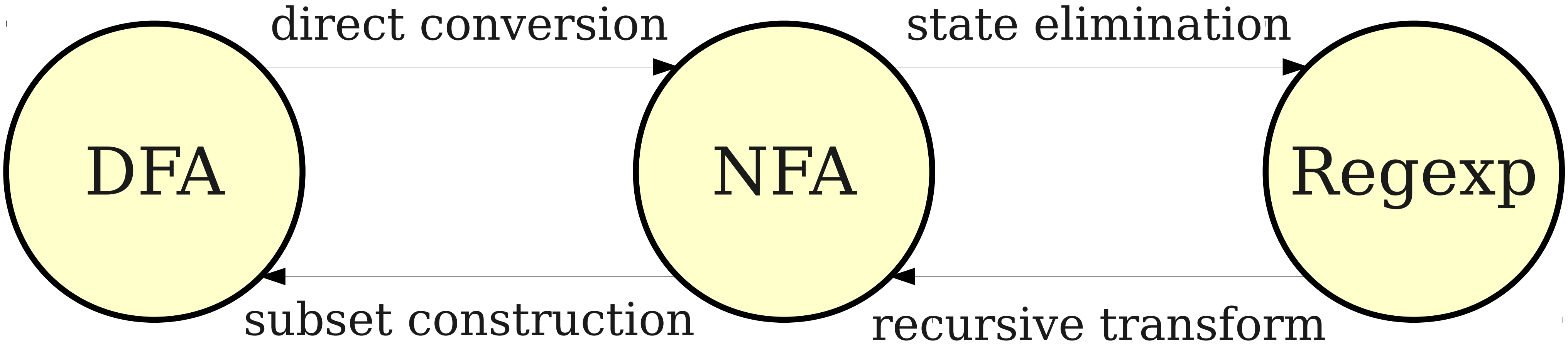




The Construction at a Glance

- Start with an NFA for the language L.
- Add a new start state q_s and accept state q_f to the NFA.
  - Add ε-transitions from each original accepting state to q_f, then mark them as not accepting.
- Repeatedly remove states other than q_s and q_f from the NFA by "shortcutting" them until only two states remain: q_s and q_f.
- The transition from q_s to q_f is then a regular expression for the NFA.

Our Transformations



Theorem: The following are all equivalent:

- L is a regular language.
- There is a DFA D such that $\mathscr{L}(D) = L$.
- There is an NFA N such that $\mathscr{L}(N) = L$.
- There is a regular expression R such that $\mathscr{L}(R) = L$.

Next Time

- **Applications of Regular Languages**
  - Answering “so what?”
- **Intuiting Regular Languages**
  - What makes a language regular?
- **The Pumping Lemma**
  - Proving languages aren't regular.