

Mathematical Induction

Part Two

Let P be some property. The **principle of mathematical induction** states that if

If it starts true...

P is true for 0

and

...and it stays true...

For any $k \in \mathbb{N}$, if P is true for k , then P is true for $k + 1$

then

P is true for every $n \in \mathbb{N}$.

...then it's always true.

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: Let $P(n)$ be the statement “the sum of the first n powers of two is $2^n - 1$.” We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

For our base case, we need to show $P(0)$ is true, meaning that the sum of the first zero powers of two is $2^0 - 1$. Since the sum of the first zero powers of two is zero and $2^0 - 1$ is zero as well, we see that $P(0)$ is true.

For the inductive step, assume that for some $k \in \mathbb{N}$ that $P(k)$ holds, meaning that

$$2^0 + 2^1 + \dots + 2^{k-1} = 2^k - 1. \quad (1)$$

We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers of two is $2^{k+1} - 1$. To see this, notice that

$$\begin{aligned} 2^0 + 2^1 + \dots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \dots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \quad (\text{via (1)}) \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Therefore, $P(k + 1)$ is true, completing the induction. ■

Induction in Practice

- Typically, a proof by induction will not explicitly state $P(n)$.
- Rather, the proof will describe $P(n)$ implicitly and leave it to the reader to fill in the details.
- Provided that there is sufficient detail to determine
 - what $P(n)$ is,
 - that $P(0)$ is true, and that
 - whenever $P(k)$ is true, $P(k+1)$ is true,the proof is usually valid.

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: By induction.

For our base case, we'll prove the theorem is true when $n = 0$. The sum of the first zero powers of two is zero, and $2^0 - 1 = 0$, so the theorem is true in this case.

For the inductive step, assume the theorem holds when $n = k$ for some arbitrary $k \in \mathbb{N}$. Then

$$\begin{aligned} 2^0 + 2^1 + \dots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \dots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

So the theorem is true when $n = k+1$, completing the induction. ■

Variations on Induction: **Starting Later**

Induction Starting at 0

- To prove that $P(n)$ is true for all natural numbers greater than or equal to 0:
 - Show that $P(0)$ is true.
 - Show that for any $k \geq 0$, that if $P(k)$ is true, then $P(k+1)$ is true.
 - Conclude $P(n)$ holds for all natural numbers greater than or equal to 0.

Induction Starting at m

- To prove that $P(n)$ is true for all natural numbers greater than or equal to m :
 - Show that $P(m)$ is true.
 - Show that for any $k \geq m$, that if $P(k)$ is true, then $P(k+1)$ is true.
 - Conclude $P(n)$ holds for all natural numbers greater than or equal to m .

n^2 versus 2^n

$$0^2 = 0 < 2^0 = 1$$

$$1^2 = 1 < 2^1 = 2$$

$$2^2 = 4 = 2^2 = 4$$

$$3^2 = 9 > 2^3 = 8$$

$$4^2 = 16 = 2^4 = 16$$

$$5^2 = 25 < 2^5 = 32$$

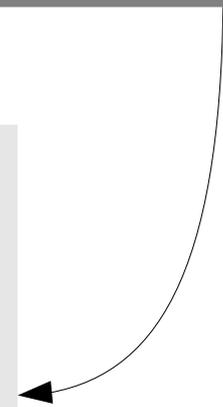
$$6^2 = 36 < 2^6 = 64$$

$$7^2 = 49 < 2^7 = 128$$

$$8^2 = 64 < 2^8 = 256$$

$$9^2 = 81 < 2^9 = 512$$

2^n is much bigger here. Does the trend continue?



Theorem: If n is a natural number and $n \geq 5$, then $n^2 < 2^n$.

Proof: Let $P(n)$ be the statement " $n^2 < 2^n$." We will prove by induction that $P(n)$ is true for all $n \in \mathbb{N}$ where $n \geq 5$, from which the theorem follows.

As our base case, we prove $P(5)$, that $5^2 < 2^5$. Note that $5^2 = 25$ and that $2^5 = 32$, so $5^2 < 2^5$.

For the inductive step, assume that for some $k \geq 5$ that $P(k)$ is true, meaning that $k^2 < 2^k$. We will prove $P(k+1)$, that $(k+1)^2 < 2^{k+1}$.

First, recall that $(k + 1)^2 = k^2 + 2k + 1$. Starting with this equality and using the fact that $k \geq 5$ (that is, that $5 \leq k$), we derive the following:

$$\begin{aligned}(k + 1)^2 &= k^2 + 2k + 1 \\ &< k^2 + 2k + k && \text{(since } 1 < 5 \leq k\text{)} \\ &= k^2 + 3k \\ &< k^2 + k \cdot k && \text{(since } 3 < 5 \leq k\text{)} \\ &= k^2 + k^2 \\ &= 2k^2\end{aligned}$$

So we see that $(k + 1)^2 < 2k^2$. Now, by our inductive hypothesis, we know $k^2 < 2^k$. Some further algebra gives us the following:

$$\begin{aligned}(k + 1)^2 &< 2k^2 && \text{(from above)} \\ &< 2(2^k) && \text{(from the inductive hypothesis)} \\ &= 2^{k+1}\end{aligned}$$

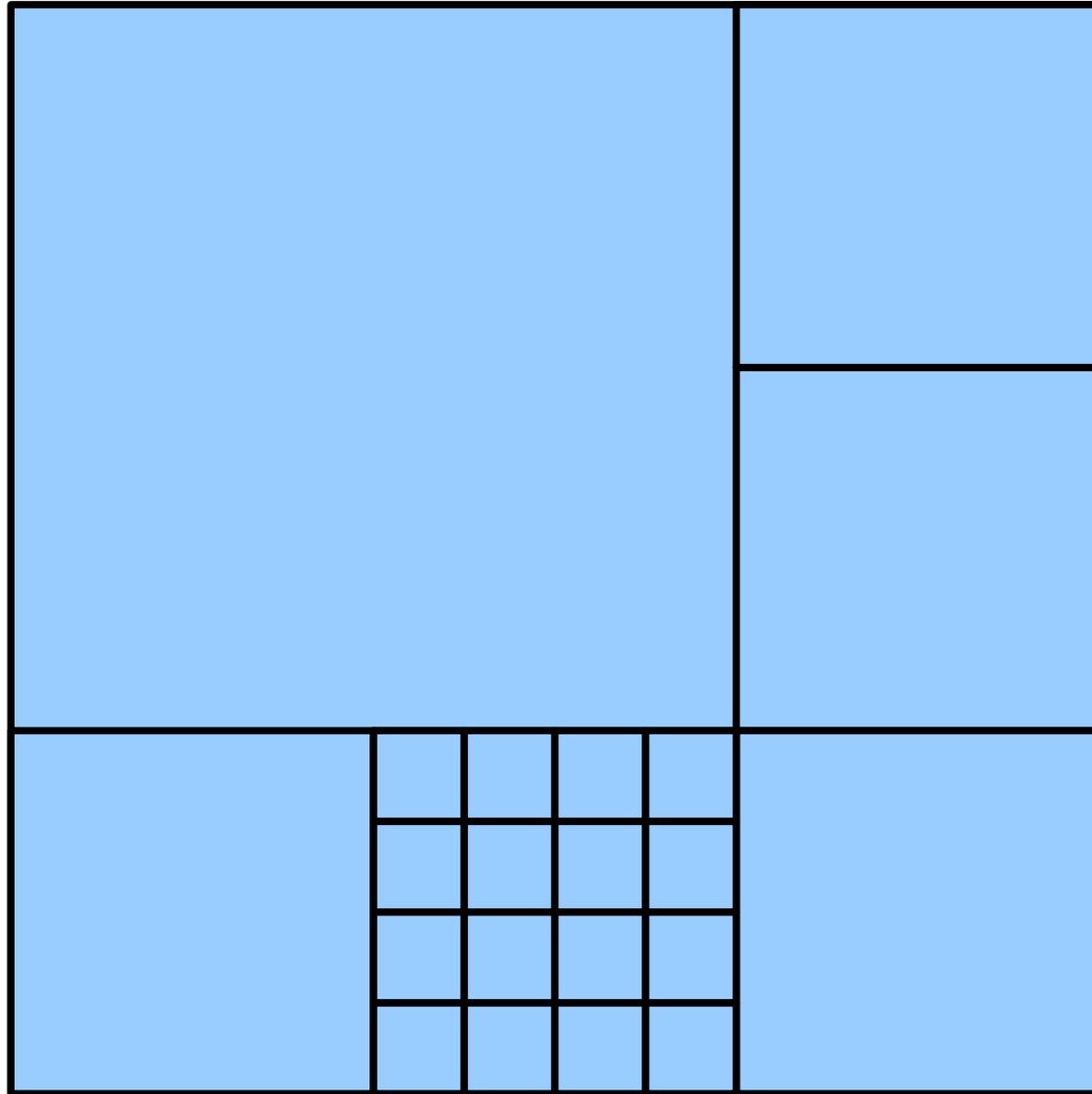
So $(k+1)^2 < 2^{k+1}$. Therefore, $P(k+1)$ is true, completing the induction. ■

A Note on the Previous Proof

- I chose this proof for a few reasons:
 - It uses induction to prove a result that is false for small n , but true for all sufficiently large n .
 - It doesn't require us to introduce any new terms and definitions.
- That said, I'm not a fan of it for a few reasons:
 - This result is not particularly deep or interesting.
 - There's a lot of algebra.
 - It's not at all obvious how to come up with this line of reasoning.
- **Challenge:** Find an inductive proof with the two “good” qualities, but without the three “bad” qualities.

Variations on Induction: **Bigger Steps**

Subdividing a Square



For what values of n can a square be subdivided into n squares?

1 2 3 4 5 6 7 8 9 10 11 12

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

1		2
		3
6	5	4

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

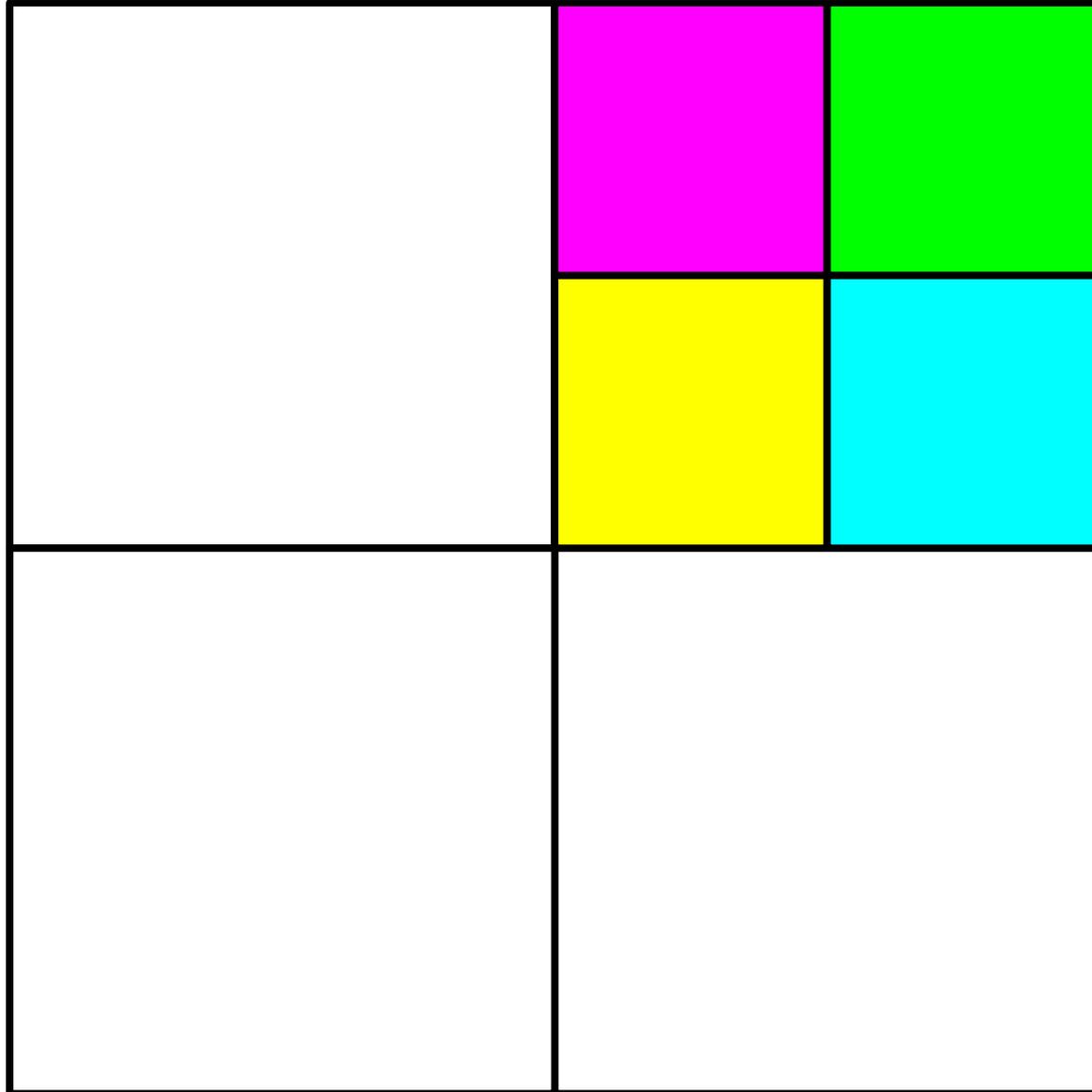
5	6	1
4	7	
3		2

1 2 3 4 5 6 7 8 9 10 11 12

1			
2	8		
3			
4	5	6	7

The Key Insight

The Key Insight



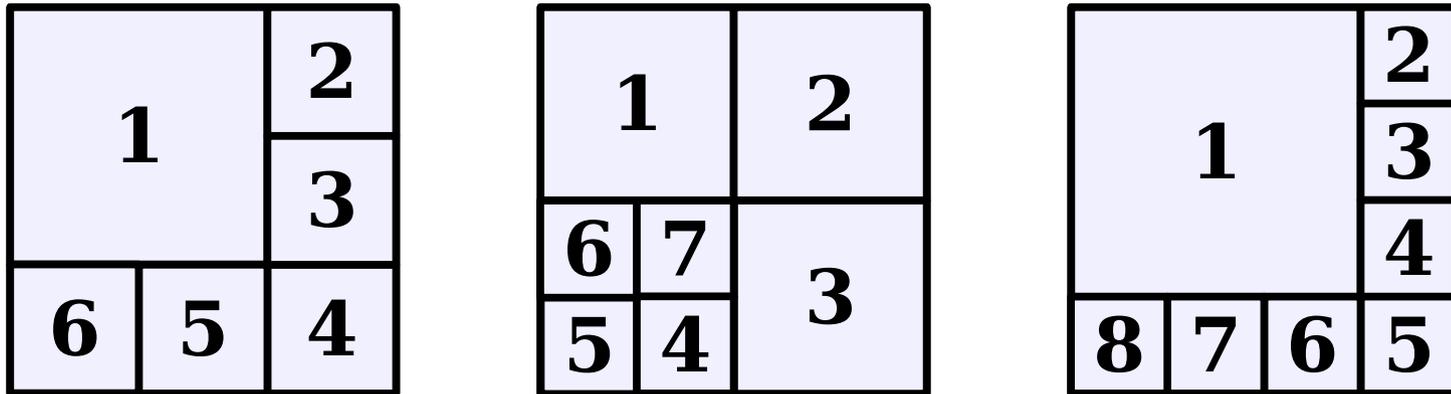
The Key Insight

- If we can subdivide a square into n squares, we can also subdivide it into $n + 3$ squares.
- Since we can subdivide a bigger square into 6, 7, and 8 squares, we can subdivide a square into n squares for any $n \geq 6$:
 - For multiples of three, start with 6 and keep adding three squares until n is reached.
 - For numbers congruent to one modulo three, start with 7 and keep adding three squares until n is reached.
 - For numbers congruent to two modulo three, start with 8 and keep adding three squares until n is reached.

Theorem: For any $n \geq 6$, it is possible to subdivide a square into n smaller squares.

Proof: Let $P(n)$ be the statement “a square can be subdivided into n smaller squares.” We will prove by induction that $P(n)$ holds for all $n \geq 6$, from which the theorem follows.

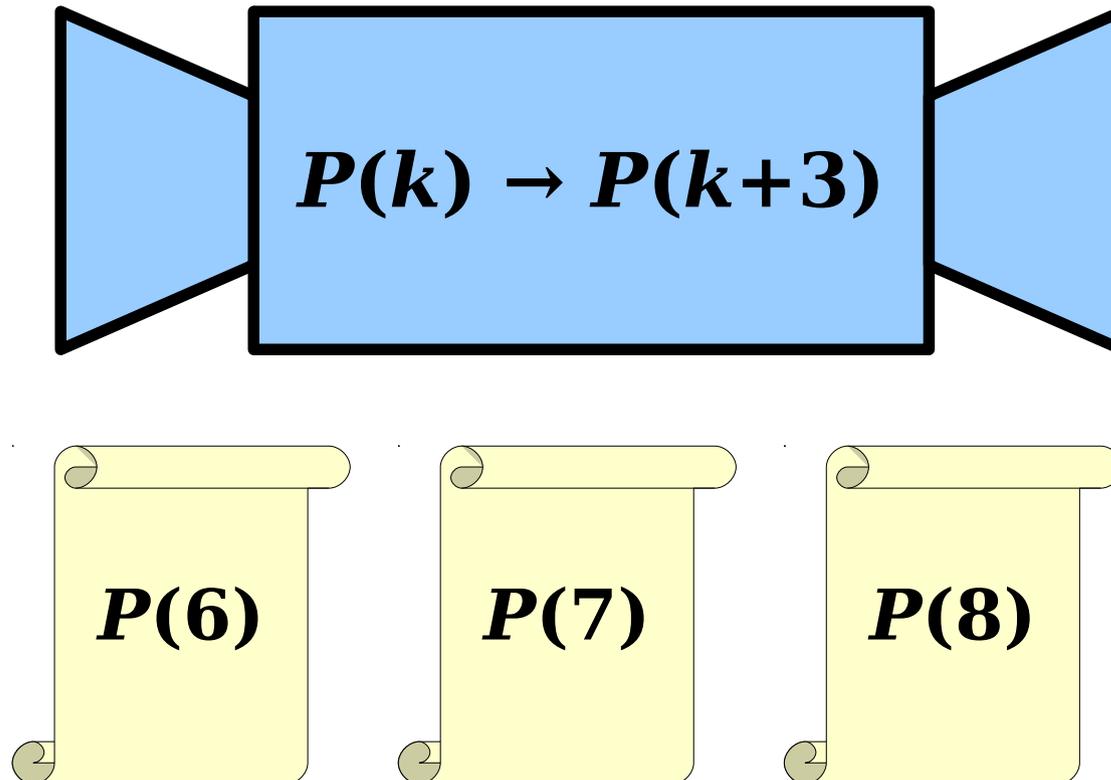
As our base cases, we prove $P(6)$, $P(7)$, and $P(8)$, that a square can be subdivided into 6, 7, and 8 squares. This is shown here:



For the inductive step, assume that for some $k \geq 6$ that $P(k)$ is true and a square can be subdivided into k squares. We prove $P(k+3)$, that a square can be subdivided into $k+3$ squares. To see this, start by obtaining (via the inductive hypothesis) a subdivision of a square into k squares. Then, choose any of the squares and split it into four equal squares. This removes one of the k squares and adds four more, so there will be a net total of $k+3$ squares. Thus $P(k+3)$ holds, completing the induction. ■

Why This Works

- This induction has three consecutive base cases and takes steps of size three.
- Thinking back to our “induction machine” analogy:



Generalizing Induction

- When doing a proof by induction:
 - Feel free to use multiple base cases.
 - Feel free to take steps of sizes other than one.
- Just be careful to make sure you cover all the numbers you think that you're covering!

Time-Out for Announcements!

Problem Set One

- Problem Set 1 due Friday, October 3 at the start of class.
 - Stop by office hours with questions!
 - Ask questions on Piazza!
 - Email the staff list (cs103-aut1415-staff@lists.stanford.edu) with questions!

Checkpoint Problems

- Problem Set 1 Checkpoints graded; feedback available online at Scoryst.
 - Please review this feedback before submitting the rest of the problems - the point of the checkpoint is to get useful feedback!
- In the future, please only submit one copy of the checkpoint per group - it *dramatically* simplifies grading.
- Submitted in the wrong category! No worries! We've got it handled.

A Quick Apology

Solution Sets

- Solution sets for the discussion problems and for the checkpoint problem are available in hardcopy at lecture or in the Gates B wing open space near Keith's office.
- SCPD students – we'll send you an email with information about solutions soon.

Piazza Questions

- We've gotten a lot of private questions on Piazza that are really interesting and probably super useful to other students.
- If you're asking a Piazza question that doesn't give away hints or answers to problem set questions, consider making it public so that everyone can learn!

GTGTC

- Girls Teaching Girls to Code (GTGTC) has an opening on their admin team.
- They run day-long tutoring sessions where high-school girls get mentorship from Stanford students. At one of their events last year, they had 40 Stanford students mentor 200 high-schoolers!
- Interested? Apply at <http://bit.ly/gtgtc-app>, or contact Jessie Duan at jduan1@stanford.edu.

Your Questions

“What are some of the most interesting/unusual proof techniques?”

“Most of the proofs in the homework can be solved algebraically without the use of words – the algebra suffices as an explanation. Especially when you use proof by contradiction. How do you know you're being superfluous with the explanations?”

“Is a program a proof? If so, suppose we have a program that is guaranteed to prove or disprove some fact. Assume that it is guaranteed to terminate, but will only do so thousands of years from now. Is it still a proof?”

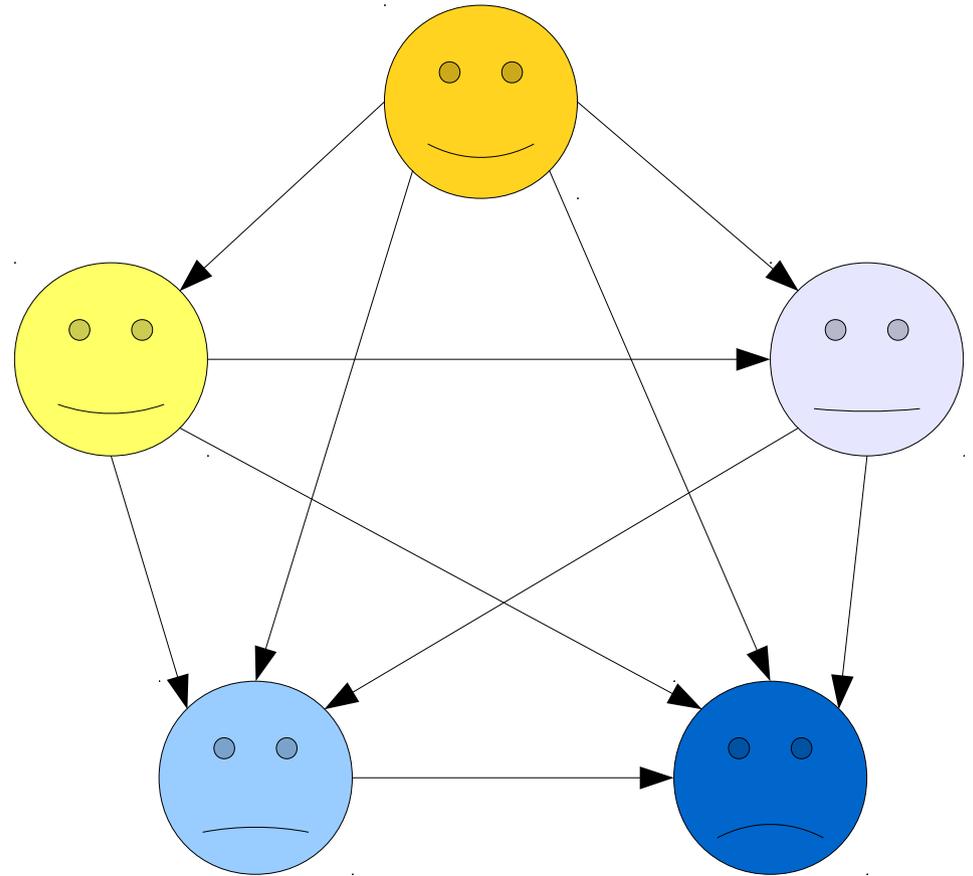
“Keith, why did you choose academia over industry?”

Back to CS103!

Example: Tournaments

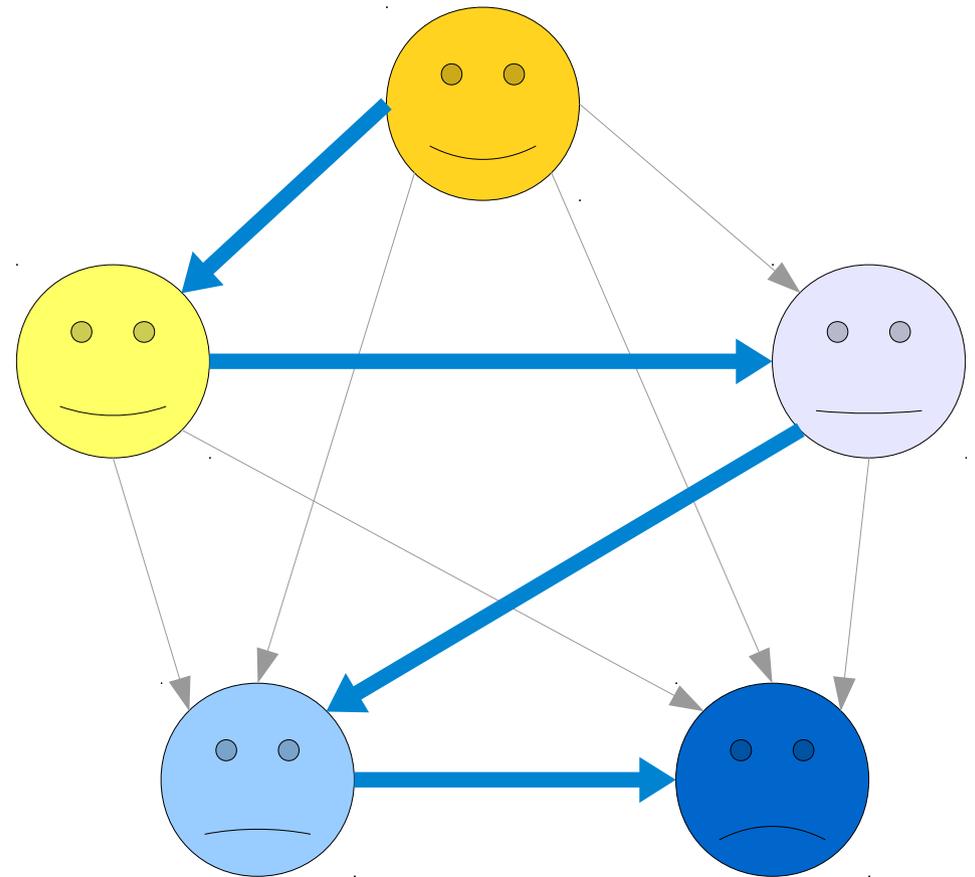
Tournaments

- A ***tournament*** is a contest for $n \geq 1$ people.
- Each person plays exactly one game against each other person, and there are no ties.
- The result can be visualized in a picture like this one, which is called a ***tournament graph***.



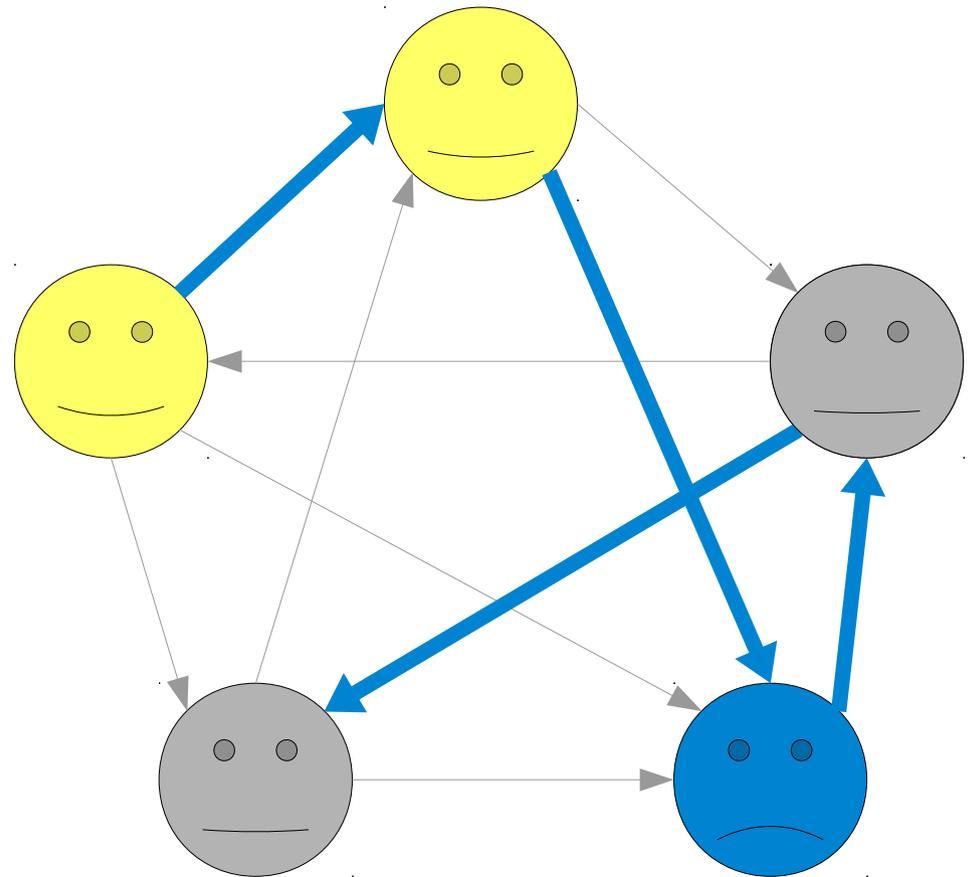
Victory Chains

- A ***victory chain*** in a tournament is a way of lining up the players so that every player beat the player that comes after them.



Victory Chains

- A **victory chain** in a tournament is a way of lining up the players so that every player beat the player that comes after them.



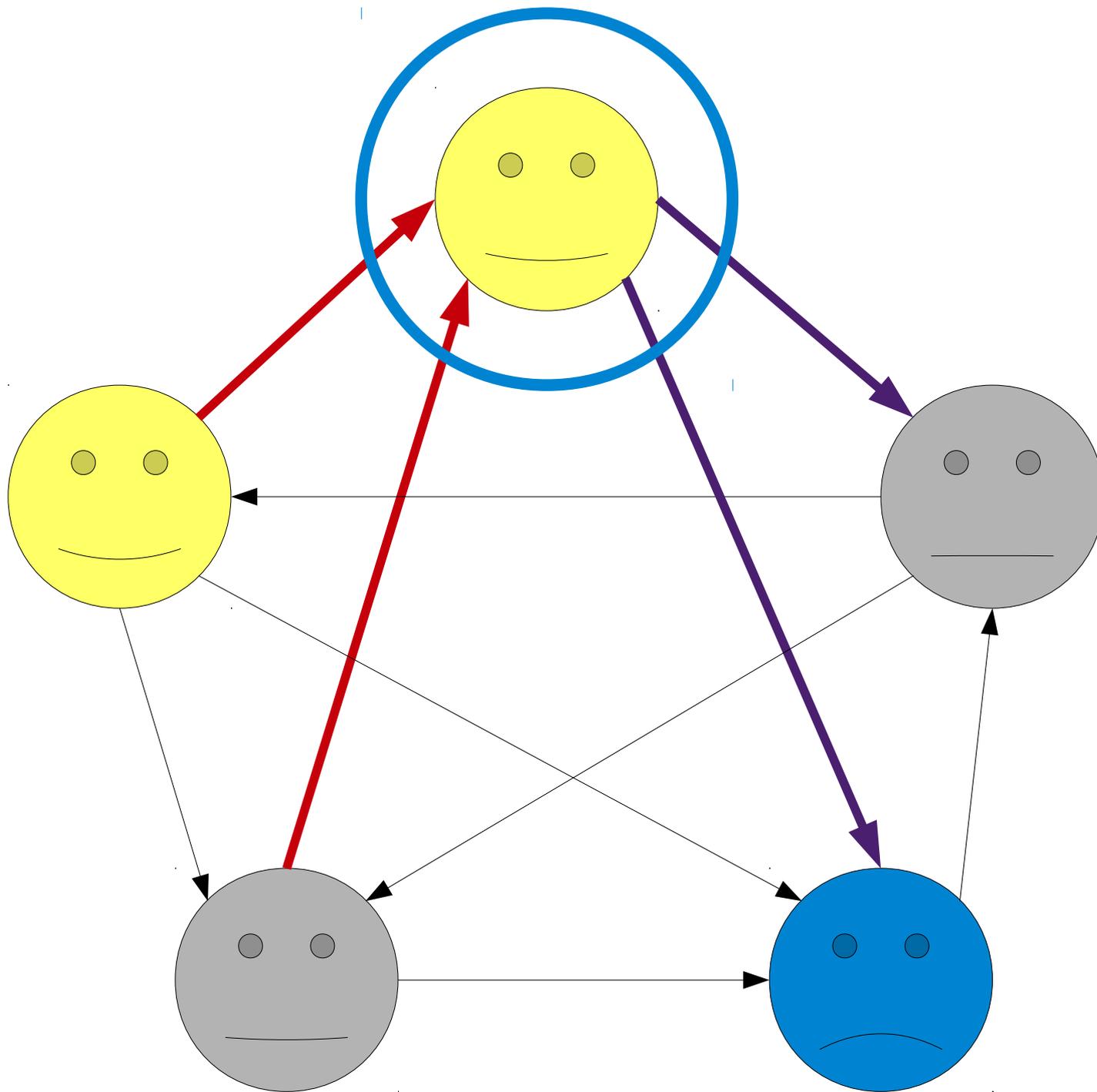
Theorem: Every tournament, regardless of the outcome has a victory chain.

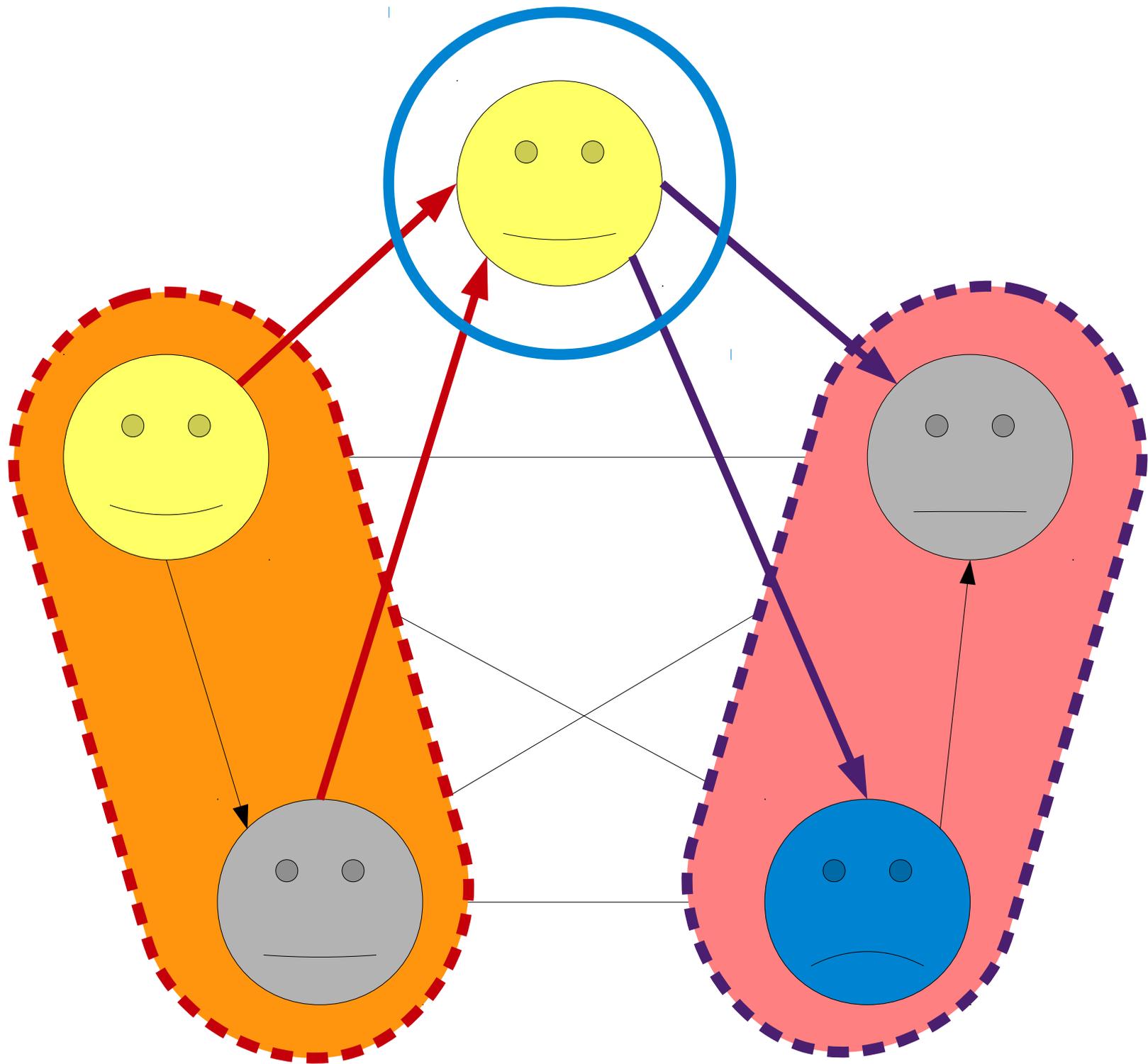
Thinking Inductively

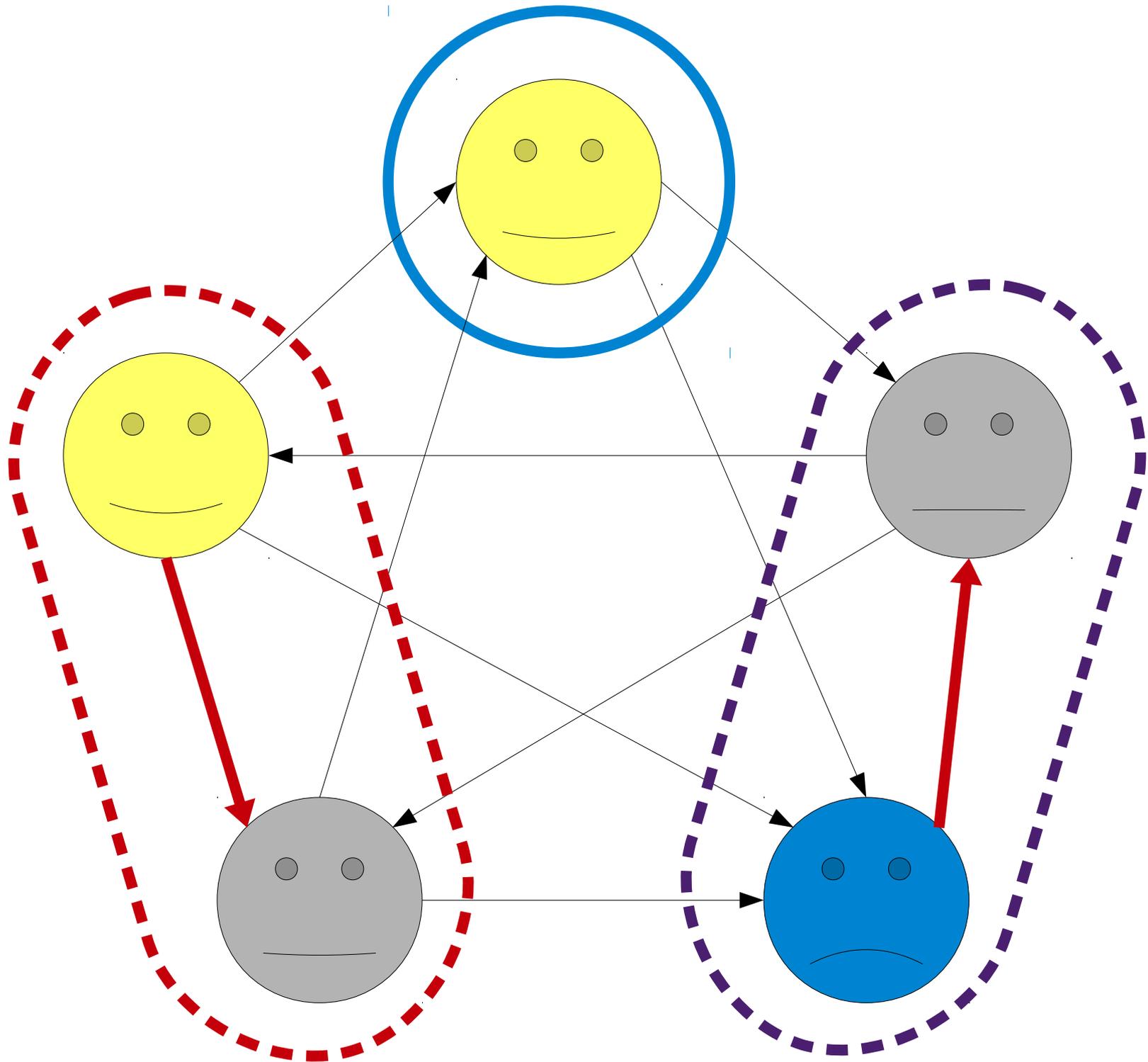
- The inductive step in an inductive proof uses the fact that the result is true for a smaller number (k) to prove that the result is true for a larger number ($k+1$).
- In most inductive proofs, the proof that the result is true for $k+1$ explicitly tries to simplify the $k+1$ case into the k case.
 - Counterfeit coins: Turn $k+1$ weighings into k weighings.
 - **MU** puzzle: Turn a sequence of $k+1$ events into a sequence of k events.
 - Triangulation: Turn a polygon of $k+1$ vertices into one with k vertices.
 - Square subdivision: Use a subdivision into k to get one for $k+3$.

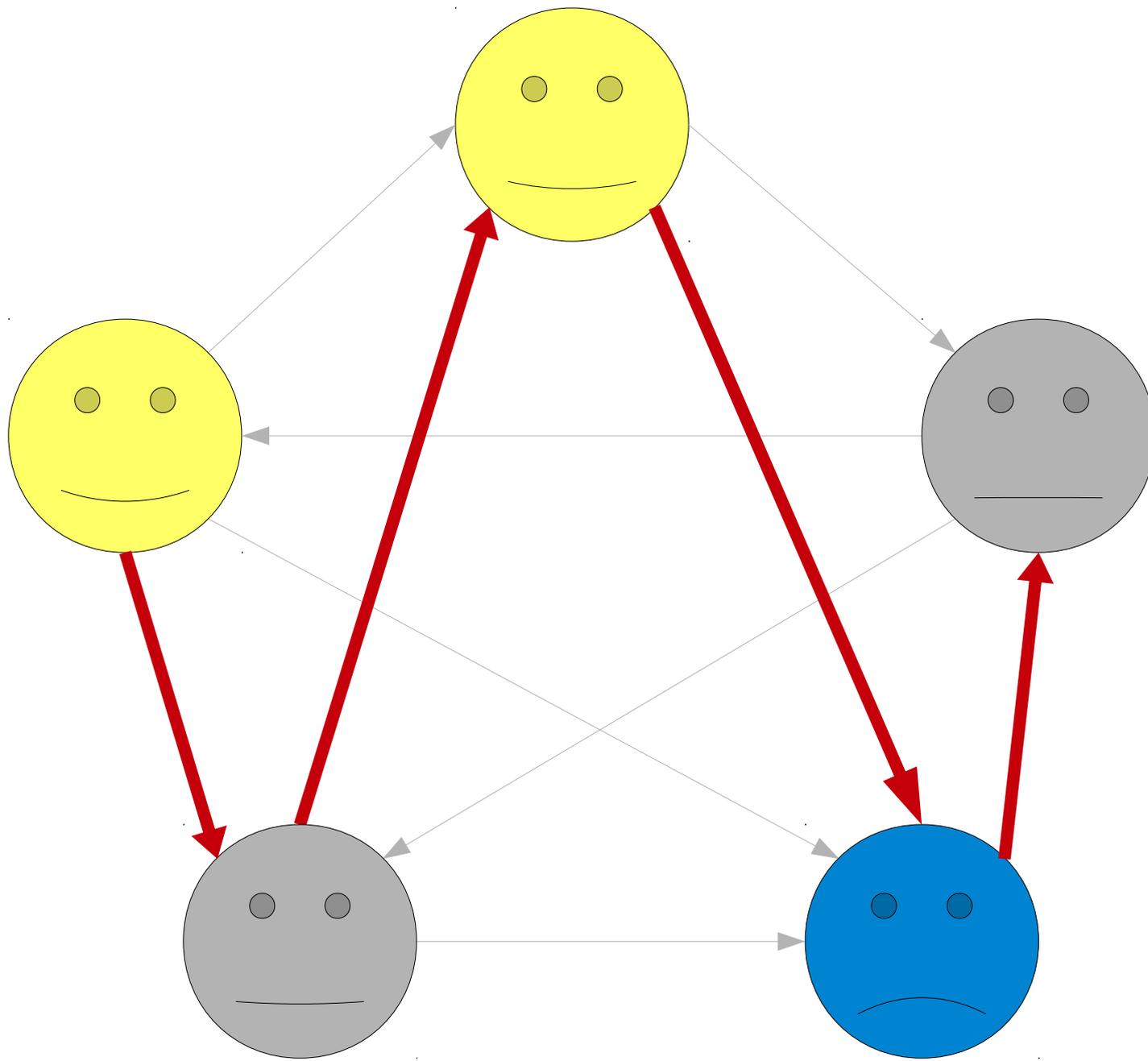
Thinking Inductively

- For our victory chain proof, we will simplify the problem by turning the larger tournament into two smaller tournaments.
- We'll inductively argue that, since those smaller tournaments each have victory chains, the larger tournament must have a victory chain.

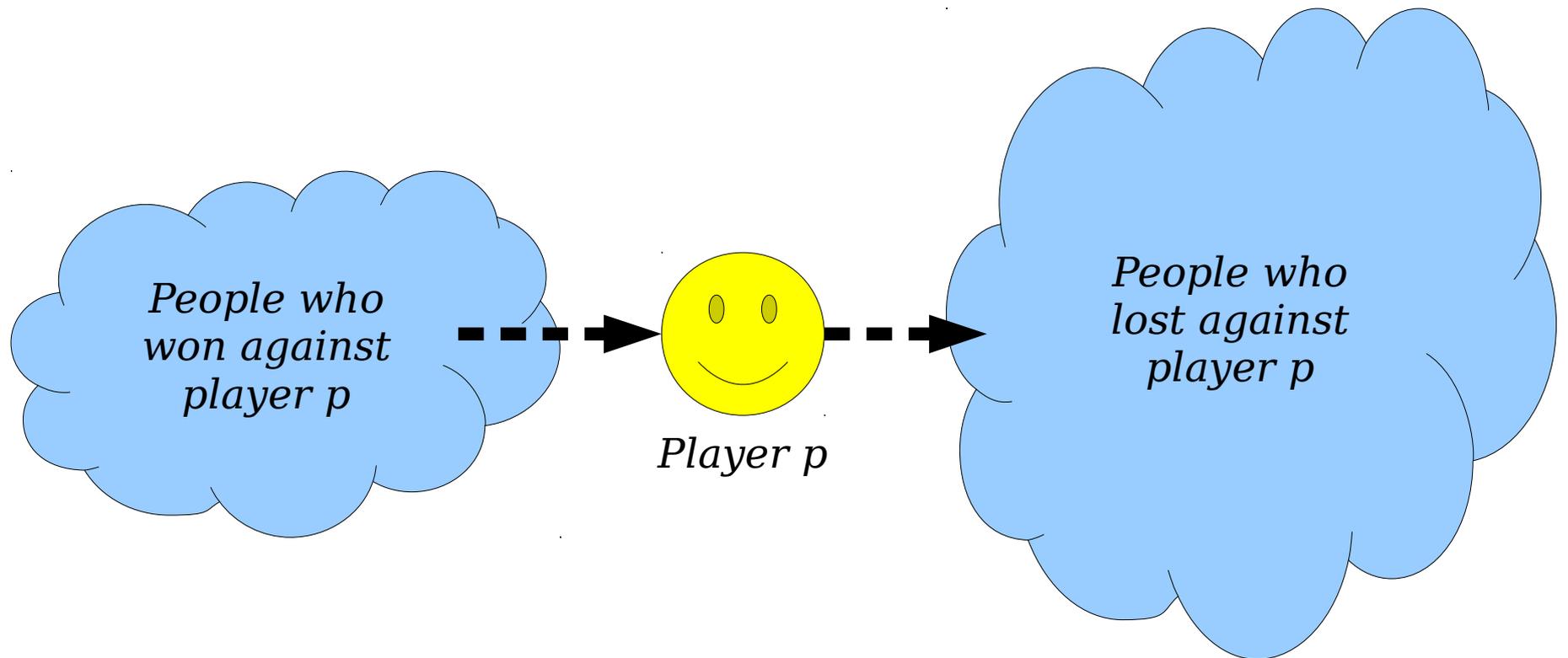








The Proof, Schematically



The Idea

- Suppose that every tournament with at most k players has a victory chain.
- Take a tournament T with $k+1$ players.
- Choose any one player p .
- Form the subtournaments T_0 and T_1 of all players who beat p and lost to p , respectively.
- Get victory chains from T_0 and T_1 .
- Splice those chains together through p .

The Idea

Suppose that every tournament with k players has a victory chain.

Take a tournament T with k players.

Choose any one player p .

This is the key idea behind an inductive proof - we're reducing the problem to smaller copies of itself.

- Form the subtournaments T_0 and T_1 of all players who beat p and lost to p , respectively.
- Get victory chains from T_0 and T_1 .

Splice those chains together through p .

Writing the Proof: A First Attempt

We're going to run into trouble in the middle of this proof. Don't worry - we'll see how to fix it.

Theorem: Every tournament has a victory chain.

Proof: Let $P(n)$ be the statement “every tournament with n players has a victory chain.” We will prove by induction that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

As a base case, we prove $P(0)$, that any tournament with no players has a victory chain. In a tournament with no players, a list of all 0 of the players vacuously satisfies the claim that every player beat the next player in the list. Therefore, $P(0)$ is true.

For the inductive step, assume that for some $k \in \mathbb{N}$ that $P(k)$ is true. In other words, we assume that any tournament with k players has a victory chain. We'll prove $P(k+1)$, that any tournament with $k+1$ players has a victory chain.

Consider any tournament T with $k+1$ players. Choose any one player p and form two subtournaments, a subtournament T_0 of all the players who beat p and a subtournament T_1 of all the players who lost to p . Both of these tournaments have between 0 and k players.

Theorem: Every

Proof: Let $P(n)$ be the statement that every tournament with n players has a victory chain. We will prove $P(n)$ for all $n \in \mathbb{N}$, from

As a base case, $P(0)$ is true because a tournament with 0 players has a victory chain consisting of all 0 of the players. For $n > 0$, let p be any player in the tournament. The next player in the victory chain is

At this point, we're stuck. We know that tournaments with exactly k players must have a victory chain, but we're not assuming anything about tournaments with $0, 1, 2, \dots, k-1$ players. Therefore, we can't necessarily say anything about these subtournaments.

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For the inductive step, assume that for some $k \in \mathbb{N}$ that $P(k)$ is true. In other words, we **assume that any tournament with k players has a victory chain**. We'll prove $P(k+1)$, that any tournament with $k+1$ players has a victory chain.

Consider any tournament T with $k+1$ players. Choose any one player p and form two subtournaments, a subtournament T_0 of all the players who beat p and a subtournament T_1 of all the players who lost to p . **Both of these tournaments have between 0 and k players.**

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What if we made some additional assumptions so that we can say something about these smaller tournaments?

As a base case, a tournament with 0 players has a victory chain. A list of all 0 of the players vacuously satisfies the claim that every player beat the next player in the list. Therefore, $P(0)$ is true.

For the inductive step, assume that for some $k \in \mathbb{N}$ that $P(k)$ is true. In other words, we **assume that any tournament with k players has a victory chain.** We'll prove $P(k+1)$, that any tournament with $k+1$ players has a victory chain.

Consider any tournament T with $k+1$ players. Choose any one player p and form two subtournaments, a subtournament T_0 of all the players who beat p and a subtournament T_1 of all the players who lost to p . **Both of these tournaments have between 0 and k players.**

Theorem: Every tournament has a victory chain.

Proof: Let $P(n)$ be the statement “every tournament with n players has a victory chain.” We will prove by induction that $P(n)$ is true for all $n \in \mathbb{N}$, from $n = 0$ onwards.

We're now assuming that the result is true for $0, 1, 2, 3, \dots, k$. Now, we can continue to make progress!

As a base case, a tournament with 0 players has a victory chain of all 0 of the players vacuously satisfies the claim that every player beat the next player in the list. Therefore, $P(0)$ is true.

For the inductive step, assume that for some $k \in \mathbb{N}$ that $P(0), P(1), P(2), \dots$, and $P(k)$ are all true. In other words, we assume that any tournament with at most k players has a victory chain. We'll prove $P(k+1)$, that any tournament with $k+1$ players has a victory chain.

Consider any tournament T with $k+1$ players. Choose any one player p and form two subtournaments, a subtournament T_0 of all the players who beat p and a subtournament T_1 of all the players who lost to p . Both of these tournaments have between 0 and k players.

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Consider any tournament T with $k+1$ players. Choose any one player p and form two subtournaments, a subtournament T_0 of all the players who beat p and a subtournament T_1 of all the players who lost to p . Both of these tournaments have between 0 and k players. Therefore, by our inductive hypothesis, these two subtournaments have victory chains, call them V_0 and V_1 . If we then splice together these chains to form the chain V_0, p, V_1 , then we have a victory chain for T : every player is present, and every player beat the player immediately after them. Therefore, this arbitrary tournament of $k+1$ players has a victory chain, so $P(k+1)$ is true, completing the induction. ■

What We Just Did

- In a normal inductive step, we assume that $P(k)$ is true and prove $P(k+1)$.
- In this type of inductive step, we assume $P(0)$, $P(1)$, ..., and $P(k)$ are true before we prove $P(k+1)$.
- That way, when we found *any* kind of smaller tournament, we knew something about its structure.
- This type of proof has a name!

Complete Induction

- If the following are true:
 - $P(0)$ is true, and
 - If $P(0), P(1), P(2), \dots, P(k)$ are true, then $P(k+1)$ is true as well.

then $P(n)$ is true for all $n \in \mathbb{N}$.

- This is called the ***principle of complete induction*** or the ***principle of strong induction***.
 - (A note: this also works starting from a number other than 0; just modify what you're assuming appropriately.)

That's a *lot* of assumptions to make!

Why is this legal?

More on Complete Induction

- This type of induction can seem too powerful, almost like it's cheating.
- It's actually perfectly safe!
- We'll do more examples when we come back next time.

Next Time

- **More on Complete Induction**
 - Building an intuition for complete induction.
 - More applications!
- **Graphs**
 - Representing relationships between objects.
 - Graph connectivity.