Regular Expressions

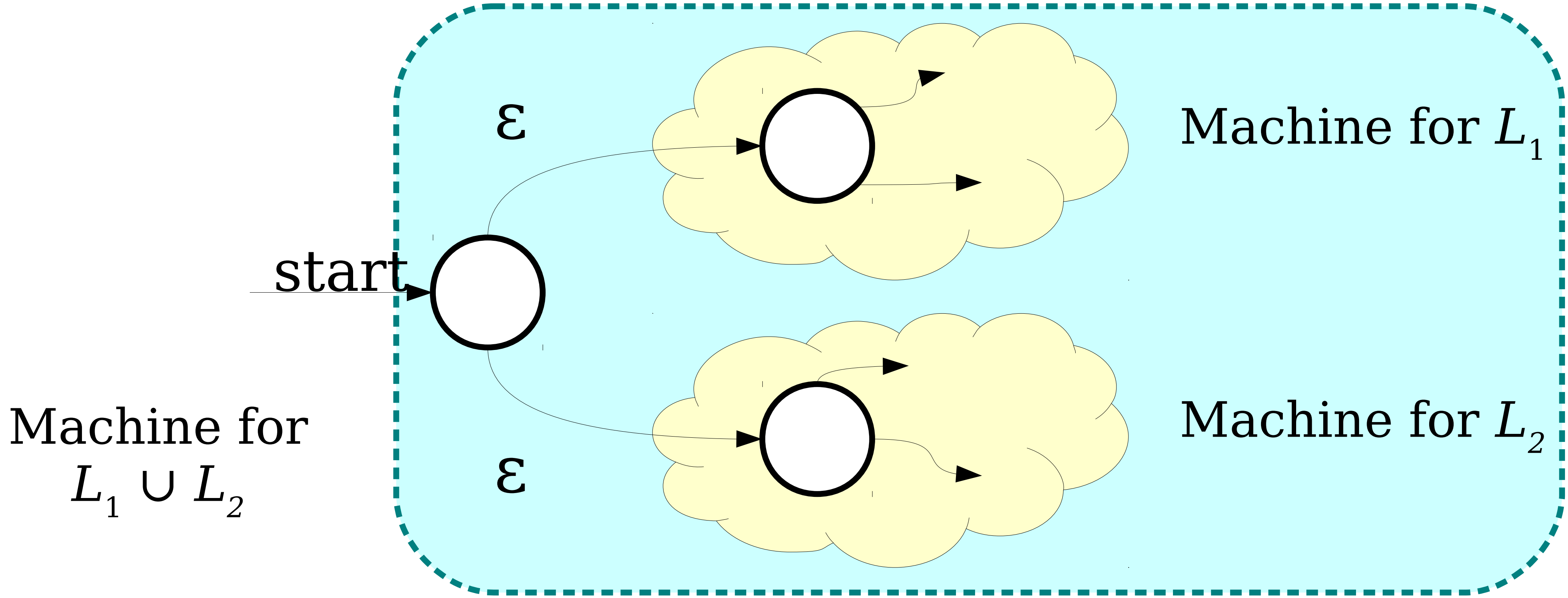
Recap from Last Time

Regular Languages

- A language L is a ***regular language*** if there is a DFA D such that $\mathscr{L}(D) = L$.
- ***Theorem:*** The following are equivalent:
  - L is a regular language.
  - There is a DFA for L.
  - There is an NFA for L.

The Union of Two Languages

- If L_1 and L_2 are languages over the alphabet Σ, the language $L_1 \cup L_2$ is the language of all strings in at least one of the two languages.
- If L_1 and L_2 are regular languages, is $L_1 \cup L_2$?



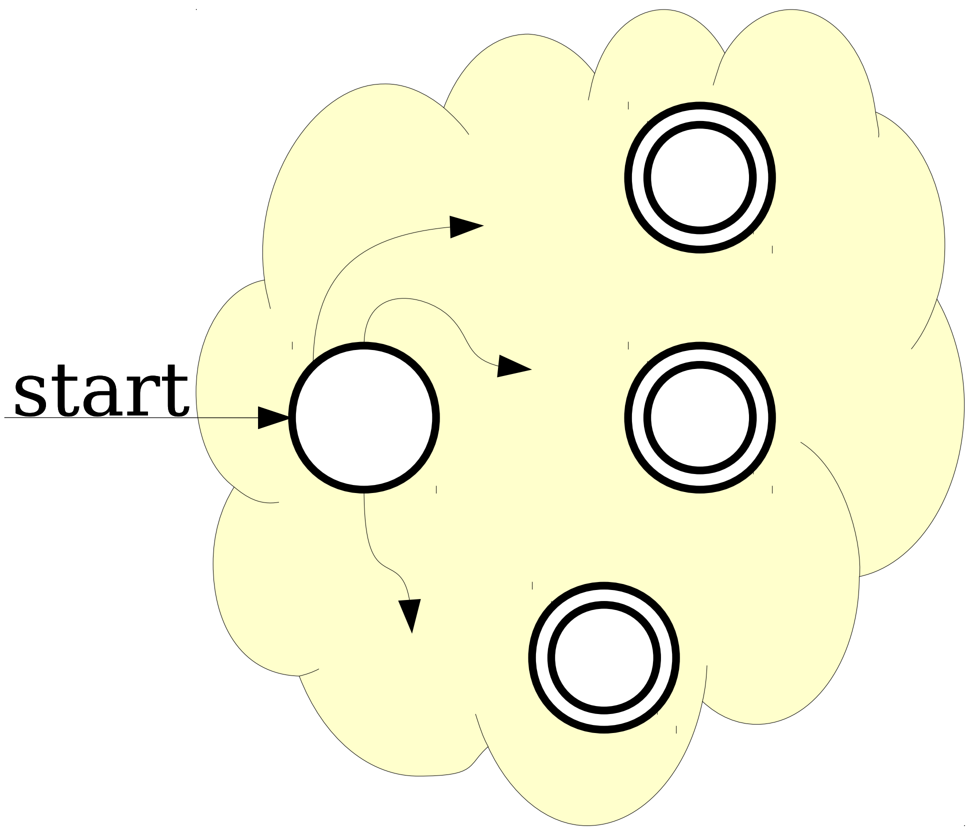
Concatenation Example

- Let Σ = { **a**, **b**, …, **z**, **A**, **B**, …, **Z** } and consider these languages over Σ:
  - ***Noun*** = { **Puppy**, **Rainbow**, **Whale**, … }
  - ***Verb*** = { **Hugs**, **Juggles**, **Loves**, … }
  - ***The*** = { **The** }
- The language ***TheNounVerbTheNoun*** is

  { **ThePuppyHugsTheWhale**,
  TheWhaleLovesTheRainbow,
  TheRainbowJugglesTheRainbow, … }

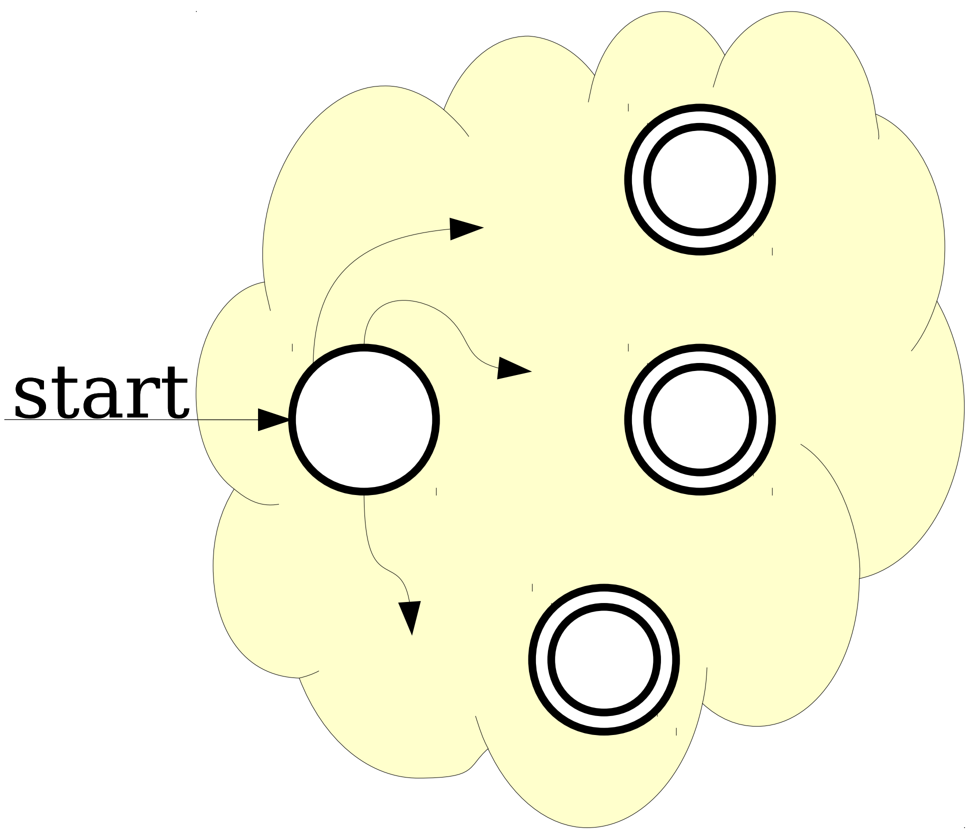
Concatenating Regular Languages

Concatenating Regular Languages

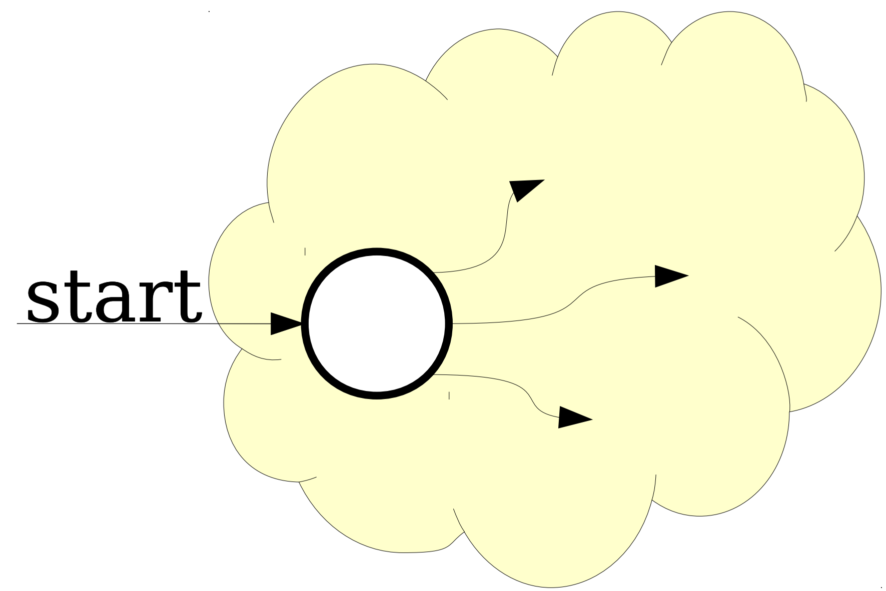


Machine for L_1

Concatenating Regular Languages

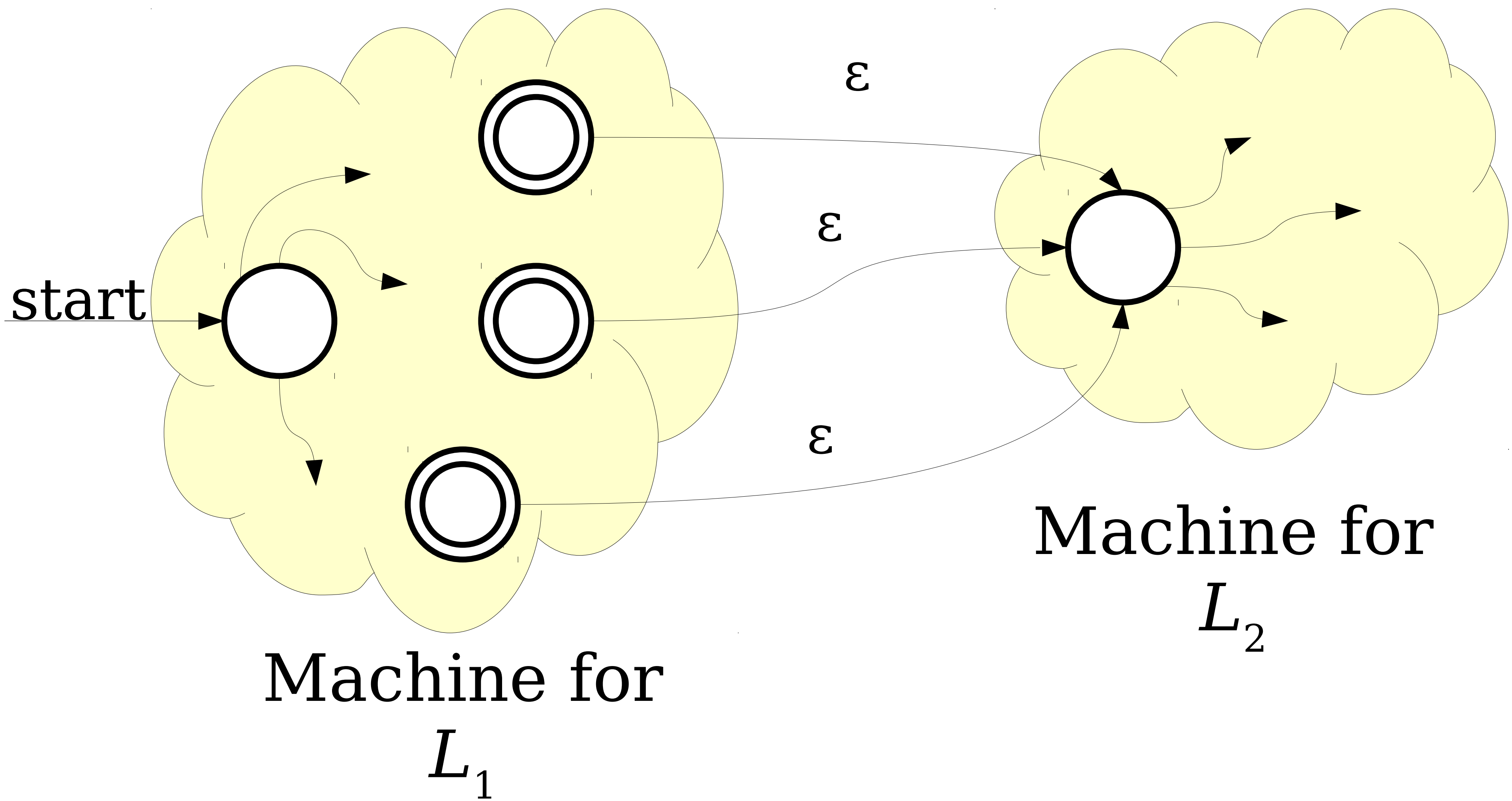


Machine for L_1

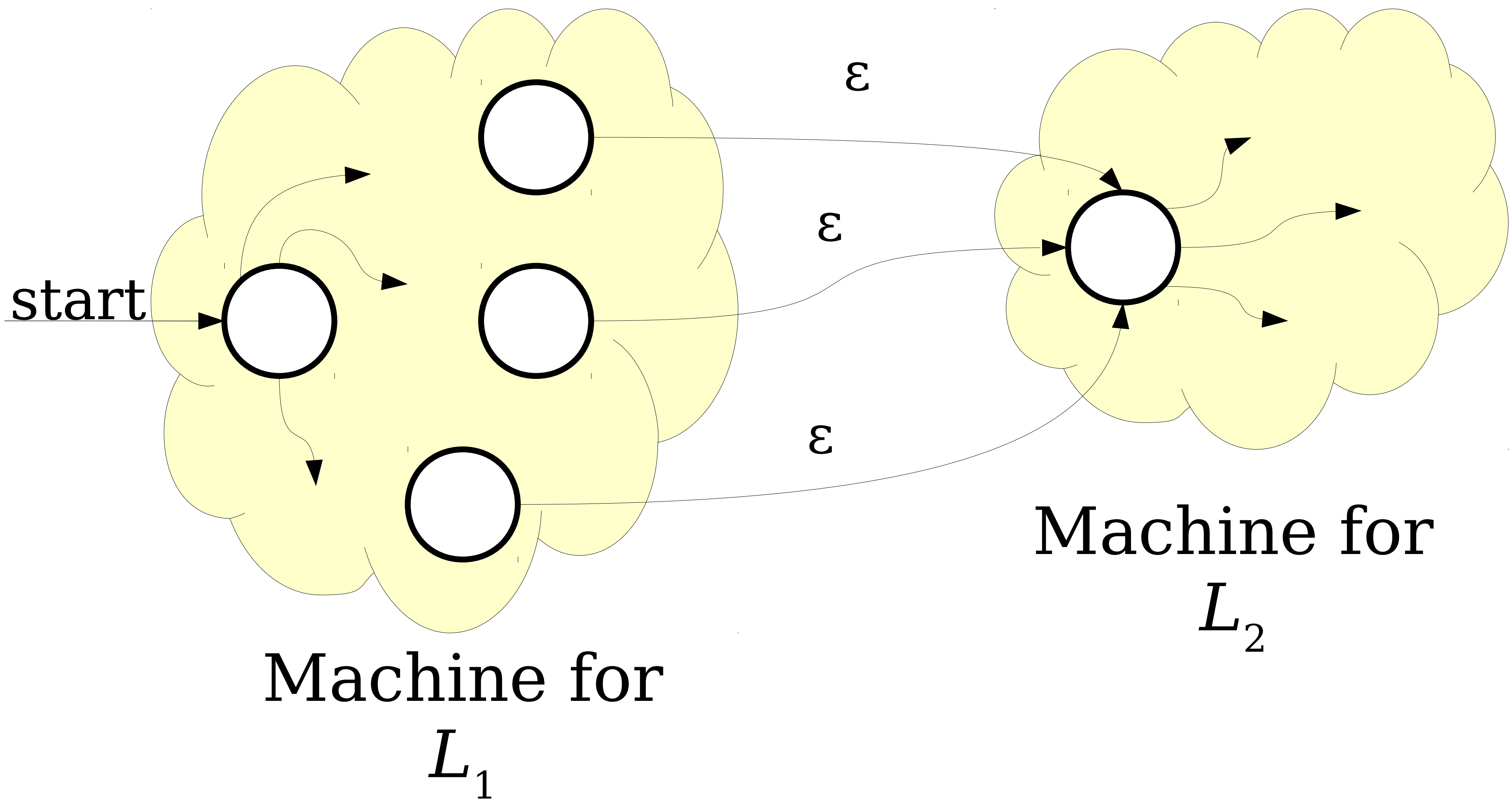


Machine for L_2

Concatenating Regular Languages



Concatenating Regular Languages



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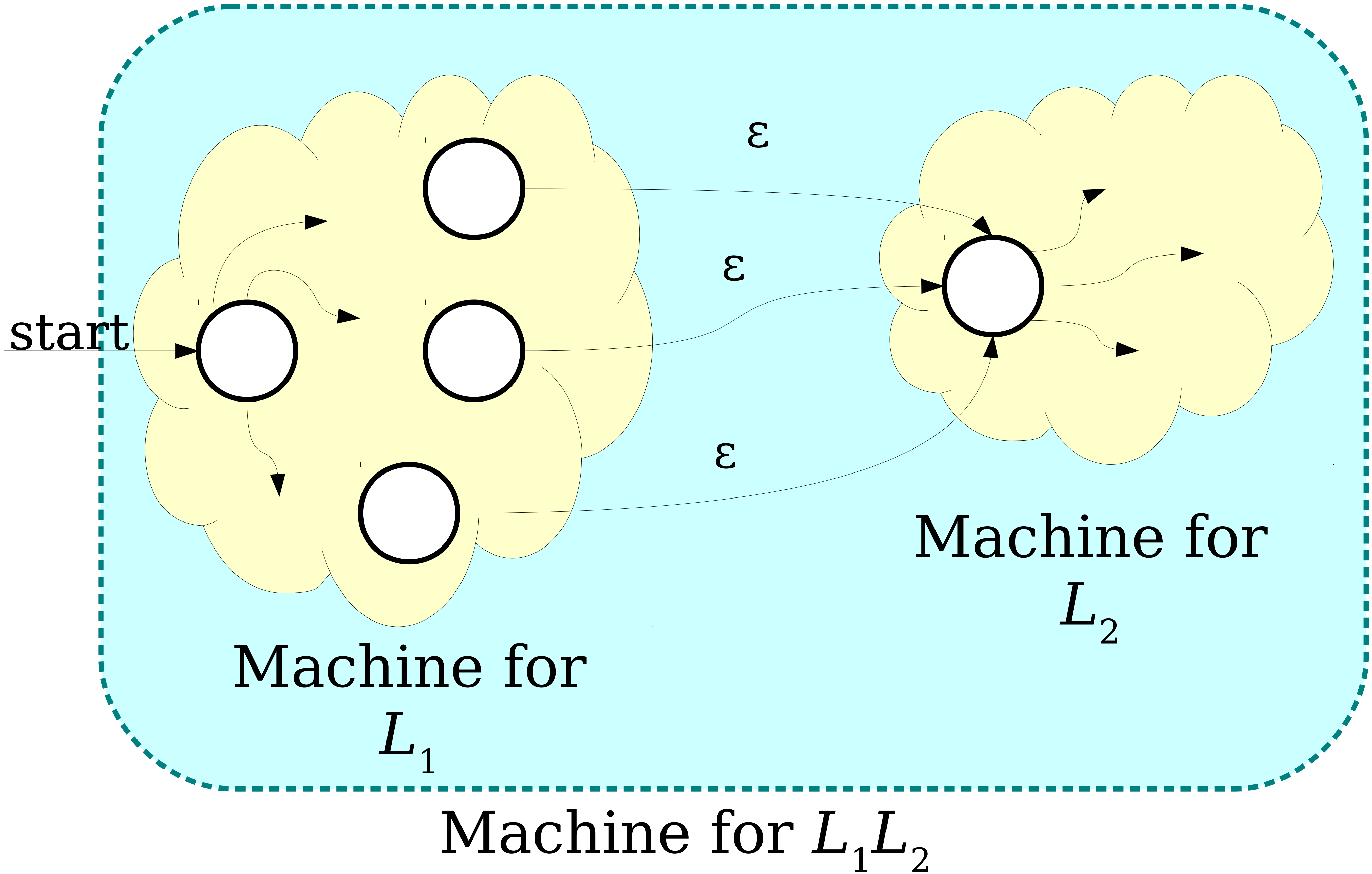


The Kleene Closure

- An important operation on languages is the ***Kleene Closure***, which is defined as

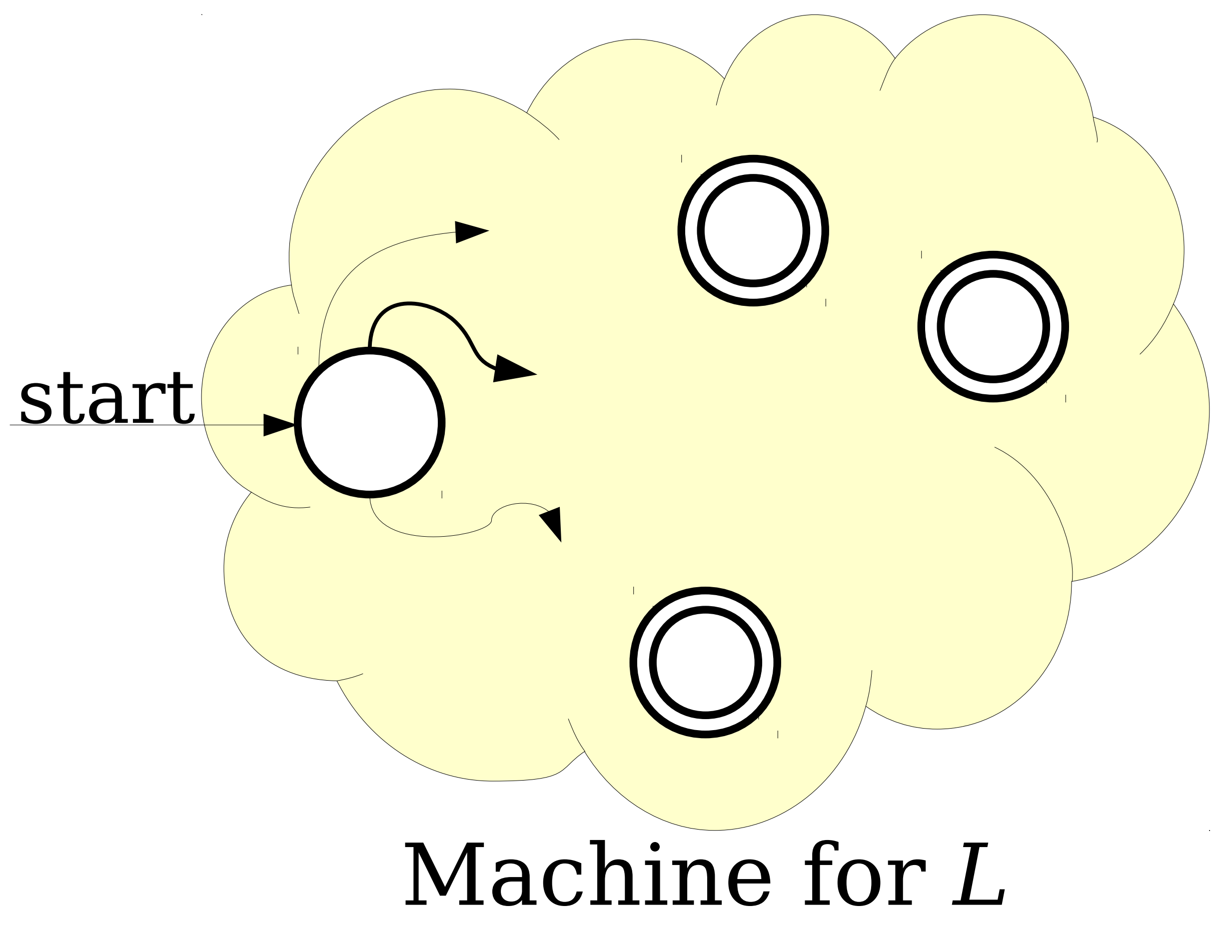
$$L^* = \bigcup_{i=0}^{\infty} L^i$$

- Mathematically:

$$\boldsymbol{w \in L^*} \quad \textbf{iff} \quad \boldsymbol{\exists n \in \mathbb{N}.\ w \in L^n}$$

- Intuitively, all possible ways of concatenating any number of copies of strings in L together.

The Kleene Star



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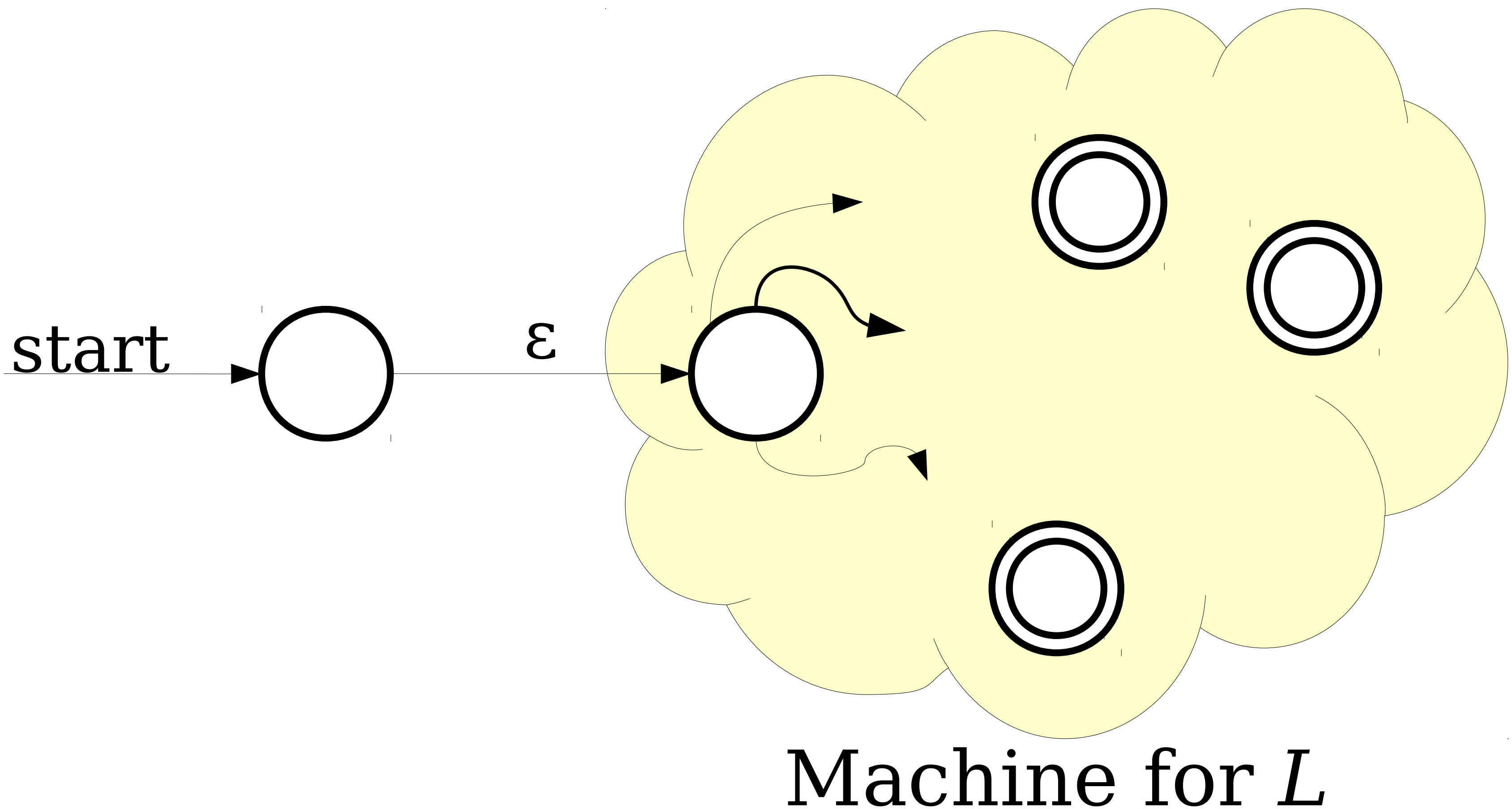

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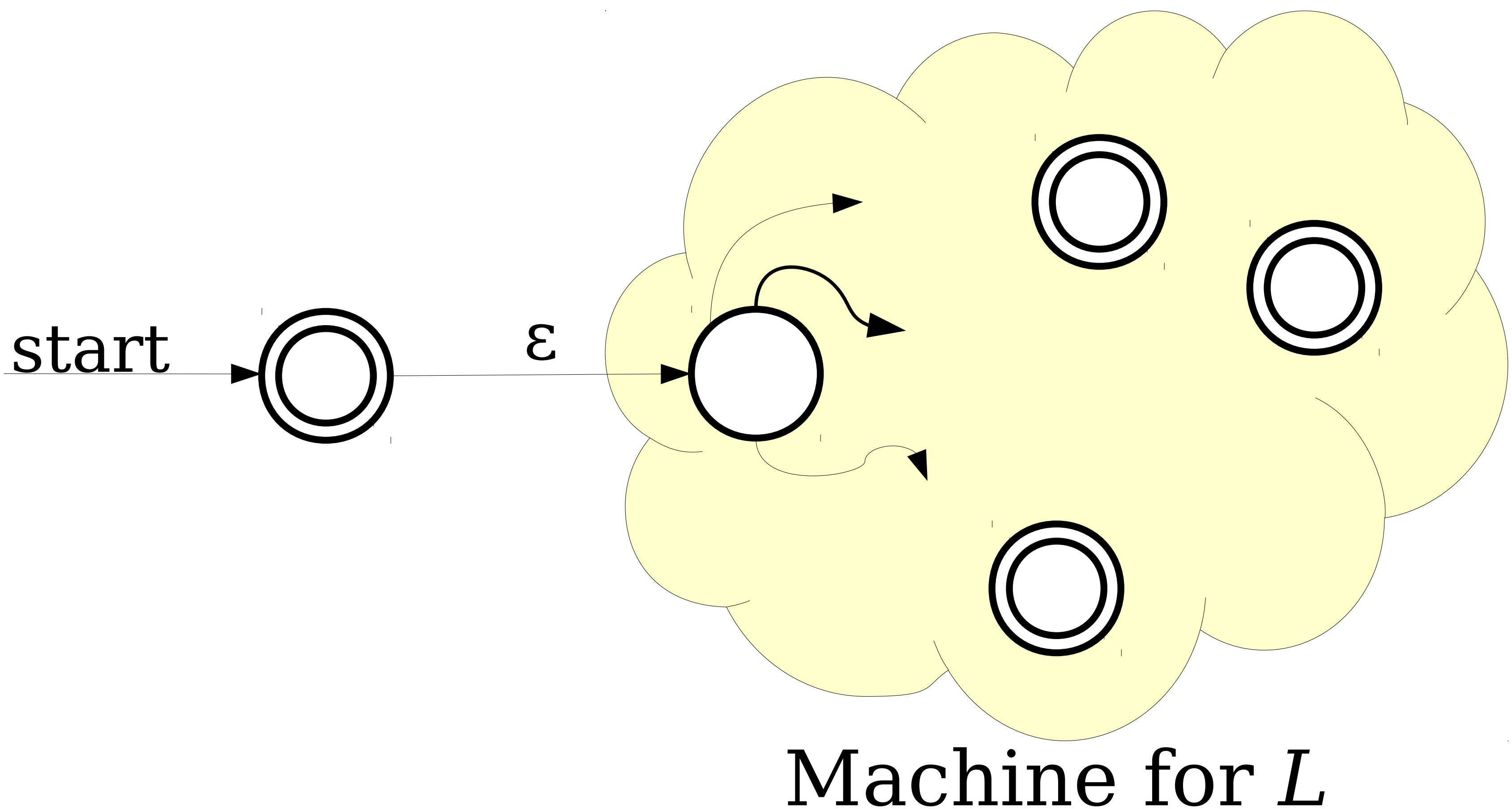

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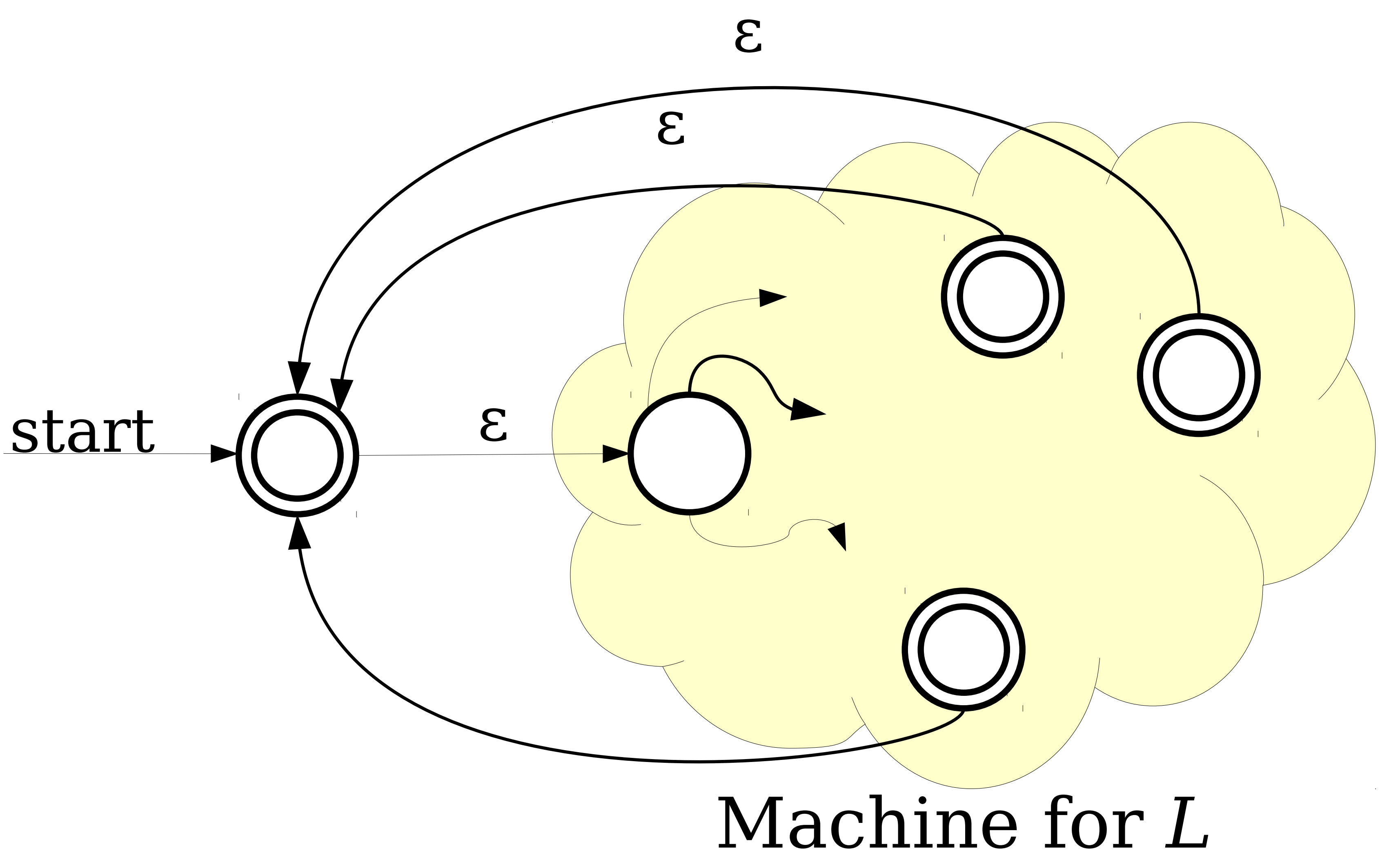

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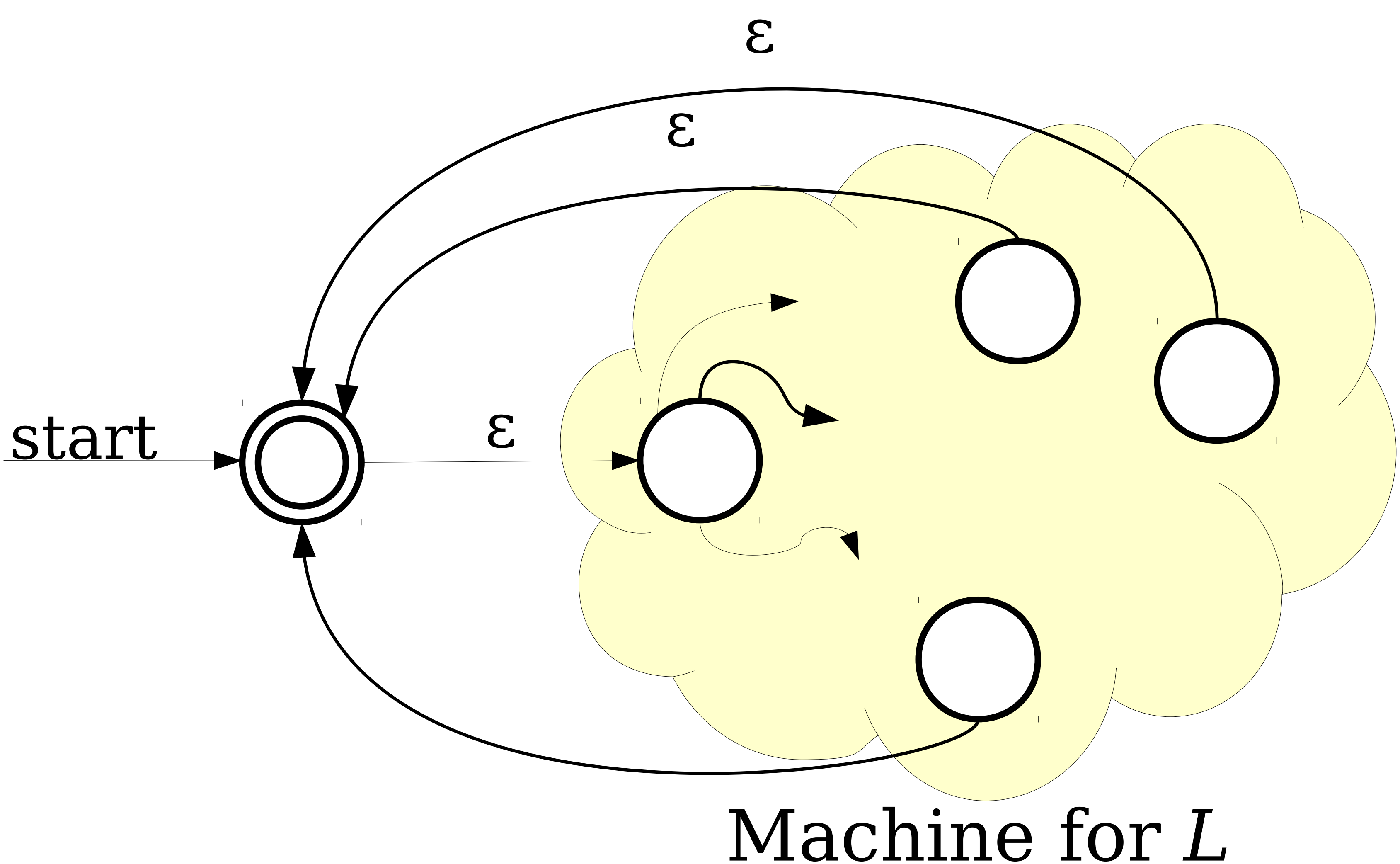

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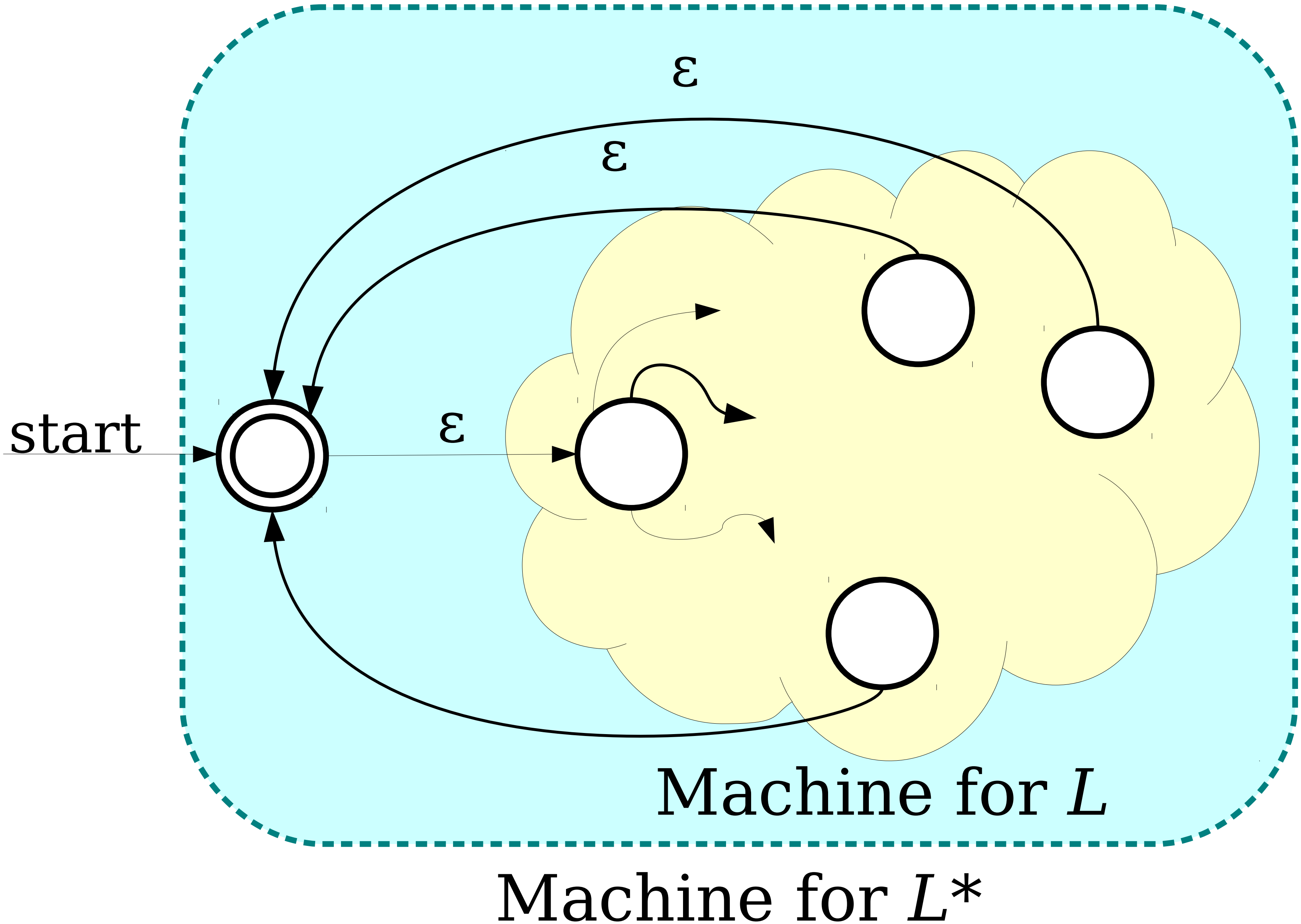

Another View of Regular Languages

Rethinking Regular Languages

- We currently have several tools for showing a language is regular.
  - Construct a DFA for it.
  - Construct an NFA for it.
  - Apply closure properties to existing languages.
- We have not spoken much of this last idea.

Constructing Regular Languages

- **Idea:** Build up all regular languages as follows:
  - Start with a small set of simple languages we already know to be regular.
  - Using closure properties, combine these simple languages together to form more elaborate languages.
- *A bottom-up approach to the regular languages.*

Regular Expressions

- ***Regular expressions*** are a descriptive format used compactly describe a language.
- Used extensively in software systems for string processing and as the basis for tools like `grep` and `flex`.
- Conceptually: regular languages are strings describing how to assemble a larger language out of smaller pieces.

Atomic Regular Expressions

- The regular expressions begin with three simple building blocks.
- The symbol **Ø** is a regular expression that represents the empty language Ø.
- The symbol **ε** is a regular expression that represents the language { ε }
  - ***This is not the same as Ø!***
- For any `a` ∈ Σ, the symbol `a` is a regular expression for the language { `a` }

Compound Regular Expressions

- We can combine together existing regular expressions in four ways.
- If R_1 and R_2 are regular expressions, $\boldsymbol{R_1R_2}$ is a regular expression for the *concatenation* of the languages of R_1 and R_2.
- If R_1 and R_2 are regular expressions, $\boldsymbol{R_1 \mid R_2}$ is a regular expression for the *union* of the languages of R_1 and R_2.
- If R is a regular expression, $\boldsymbol{R^*}$ is a regular expression for the *Kleene closure* of the language of R.
- If R is a regular expression, $\boldsymbol{(R)}$ is a regular expression with the same meaning as R.

Operator Precedence

- Regular expression operator precedence:

$$(R)$$

$$R^*$$

$$R_1 R_2$$

$$R_1 \mid R_2$$

- So `ab*c|d` is parsed as `((a(b*))c)|d`

Regular Expression Examples

- The regular expression **`trick|treat`** represents the regular language { **`trick`**, **`treat`** }
- The regular expression **`booo*`** represents the regular language { **`boo`**, **`booo`**, **`boooo`**, … }
- The regular expression **`candy!(candy!)*`** represents the regular language { **`candy!`**, **`candy!candy!`**, **`candy!candy!candy!`**, … }

Regular Expressions, Formally

- The ***language of a regular expression*** is the language described by that regular expression.
- Formally:
  - $\mathscr{L}(\varepsilon) = \{\varepsilon\}$
  - $\mathscr{L}(\emptyset) = \emptyset$
  - $\mathscr{L}(\mathbf{a}) = \{\mathbf{a}\}$
  - $\mathscr{L}(R_1 R_2) = \mathscr{L}(R_1)\ \mathscr{L}(R_2)$
  - $\mathscr{L}(R_1 \mid R_2) = \mathscr{L}(R_1) \cup \mathscr{L}(R_2)$
  - $\mathscr{L}(R^*) = \mathscr{L}(R)^*$
  - $\mathscr{L}((R)) = \mathscr{L}(R)$

Worthwhile activity: Apply this recursive definition to

`a(b|c)((d))`

and see what you get.

Regular Expressions are Awesome

- Let $\Sigma = \{\texttt{0}, \texttt{1}\}$
- Let $L = \{\ w \in \Sigma^* \mid w$ contains `00` as a substring $\}$

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11011100101
0000
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The length of a string w is denoted $|w|$

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1*0?1*

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Regular Expressions are Awesome

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Regular expression for email addresses:

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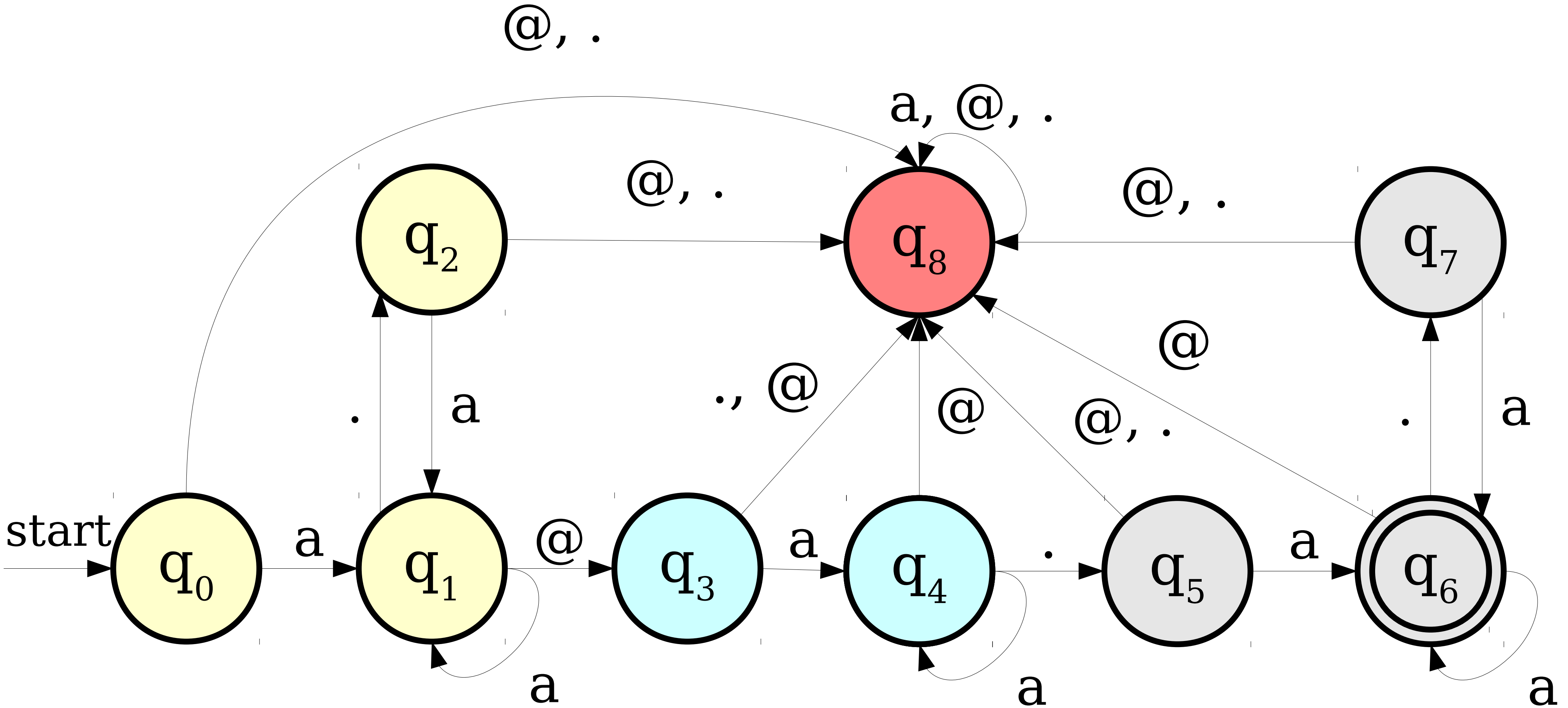


Shorthand Summary

- R^n is shorthand for $RR \ldots R$ (n times).
- Σ is shorthand for "any character in Σ."
- R? is shorthand for (R | ε), meaning "zero or one copies of R."
- R^+ is shorthand for RR^*, meaning "one or more copies of R."

Time-Out for Announcements!

Upcoming Talk

- WiCS is hosting a talk by Yoky Matsuoka, VP of Technology at Nest Labs and cofounder of Google[*x*].
- Before that, she was a professor of CS, neuroscience, and ME.
- Coming up next Thursday, November 6 at 4PM at Braun Auditorium.
- RSVP requested: http://goo.gl/forms/KwDpCiUfHn

STEM Fellows Program

- “[T]he Stanford Undergraduate STEM Fellows Program provides support to students who will promote the diversity (broadly defined) of the future professoriate. The program [… seeks] to increase the number of PhDs earned by under-represented groups in the areas of science, technology, engineering, and math.”
- Deadline is December 8, but I'd encourage applying early.
- Details online at

  https://undergrad.stanford.edu/opportunities-research/fellowships/fellowships-listing/stanford-undergraduate-stem-fellows-program

Problem Set Four Solutions

- PS4 solutions will be available outside of class today and in the filing cabinet after that.
  - SCPD students: you should receive them soon.
- Want older solution sets? Pick them up soon before they get recycled!

Order in a World of Chaos

- *Please keep the box of exams alphabetized*. Everyone pays if even a small number of people scramble the exams.

Your Questions

“I've noticed that you're very passionate about diversity in CS. What advice would you have for someone who is part of a minority and is about to have a job/internship in an environment that is not likely to be diverse?”

“In your opinion, what's the biggest number?”

“What do you do in your free time?”

“Why don't we have a class called Data Structures? Is 106B/X our equivalent of this? Recruiters and interviewers ask me why I haven't taken data structures all the time and I never know how to respond.”

Back to CS103!

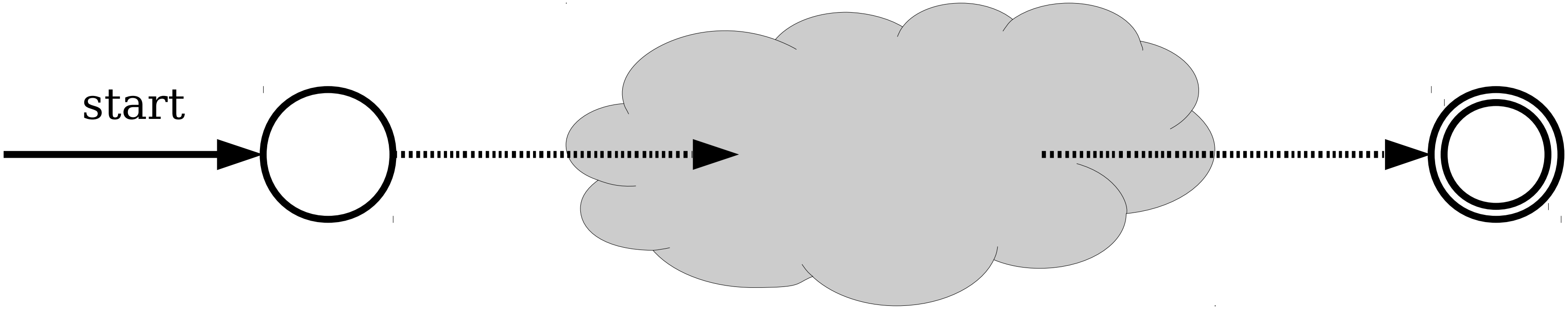
The Power of Regular Expressions

Theorem: If R is a regular expression, then $\mathscr{L}(R)$ is regular.

Proof idea: Show how to convert a regular expression into an NFA.

A Marvelous Construction

- The following theorem proves the language of any regular expression is regular:
- ***Theorem:*** For any regular expression R, there is an NFA N such that
  - $\mathscr{L}(R) = \mathscr{L}(N)$
  - N has exactly one accepting state.
  - N has no transitions into its start state.
  - N has no transitions out of its accepting state.



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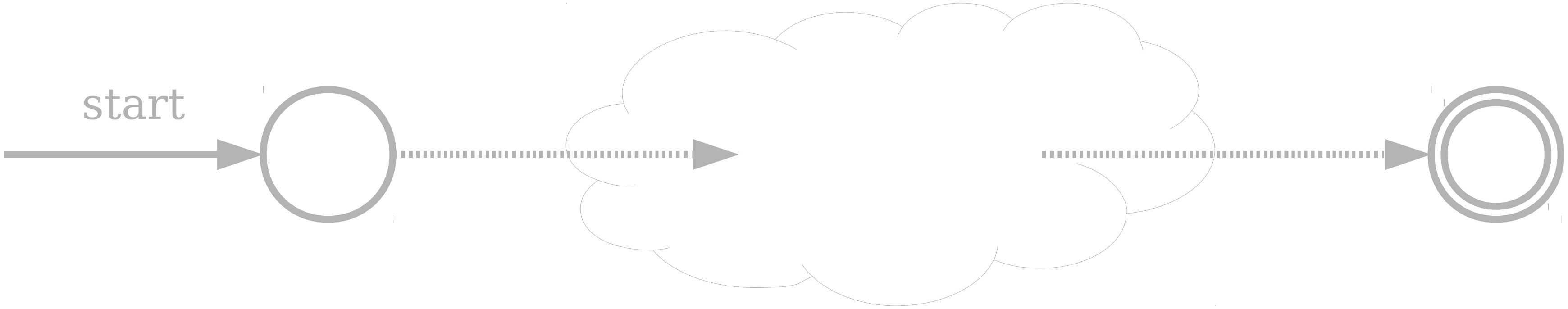

A Marvelous Construction

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These are stronger requirements than are necessary for a normal NFA. We enforce these rules to simplify the construction.

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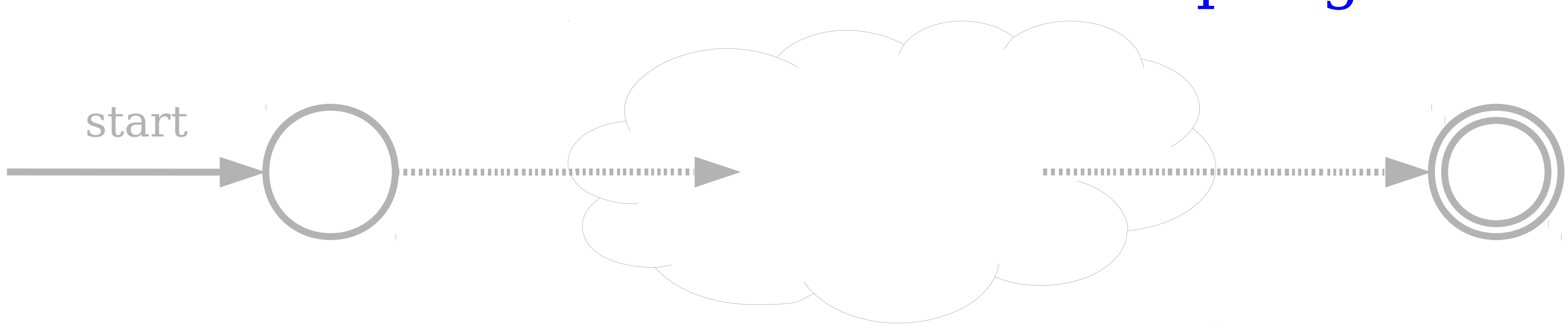
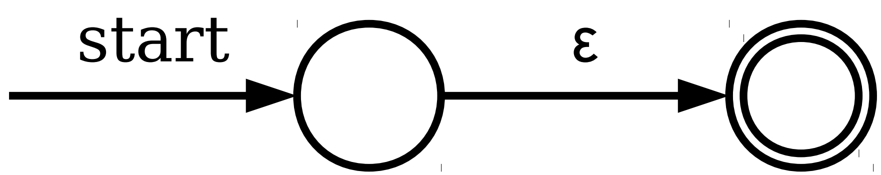
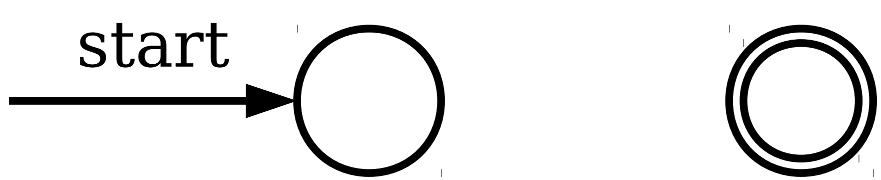


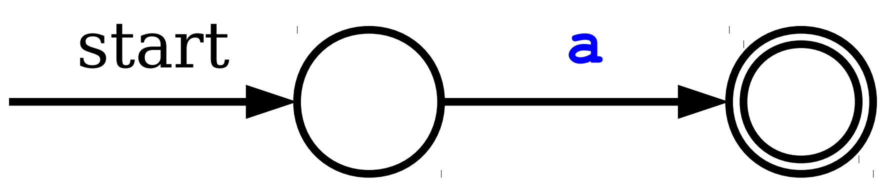
Base Cases



Automaton for ε



Automaton for Ø



Automaton for single character **`a`**

Construction for R_1R_2

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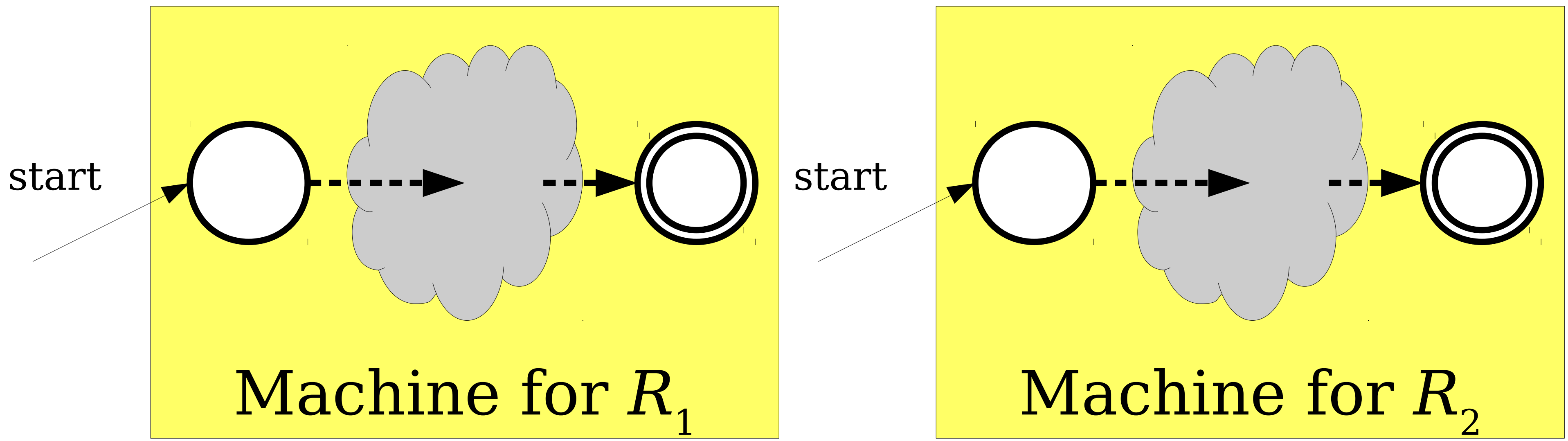

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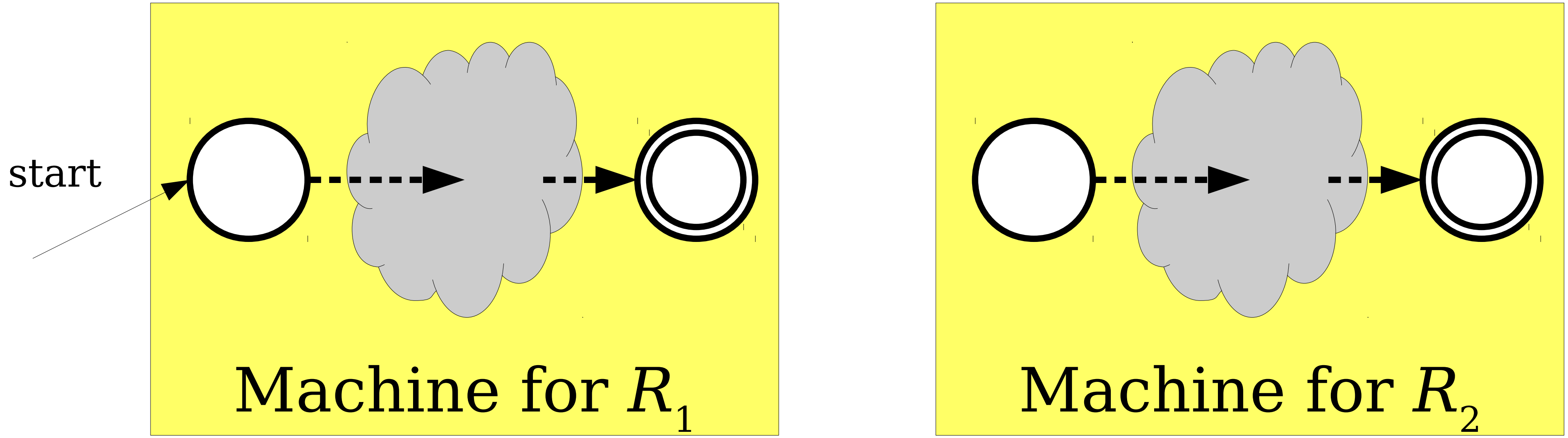

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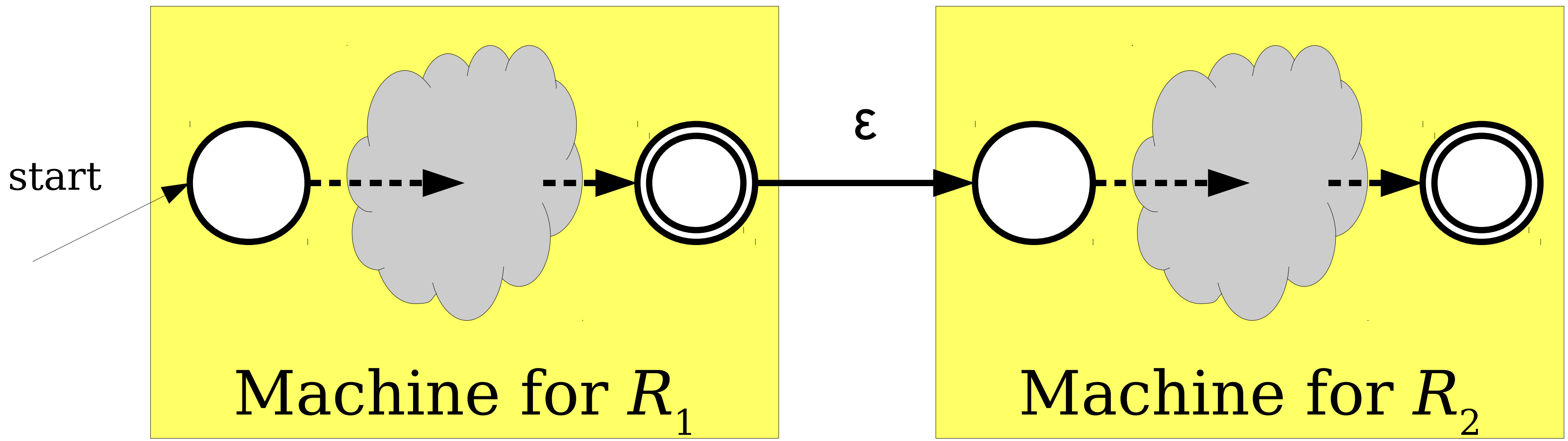

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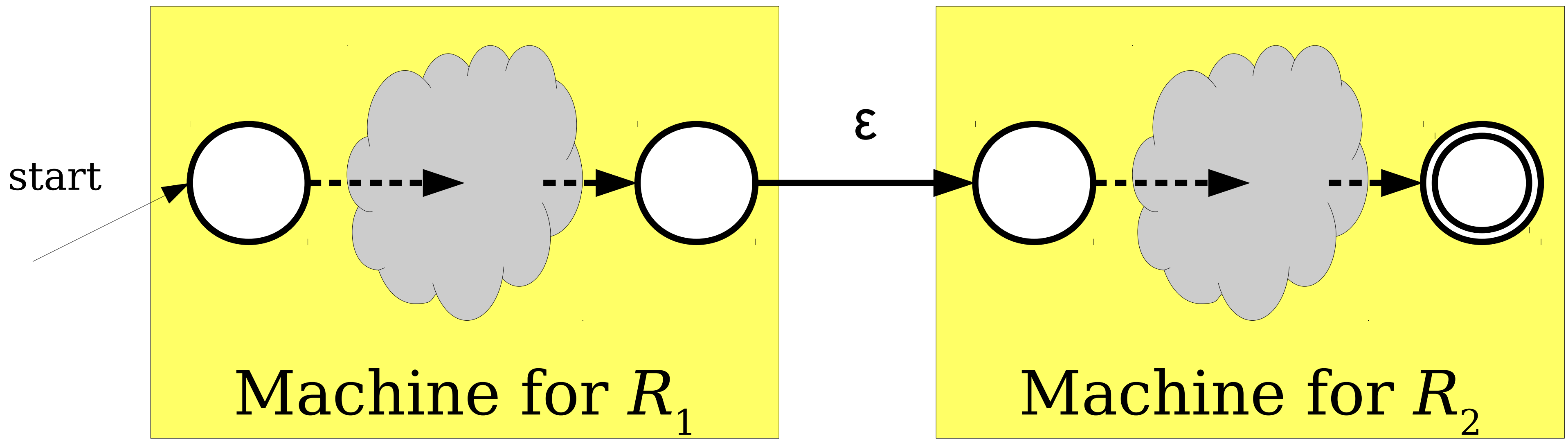
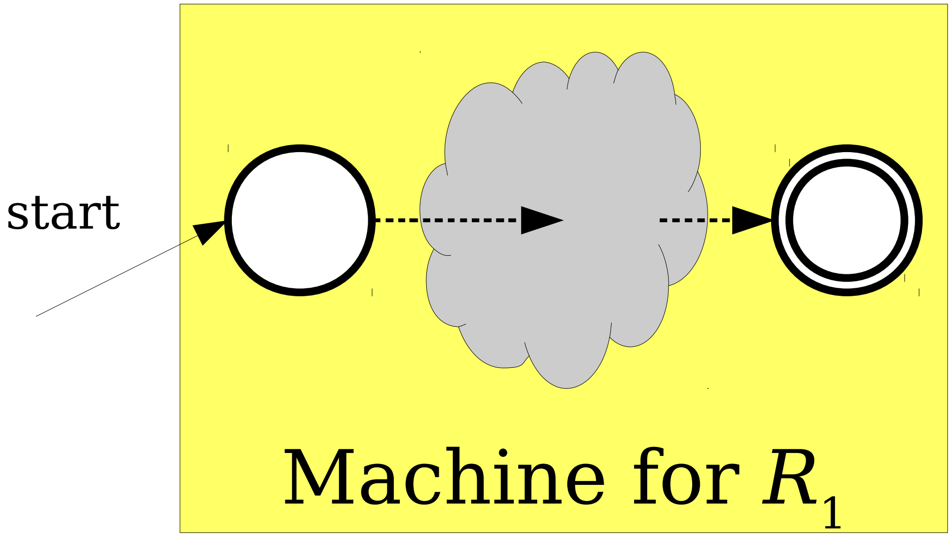
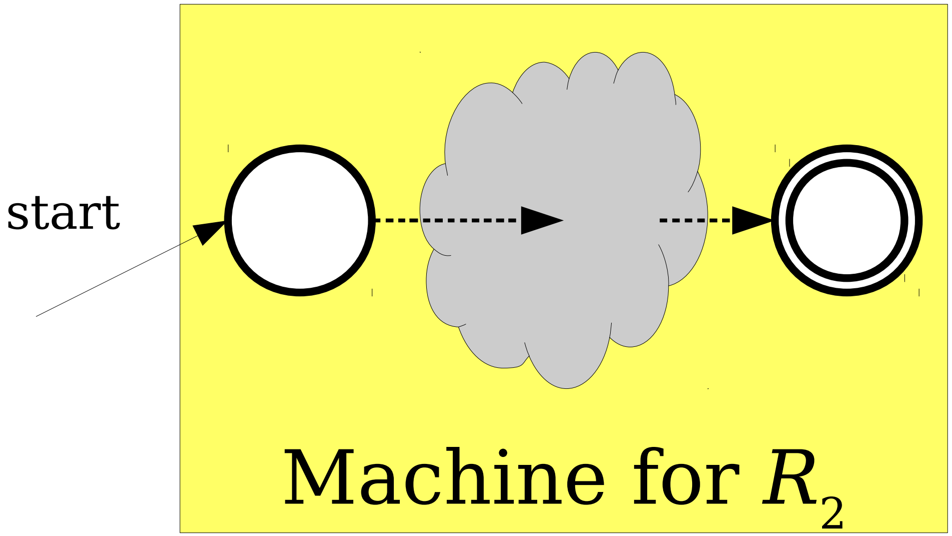


Construction for $R_1 \mid R_2$

Construction for R_1 | R_2





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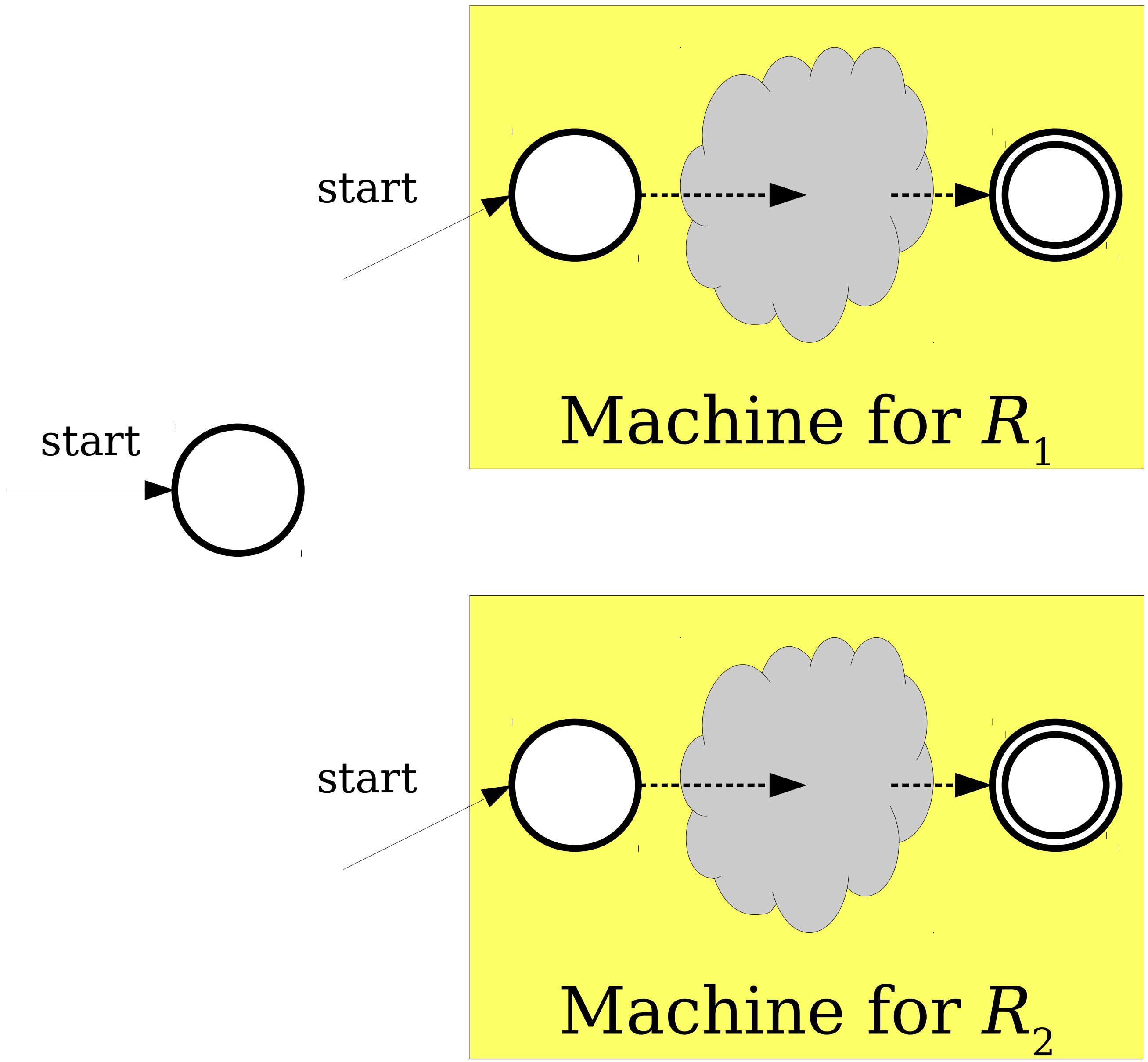

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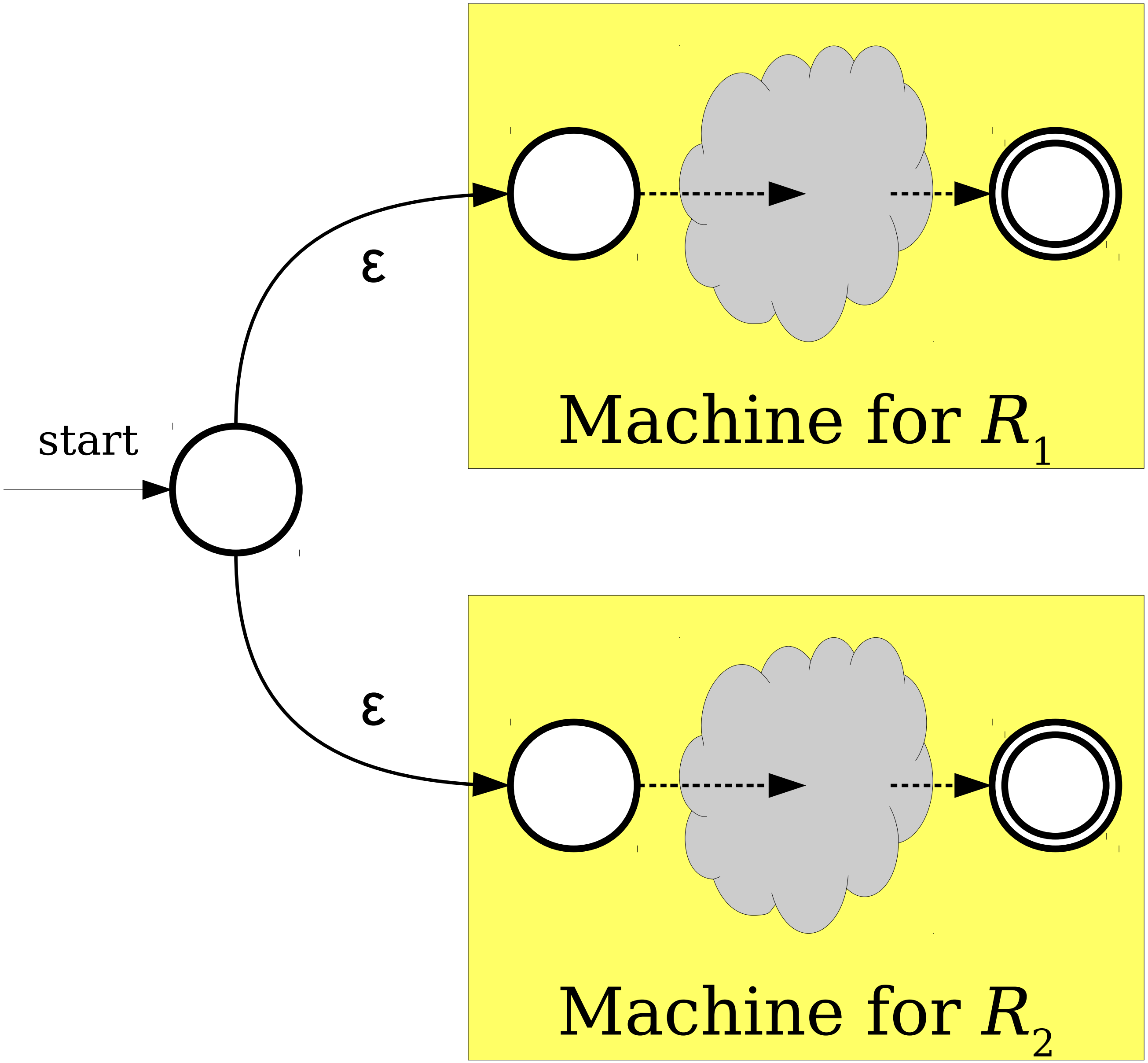

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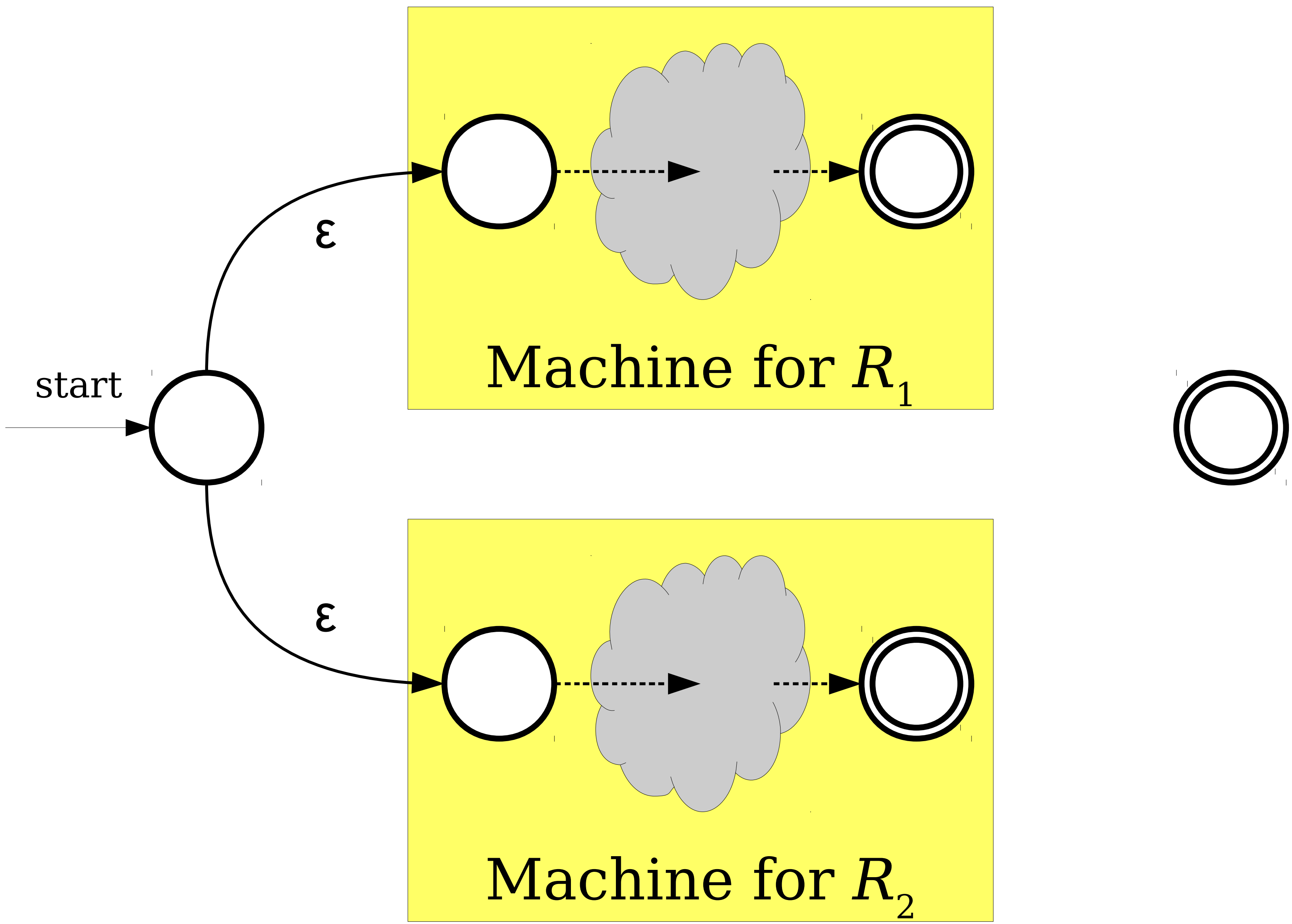

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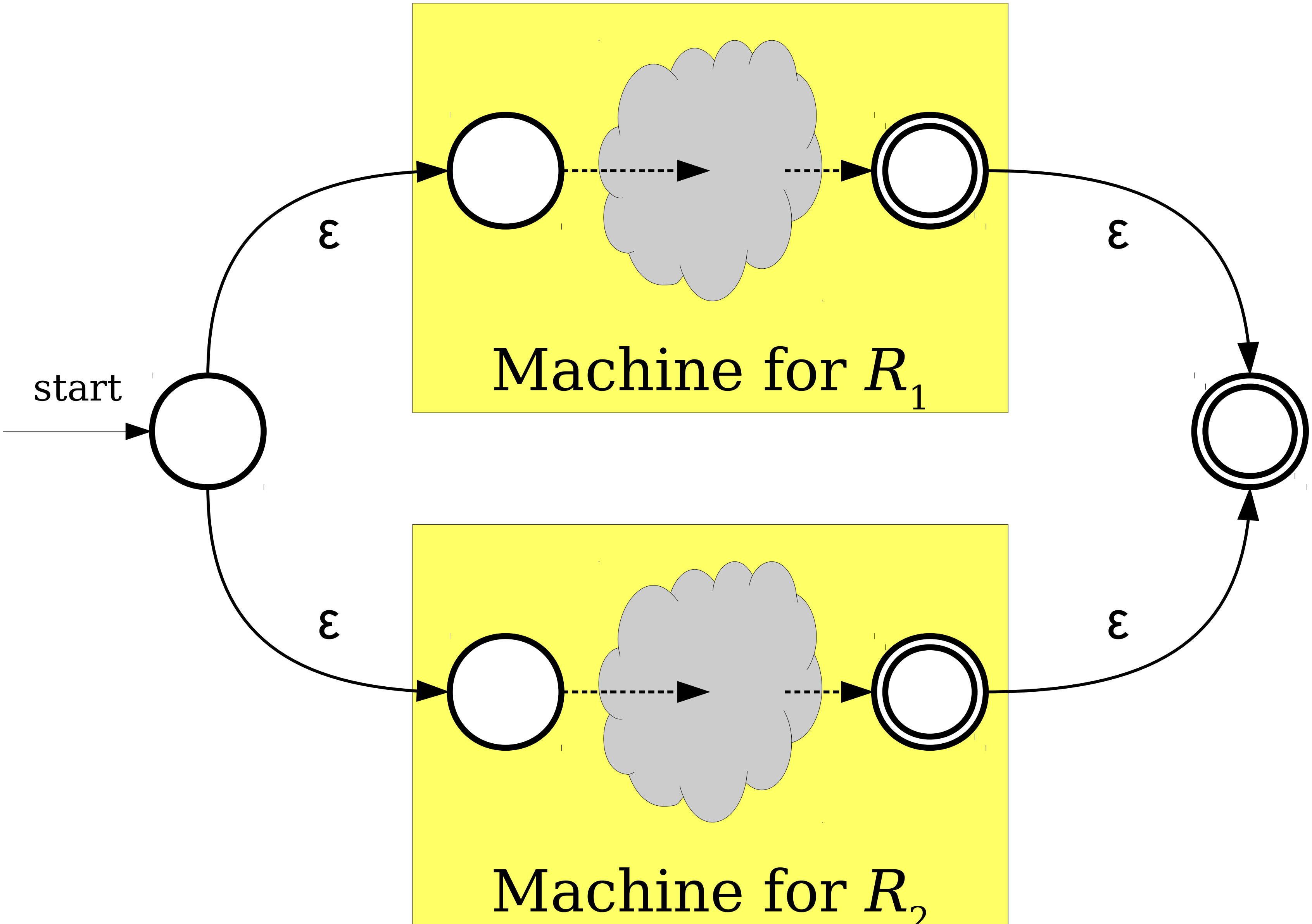
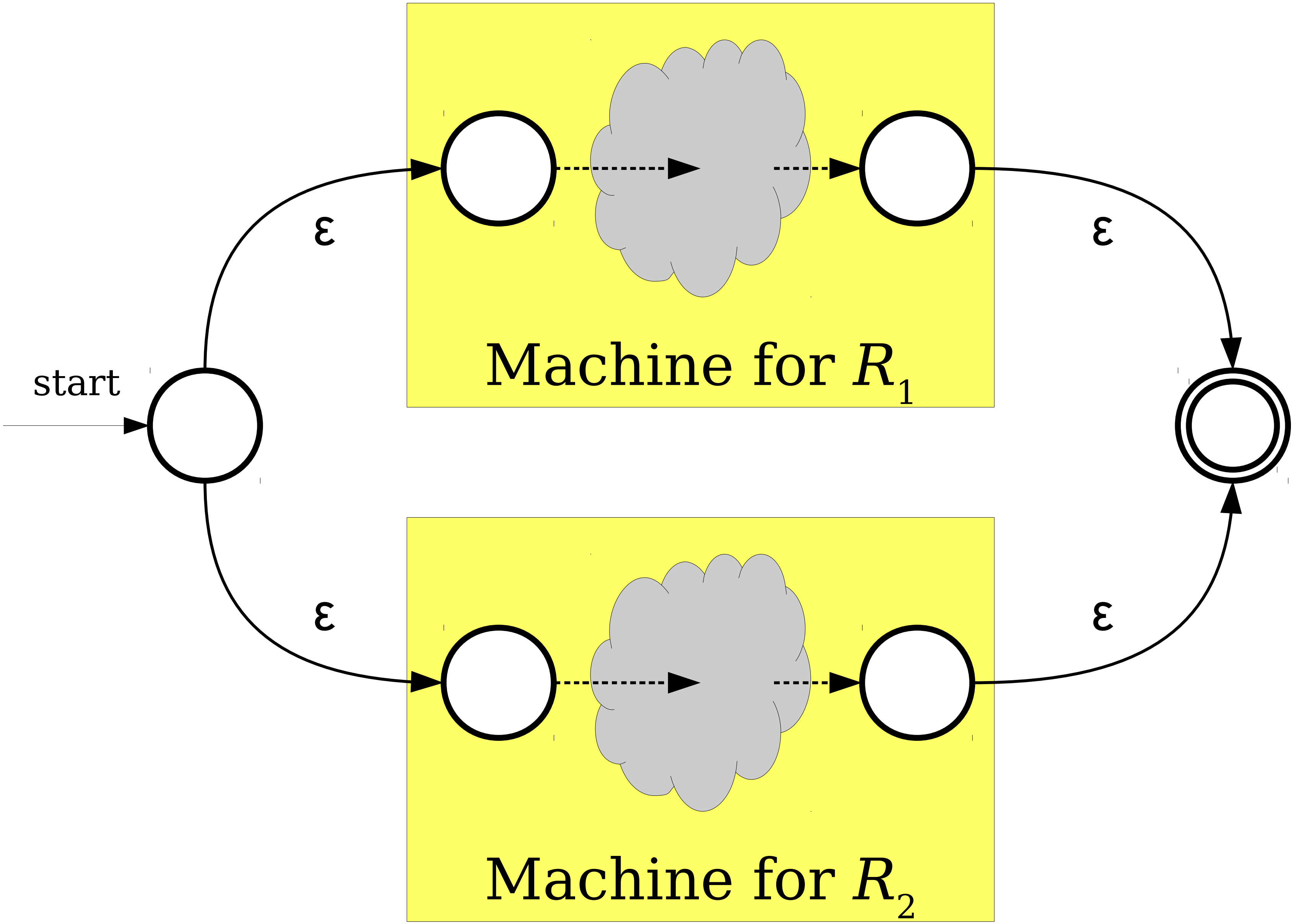
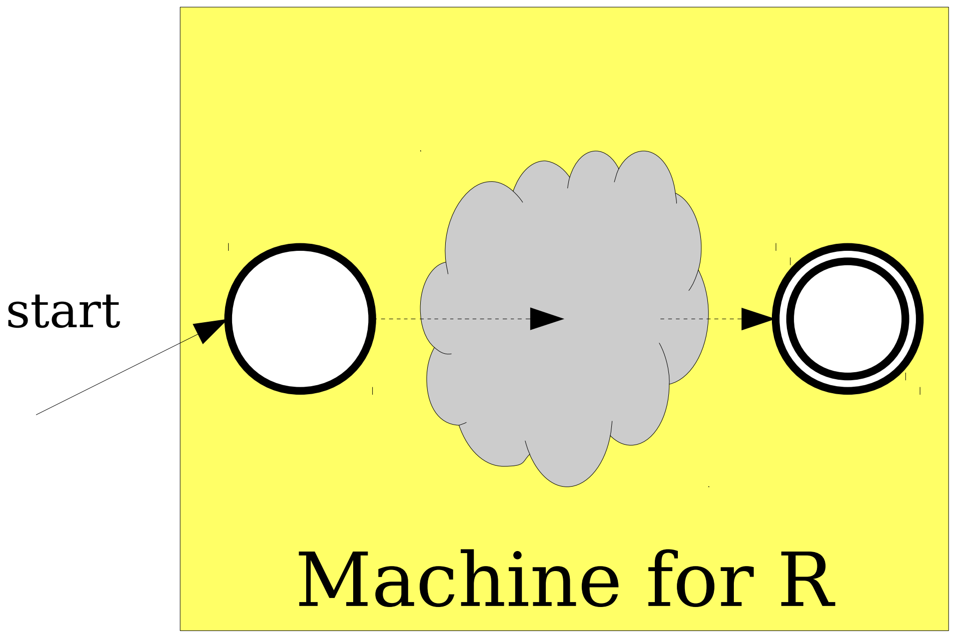


Construction for $R_1 \mid R_2$

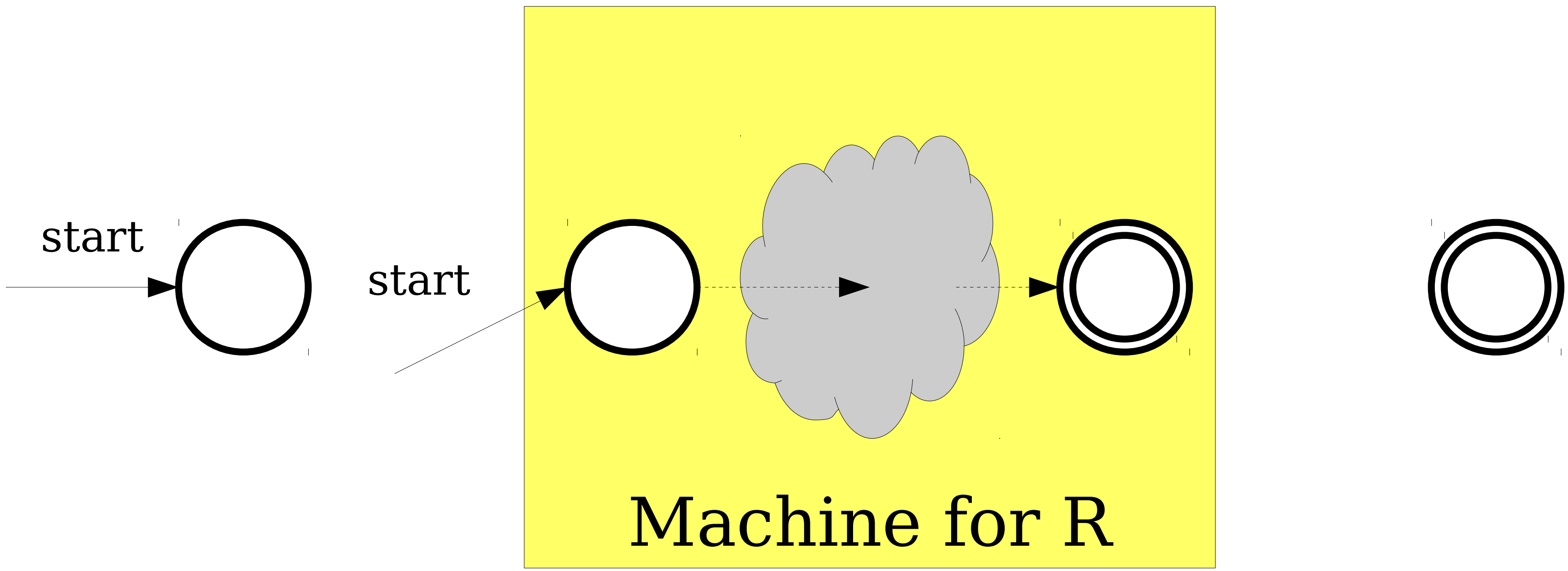


Construction for $R*$

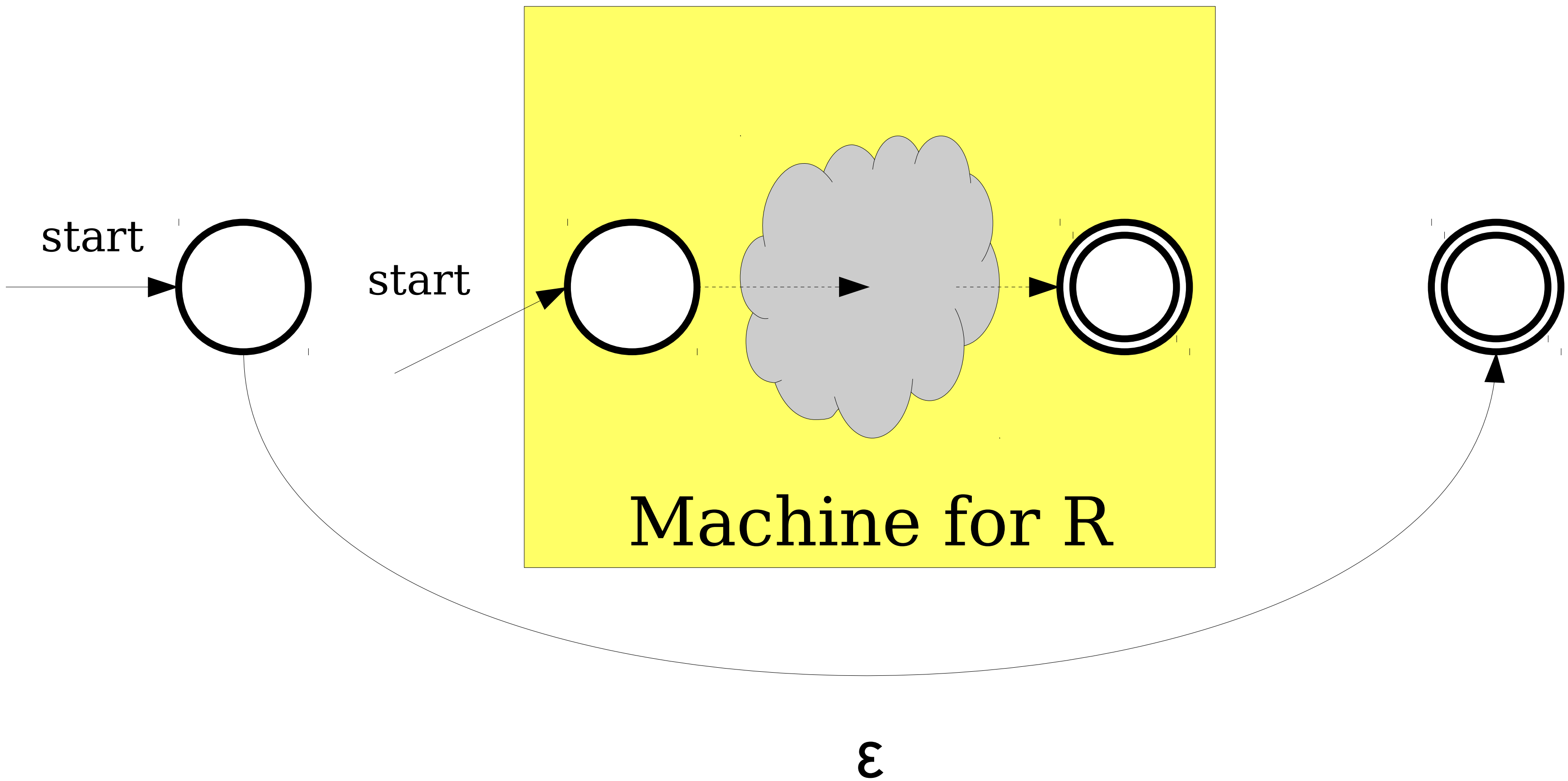
Construction for R^*



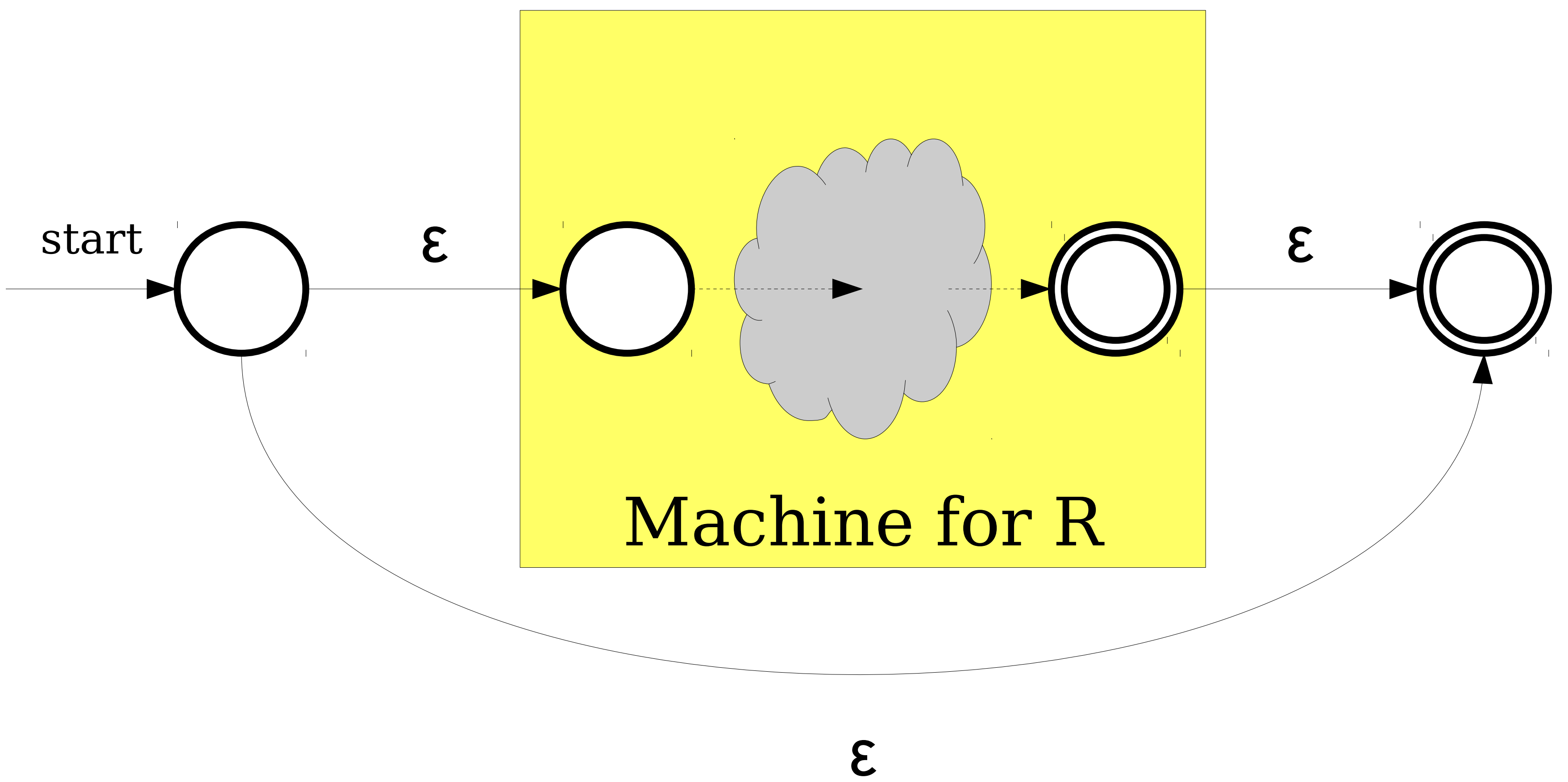
Construction for R^*



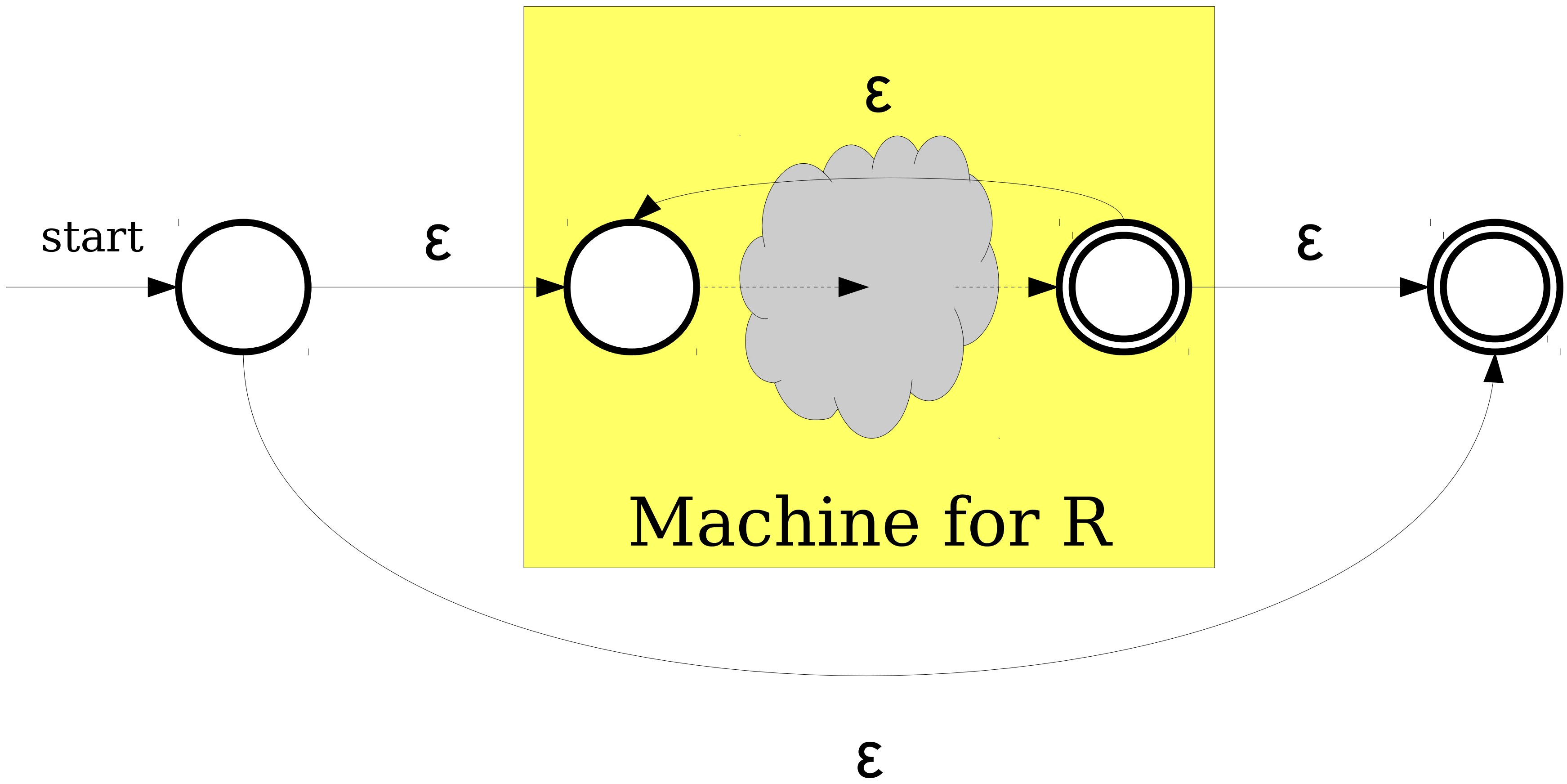
Construction for R*



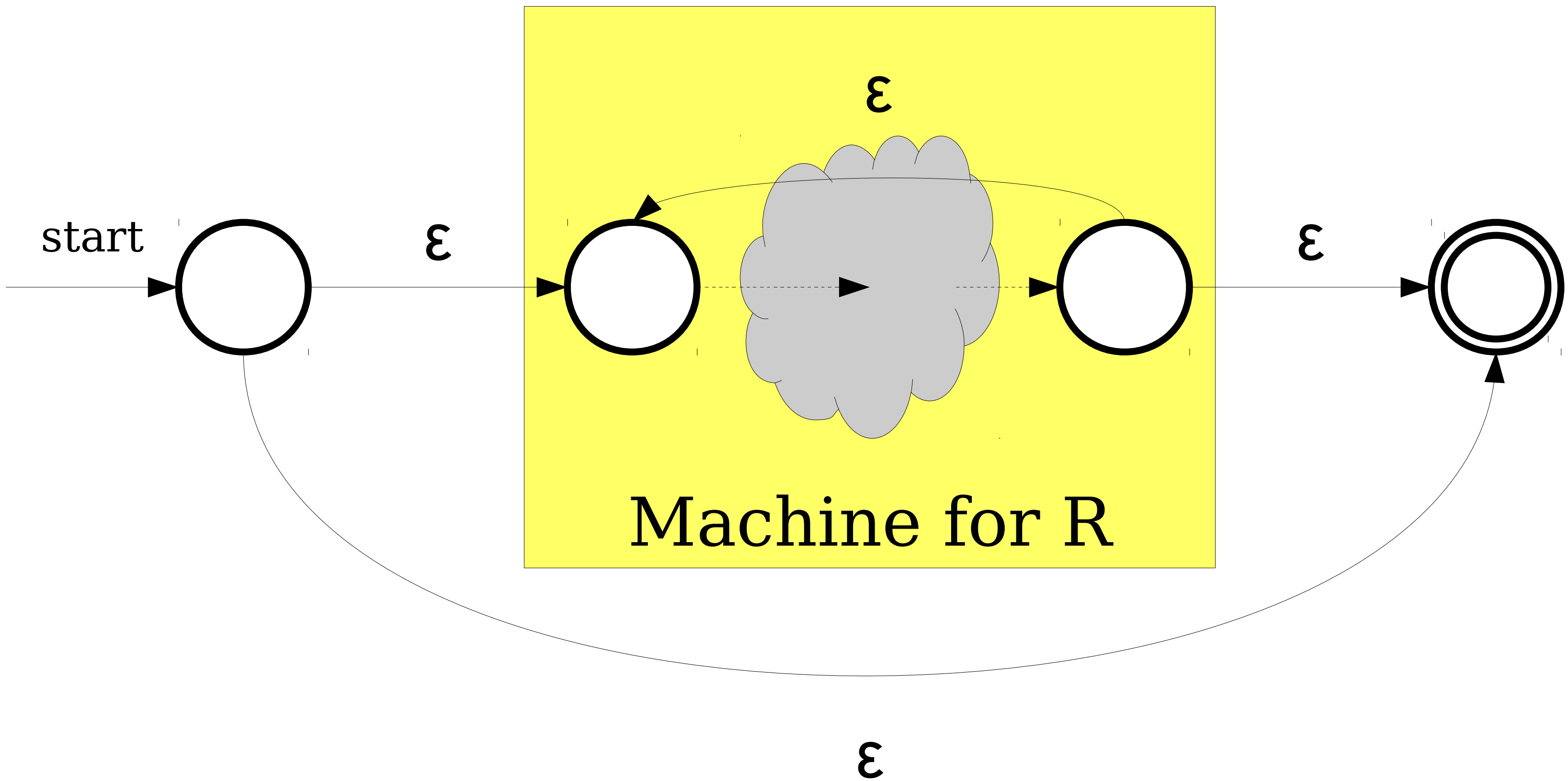
Construction for R*



Construction for *R**



Construction for $R*$



Why This Matters

- Many software tools work by matching regular expressions against text.
- One possible algorithm for doing so:
  - Convert the regular expression to an NFA.
  - (Optionally) Convert the NFA to a DFA using the subset construction.
  - Run the text through the finite automaton and look for matches.
- Runs extremely quickly!

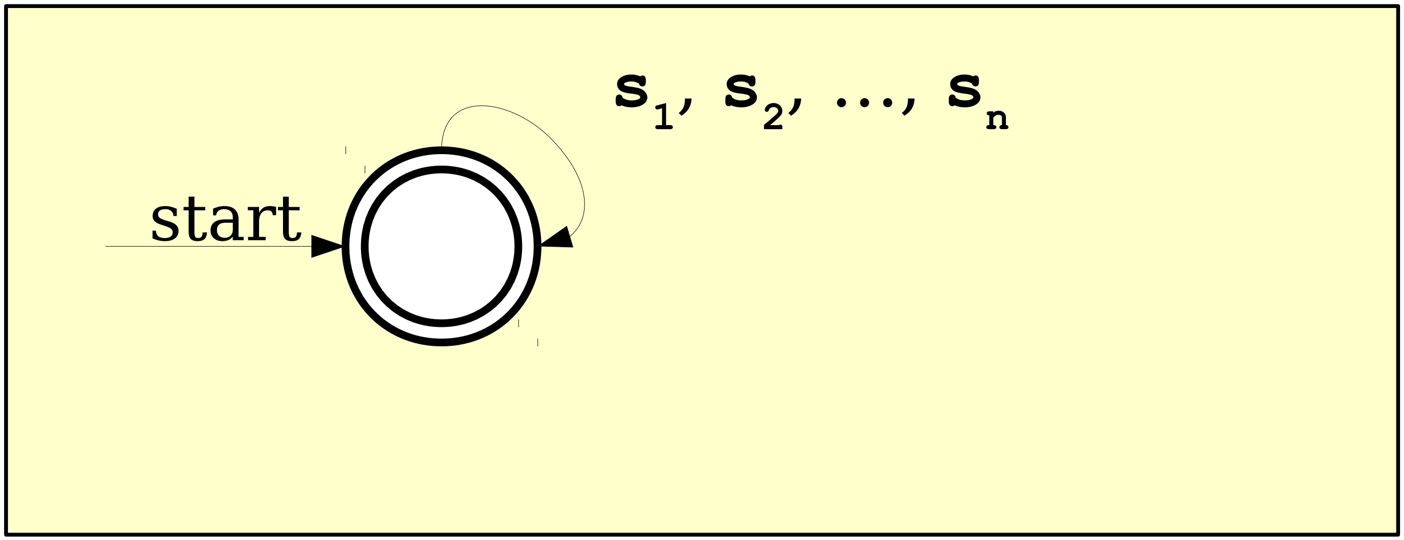
The Power of Regular Expressions

Theorem: If L is a regular language, then there is a regular expression for L.

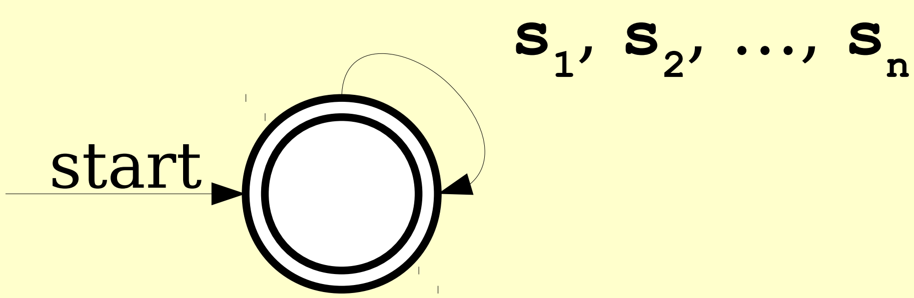
This is not obvious!

Proof idea: Show how to convert an arbitrary NFA into a regular expression.

From NFAs to Regular Expressions

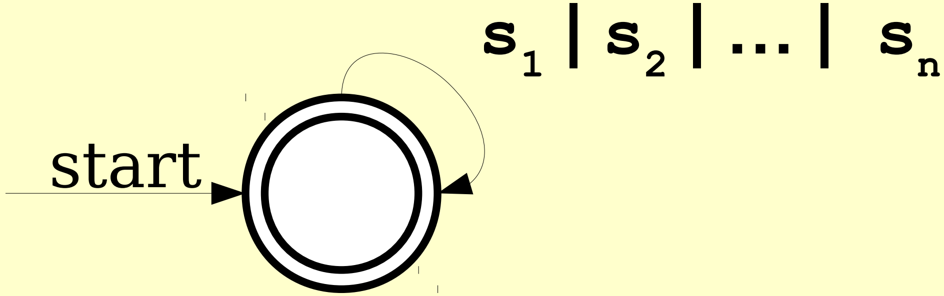


From NFAs to Regular Expressions



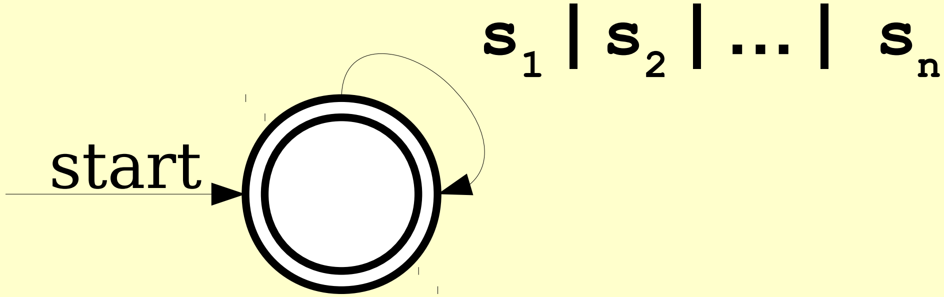
Regular expression: $(s_1 \mid s_2 \mid \ldots \mid s_n)^*$

From NFAs to Regular Expressions



Regular expression: $(s_1 \mid s_2 \mid \ldots \mid s_n)^*$

From NFAs to Regular Expressions

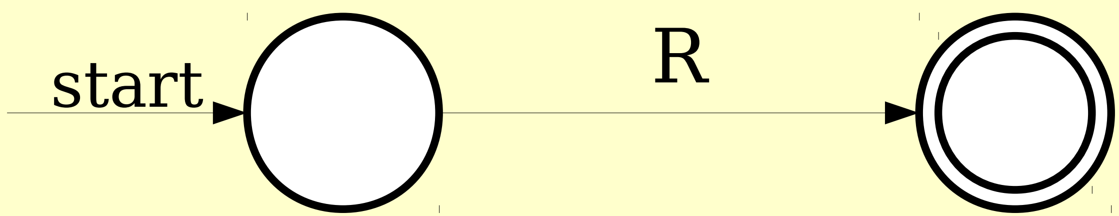


Regular expression: $(s_1 \mid s_2 \mid \ldots \mid s_n)^*$

Key idea: Label transitions with arbitrary regular expressions.

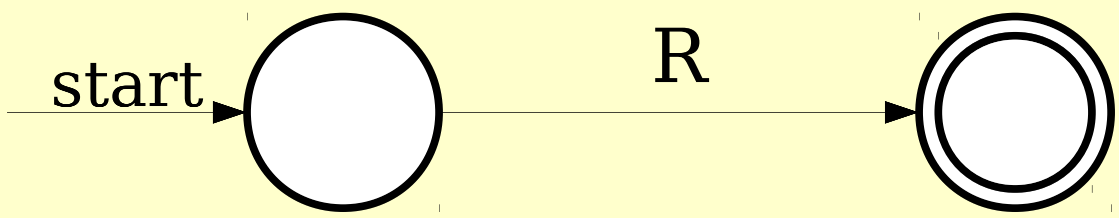
From NFAs to Regular Expressions

From NFAs to Regular Expressions



Regular expression: **R**

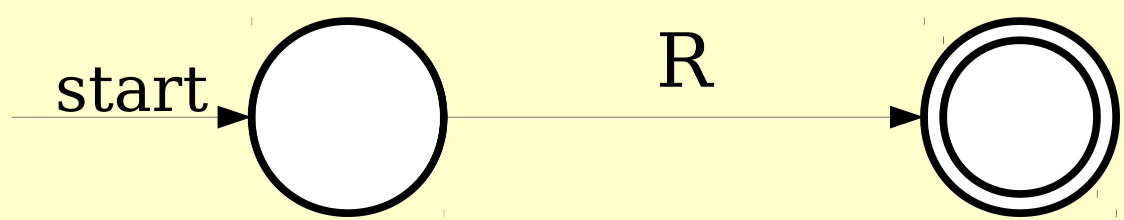
From NFAs to Regular Expressions



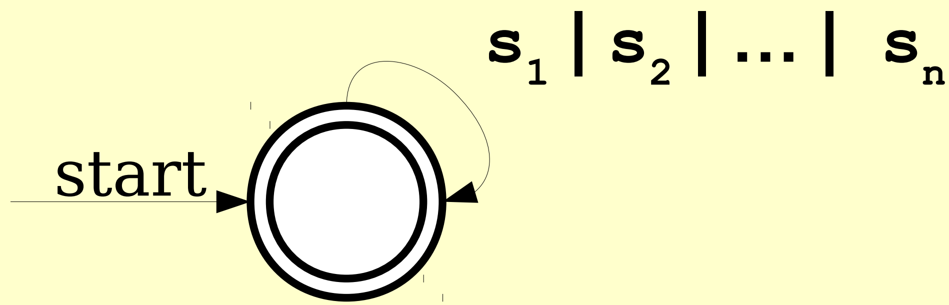
Regular expression: **R**

Key idea: If we can convert any NFA into something that looks like this, we can easily read off the regular expression.

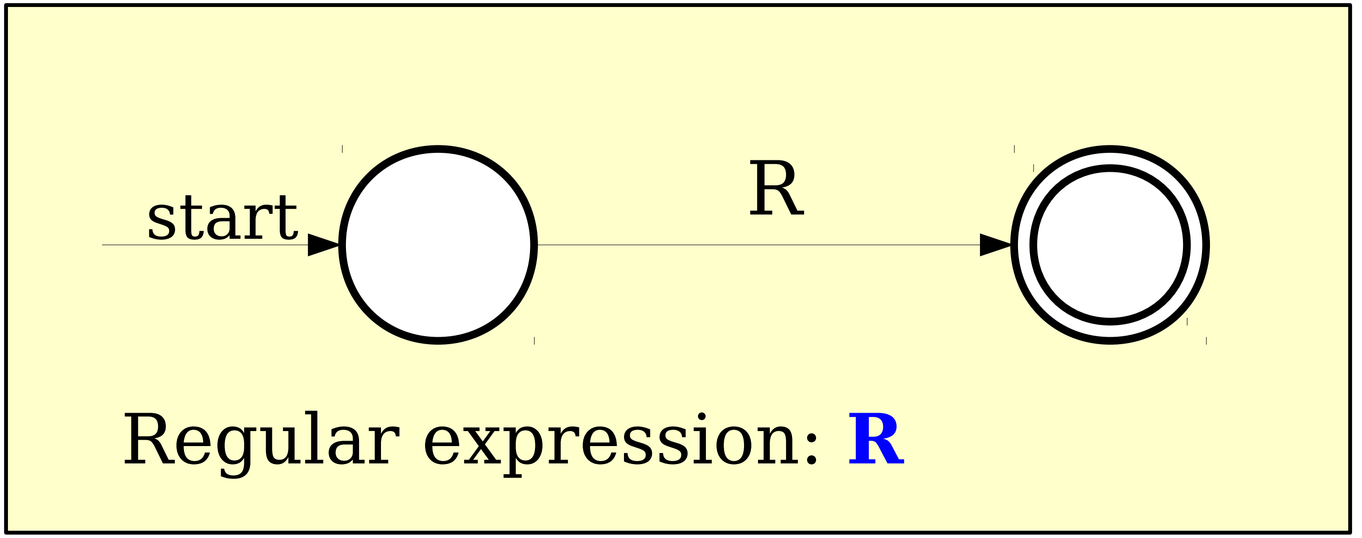
From NFAs to Regular Expressions

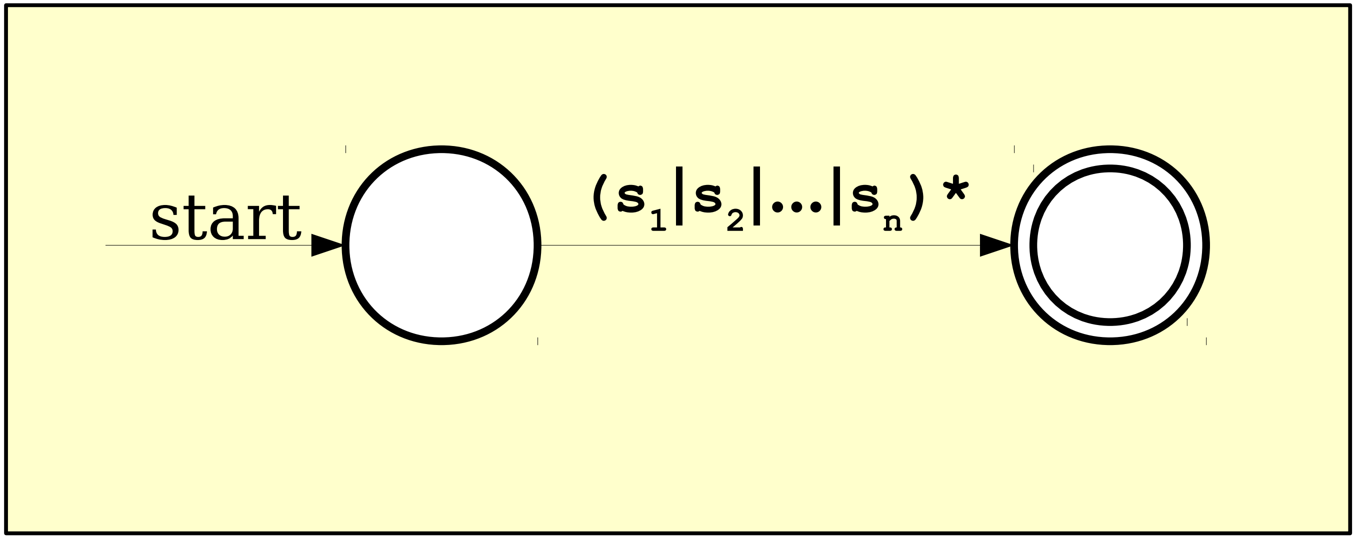


Regular expression: **R**



From NFAs to Regular Expressions





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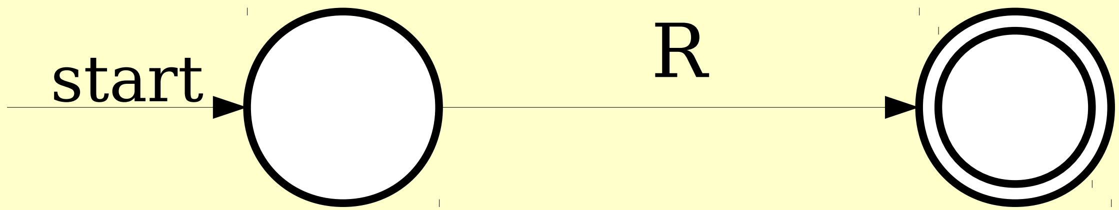
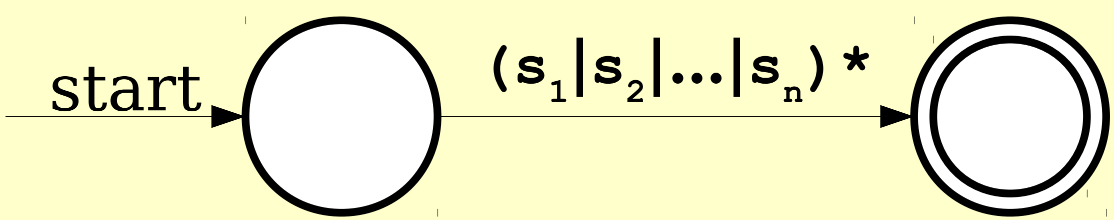
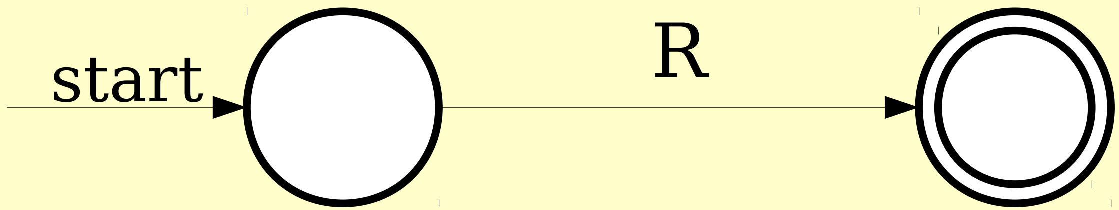


Regular expression: **R**



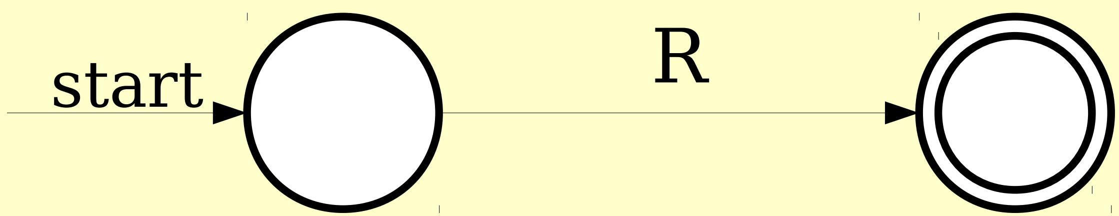
Regular expression: $(s_1|s_2| \ldots |s_n)*$

From NFAs to Regular Expressions

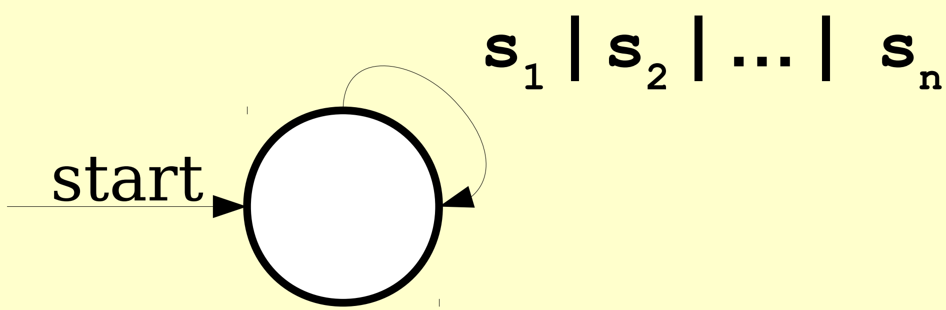


Regular expression: **R**

From NFAs to Regular Expressions



Regular expression: **R**



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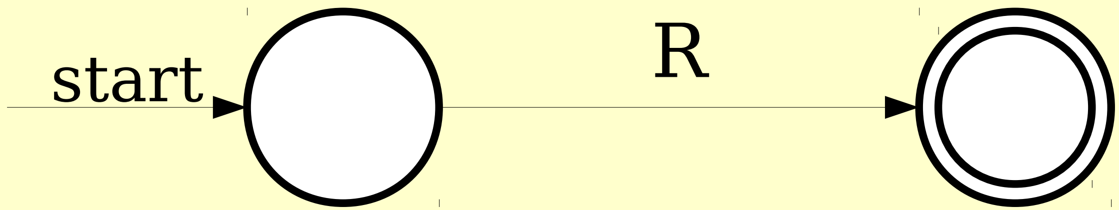
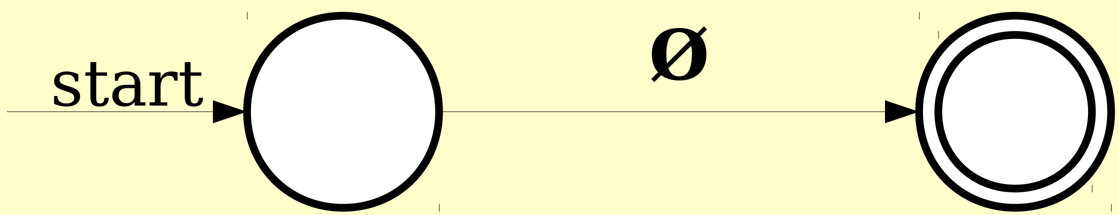
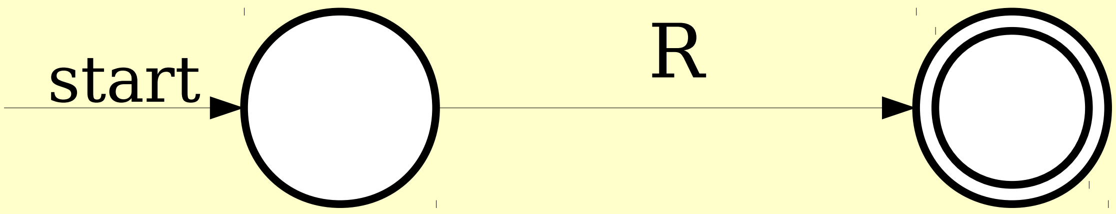


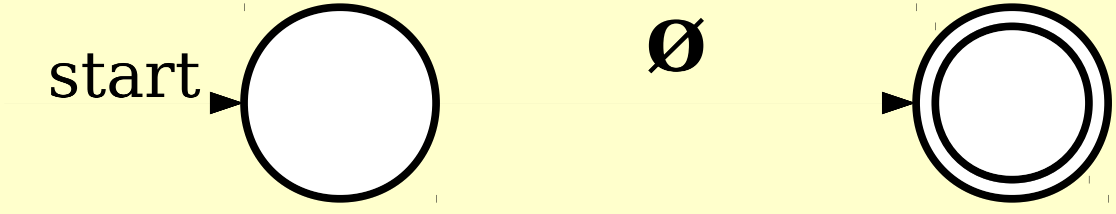
Regular expression: **R**



From NFAs to Regular Expressions

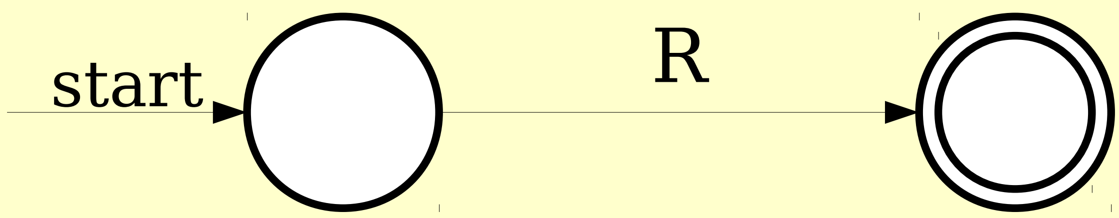


Regular expression: **R**



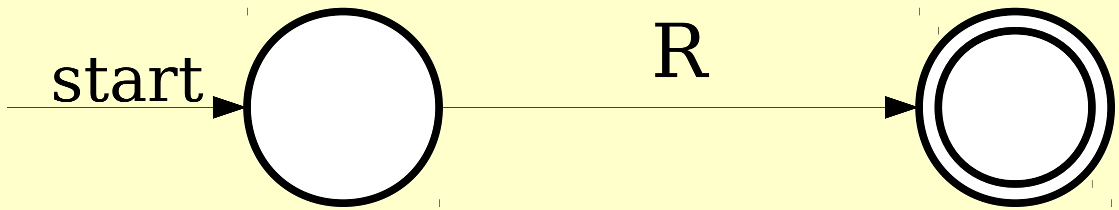
Regular expression: **Ø**

From NFAs to Regular Expressions

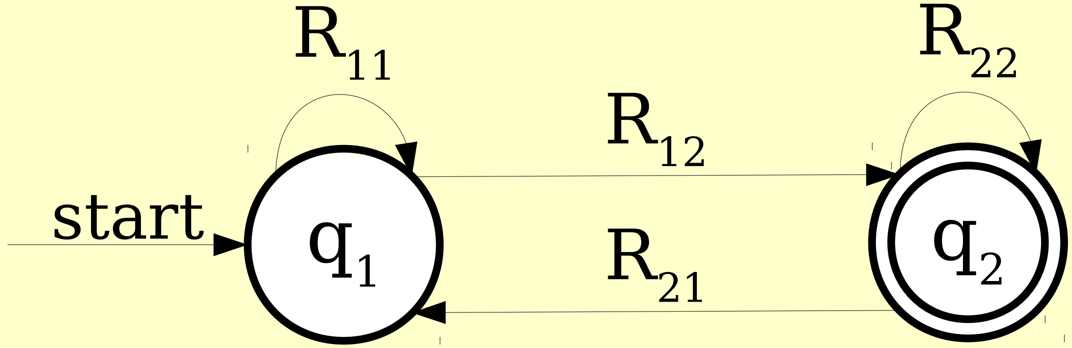


Regular expression: **R**

From NFAs to Regular Expressions



Regular expression: **R**



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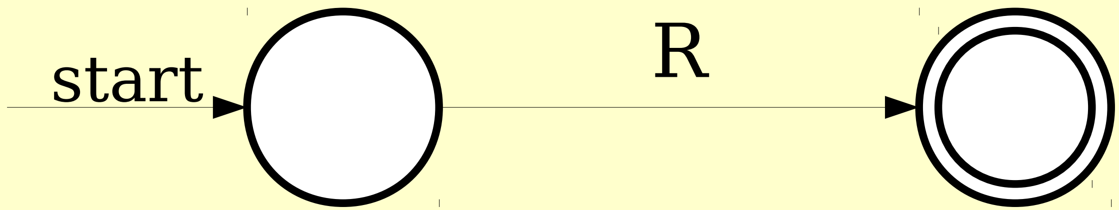
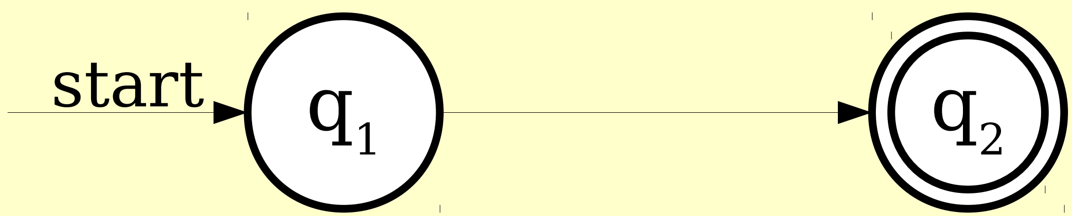
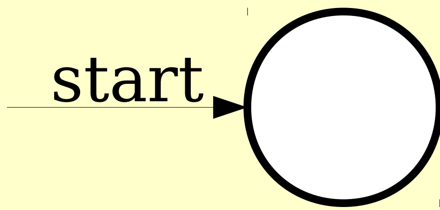


Regular expression: **R**

R_{11}* R_{12} (R_{22} | $R_{21}R_{11}$*R_{12})*

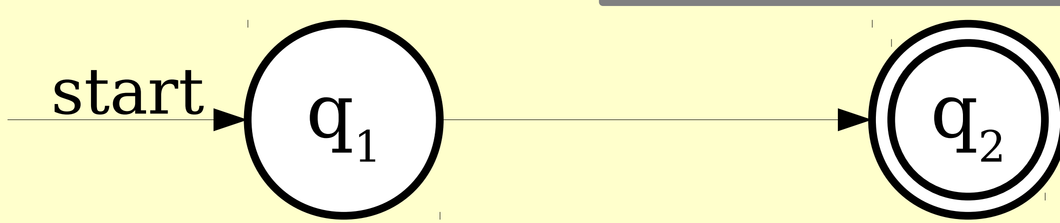


From NFAs to Regular Expressions



Regular expressi

R_{11}* R_{12} (R



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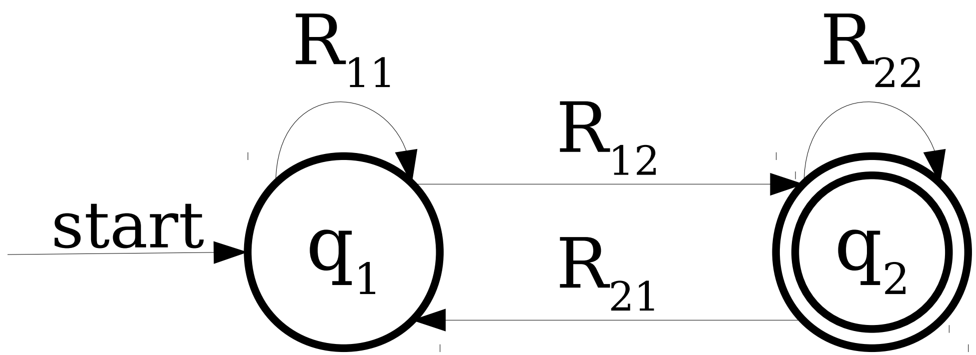

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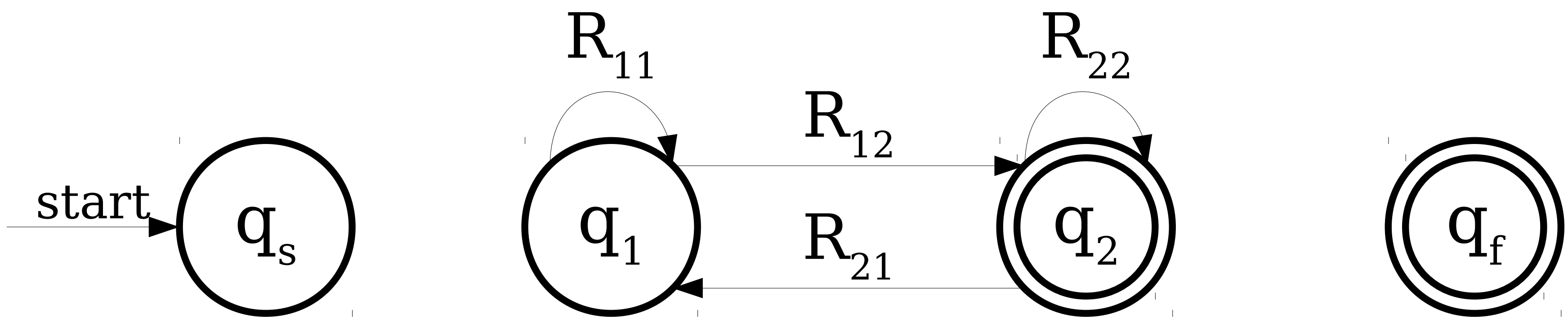

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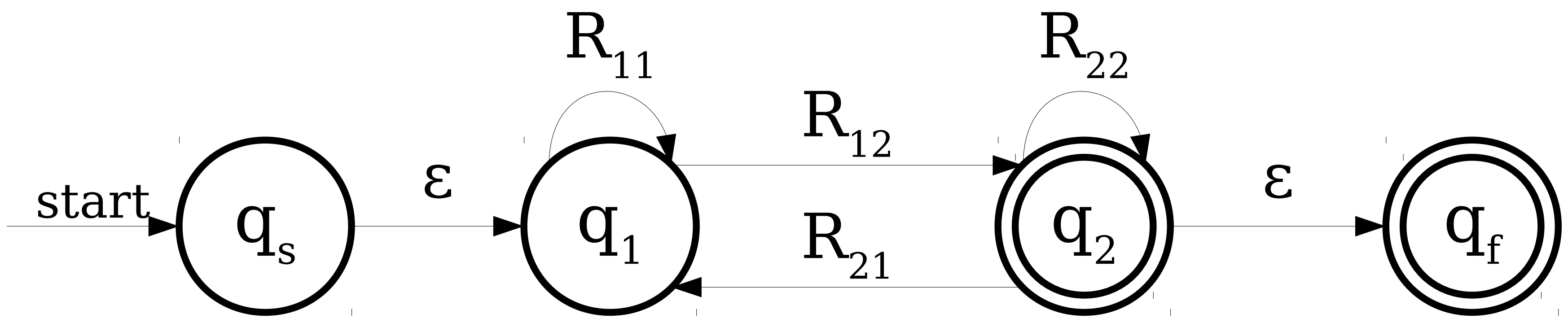

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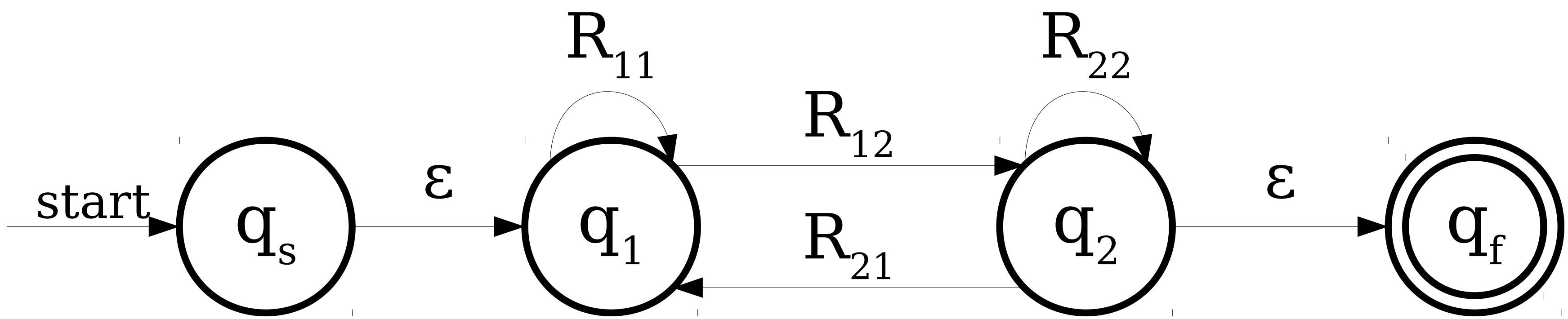

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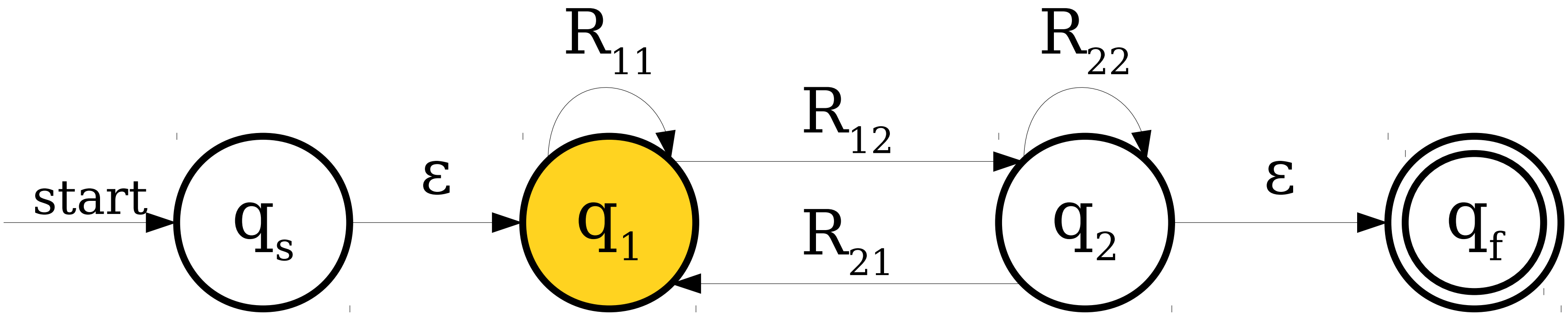

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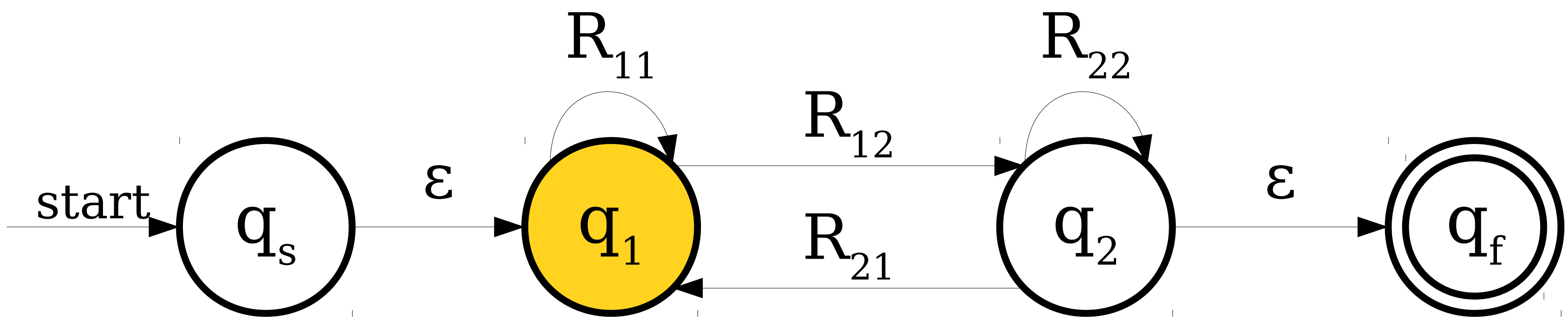
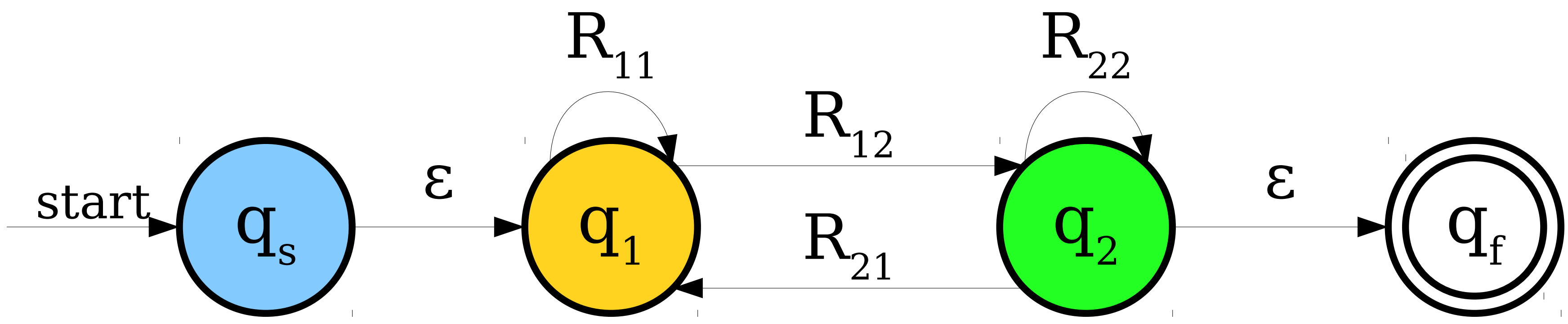
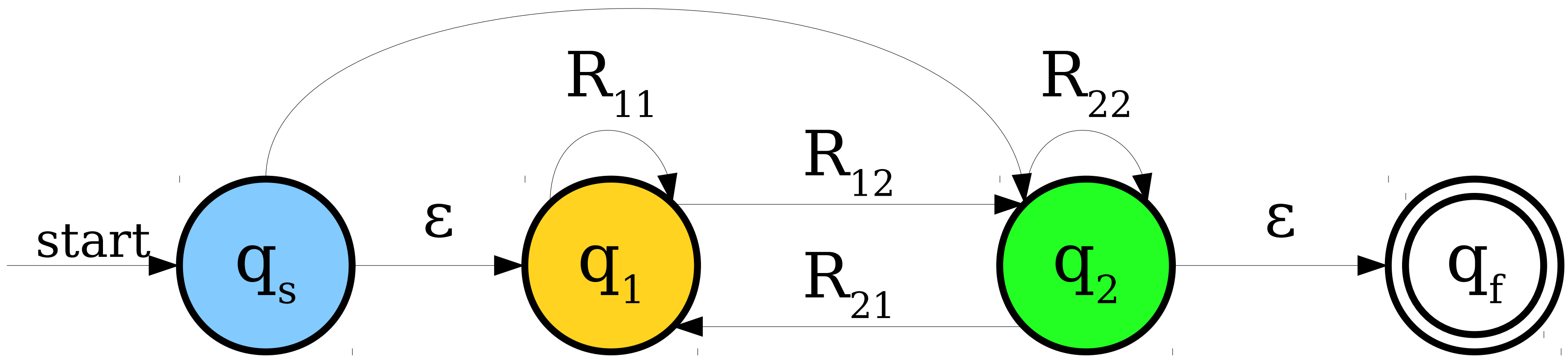


Could we eliminate this state from the NFA?

From NFAs to Regular Expressions



From NFAs to Regular Expressions



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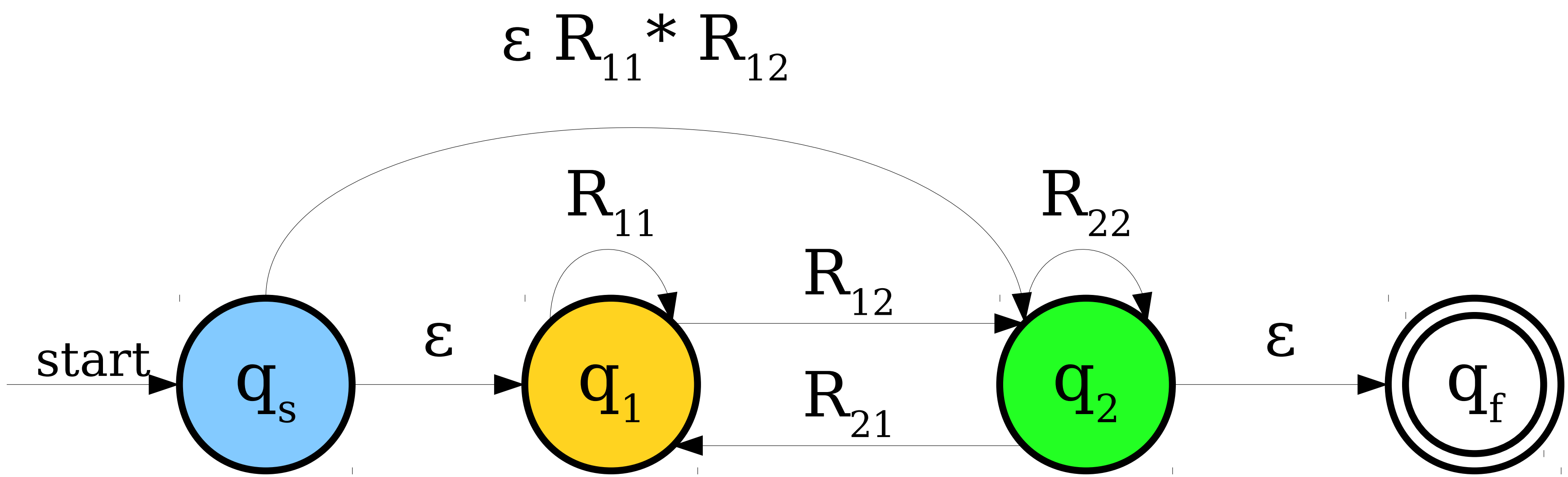
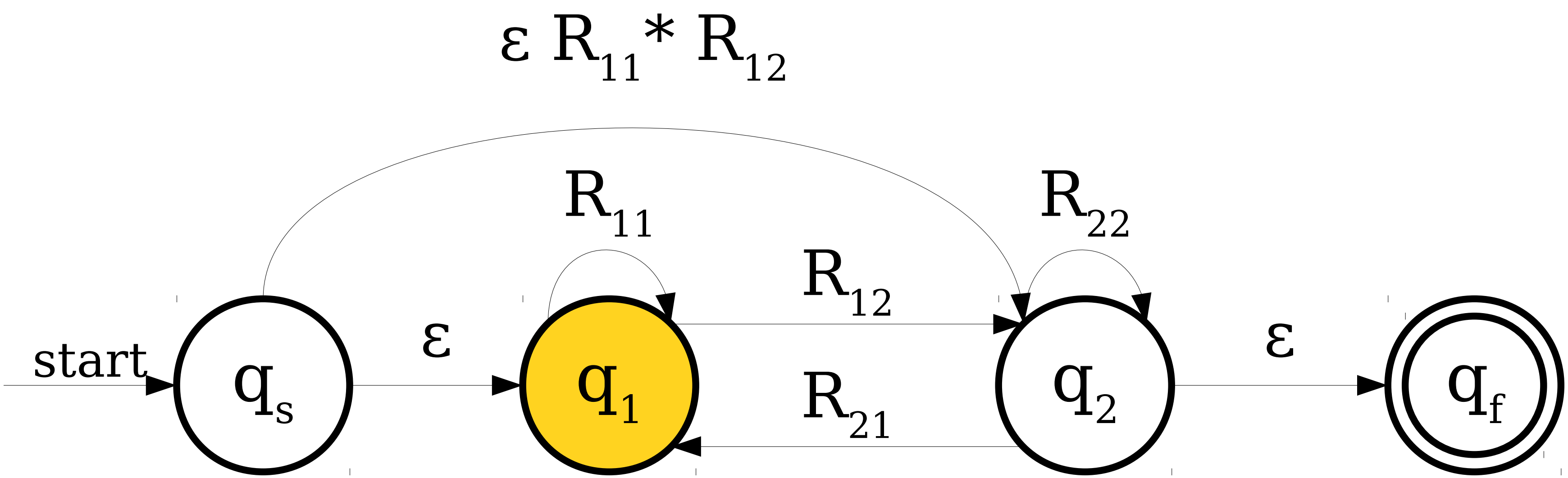
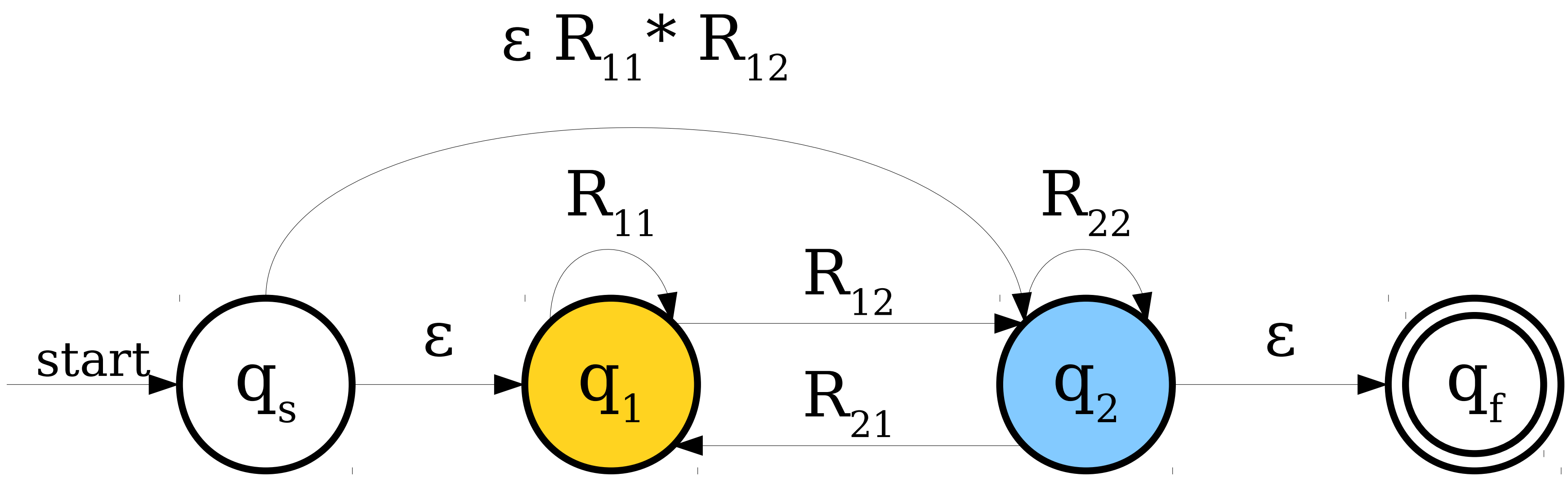


Note: We're using **concatenation** and **Kleene closure** in order to skip this state.

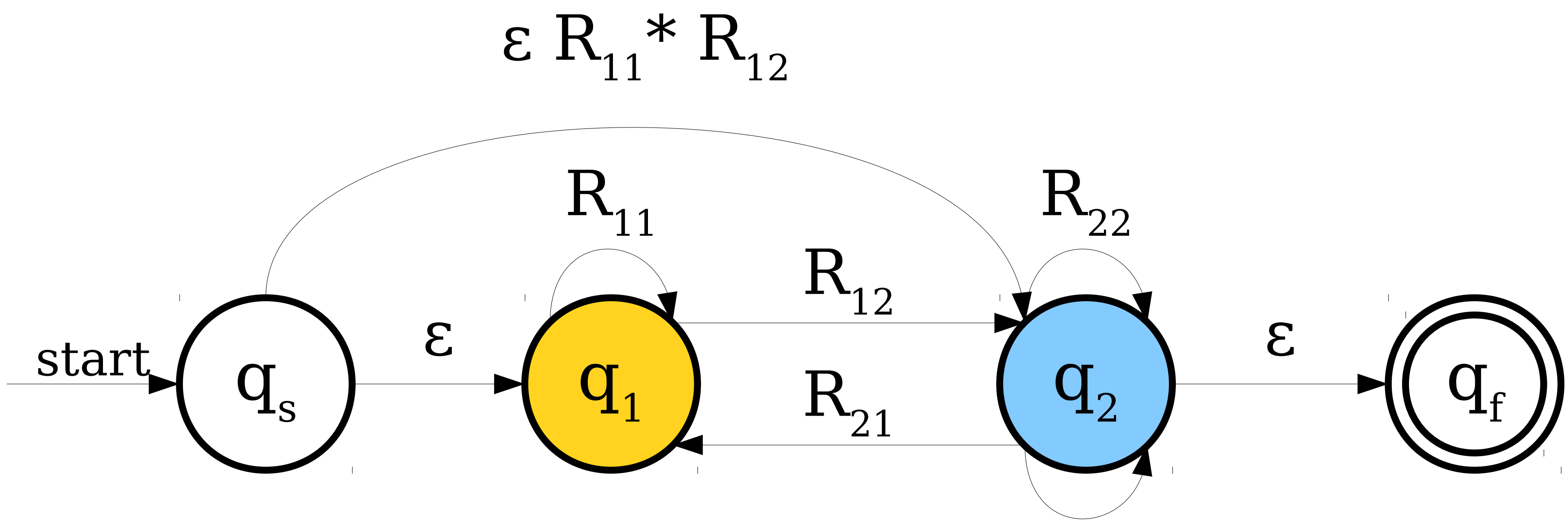
From NFAs to Regular Expressions



From NFAs to Regular Expressions



From NFAs to Regular Expressions



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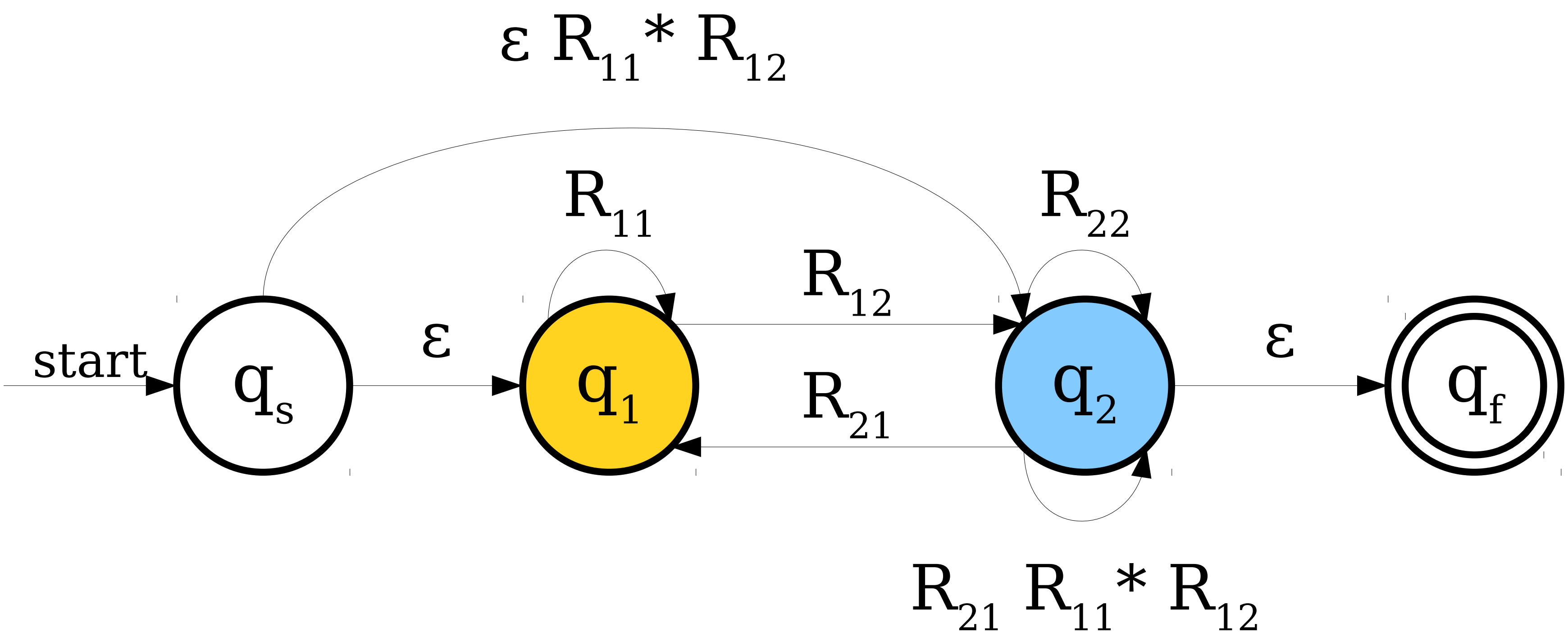

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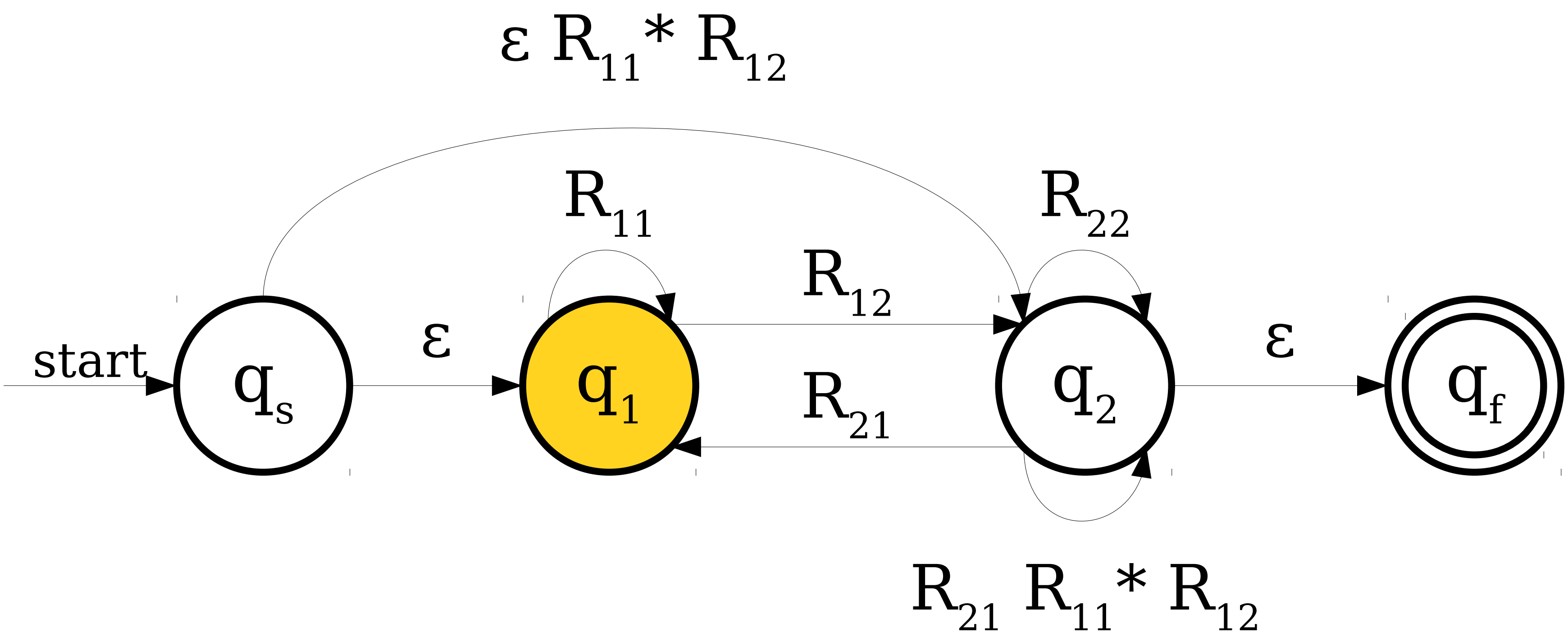

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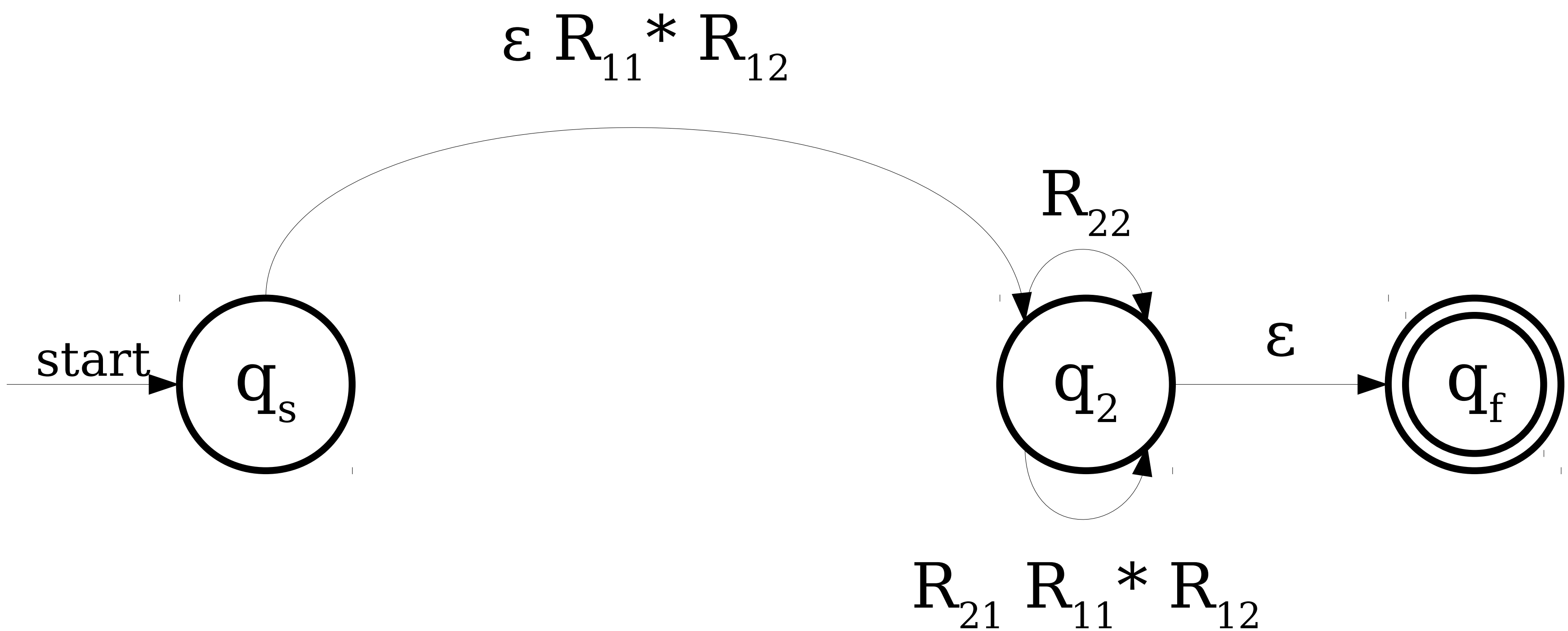

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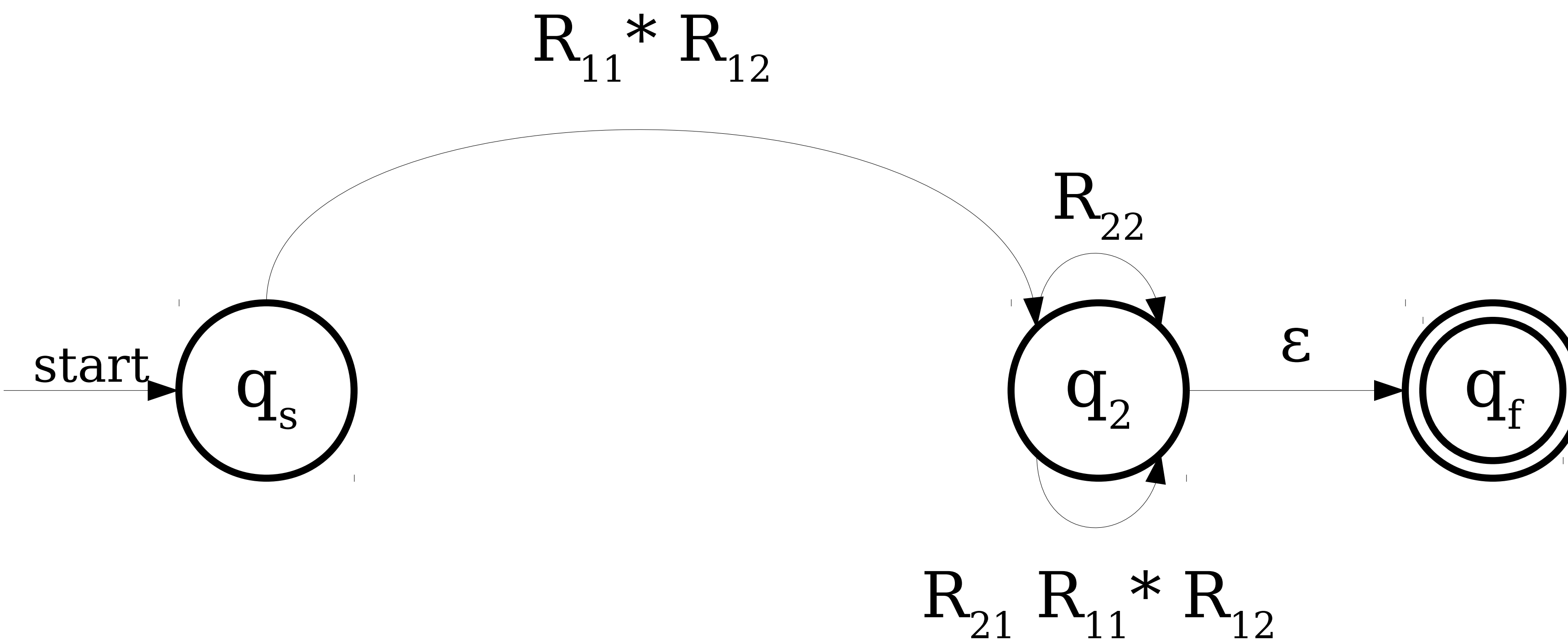

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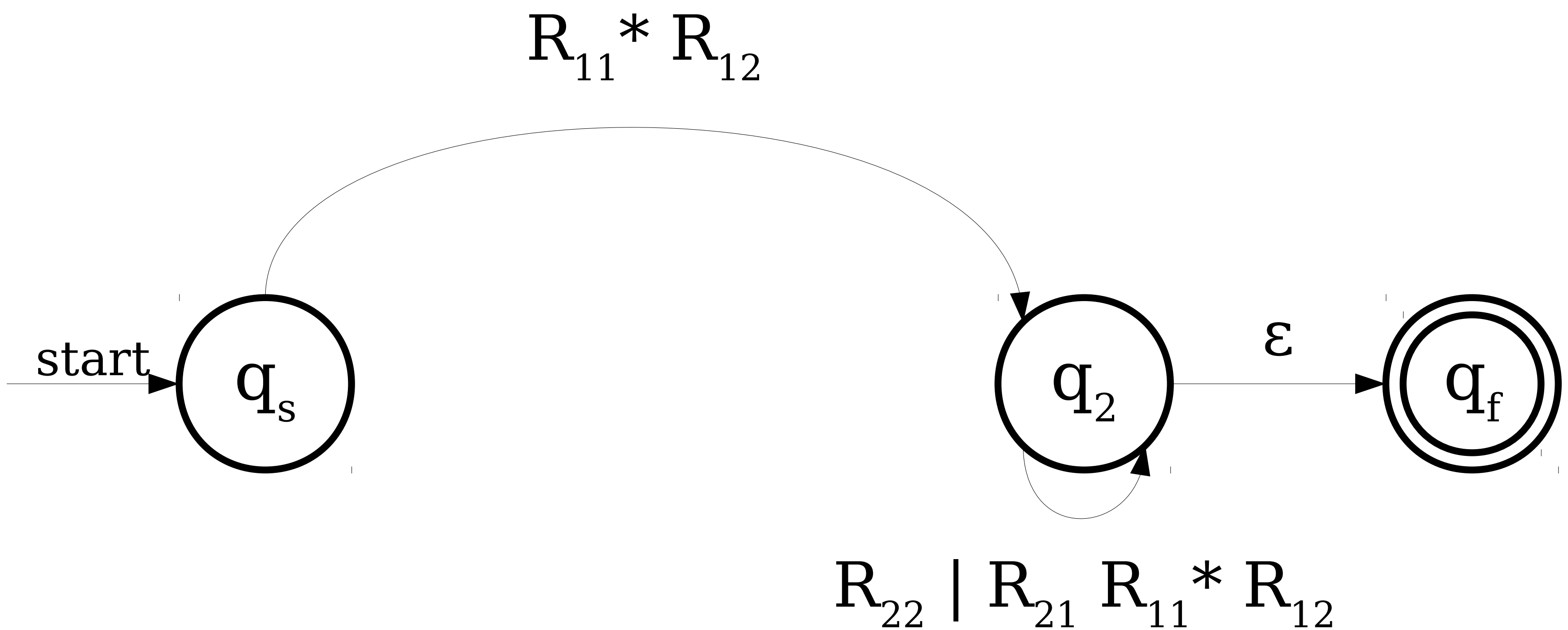
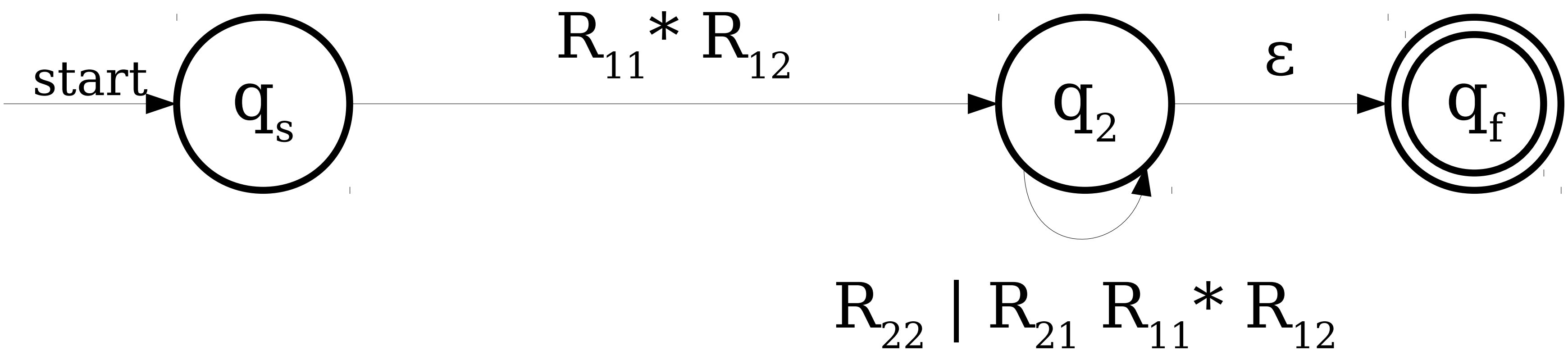


Note: We're using union to combine these transitions together.

From NFAs to Regular Expressions



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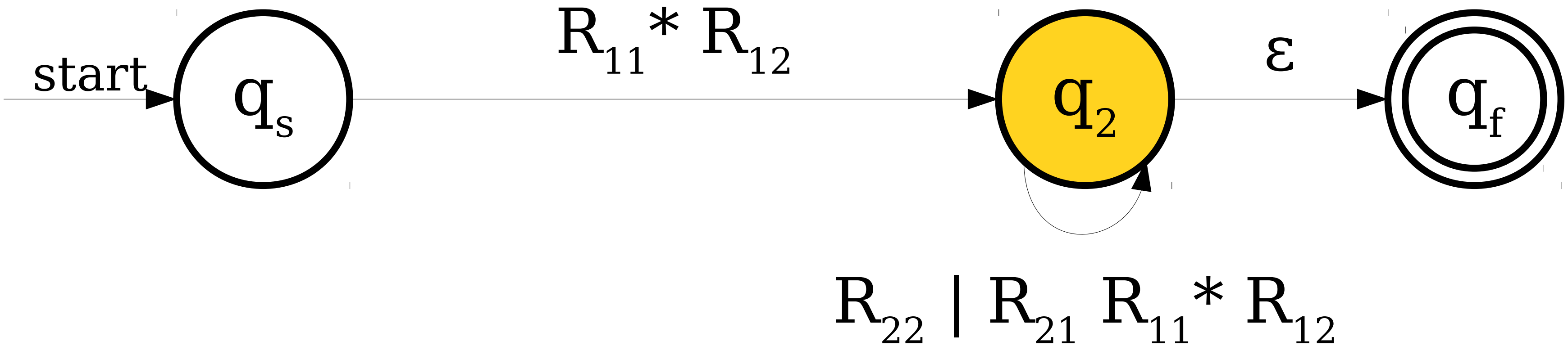

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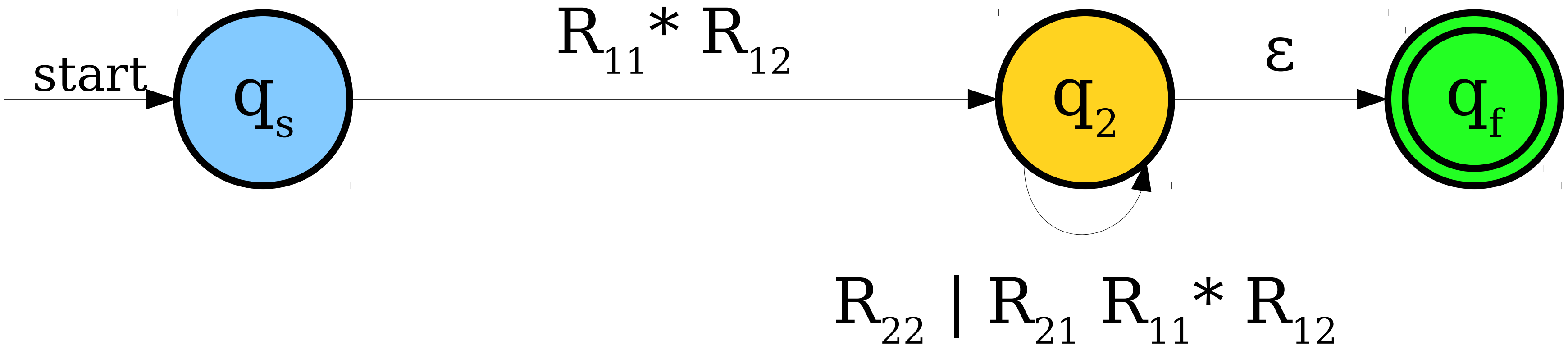

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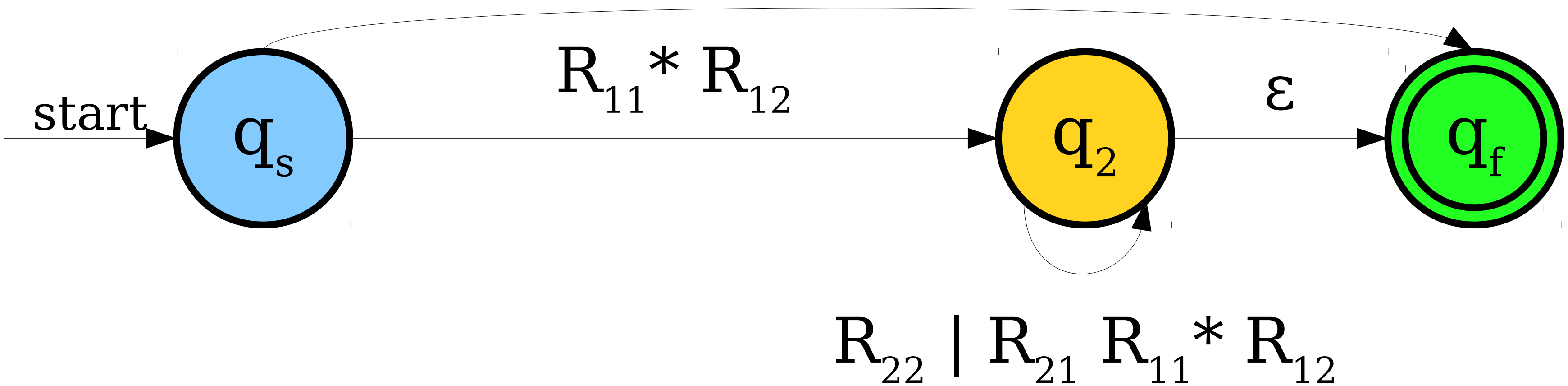

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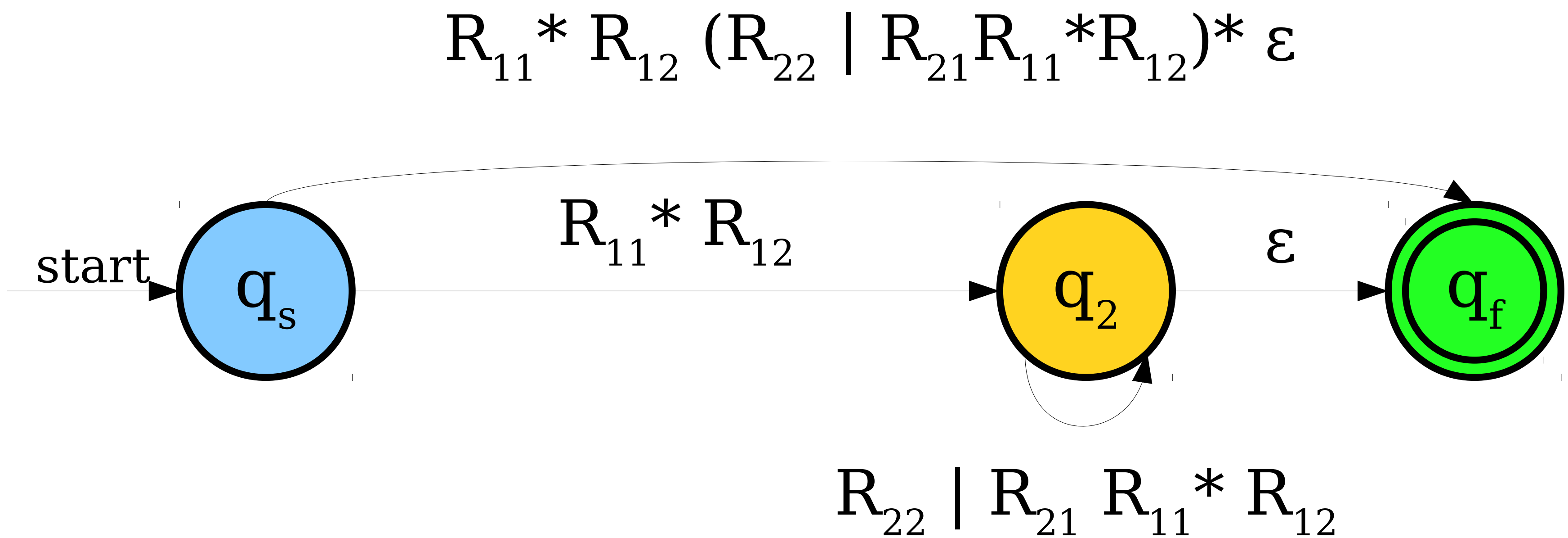

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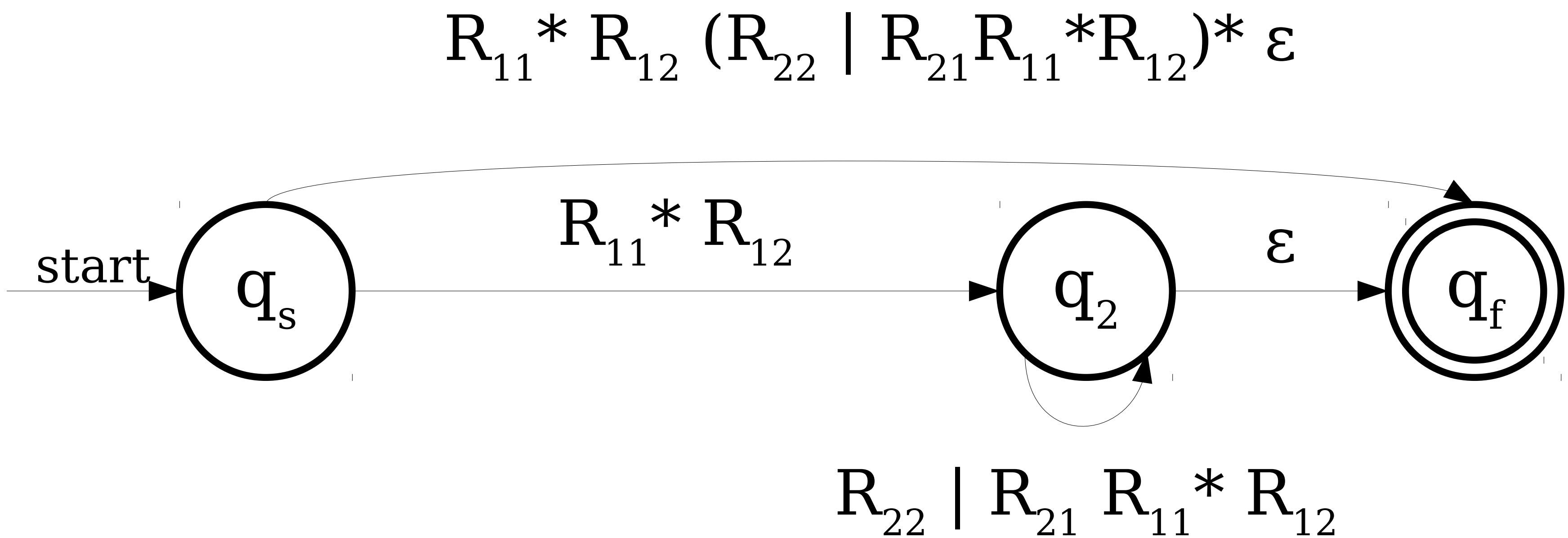

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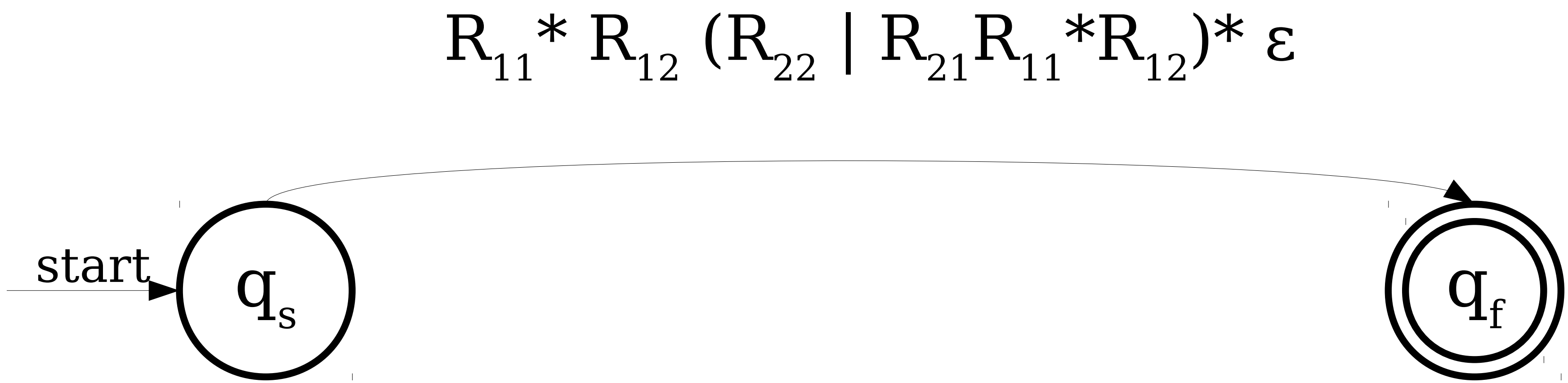

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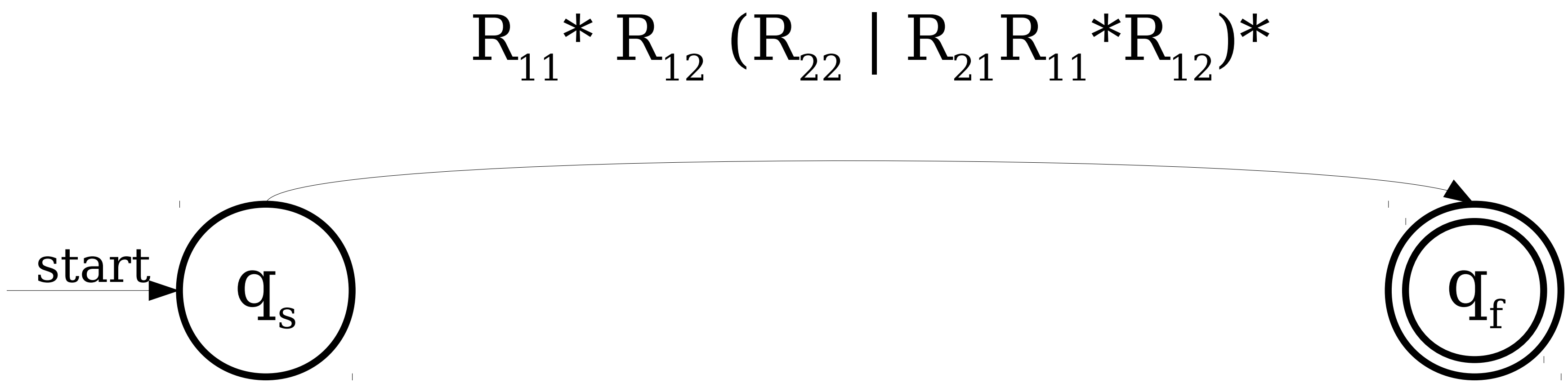

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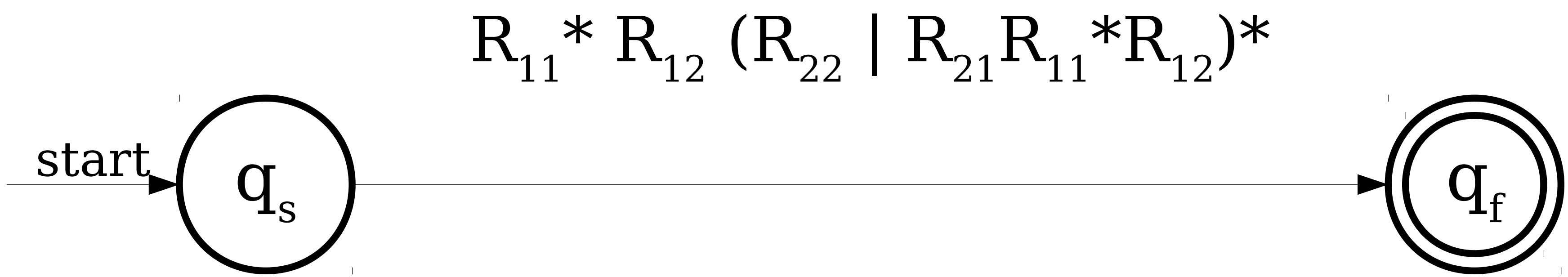

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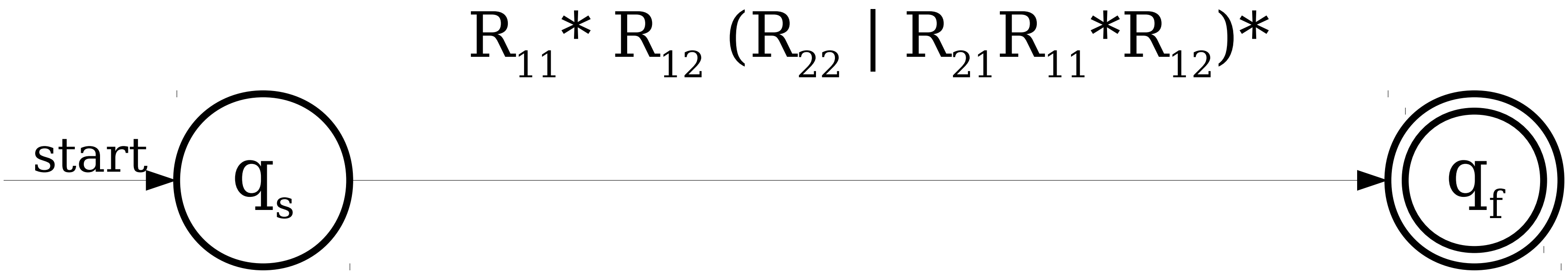
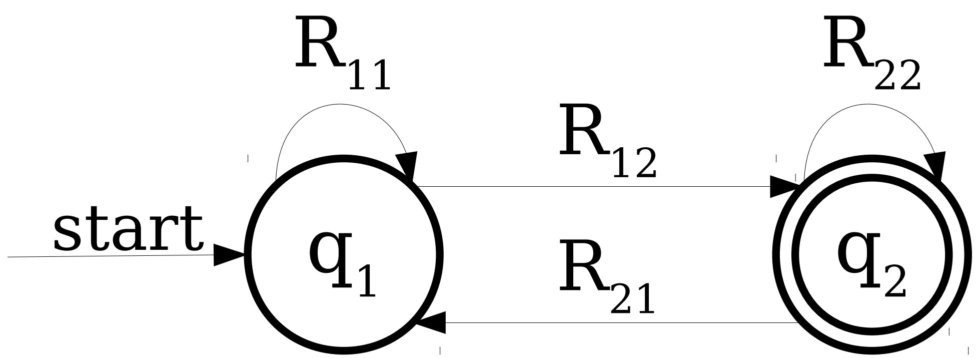
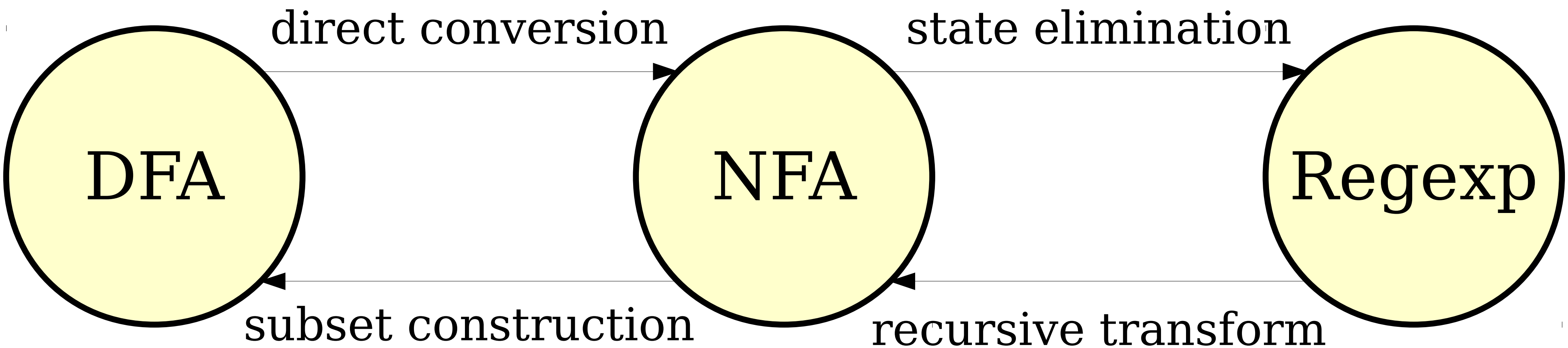




The Construction at a Glance

- Start with an NFA for the language L.
- Add a new start state q_s and accept state q_f to the NFA.
  - Add ε-transitions from each original accepting state to q_f, then mark them as not accepting.
- Repeatedly remove states other than q_s and q_f from the NFA by "shortcutting" them until only two states remain: q_s and q_f.
- The transition from q_s to q_f is then a regular expression for the NFA.

Our Transformations



Theorem: The following are all equivalent:

- L is a regular language.
- There is a DFA D such that $\mathscr{L}(D) = L$.
- There is an NFA N such that $\mathscr{L}(N) = L$.
- There is a regular expression R such that $\mathscr{L}(R) = L$.

Next Time

- **Applications of Regular Languages**
  - Answering “so what?”
- **Intuiting Regular Languages**
  - What makes a language regular?
- **The Myhill-Nerode Theorem**
  - The limits of regular languages.