Extra Practice Problems 3

You asked for some extra practice problems on a few specific topics, so we've put together some questions on precisely those topics. Hope this helps! We'll release solutions on Wednesday.

Propositional Logic

On Problem Set Three, we mentioned that any formula in propositional logic can be rewritten as an equivalent propositional formula that uses only the $\neg$ and $\to$ connectives. It turns out that you can also rewrite any propositional formula using the $\to$ and $\bot$ connectives. This problem explores how.

i. Find a formula that's logically equivalent to $\neg p$ that uses only the variable p and the $\to$ and $\bot$ connectives. No justification is necessary.

ii. Find a formula that's logically equivalent to $\top$ that uses only the $\to$ and $\bot$ connectives. No justification is necessary.

iii. Find a formula that's logically equivalent to $p \land q$ that uses only the variables p and q and the $\to$ and $\bot$ connectives. No justification is necessary.

Since you can express $\neg$, $\land$, and $\top$ using just $\to$ and $\bot$, every possible formula in propositional logic can be expressed using purely the $\to$ and $\bot$ connectives. Isn't that nifty?

First-Order Logic

i. Given only the predicates

   Set(S), which states that S is a set, and
   $x \in y$, which states that x is an element of y,

   write a sentence in first-order logic that says "if S and T are arbitrary sets, then $S \Delta T$ exists." (Recall that the symmetric difference $S \Delta T$ is the set of all elements that belong to exactly one of S and T).

ii. Given only the predicates

   Set(S), which states that S is a set;
   $x \in y$, which states that x is an element of y;
   Natural(x), which states that x is a natural number; and
   $x < y$, which states that x is less than y,

   Write a statement in first order logic that says "if $S \subseteq \mathbb{N}$ is finite, then $\mathbb{N} - S$ is infinite."
   (Hint: There's a reason we've given you the $<$ predicate.)

Graph Theory

Let's talk a bit more about graph colorability.

i. 2-colorability, 3-colorability, and 4-colorability commonly come up in computer science and discrete mathematics, but 1-colorability is rarely talked about. Why do you think that is?

ii. Let G be a graph that is *not* 2-colorable. Prove that G contains a cycle of odd length. *(Hint: Prove the contrapositive. What do you know about graphs with no odd-length cycles?)*

Tournaments

Problem Set Two and Problem Set Three explored ***tournaments***, graphs in which every pair of distinct nodes has exactly one edge between them and there are no self-loops.

A ***pseudotournament*** is a graph with $n \geq 2$ formed by starting with a tournament and deleting exactly one edge. Equivalently, a pseudotournament is a graph where every pair of distinct nodes except for some specific pair $\{u, v\}$ has exactly one edge between them and there are no self-loops.

Prove that for any $n \geq 2$, there is a pseudotournament with no tournament winners. This shows that the result you proved about tournament winners in Problem Set Two is specific to tournaments and doesn't work in general directed graphs.

Induction

Recall that the Fibonacci numbers are defined with the following recurrence relation:

$$F_0 = 0 \qquad F_1 = 1 \qquad F_{n+2} = F_n + F_{n+1}$$

The first few Fibonacci numbers are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

On Problem Set Two, you found a formula for the sum $F_0^2 + F_1^2 + \ldots + F_n^2$. Find a formula for the summation $F_0 + F_1 + \ldots + F_n$, then prove it correct using induction.

Indirect Proofs

A ***quadratic equation*** is an equation of the form $ax^2 + bx + c = 0$. A ***root*** of a quadratic equation is a real number x for which the equation is true.

Prove that if a, b, and c are all odd numbers, then $ax^2 + bx + c = 0$ has no rational roots. As a hint, don't use the quadratic formula. Instead, proceed by contradiction and think about what happens if $x = p/q$ is a root of the equation.