Indirect Proofs

Outline for Today

- ***Today will be pretty packed!***
- **Preliminaries**
  - Disproving statements
  - Mathematical implications
- **Proof by Contrapositive**
  - The basic method.
  - An interesting application.
- **Proof by Contradiction**
  - The basic method.
  - Contradictions and implication.

Disproving Statements

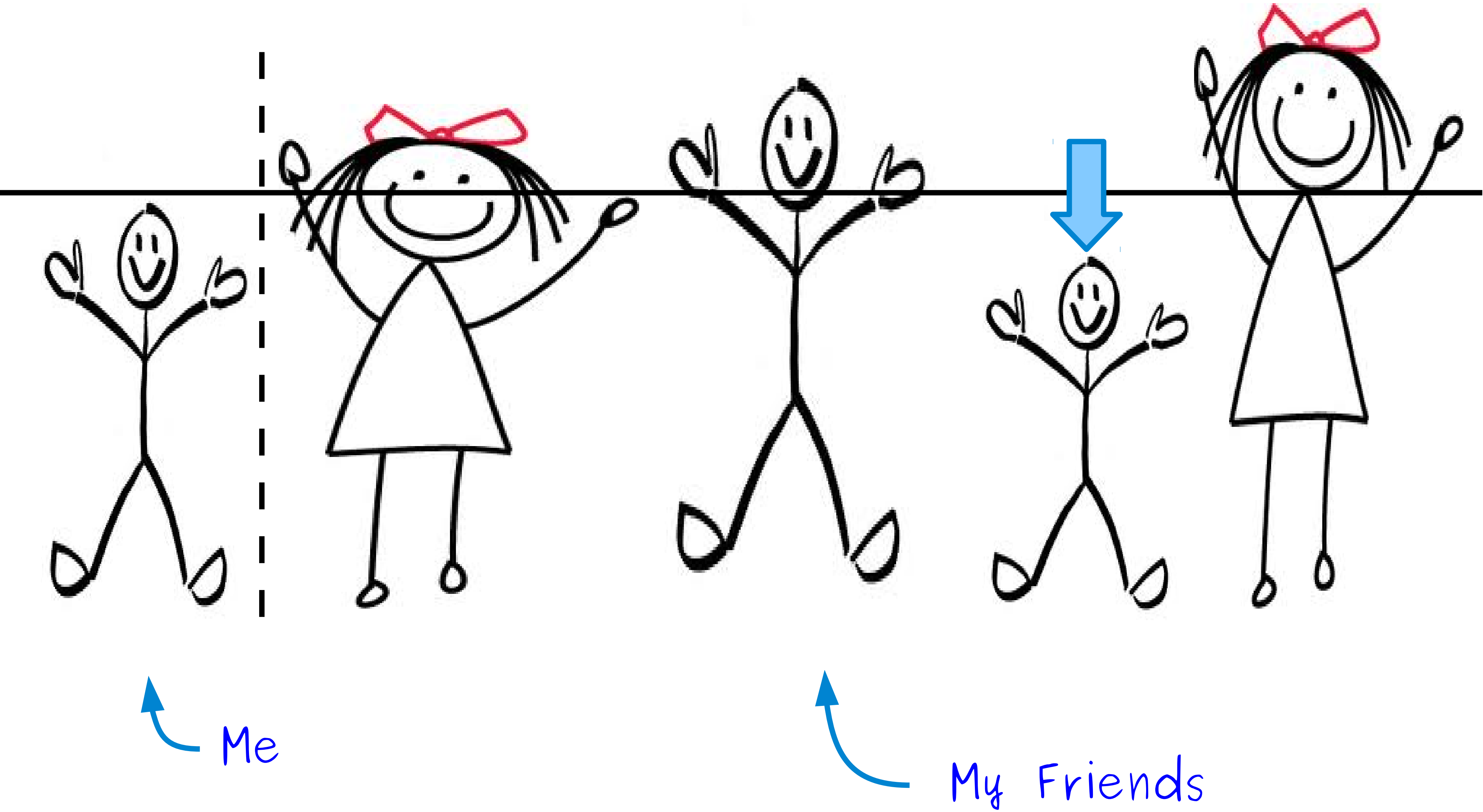
Proofs and Disproofs

- A ***proof*** is an argument establishing why a statement is true.
- A ***disproof*** is an argument establishing why a statement is *false*.
- Although proofs generally are more famous than disproofs, many important results in mathematics have been disproofs.
  - We'll see some later this quarter!

Writing a Disproof

- The easiest way to disprove a statement is to write a proof of the opposite of that statement.
  - The opposite of a statement X is called the ***negation*** of statement X.
- A typical disproof is structured as follows:
  - Start by stating that you're going to disprove some statement X.
  - Write out the negation of statement X.
  - Write a normal proof that statement X is false.

"All My Friends Are Taller Than Me"



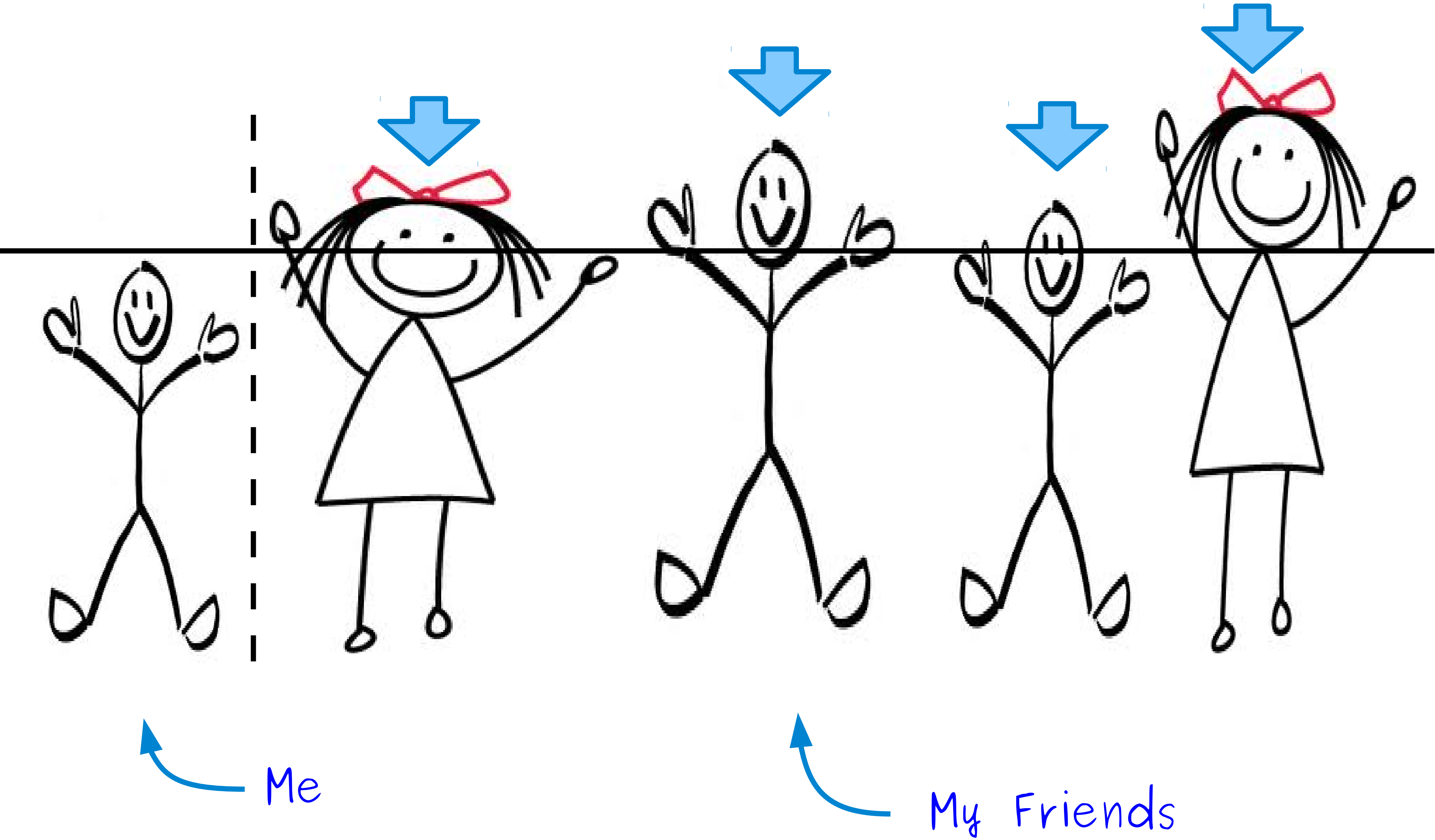
The negation of the *universal* statement

For all x, $P(x)$ is true.

is the *existential* statement

There exists an x where $P(x)$ is false.

“Some Friend Is Shorter Than Me”



The negation of the *existential* statement

There exists an x where $P(x)$ is true.

is the *universal* statement

For all x, $P(x)$ is false.

What would we have to show to disprove the following statement?

"Some set is the same size as its power set."

First, is this an existential statement or a universal statement?

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"There is a set S where S is the same size as its power set."

What happens when you negate an existential statement?

"There is a set *S* where *S* is the same size as its power set."

What happens when you negate an existential statement?

"For any set S, the set S is ***<u>not</u>*** the same size as its power set."

This is what we need to prove
to disprove the original statement.

"For any set *S*, the set *S* is ***not*** the same size as its power set."

Logical Implication

Implications

- An ***implication*** is a statement of the form

  If *P* is true, then *Q* is true.

- Some examples:
  - If n is an even integer, then n^2 is an even integer.
  - If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
  - If you liked it, then you should've put a ring on it.

Implications

- An ***implication*** is a statement of the form

If *P* is true, then *Q* is true.

- In the above statement, the term "*P* is true" is called the ***antecedent*** and the term "*Q* is true" is called the ***consequent***.

What Implications Mean

- Consider the simple statement

  If I put fire near cotton, it will burn.

- Some questions to consider:
  - Does this apply to all fire and all cotton, or just some types of fire and some types of cotton? *(Scope)*
  - Does the fire cause the cotton to burn, or does the cotton burn for another reason? *(Causality*)
- These are deeper questions than they might seem.
- To mathematically study implications, we need to formalize what implications really mean.

What Implications Mean

- In mathematics, the statement

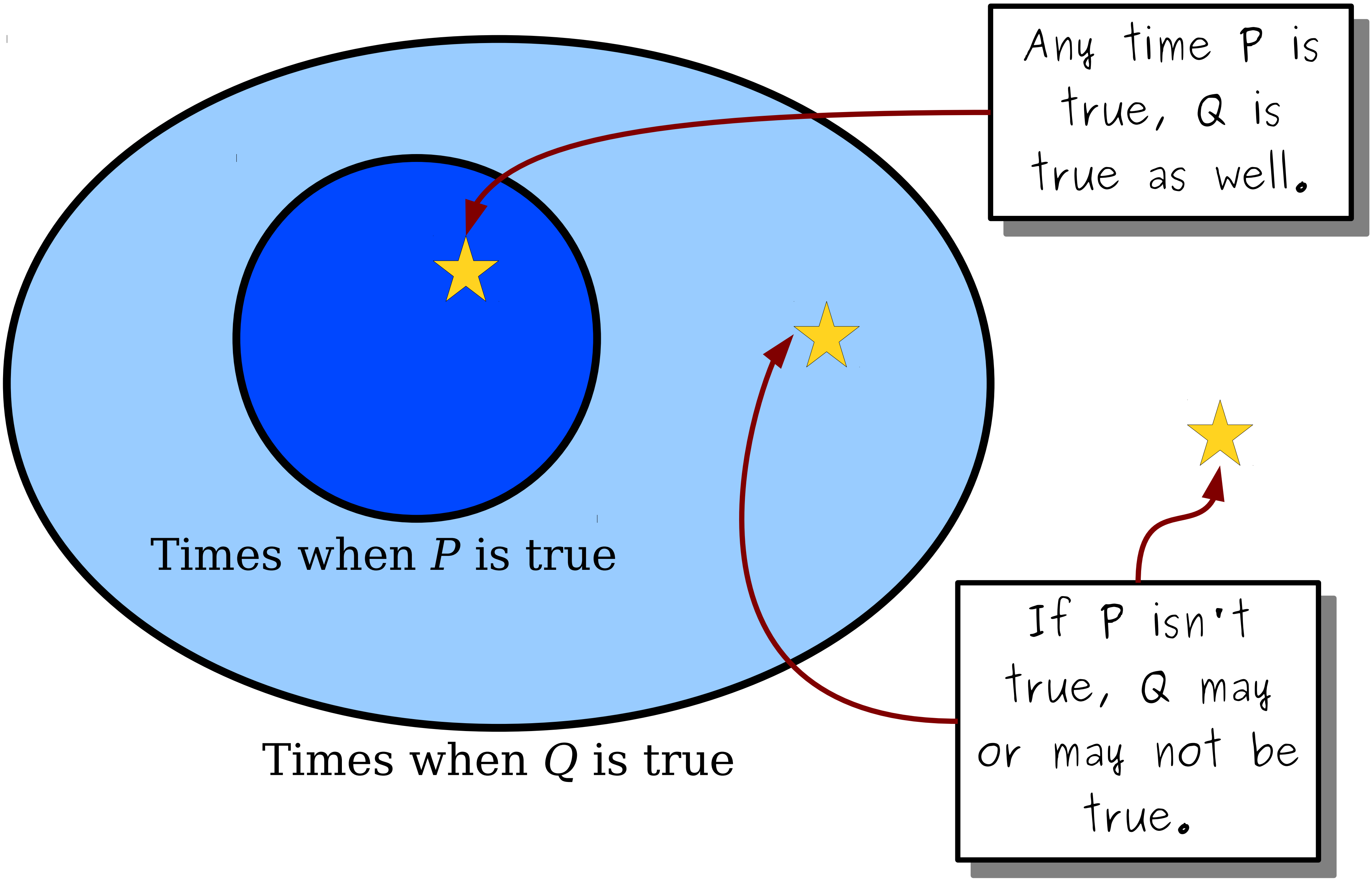
  If *P* is true, then *Q* is true.

  means *exactly* the following:

  Any time *P* is true, we are guaranteed that *Q* must also be true.

- There is no discussion of correlation or causation here. It simply means that if you find that *P* is true, you'll find that *Q* is true.

Implication, Diagrammatically



What Implication Doesn't Mean

- Implication is directional.
  - "If you die in Canada, you die in real life" doesn't mean that if you die in real life, you die in Canada.
- Implication only cares about cases where the antecedent is true.
  - "If an animal is a puppy, you should hug it" doesn't mean that if the animal is *not* a puppy, you *shouldn't* hug it.
- Implication says nothing about causality.
  - "If I like math, then 2 + 2 = 4" is true because any time I like math, we'll find that 2 + 2 = 4.
  - "If I hate math, then 2 + 2 = 4" is also true because any time I hate math, we'll find that 2 + 2 = 4.

Puppies Are Adorable

- Consider the statement

If *x* is a puppy, then I love *x*.

- Can you explain why the following statement is *not* the negation of the original statement?

If *x* is a puppy, then I don't love *x*.

- This second statement is too strong.
- Here's the correct negation:

There is some puppy *x* that I don't love.

The negation of the statement

“If *P* is true, then *Q* is true”

is the statement

“There is at least one case where *P* is true and *Q* is false.”

Proof by Contrapositive

The Contrapositive

- The ***contrapositive*** of the implication "If P, then Q" is the implication "If ***not*** Q, then ***not*** P."
- For example:
  - "If Harry had opened the right book, then Harry would have learned about Gillyweed."
  - Contrapositive: "If Harry didn't learn about Gillyweed, then Harry didn't open the right book."
- Another example:
  - "If I store the cat food inside, then wild raccoons will not steal my cat food."
  - Contrapositive: "If wild raccoons stole my cat food, then I didn't store it inside."

To prove the statement

If *P* is true, then *Q* is true

You may instead prove the statement

If *Q* is false, then *P* is false.

This is called a ***proof by contrapositive***.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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Proof: By contrapositive;

> We're starting this proof by telling the reader that it's a proof by contrapositive. This helps cue the reader into what's about to come next.

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Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove. It also acts as a sanity check by forcing us to write out what we think the contrapositive is.

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We've said that we're going to prove this new implication, so let's go do it! The rest of this proof will look a lot like a standard direct proof.

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Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Since n is odd, there is some integer k such that $n = 2k + 1$.

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Since n is odd, there is some integer k such that $n = 2k + 1$. Squaring both sides of this equality and simplifying gives the following:

$$n^2 = (2k + 1)^2$$

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Since n is odd, there is some integer k such that $n = 2k + 1$. Squaring both sides of this equality and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \end{aligned}$$

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Since n is odd, there is some integer k such that $n = 2$
and s

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The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.

2. Explicitly state the contrapositive of what we want to prove.

3. Go prove the contrapositive.

Biconditionals

- Combined with what we saw on Wednesday, we have proven that, if n is an integer:

If n is even, then n^2 is even.

If n^2 is even, then n is even.

- Therefore, if n is an integer:

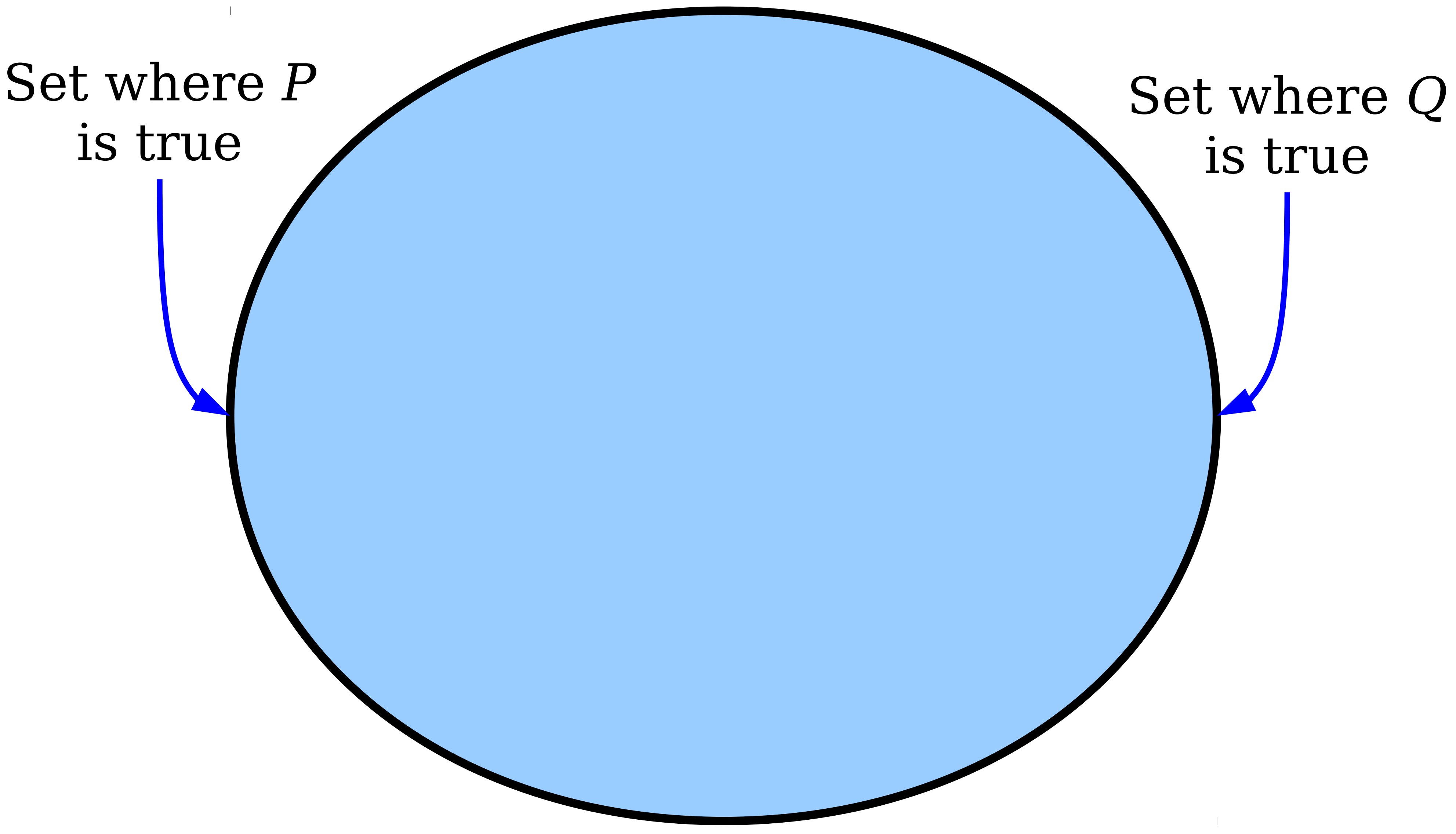
n is even if and only if n^2 is even.

- "If and only if" is often abbreviated ***iff***:

n is even iff n^2 is even.

- This is called a ***biconditional***.

P iff Q



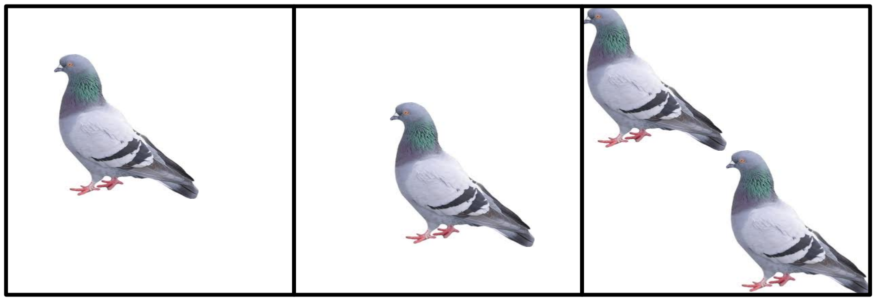
Proving Biconditionals

- To prove ***P*** **iff** ***Q***, you need to prove that P implies Q and that Q implies P.
- You can use any proof techniques you'd like to show each of these statements.
  - In our case, we used a direct proof and a proof by contrapositive.

The Pigeonhole Principle

The Pigeonhole Principle

- Suppose that you have *n* pigeonholes.
- Suppose that you have $m > n$ pigeons.
- If you put the pigeons into the pigeonholes, some pigeonhole will have more than one pigeon in it.



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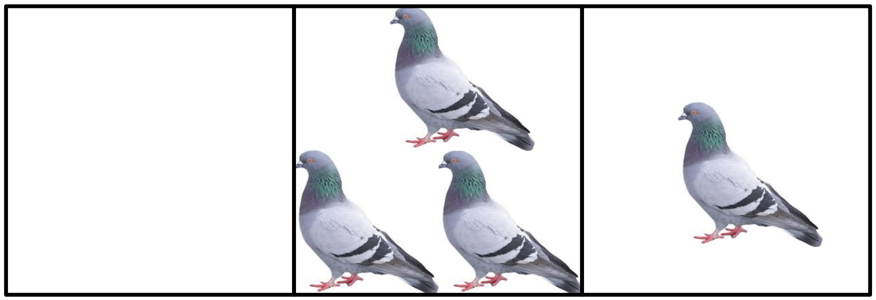

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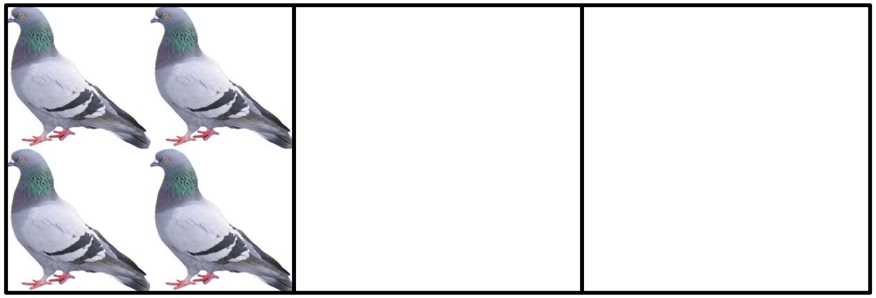

The Pigeonhole Principle

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- We want to prove the statement

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If "some bin contains at least two objects" is false, then "$m > n$" is false.

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Is this a universal statement or an existential statement?

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The Pigeonhole Principle

- Suppose that *m* objects are distributed into *n* bins.
- We want to prove the statement

  If $m > n$, then some bin contains at least two objects.

- What is the contrapositive of this statement?

  If every bin contains at most one object, then $m \leq n$.

- Look at the definitions of *m* and *n*. Does this make sense?

Theorem: Let m objects be distributed into n bins. If $m > n$, then some bin contains at least two objects.

Proof: By contrapositive; we prove that if every bin contains at most one object, then $m \leq n$.

Let x_i denote the number of objects in bin i. Since m is the number of total objects, we see that

$$m = x_1 + x_2 + \ldots + x_n.$$

We're assuming every bin has at most one object. In our notation, this means that $x_i \leq 1$ for all i. Using this inequality, we get the following:

$$\begin{aligned} m &= x_1 + x_2 + \ldots + x_n \\ &\leq 1 + 1 + \ldots + 1 \qquad (n \textit{ times}) \\ &= n. \end{aligned}$$

So $m \leq n$, as required. ■

Some Simple Applications

- Any group of 367 people must have a pair of people that share a birthday.
  - 366 possible birthdays (pigeonholes)
  - 367 people (pigeons)
- Two people in San Francisco have the exact same number of hairs on their head.
  - Maximum number of hairs ever found on a human head is no greater than 500,000.
  - There are over 800,000 people in San Francisco.
- Each day, two people in New York City drink the same amount of water, to the thousandth of a fluid ounce.
  - No one can drink more than 50 gallons of water each day.
  - That's 6,400 fluid ounces. This gives 6,400,000 possible numbers of thousands of fluid ounces.
  - There are about 8,000,000 people in New York City proper.

Time-Out for Announcements!

Announcements

- Problem Set 1 out.
- **Checkpoint** due Monday, April 6.
  - Grade determined by attempt rather than accuracy. It's okay to make mistakes – we want you to give it your best effort, even if you're not completely sure what you have is correct.
  - We will get feedback back to you with comments on your proof technique and style.
  - The more effort you put in, the more you'll get out.
- **Remaining problems** due Friday, April 10.
  - Feel free to email us with questions, stop by office hours, or ask questions on Piazza!

Submitting Assignments

- As a pilot for this quarter, we'll be using ***Scoryst*** for assignment submissions.
  - All submissions should be electronic. If you're having trouble submitting, please contact the course staff.
- Signup link available at the course website.
- Late policy:
  - Everyone has *three* 24-hour late days.
  - You may use at most two per assignment (and can't use late days on checkpoints.)
  - Nothing may be submitted more than one class period past the due date.
- ***We are strict about deadlines***. Anything submitted past the deadline will consume a late day, even if it's only a fraction of a second late.

Working in Groups

- You can work on the problem sets individually, in a pair, or in a group of three.
- Each group should only submit a single problem set, and should submit it only once.
- Full details about the problem sets, collaboration policy, and Honor Code can be found in Handouts 04 and 05.

Office hours start Monday.

Schedule will be available
on the course website.

Back to CS103!

Proof by Contradiction

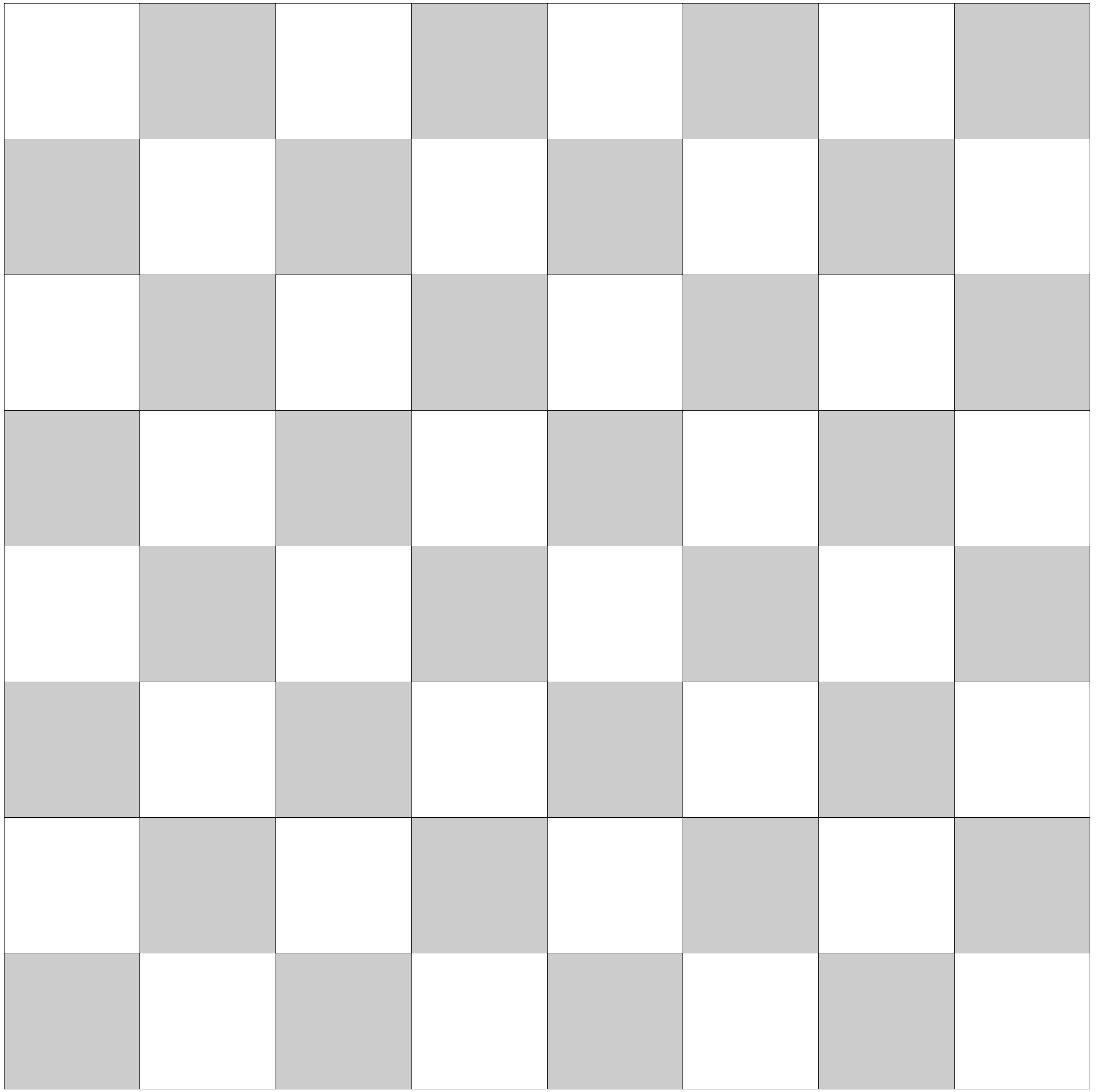
“When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth.”

- Sir Arthur Conan Doyle, *The Adventure of the Blanched Soldier*

Proof by Contradiction

- A ***proof by contradiction*** is a proof that works as follows:
  - To prove that P is true, assume that P is *not* true.
  - Based on the assumption that P is not true, conclude something impossible.
  - Assuming the logic is sound, the only valid explanation is that the original assumption must have been wrong.
  - Therefore, P can't be false, so it must be true.

Tiling a Checkerboard



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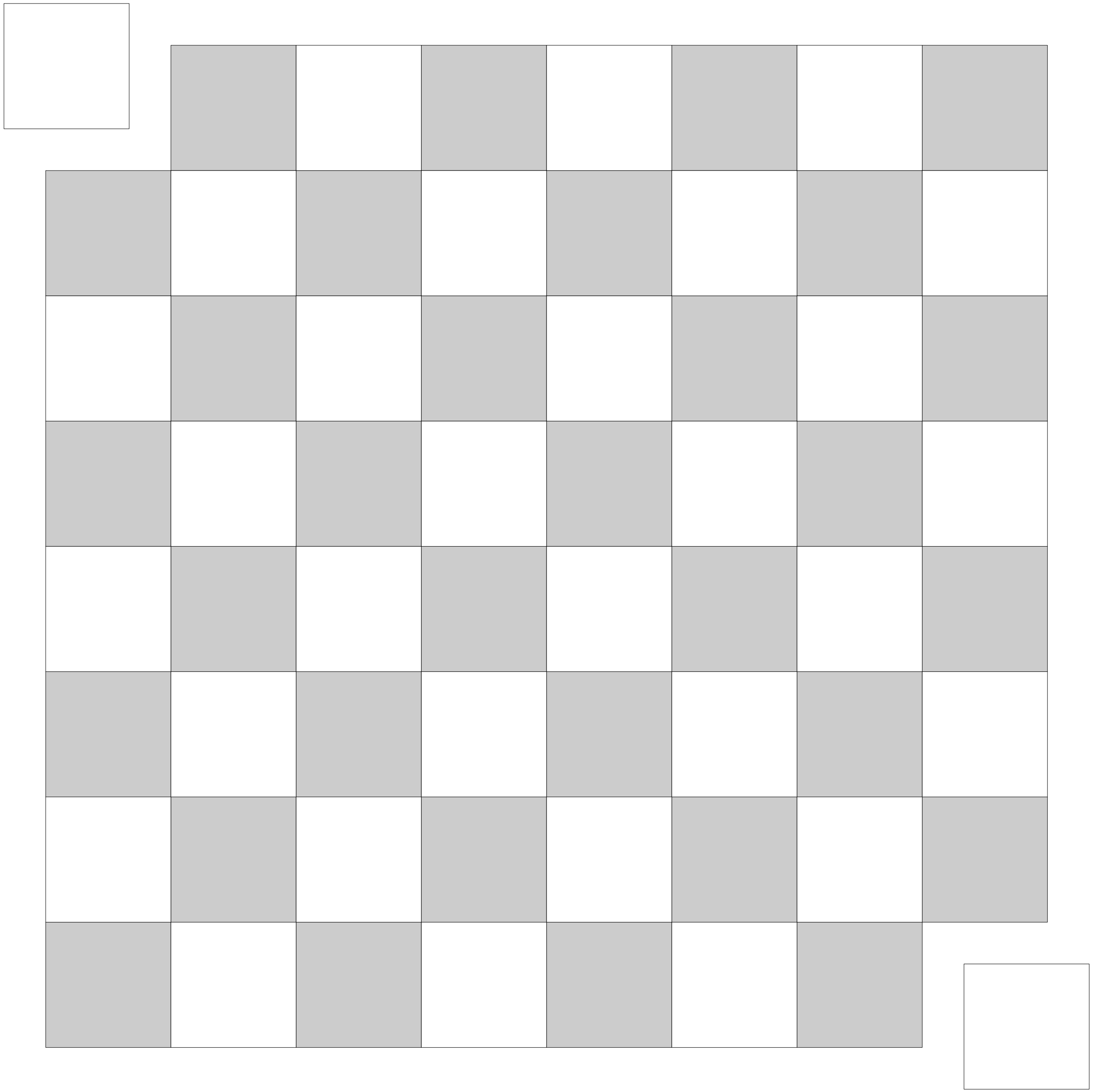

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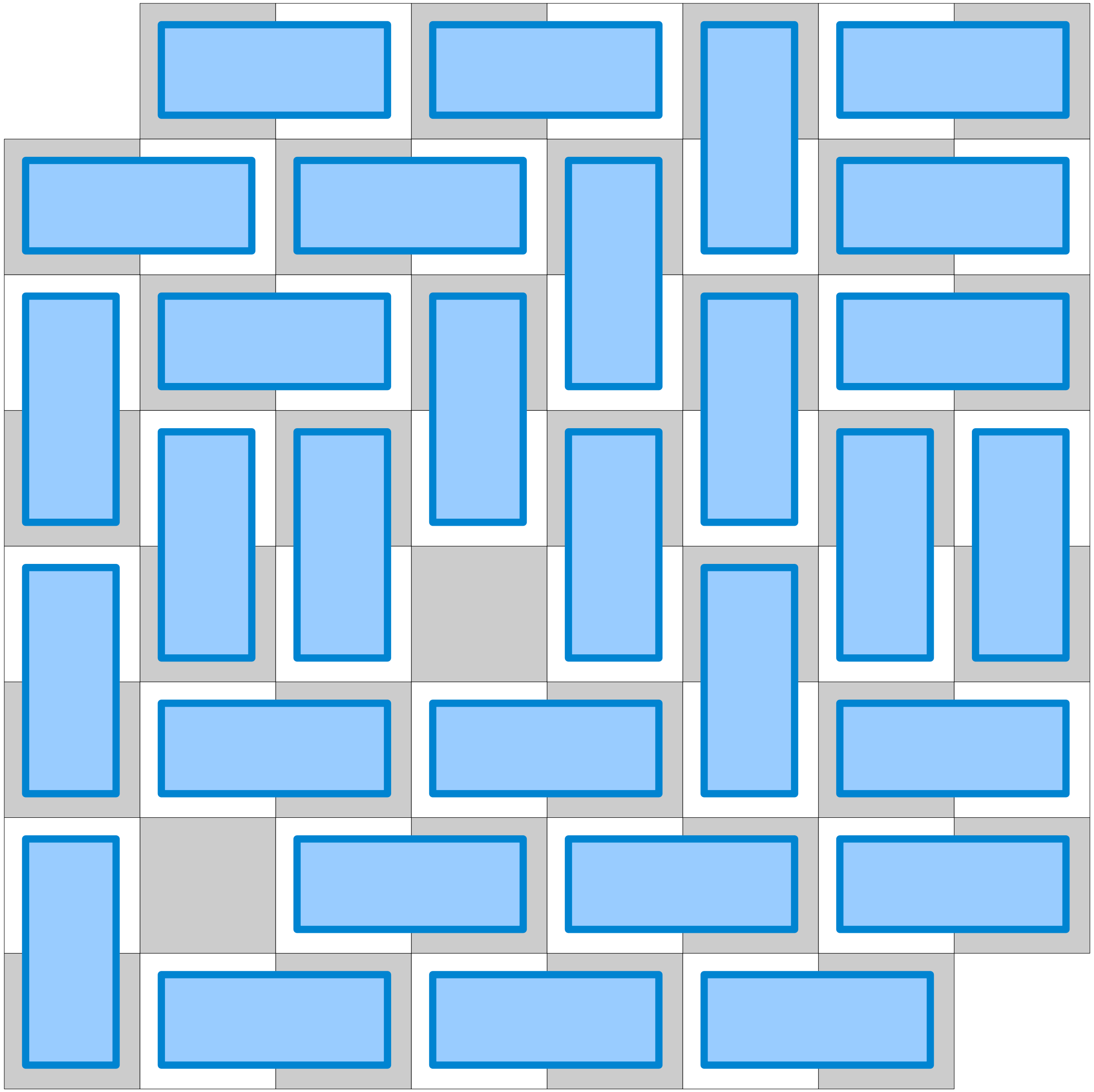

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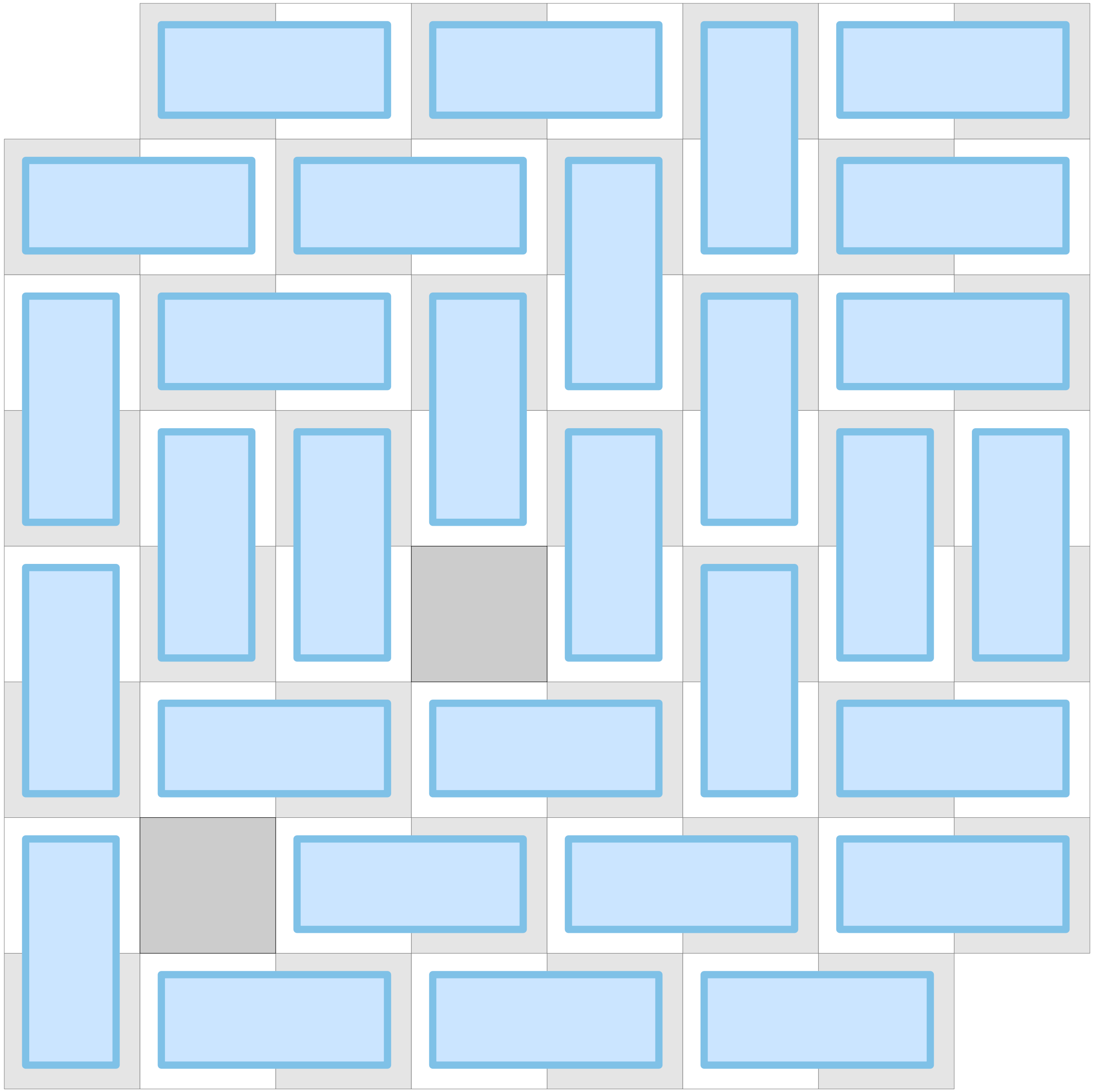

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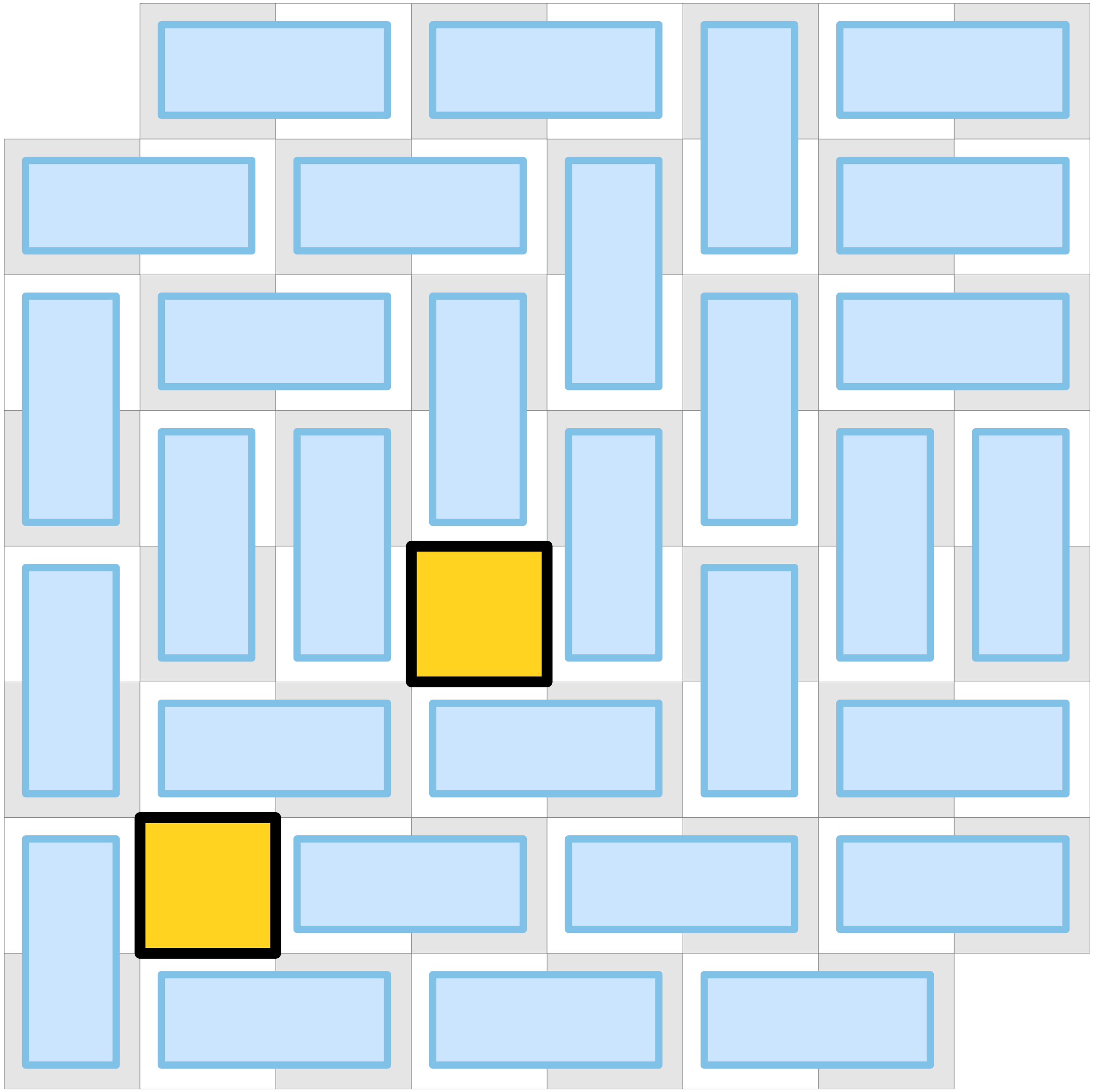

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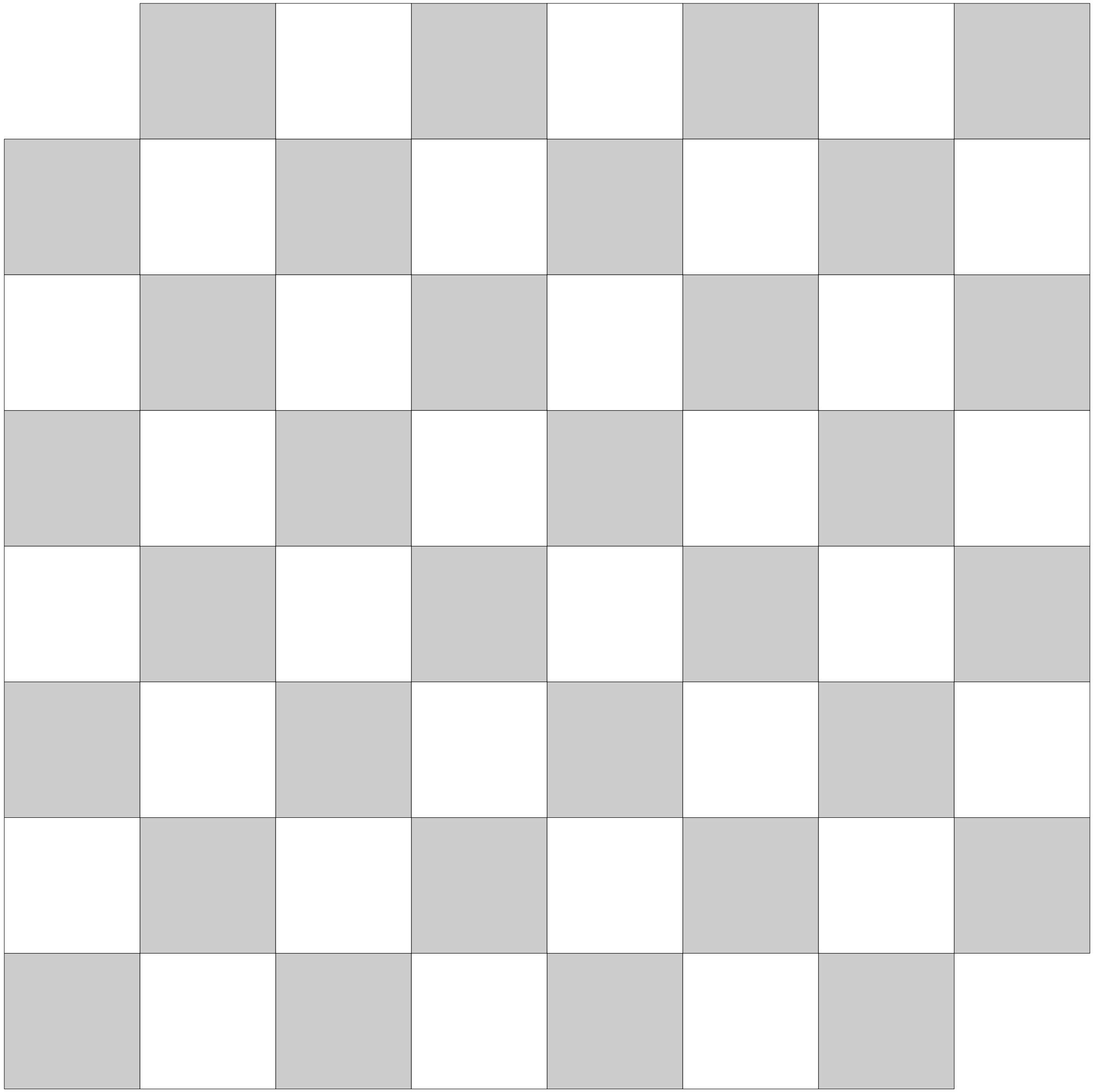

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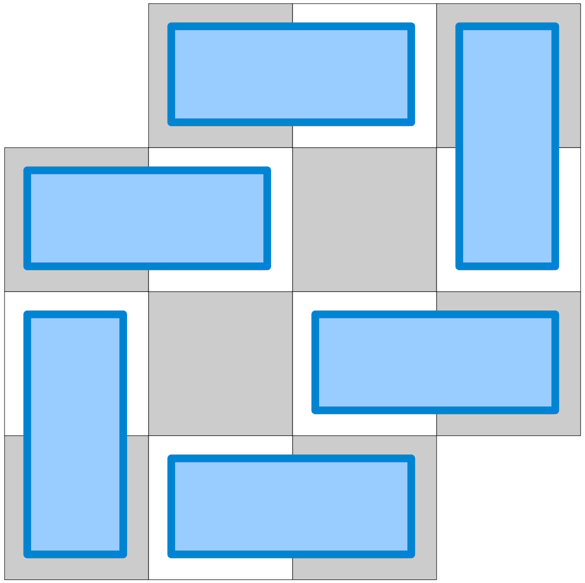

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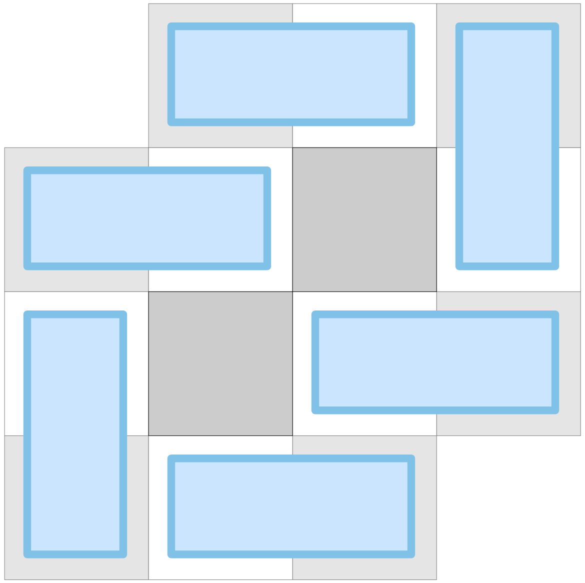

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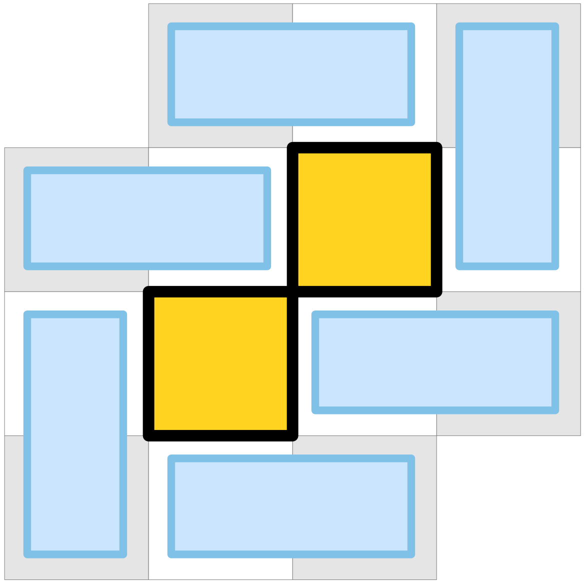
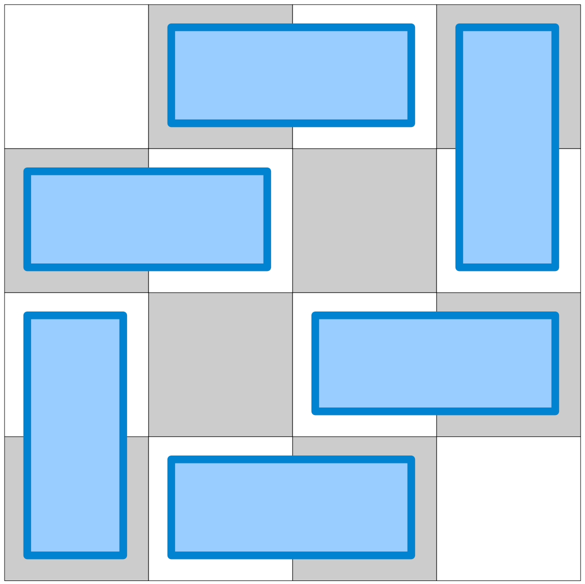


An Explanation



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Proof: Assume for the sake of contradiction that it is possible to tile an 8×8 checkerboard missing two opposite corners with 2×1 dominoes. This means that there is a way to cover the board with dominoes such that no two dominoes overlap and every domino covers exactly two squares.

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Proof: Assume for the sake of contradiction that it is possible to tile an 8×8 checkerboard missing two opposite corners with 2×1 dominoes. This means that there is a way to cover the board with dominoes such that no two dominoes overlap and every domino covers exactly two squares.

An 8×8 checkerboard has 64 squares, of which 32 are white and 32 are black.

Theorem: It is impossible to tile an 8×8 checkerboard missing two opposite corners with 2×1 dominoes.

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Rational and Irrational Numbers

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- A number r is called a ***rational number*** if it can be written as

$$r = \frac{p}{q}$$

  where p and q are integers and $q \neq 0$.

- A number that is not rational is called ***irrational***.
- Useful theorem: If r is rational, r can be written as p / q where $q \neq 0$ and where p and q have no common factors other than ± 1.

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Vi Hart on Pythagoras and
the Square Root of Two:

http://www.youtube.com/watch?v=X1E7I7_r3Cw

Proving Implications

- To prove the implication

  "If *P* is true, then *Q* is true."

- you can use these three techniques:
  - **Direct Proof.**
    - Assume P and prove Q.
  - **Proof by Contrapositive**
    - Assume not Q and prove not P.
  - **Proof by Contradiction**
    - … what does this look like?

Negating Implications

- To prove the statement

  “If *P* is true, then *Q* is true”

  by contradiction, we do the following:

  - Assume this statement is false.
  - Derive a contradiction.
  - Conclude that the statement is true.
- What is the negation of this statement?

  “*P* is true and *Q* is false”

Contradictions and Implications

- To prove the statement

  “If *P* is true, then *Q* is true”

  using a proof by contradiction, do the following:

  - Assume that *P* is true and that *Q* is false.
  - Derive a contradiction.
  - Conclude that if *P* is true, *Q* must be as well.

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We have reached a contradiction, so our assumption must have been incorrect. Thus if n is an integer and n^2 is even, n is even as well. ■

Skills from Today

- Disproving statements
- Negating universal and existential statements.
- Negating implications.
- Determining the contrapositive of a statement.
- Writing a proof by contrapositive.
- Writing a proof by contradiction.

Next Time

- **Proof by Induction**
  - Proofs on sums, programs, algorithms, etc.

Appendices: Helpful References

Negating Implications

“If *P*, then *Q*”

becomes

“*P* but not *Q*”

Negating Universal Statements

“For all *x*, P(*x*) is true”

becomes

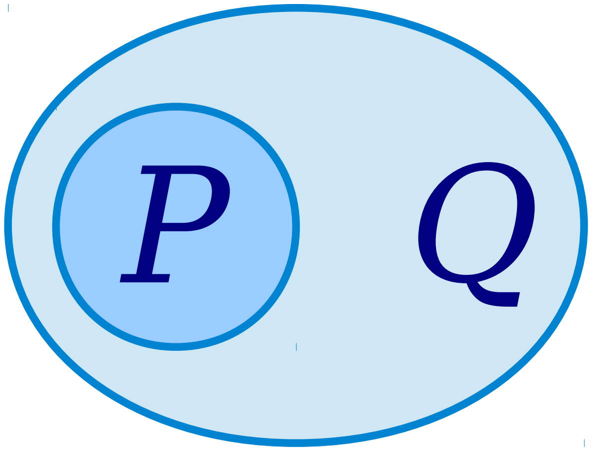
“There is an *x* where P(*x*) is false.”

Negating Existential Statements

“There exists an *x* where P(*x*) is true”

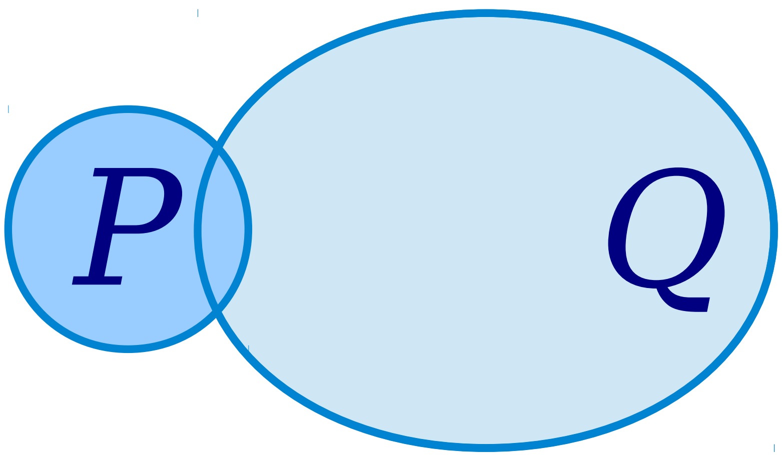
becomes

“For all *x*, P(*x*) is false.”



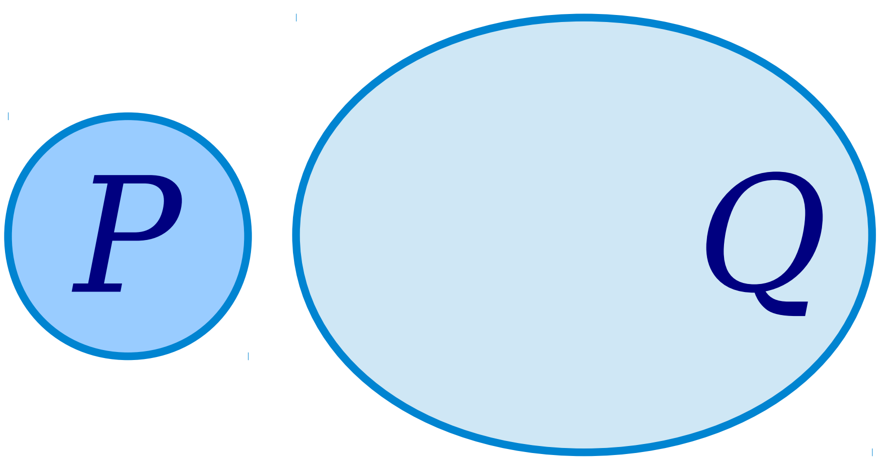
P **implies** ***Q***

“If P is true, then Q is true.”



P **does not imply** ***Q***
-and-
P **does not imply not** ***Q***

“Sometimes P is true and Q is true,
-and-
sometimes P is true and Q is false.”



P **implies not** ***Q***

If P is true, then Q is false