Functions

Outline for Today

- **Functions**
  - Modeling transformations between sets.
- **Special Classes of Functions**
  - Injections, surjections, and bijections.
- **Cardinality**
  - Revisiting our first lecture with our new techniques!

What is a function?

Functions, High-School Edition

- In high school, functions are usually given as objects of the form

$$f(x) = \frac{x^3+3x^2+15x+7}{1-x^{137}}$$

- Typically, these types of functions take in real numbers as input and produce real numbers as output.
- They might not be defined for all real numbers.

Functions, CS Edition

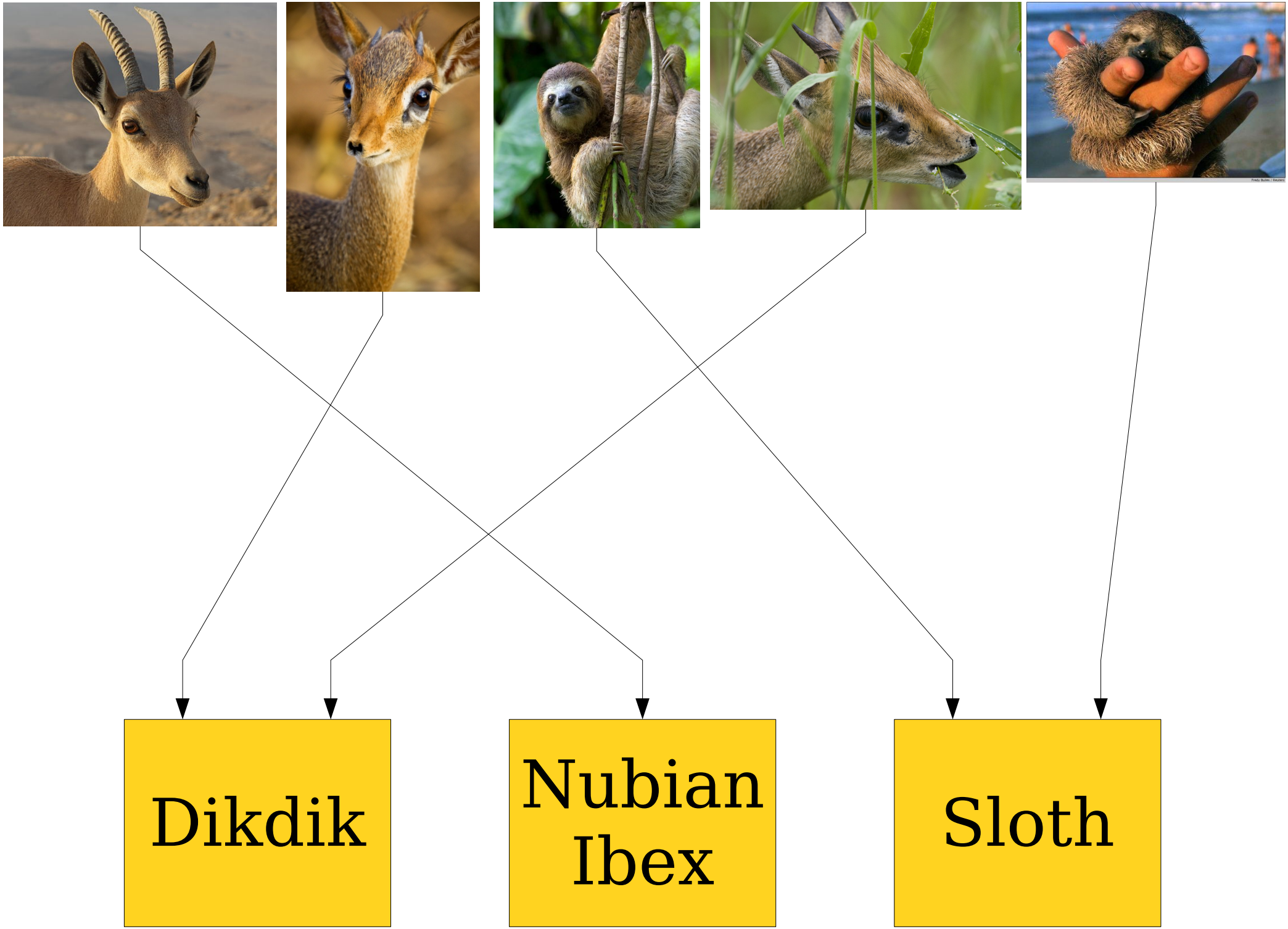
- In software, functions are usually things of the form

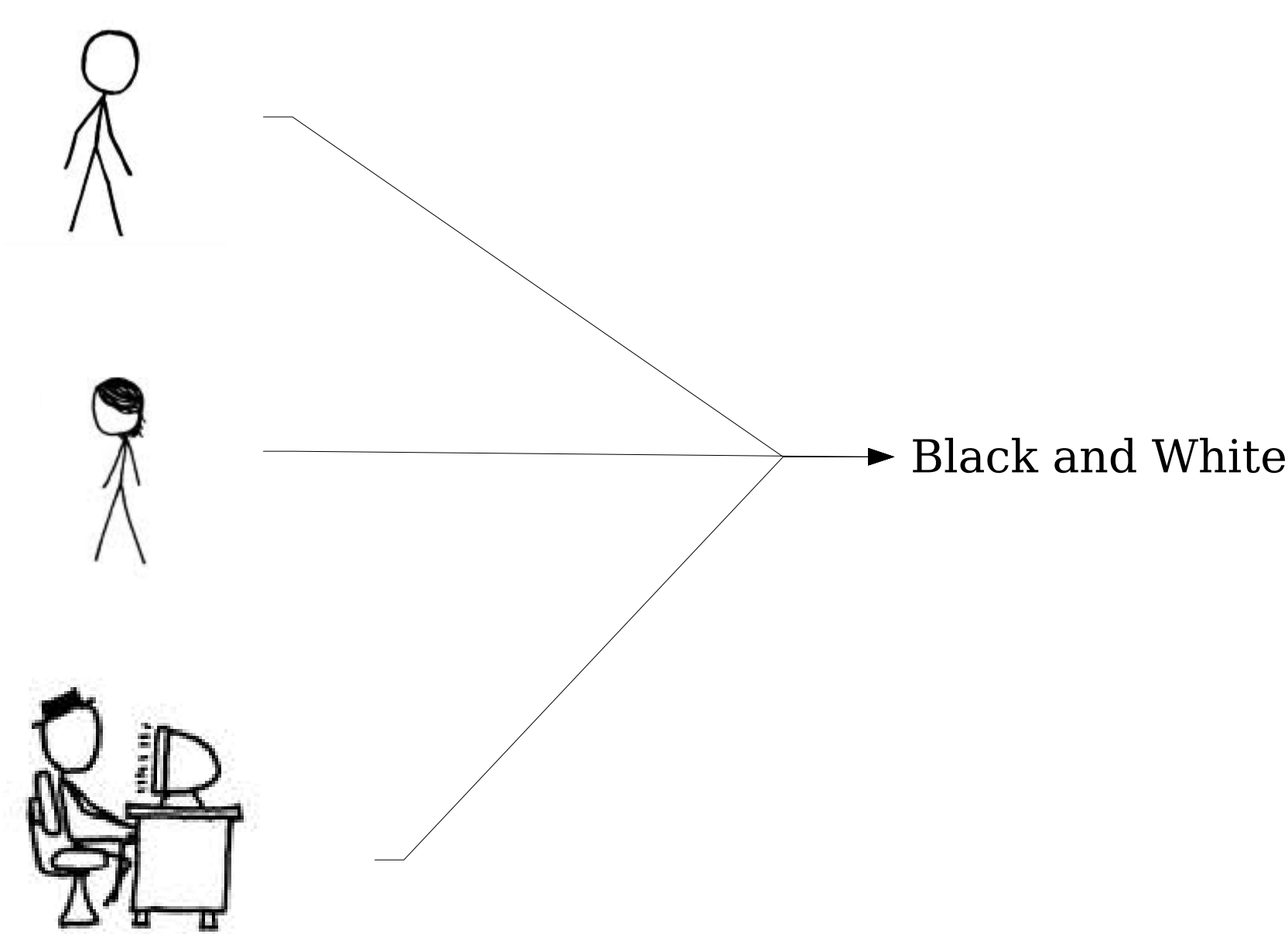
```
int flipUntil(int n) {
  int numHeads = 0;
  int numTries = 0;

  while (numHeads < n) {
    if (randomBoolean()) numHeads++;
    numTries++;
  }

  return numTries;
}
```

- They can take in any number of arguments of any type, and optionally might return something.
- They might not return the same inputs given the same outputs.

Functions, Math Edition

A ***function*** is an object that associates every object in some set A with a corresponding object in some set B.


Dikdik
Nubian Ibex
Sloth


Black and White

Terminology

- A ***function*** f is a mapping from one set A to another set B such that every element of A is associated with a single element of B.
  - For each $a \in A$, there is some $b \in B$ with $f(a) = b$.
  - Evaluating the same function on the same inputs always produces the same output: if $f(a) = b_0$ and $f(a) = b_1$, then $b_0 = b_1$.
- If f is a function from A to B, we say that f is a ***mapping*** from A to B.
  - We call A the ***domain*** of f.
  - We call B the ***codomain*** of f.

Domains and Codomains

- f f is a function whose domain is A and whose codomain is B, we write $\boldsymbol{f : A \to B}$.
- This notation just says what the domain and codomain of the function is. It doesn't say how the function is evaluated.
- Think of it like a “function prototype” in C or C++. The notation $f : A \to B$ is like writing

  B f(A `argument`);

  We know that f takes in an A and returns a B, but we don't know exactly which B it's going to return for a given A.

Domains and Codomains

- A function f must be defined for every element of the domain.
  - For example, if $f : \mathbb{R} \to \mathbb{R}$, then the following function is ***not*** a valid choice for f:

$$f(x) = 1 / x$$

- The output of f on any element of its domain must be an element of the codomain.
  - For example, if $f : \mathbb{R} \to \mathbb{N}$, then the following function is ***not*** a valid choice for f:

$$f(x) = x$$

- However, a function f does not have to produce all possible values in its codomain.
  - For example, if $f : \mathbb{N} \to \mathbb{N}$, then the following function is a valid choice for f:

$$f(n) = n^2$$

Defining Functions

- Typically, we specify a function by describing a rule that maps every element of the domain to some element of the codomain.
- Examples:
  - $f(n) = n + 1$, where $f : \mathbb{Z} \to \mathbb{Z}$
  - $f(x) = \sin x$, where $f : \mathbb{R} \to \mathbb{R}$
  - $f(x) = \lceil x \rceil$, where $f : \mathbb{R} \to \mathbb{Z}$
- Notice that we're giving both a rule and the domain/codomain.

Defining Functions

Typically, we specify a function by describing a rule that maps every element of the domain to some codomain.

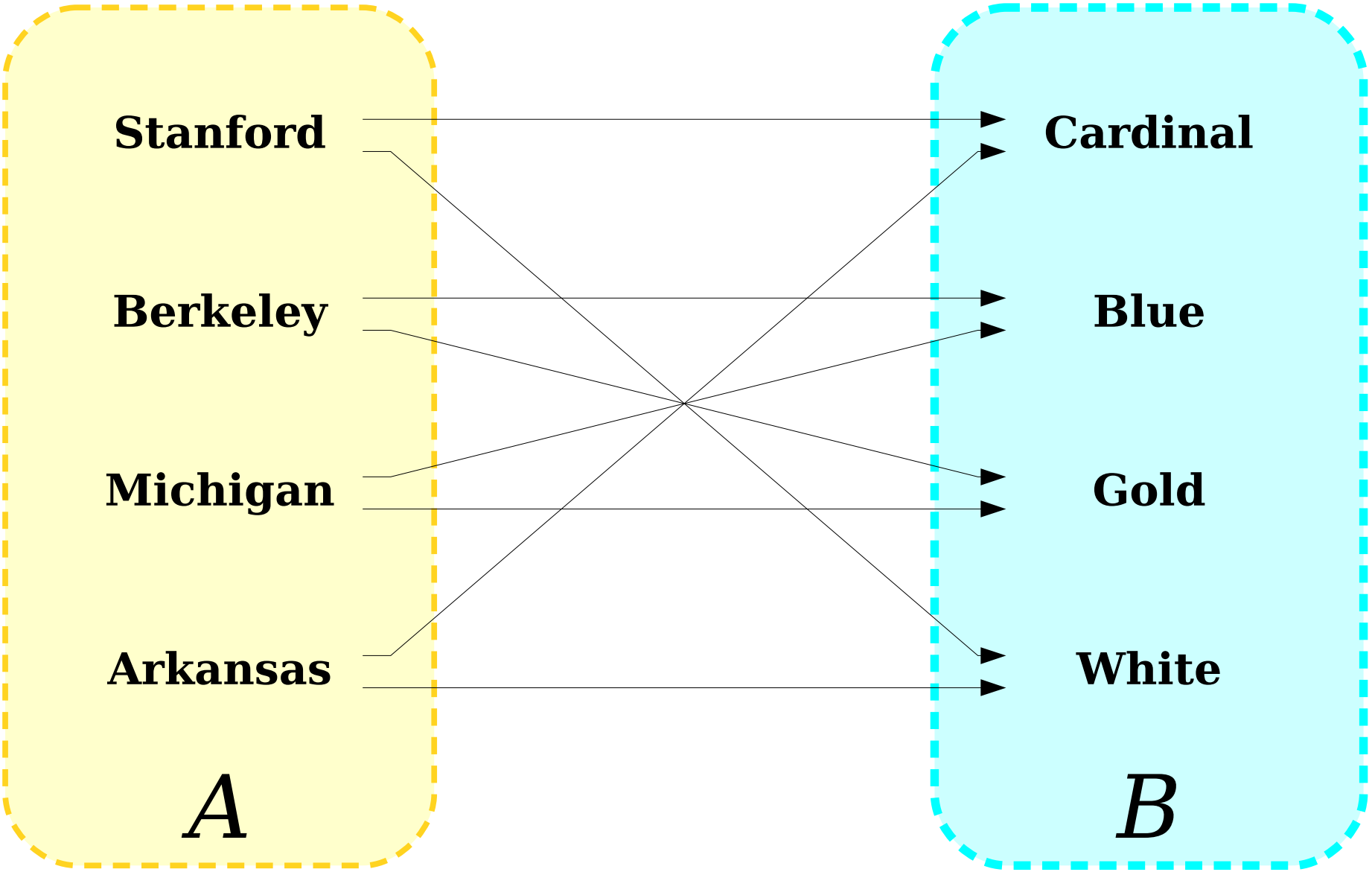
This is the ceiling function – the smallest integer greater than or equal to x. For example, $\lceil 1 \rceil = 1$, $\lceil 1.37 \rceil = 2$, and $\lceil \pi \rceil = 4$.

Examples:

- $f(n) = n + 1$, where $f : \mathbb{Z}$
- $f(x) = \sin x$, where $f : \mathbb{R} \to \mathbb{R}$
- $f(x) = \lceil x \rceil$, where $f : \mathbb{R} \to \mathbb{Z}$

Notice that we're giving both a rule and the domain/codomain.

Is this a function from A to B?



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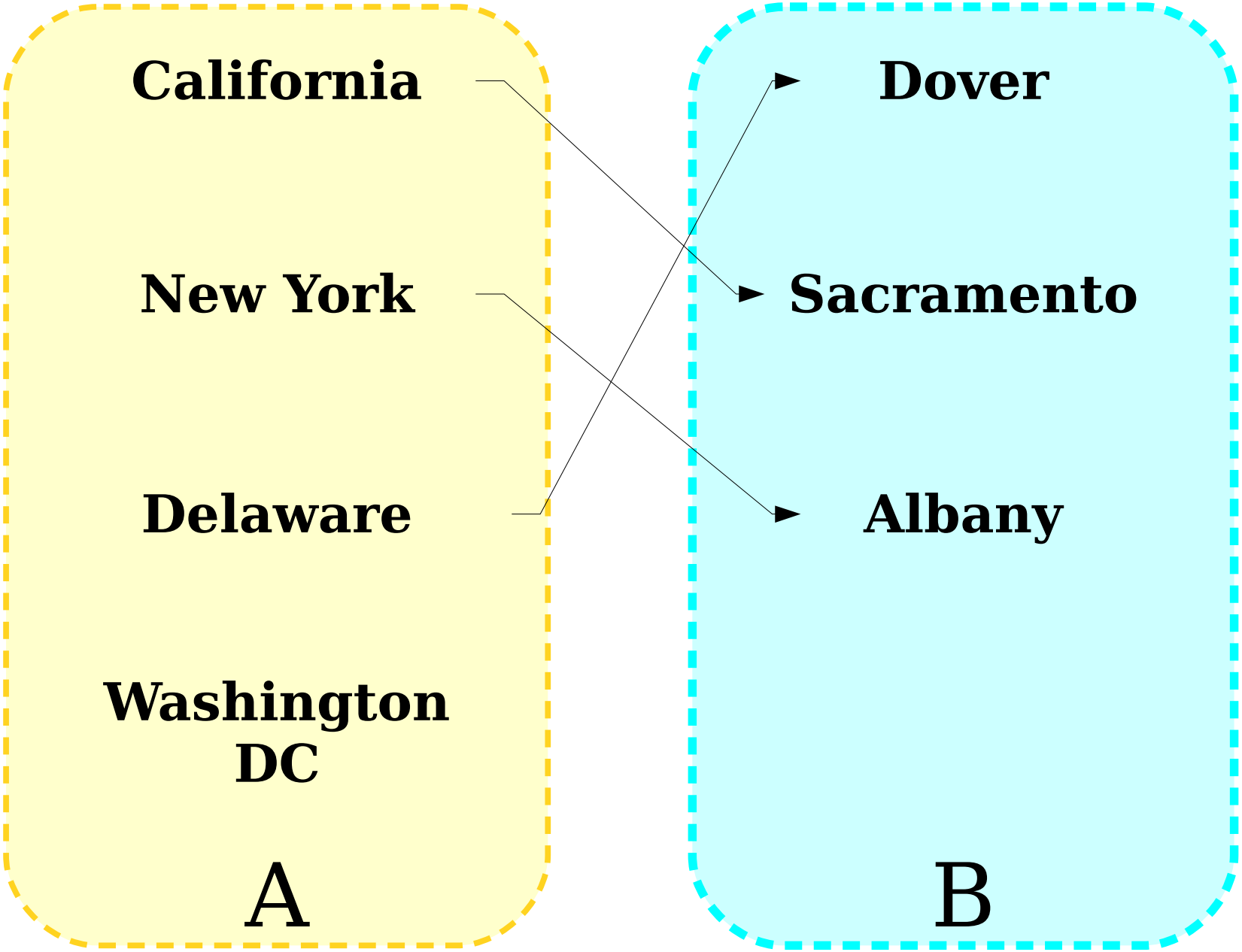

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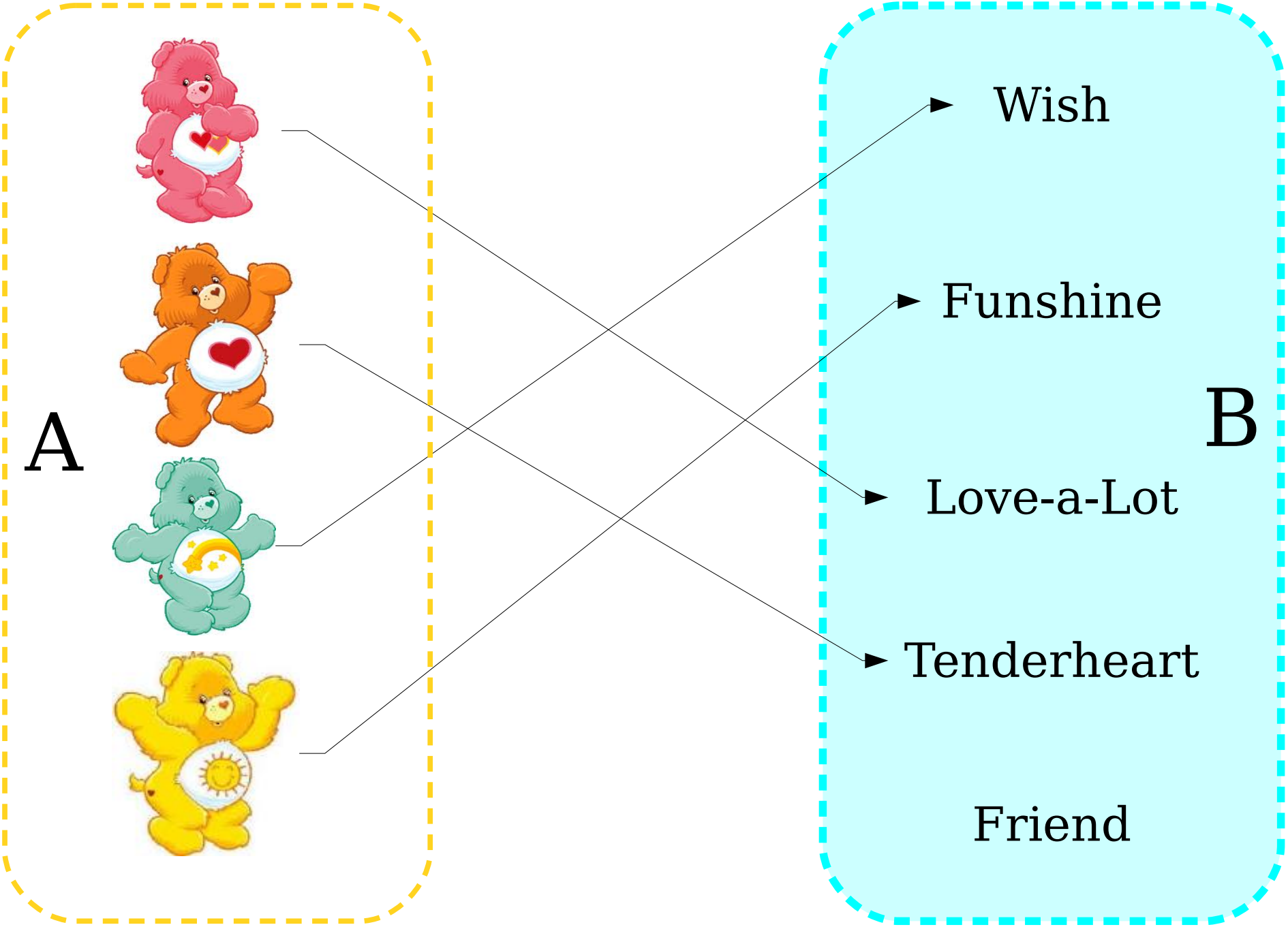

Piecewise Functions

- Functions may be specified ***piecewise***, with different rules applying to different elements.
- As an example, consider this function $f : \mathbb{N} \to \mathbb{Z}$ defined as follows:

$$f(n) = \begin{cases} -n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{otherwise} \end{cases}$$

- When defining a function piecewise, it's up to you to confirm that it defines a legal function!

☿	Mercury
♀	Venus
♁	Earth
♂	Mars
♃	Jupiter
♄	Saturn
♅	Uranus
♆	Neptune
♇	Pluto

☿	Mercury
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♆	Neptune
	~~*Pluto*~~

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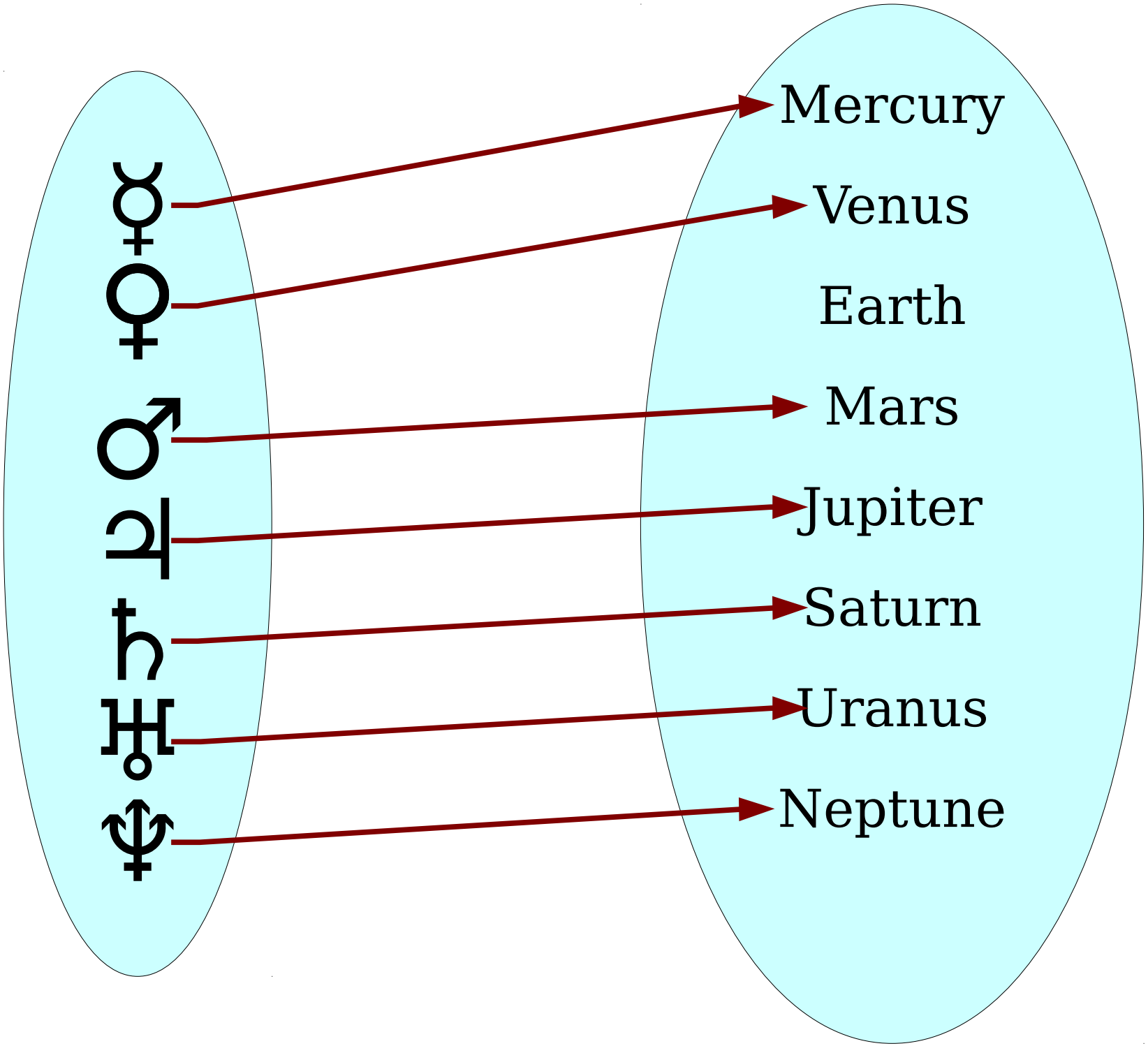
☿
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Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) if each element of the codomain has at most one element of the domain that maps to it.
  - A function with this property is called an ***injection***.
- Formally, $f : A \to B$ is an injection if this statement is true:

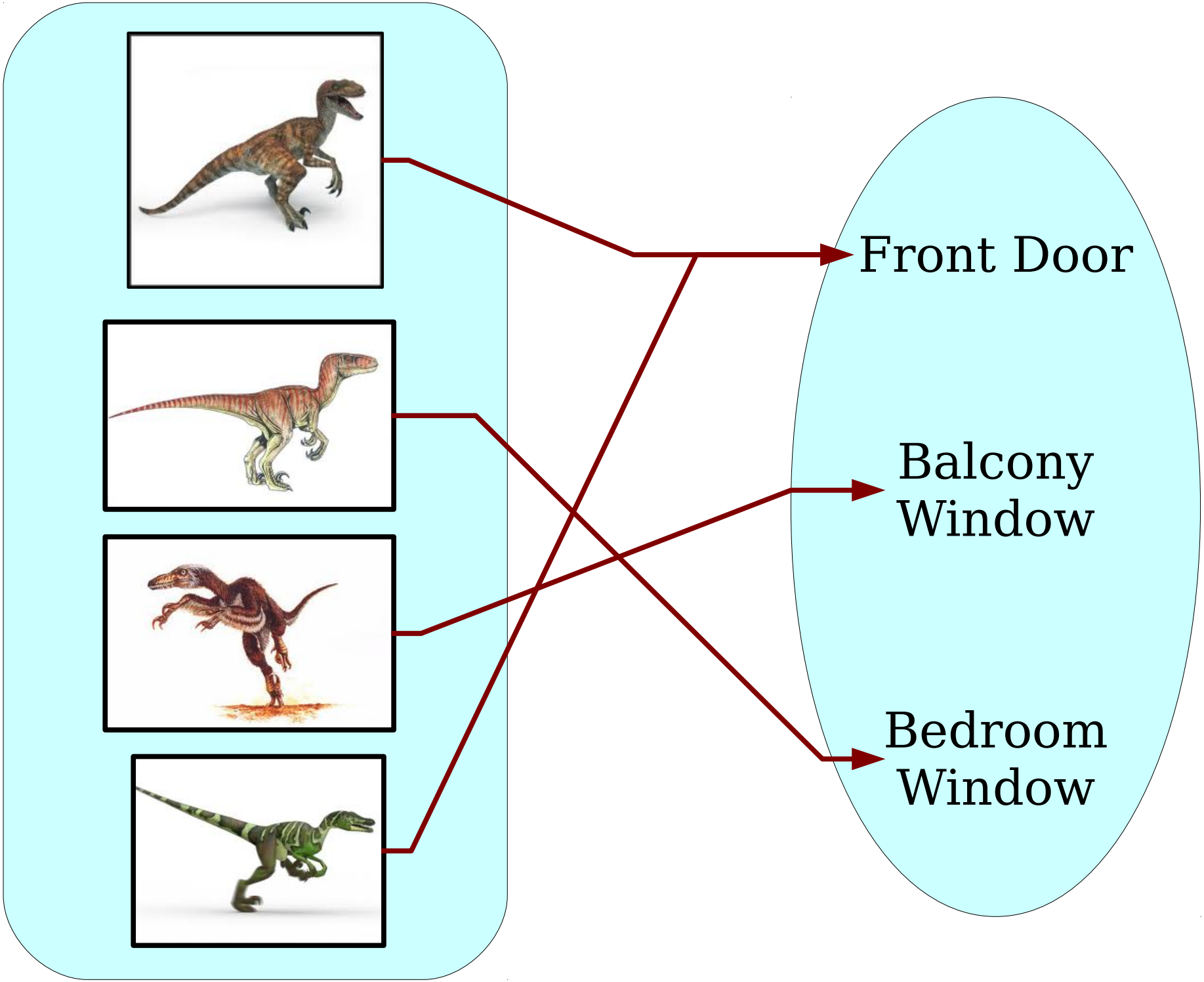
$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same"*)

- Equivalently:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different"*)


Front Door
Balcony Window
Bedroom Window

Surjective Functions

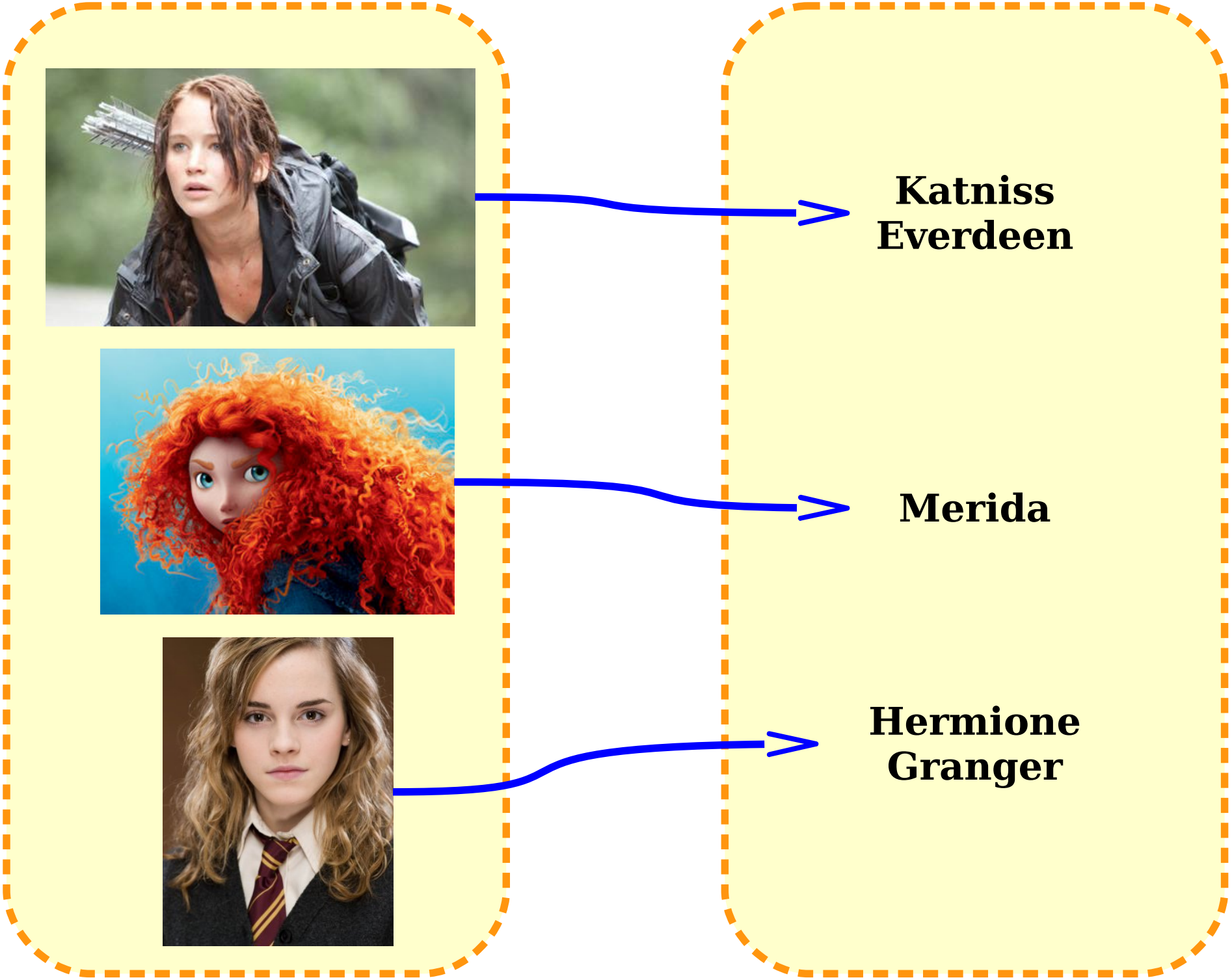
- A function $f : A \to B$ is called ***surjective*** (or ***onto***) if each element of the codomain is "covered" by at least one element of the domain.
  - A function with this property is called a ***surjection***.
- Formally, $f : A \to B$ is a surjection if this statement is true:

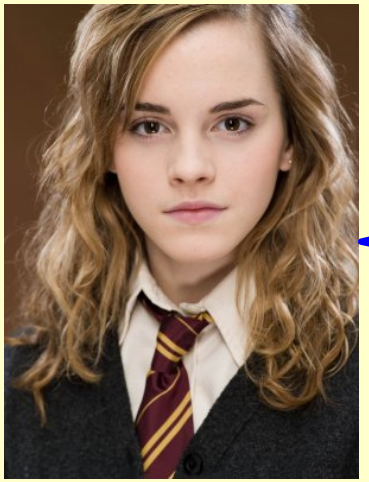
$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every possible output, there's at least one possible input that produces it"*)

Injections and Surjections

- An injective function associates ***at most*** one element of the domain with each element of the codomain.
- A surjective function associates ***at least*** one element of the domain with each element of the codomain.
- What about functions that associate ***exactly one*** element of the domain with each element of the codomain?


Katniss Everdeen
Merida
Hermione Granger



Katniss Everdeen

Merida

Hermione Granger

Bijections

- A function that associates each element of the codomain with a unique element of the domain is called ***bijective***.
  - Such a function is a ***bijection***.
- Formally, a bijection is a function that is both ***injective*** and ***surjective***.
- Bijections are sometimes called ***one-to-one correspondences***.
  - Not to be confused with “one-to-one functions.”

Time-Out for Announcements!

Problem Set Three

- PS3 Checkpoint was due at 12:50PM today. We'll get it graded and returned by Wednesday.
- Remaining PS3 questions are due on Friday at 12:50PM.
  - Have a question? Ask on Piazza, stop by office hours, or email the staff list!
- PS2 will be graded and returned by Wednesday at the start of class.

Upcoming Talk

- Defense Secretary Ashton Carter will be speaking about cybersecurity and technological innovation this Thursday from 11:10AM – 12:30PM in Cemex.
- Promises to be an interesting talk!
- RSVP using **this link**.

Your Questions

“When you were a Stanford student, what methods did you use to learn some of the harder concepts in programming and CS theory? Do you try to use these methods in the way you teach students today?”

I've found it super helpful to ask questions of the form “does it have to be this way?” to make sure I understand how everything fits together.

“What project (either programming or not) are you most proud of?”

Oof, that's hard to say. It's probably either building CS166 or trying to code up a huge number of algorithms and data structures.

Back to CS103!

Cardinality Revisited

Cardinality

- Recall *(from our first lecture!)* that the ***cardinality*** of a set is the number of elements it contains.
- If S is a set, we denote its cardinality by $|S|$.
- For finite sets, cardinalities are natural numbers:
  - $|\{1, 2, 3\}| = 3$
  - $|\{100, 200\}| = 2$
- For infinite sets, we introduced ***infinite cardinals*** to denote the size of sets:

$$|\mathbb{N}| = \aleph_0$$

Defining Cardinality

- It is difficult to give a rigorous definition of what cardinalities actually are.
  - What is 4? What is $\aleph_0$?
- ***Idea:*** Define cardinality as a *relation* between two sets rather than as an absolute quantity.
- We'll define what these relations between sets mean without actually defining what "a cardinality" actually is:

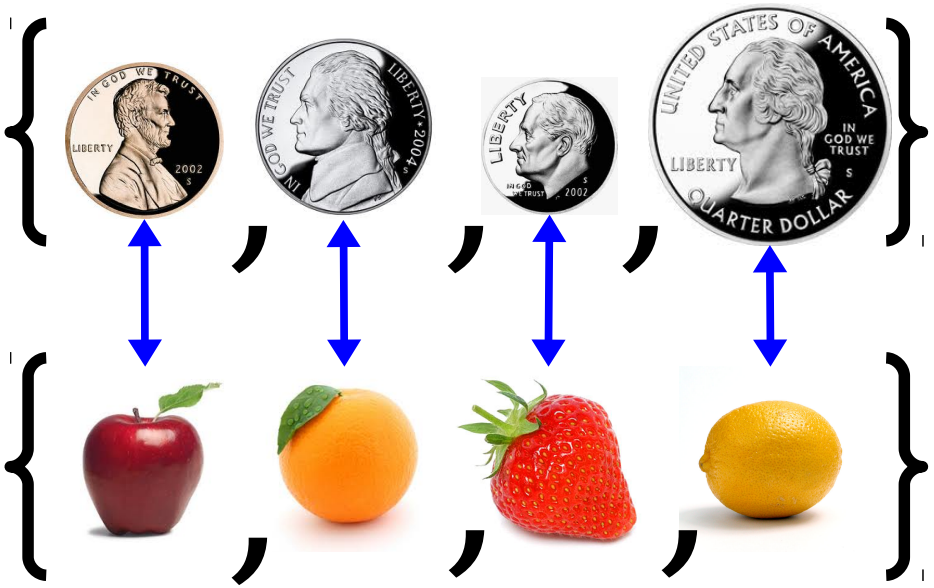
$$|S|=|T| \qquad |S|\neq|T| \qquad |S|\leq|T| \qquad |S|<|T|$$

- Cardinality exists *between* sets!

Comparing Cardinalities

- The relationships between set cardinalities are defined in terms of functions between those sets.
- $|S| = |T|$ is defined using bijections.

$|S| = |T|$ if there exists a *bijection* $f : S \to T$



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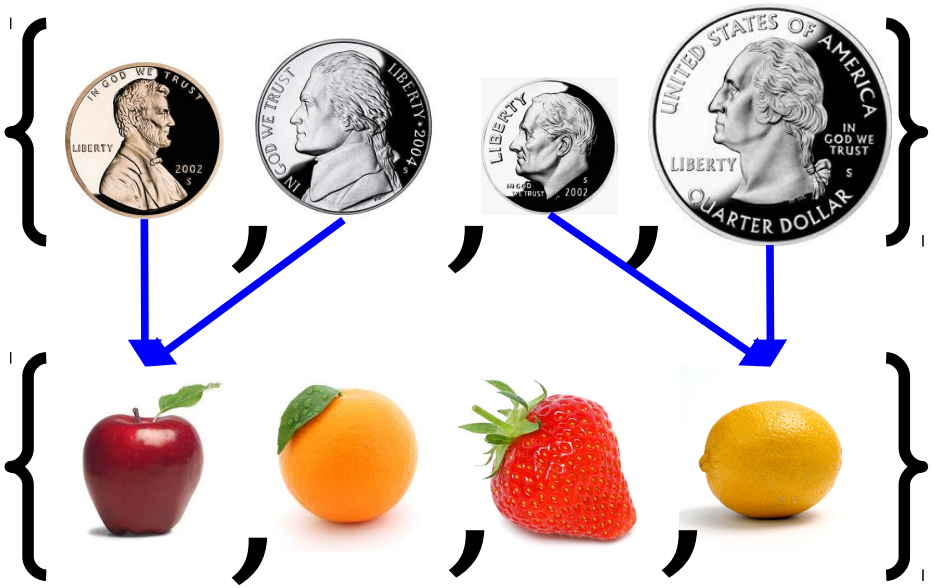
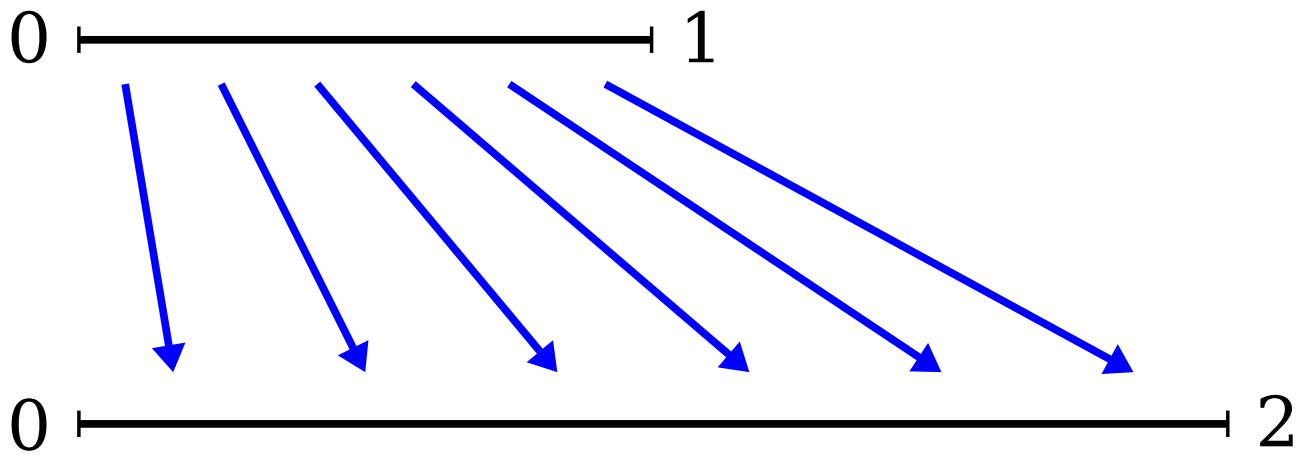


Properties of Cardinality

- For any sets R, S, and T, the following are true:
  - $|S| = |S|$.
  - If $|S| = |T|$, then $|T| = |S|$.
  - If $|R| = |S|$ and $|S| = |T|$, then $|R| = |T|$.
- ***Read the course notes for proofs of these results!***

Infinity is Weird...

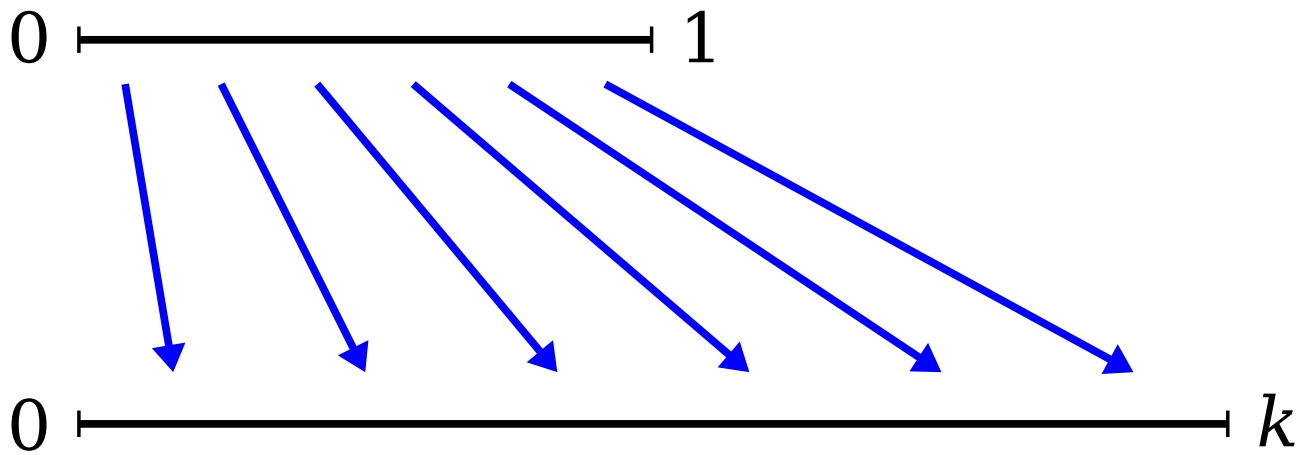
Home on the Range



$$f : [0, 1] \to [0, 2]$$
$$f(x) = 2x$$

$$|[0, 1]| = |[0, 2]|$$

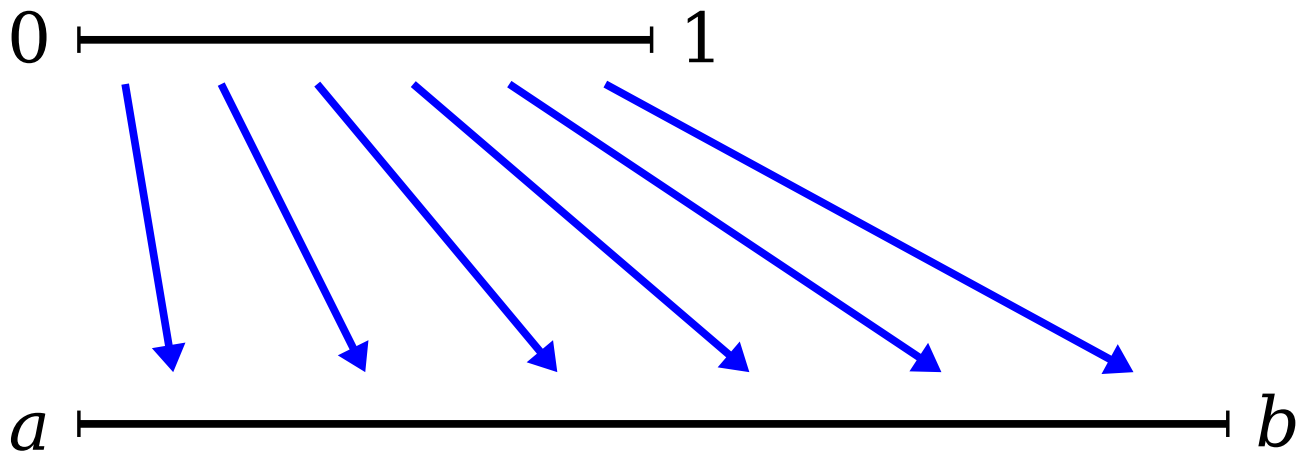
Home on the Range



$$f : [0,\ 1] \to [0,\ k]$$
$$f(x) = kx$$

$$|[0,\ 1]| = |[0,\ k]|$$

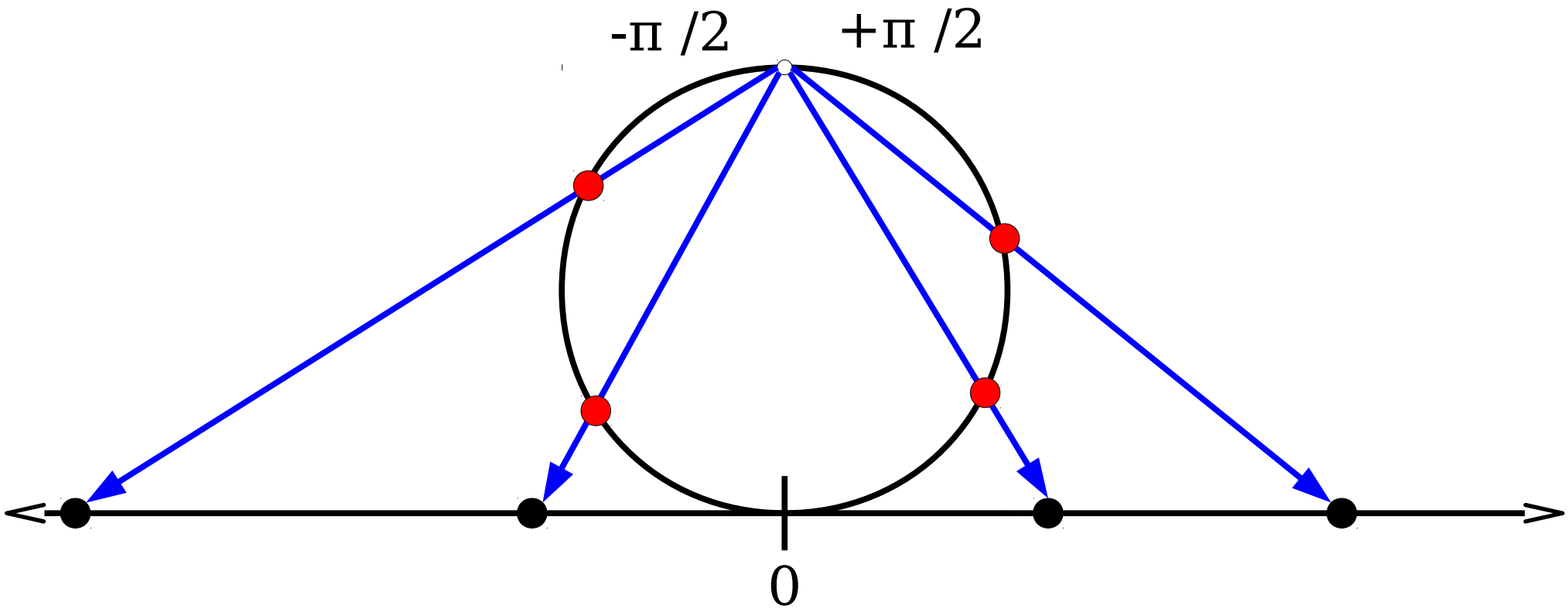
Home on the Range



$$f : [0, 1] \to [a, b]$$
$$f(x) = (b - a)x + a$$

$$|[0, 1]| = |[a, b]|$$

Put a Ring On It



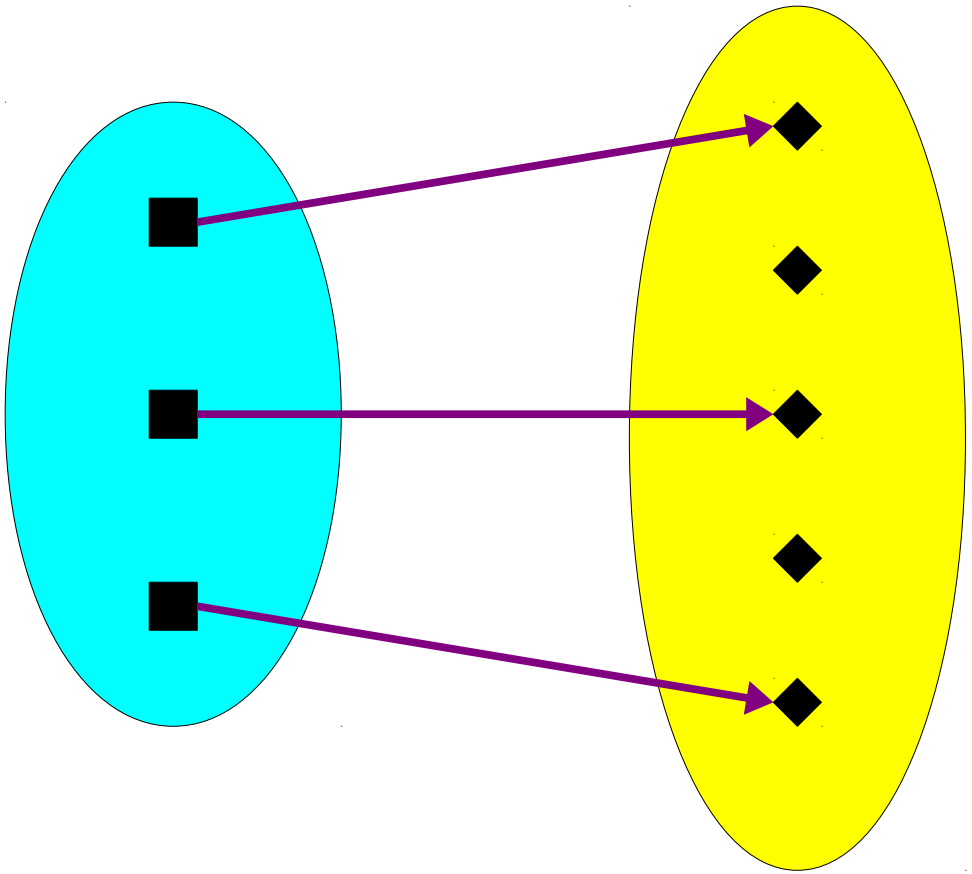
$$f : (-\pi/2, \pi/2) \to \mathbb{R}$$
$$f(x) = \tan x$$

$$|(-\pi/2, \pi/2)| = |\mathbb{R}|$$

Ranking Cardinalities

- We define $|S| \leq |T|$ as follows:

  $|S| \leq |T|$ if there is an injection $f : S \to T$



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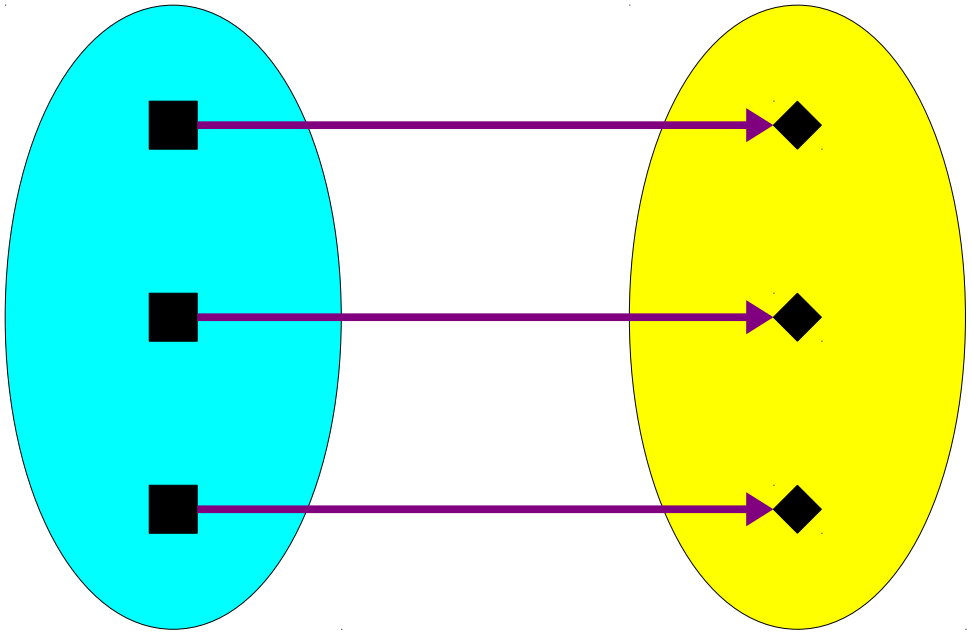

Ranking Cardinalities

- We define $|S| \leq |T|$ as follows:

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- For any sets R, S, and T:
  - $|S| \leq |S|$.
  - If $|R| \leq |S|$ and $|S| \leq |T|$, then $|R| \leq |T|$.
  - Either $|S| \leq |T|$ or $|T| \leq |S|$.

Theorem (Cantor-Bernstein-Schroeder): If S and T are sets where $|S| \leq |T|$ and $|T| \leq |S|$, then $|S| = |T|$

(This was first proven by Richard Dedekind.)

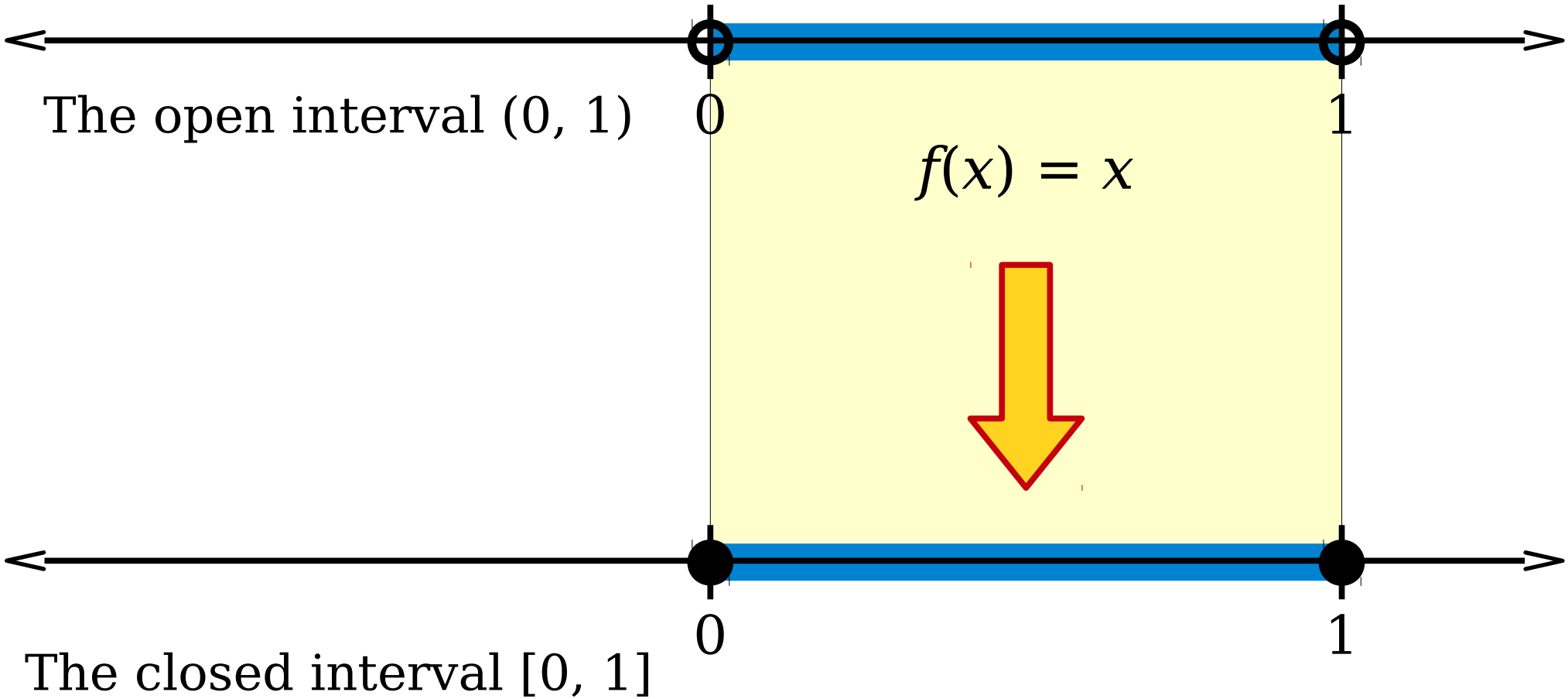
The CBS Theorem

- ***Theorem:*** If $|S| \leq |T|$ and $|T| \leq |S|$, then $|S| = |T|$.
- Isn't this, kinda, you know, obvious?
- Look at the definitions. What does the above theorem actually say?

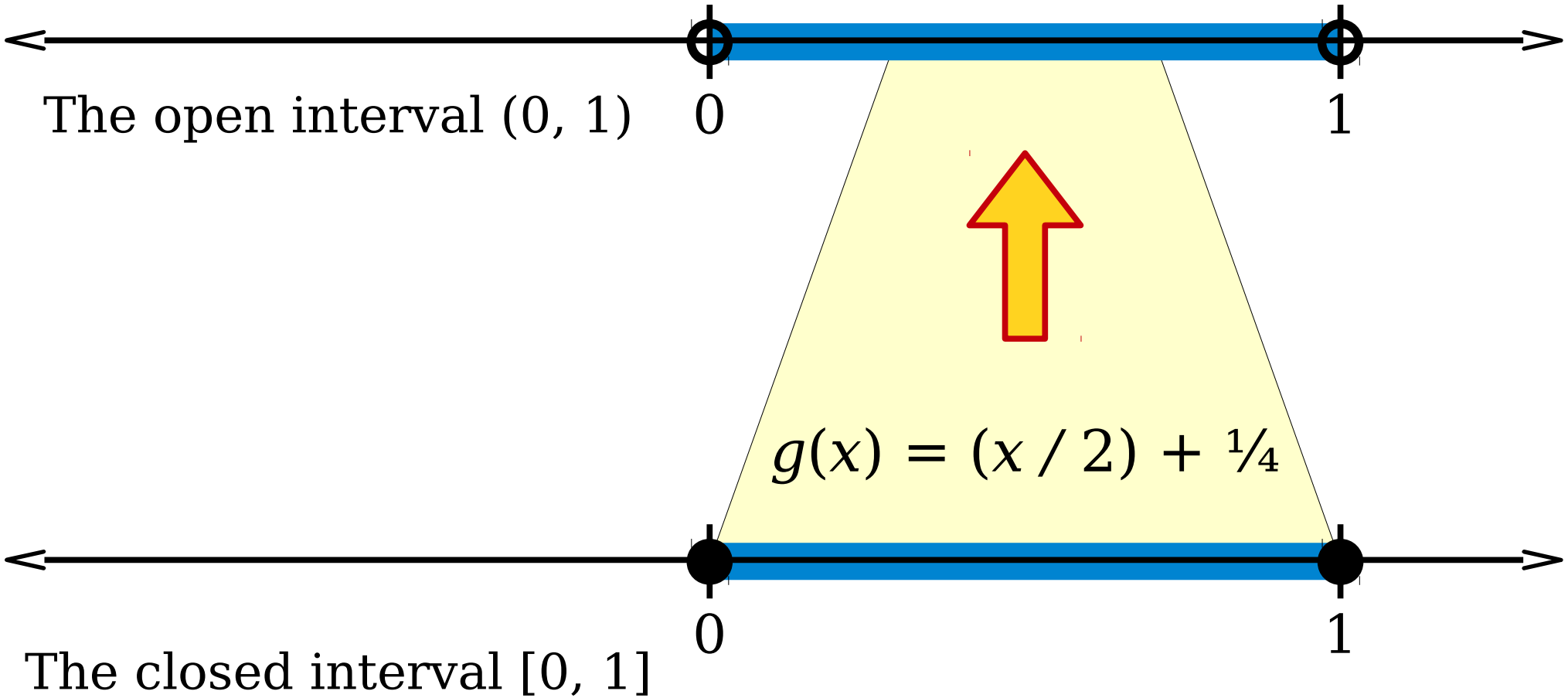
If there is an injection $f : S \to T$ and an injection $g : T \to S$, then there must be some bijection $h : S \to T$.

- This is much less obvious than it looks.

Why CBS is Tricky



Why CBS is Tricky



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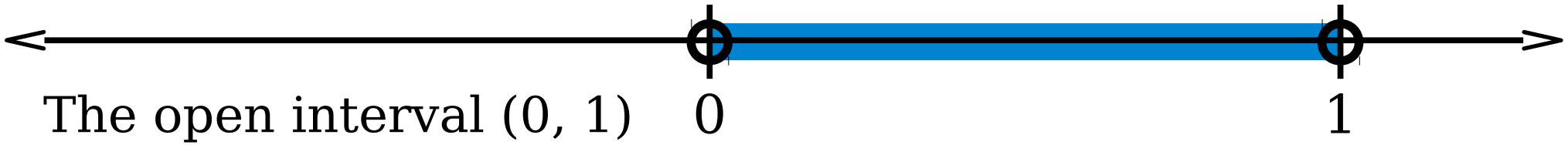
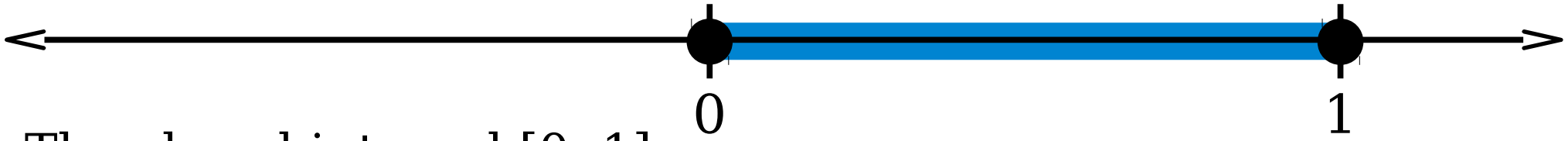


The open interval (0, 1)

There has to be a bijection between these two sets… so what is it?



The closed interval [0, 1]

Proving CBS, Intuitively

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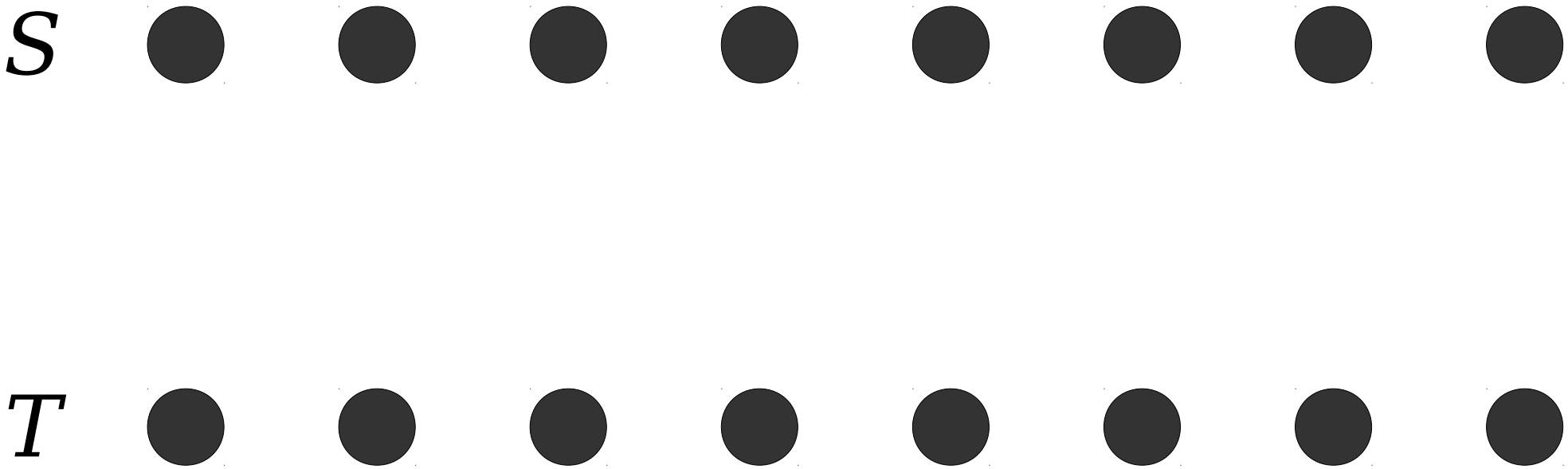
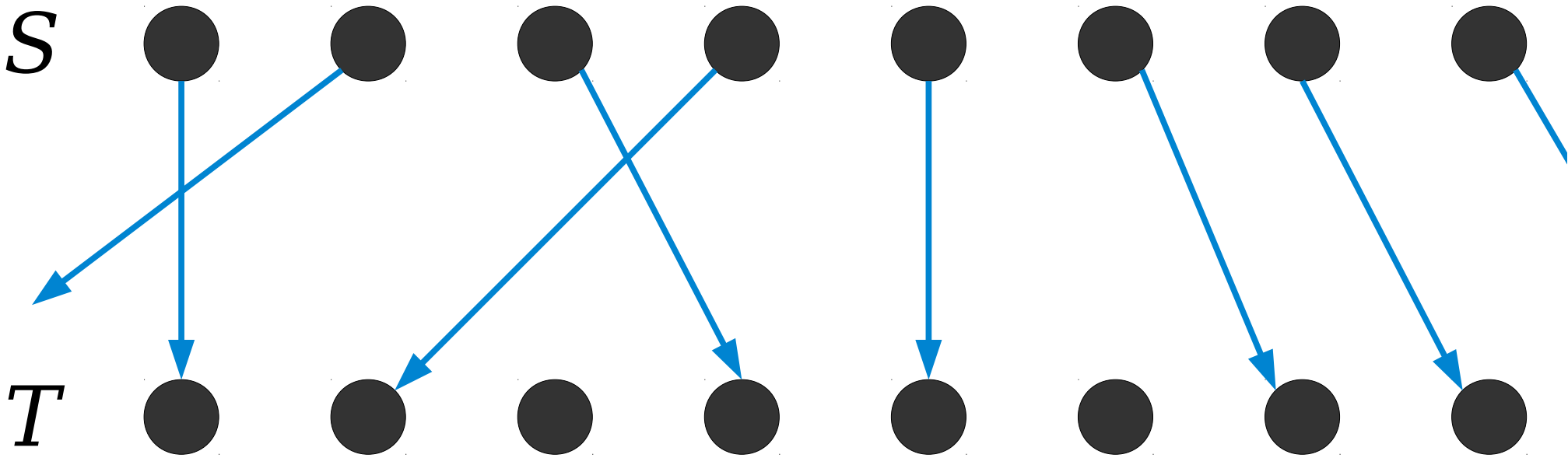


Proving CBS, Intuitively



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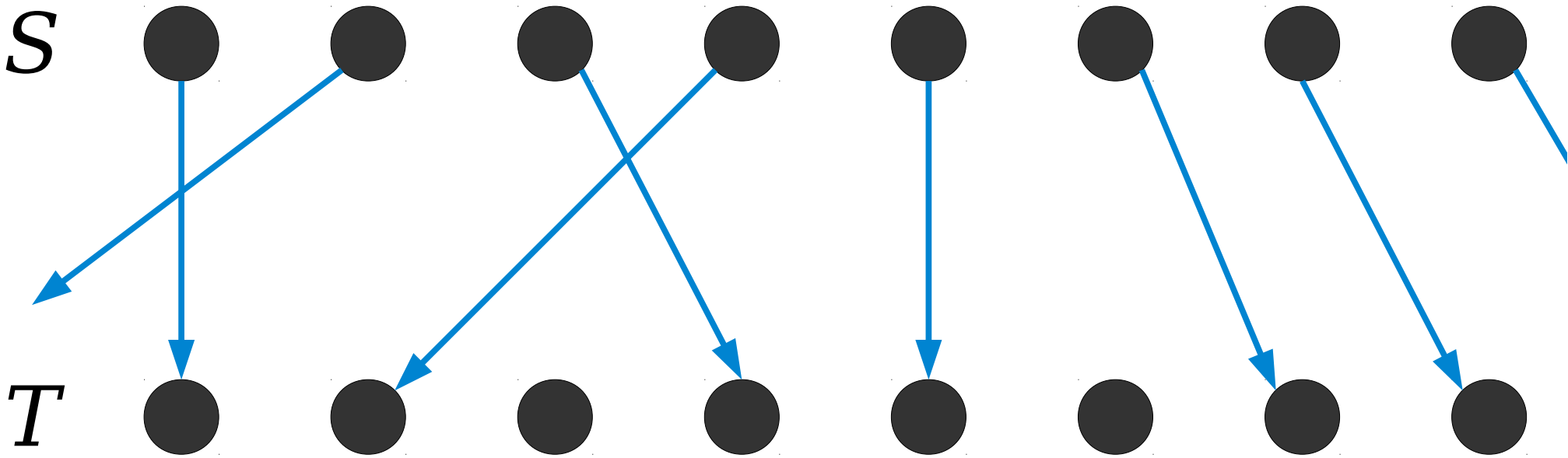
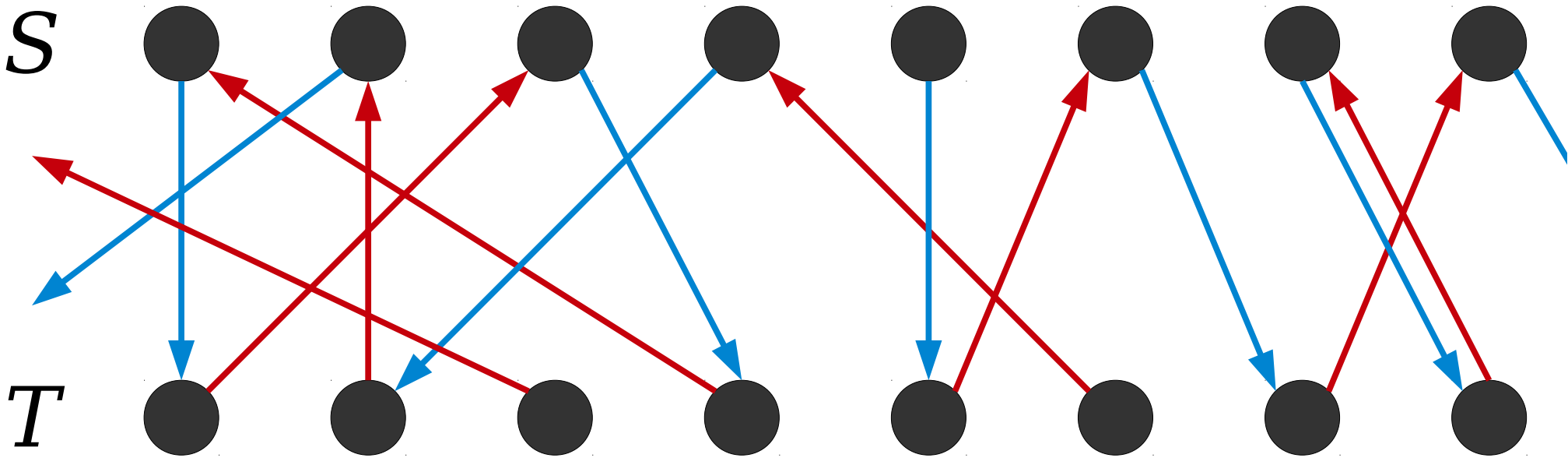


Blue lines represent the injection $f : S \to T$

Proving CBS, Intuitively



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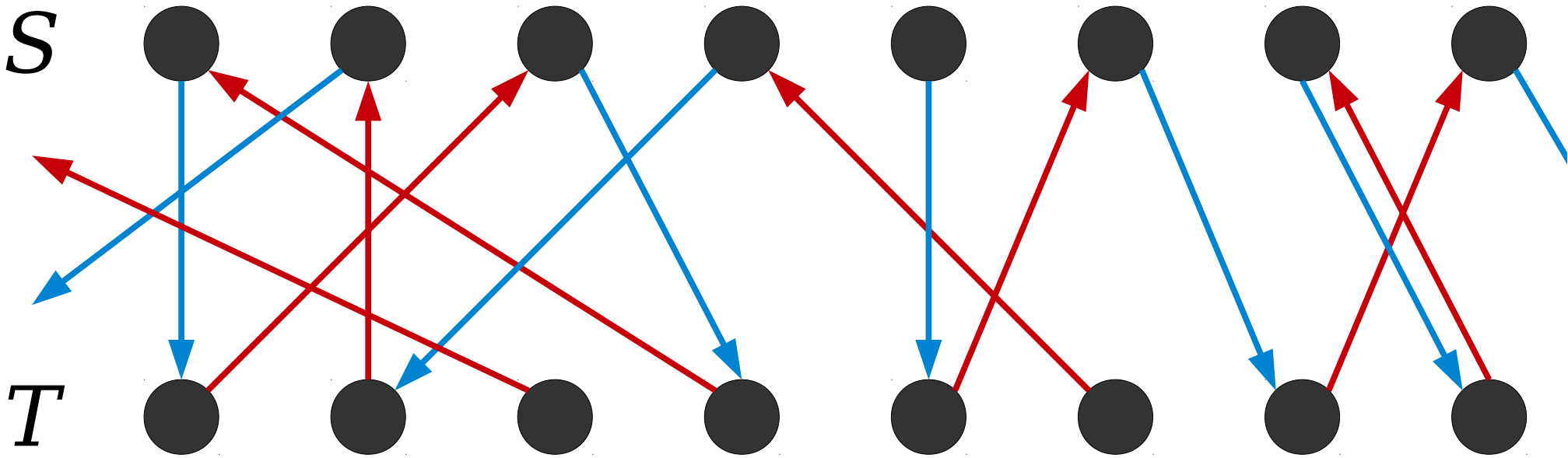
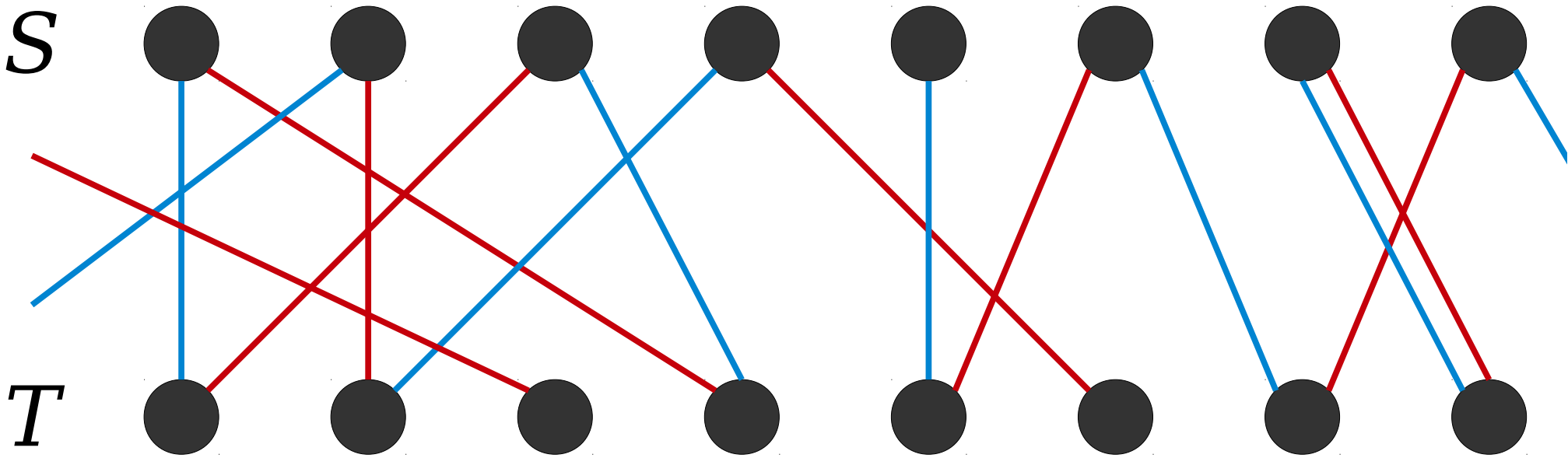
Proving CBS, Intuitively



Blue lines represent the injection $f : S \to T$
Red lines represent the injection $g : T \to S$

Proving CBS, Intuitively



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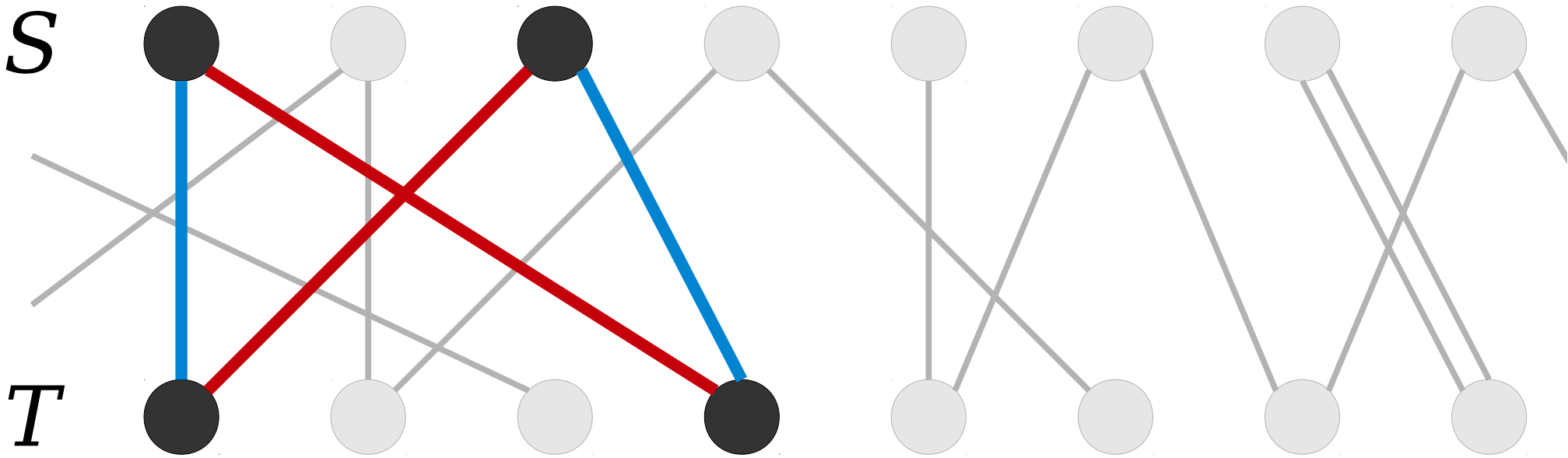

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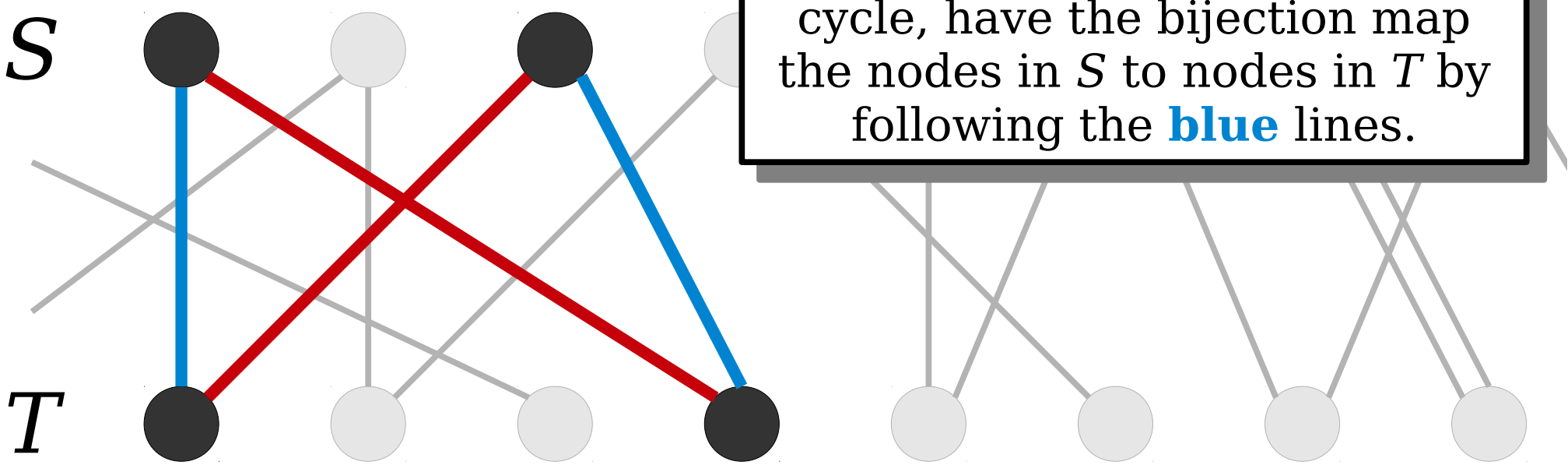
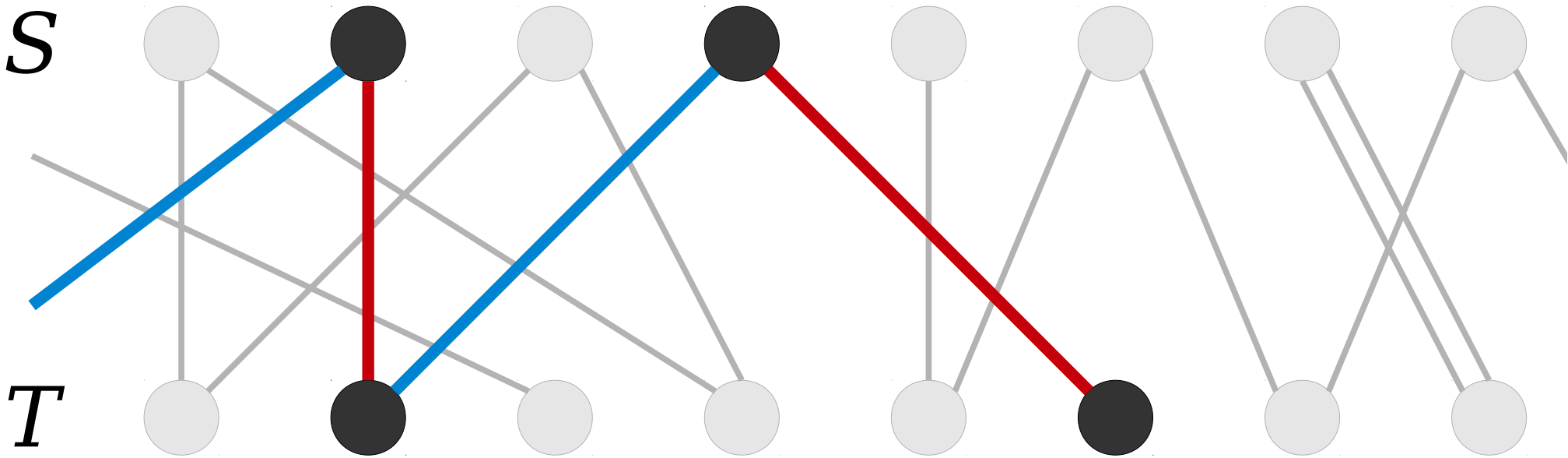
Proving CBS, Intuitively



If the connected component is a cycle, have the bijection map the nodes in S to nodes in T by following the **blue** lines.

Blue lines represent the injection $f : S \to T$
Red lines represent the injection $g : T \to S$

Proving CBS, Intuitively



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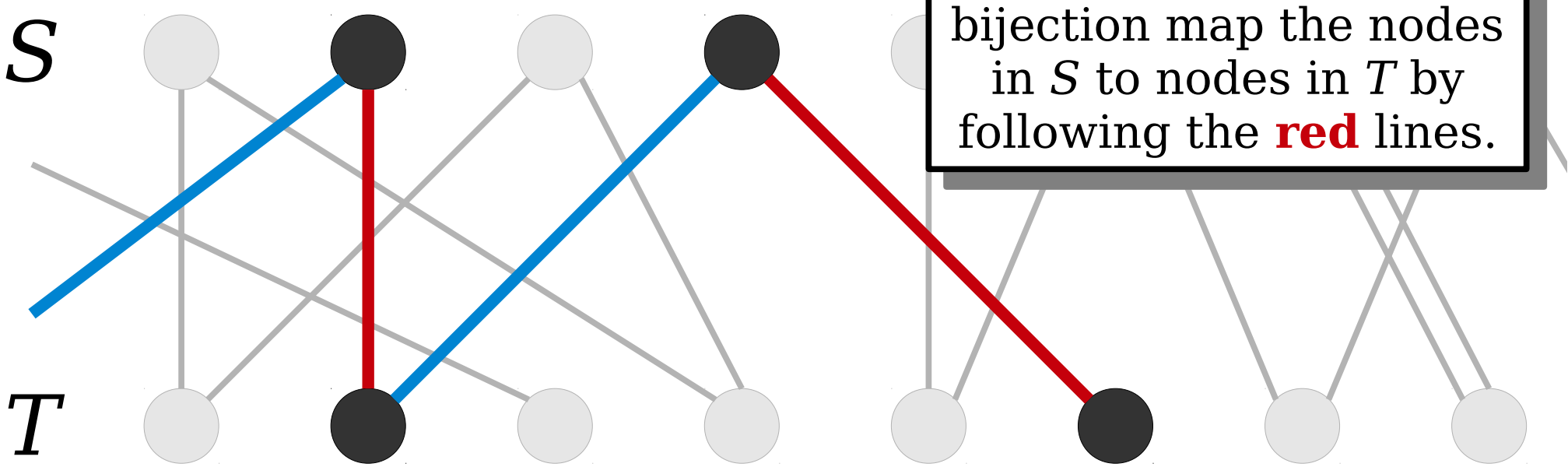
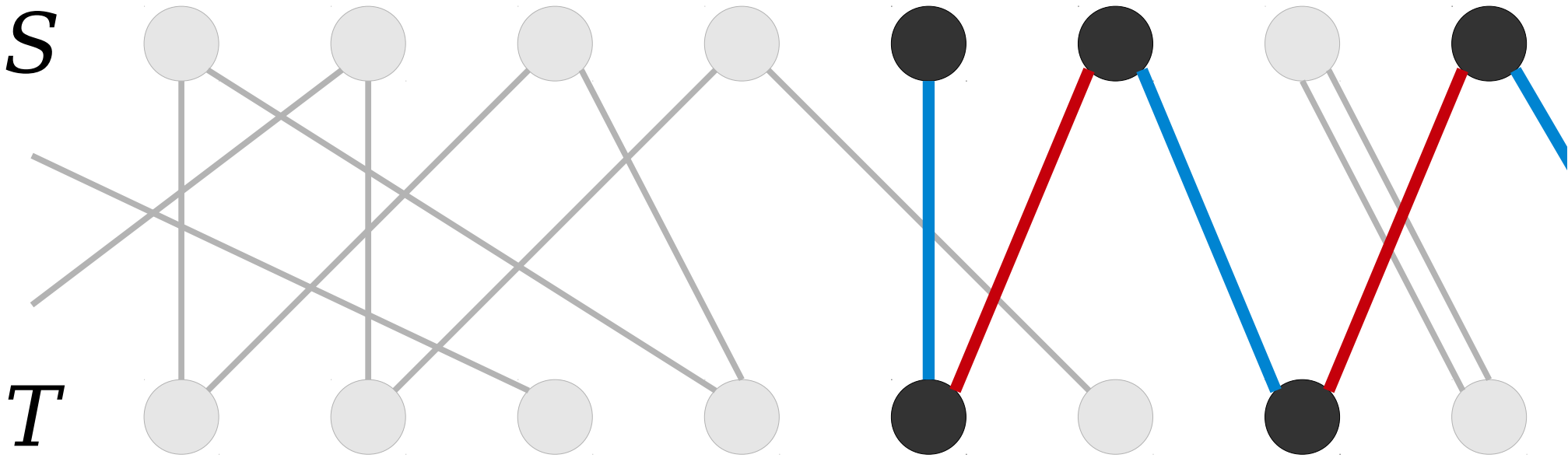


If the CC is an infinite path starting with a node in T, have the bijection map the nodes in S to nodes in T by following the **red** lines.

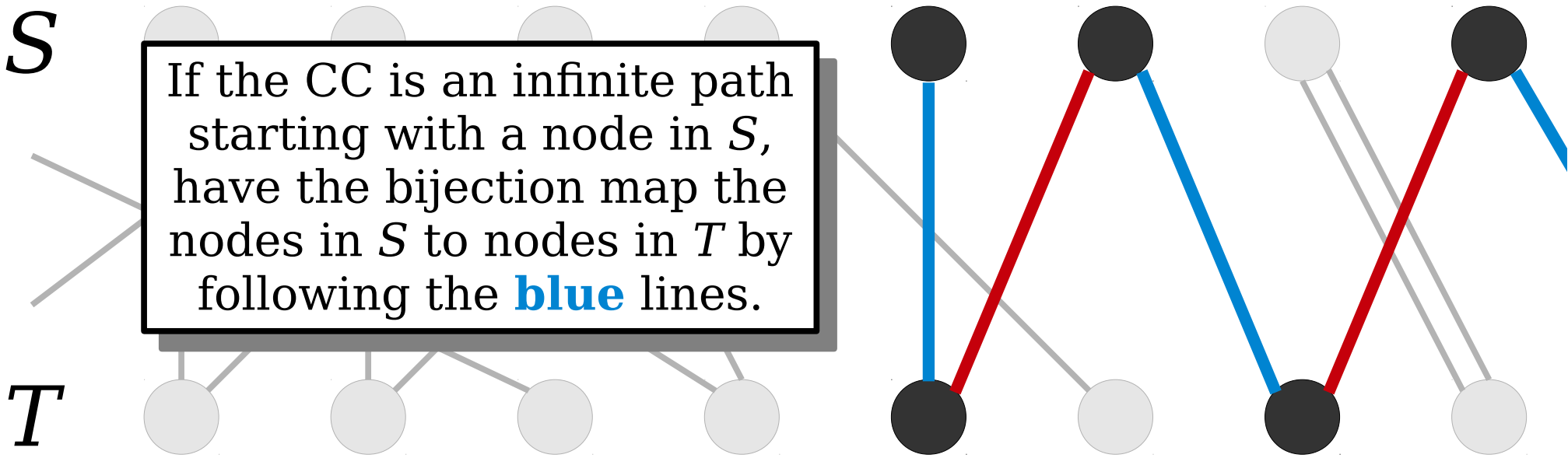
Blue lines represent the injection $f : S \to T$
Red lines represent the injection $g : T \to S$

Proving CBS, Intuitively



Blue lines represent the injection $f : S \to T$
Red lines represent the injection $g : T \to S$

Proving CBS, Intuitively



If the CC is an infinite path starting with a node in S, have the bijection map the nodes in S to nodes in T by following the **blue** lines.

Blue lines represent the injection $f : S \to T$
Red lines represent the injection $g : T \to S$

Why This Matters

- I chose to sketch out the proof of the CBS theorem because it combines so many different pieces:
  - Bipartite graphs, connected components, paths, cycles, etc.
- Don't worry too much about the specifics of this proof. Think of it as more of a "math symphony." ☺

An Application

The Cartesian Product

- The ***Cartesian product*** of $A \times B$ of two sets is defined as

$$A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$$

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	a	***b***	***c***
0			
1			
2			

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	a	b	c
0	$(0, a)$	$(0, b)$	$(0, c)$
1	$(1, a)$	$(1, b)$	$(1, c)$
2	$(2, a)$	$(2, b)$	$(2, c)$

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- We denote $A^2 = A \times A$

$$\underbrace{\{0, 1, 2\}^2}_{A^2} = \left\{ \begin{array}{l} (0, 0),(0, 1),(0, 2), \\ (1, 0),(1, 1),(1, 2), \\ (2, 0),(2, 1),(2, 2) \end{array} \right\}$$

What is $|\mathbb{N}^2|$?

$\mathbb{N}$	0	1	2	3	4	5	6	...

$\mathbb{N}$	0	1	2	3	4	5	6	…

	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)	(0, 6)	(0, …)
	(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, …)
	(2, 0)	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, …)
$\mathbb{N}^2$	(3, 0)	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)	(3, …)
	(4, 0)	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)	(4, …)
	(5, 0)	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)	(5, …)
	(…, 0)	(…, 1)	(…, 2)	(…, 3)	(…, 4)	(…, 5)	(…, 6)	…

$\mathbb{N}$	0	1	2	3	4	5	6	…

	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)	(0, 6)	(0, …)
	(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, …)
	(2, 0)	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, …)
$\mathbb{N}^2$	(3, 0)	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)	(3, …)
	(4, 0)	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)	(4, …)
	(5, 0)	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)	(5, …)
	(…, 0)	(…, 1)	(…, 2)	(…, 3)	(…, 4)	(…, 5)	(…, 6)	…

$$|\mathbb{N}| \leq |\mathbb{N}^2|$$

$\mathbb{N}$	0	1	2	3	4	5	6	…

	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)	(0, 6)	(0, …)
	(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, …)
	(2, 0)	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, …)
$\mathbb{N}^2$	(3, 0)	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)	(3, …)
	(4, 0)	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)	(4, …)
	(5, 0)	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)	(5, …)
	(…, 0)	(…, 1)	(…, 2)	(…, 3)	(…, 4)	(…, 5)	(…, 6)	…

$|\mathbb{N}| \leq |\mathbb{N}^2|$ Find an injection $f : \mathbb{N} \to \mathbb{N}^2$

$\mathbb{N}$	0	1	2	3	4	5	6	…

	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)	(0, 6)	(0, …)
	(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, …)
	(2, 0)	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, …)
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	(4, 0)	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)	(4, …)
	(5, 0)	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)	(5, …)
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	(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, …)
	(2, 0)	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, …)
$\mathbb{N}^2$	(3, 0)	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)	(3, …)
	(4, 0)	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)	(4, …)
	(5, 0)	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)	(5, …)
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	(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, …)
	(2, 0)	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, …)
$\mathbb{N}^2$	(3, 0)	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)	(3, …)
	(4, 0)	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)	(4, …)
	(5, 0)	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)	(5, …)
	(…, 0)	(…, 1)	(…, 2)	(…, 3)	(…, 4)	(…, 5)	(…, 6)	…

$|\mathbb{N}| \leq |\mathbb{N}^2|$ Find an injection $f : \mathbb{N} \to \mathbb{N}^2$ $\boldsymbol{f(n) = (0, n)}$

$\mathbb{N}$	0	1	2	3	4	5	6	…

	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)	(0, 6)	(0, …)
	(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, …)
	(2, 0)	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, …)
$\mathbb{N}^2$	(3, 0)	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)	(3, …)
	(4, 0)	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)	(4, …)
	(5, 0)	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)	(5, …)
	(…, 0)	(…, 1)	(…, 2)	(…, 3)	(…, 4)	(…, 5)	(…, 6)	…

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	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)	(0, 6)	(0, …)
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	(4, 0)	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)	(4, …)
	(5, 0)	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)	(5, …)
	(…, 0)	(…, 1)	(…, 2)	(…, 3)	(…, 4)	(…, 5)	(…, 6)	…

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$|\mathbb{N}^2| \leq |\mathbb{N}|$

$\mathbb{N}$	0	1	2	3	4	5	6	…

	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)	(0, 6)	(0, …)
	(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, …)
	(2, 0)	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, …)
$\mathbb{N}^2$	(3, 0)	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)	(3, …)
	(4, 0)	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)	(4, …)
	(5, 0)	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)	(5, …)
	(…, 0)	(…, 1)	(…, 2)	(…, 3)	(…, 4)	(…, 5)	(…, 6)	…

$|\mathbb{N}| \leq |\mathbb{N}^2|$ Find an injection $f : \mathbb{N} \to \mathbb{N}^2$ $\boldsymbol{f(n) = (0, n)}$

$|\mathbb{N}^2| \leq |\mathbb{N}|$ Find an injection $f : \mathbb{N}^2 \to \mathbb{N}$

$\mathbb{N}$	0	1	2	3	4	5	6	…

	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)	(0, 6)	(0, …)
	(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, …)
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	(4, 0)	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)	(4, …)
	(5, 0)	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)	(5, …)
	(…, 0)	(…, 1)	(…, 2)	(…, 3)	(…, 4)	(…, 5)	(…, 6)	…

$|\mathbb{N}| \leq |\mathbb{N}^2|$ Find an injection $f : \mathbb{N} \to \mathbb{N}^2$ $\boldsymbol{f(n) = (0, n)}$

$|\mathbb{N}^2| \leq |\mathbb{N}|$ Find an injection $f : \mathbb{N}^2 \to \mathbb{N}$ $\boldsymbol{f(a, b) = 2^a 3^b}$

Surprisingly counterintuitive result:

$$|\mathbb{N}| = |\mathbb{N}^2|$$