Problem Set 3

This third problem set explores binary relations, functions, and their applications. We've chosen these problems to help you get a sense for how to reason about these structures, how to write proofs using formal mathematical definitions, and what their applications are in practical programming. By the time you're done with these problems, you'll have a much more nuanced understanding of these structures and how to use them!

Good luck, and have fun!

Checkpoint due Monday, October 12 at the start of lecture.

Assignment due Friday, October 16 at the start of lecture.

This checkpoint problem is due on Monday at the start of lecture and should be submitted on GradeScope.

Checkpoint Problem: Strict Orders (2 Points)

Recall that we defined strict orders to be relations that are irreflexive, asymmetric, and transitive. We proved in lecture that any relation that is asymmetric and transitive is also a strict order, meaning that we could have potentially left irreflexivity out of our definition of strict orders.

Interestingly enough, it turns out that we could have also left asymmetry out of our definition and just gone with irreflexivity and transitivity.

Prove that a binary relation R over a set A is a strict order if and only if R is irreflexive and transitive.

The remaining problems should be completed by Friday, October 16 and submitted on GradeScope.

Problem One: The Vitali Relation (5 Points)

Let $\mathbb{Q}$ be the set of all rational numbers. Formally, $\mathbb{Q} = \{ {}^{p}/_{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z}, \text{and } q \neq 0 \}$.

Consider the following binary relation V over the set $\mathbb{R}$:

$$xVy \quad \text{if} \quad y - x \in \mathbb{Q}$$

i. Prove that V is an equivalence relation.

ii. What is $[0]_V$? Prove it.

The name V for this relation comes from the fact that it's used in a result called *Vitali's theorem*, a landmark result in the mathematical discipline of measure theory. To learn more, take Math 115 or Math 171.

Problem Two: Euclidean Relations (4 Points)

In Euclid's *The Elements*, considered a landmark work in ancient mathematics, the ancient Greek mathematician Euclid states that "things which equal the same thing also equal one another." In honor of Euclid, we say that a binary relation R over a set A is ***Euclidean*** if

$$\forall x \in A.\ \forall y \in A.\ \forall z \in A.\ (xRy \land xRz \to yRz).$$

Let R be an arbitrary binary relation over some set A. Prove that R is an equivalence relation if and only if it is reflexive and Euclidean.

Problem Three: Covering Relations (8 Points)

Let $<_A$ be a strict order relation over a set A. We can define a new binary relation $\lessdot_A$, called the ***covering relation for*** $<_A$, as follows:

$$x \lessdot_A y \quad \text{if} \quad x <_A y \text{ and there is no } z \in A \text{ where } x <_A z \text{ and } z <_A y$$

This question explores properties of covering relations.

i. Consider the $<$ relation over the set $\mathbb{N}$. What is the relation $\lessdot$? Briefly justify your answer.

ii. Prove that the relation $\lessdot$ you found in part (i) isn't a strict order.

iii. Let $<_A$ be a strict order over a set A. There is a close connection between the Hasse diagram of $<_A$ and the covering relation $\lessdot_A$. What is it? No formal proof is required.

Let A be a set and $<_A$ be a strict order over A. A ***chain in*** $<_A$ is a series of elements $x_1, \ldots, x_k$ drawn from A such that

$$x_1 <_A x_2 <_A \ldots <_A x_k.$$

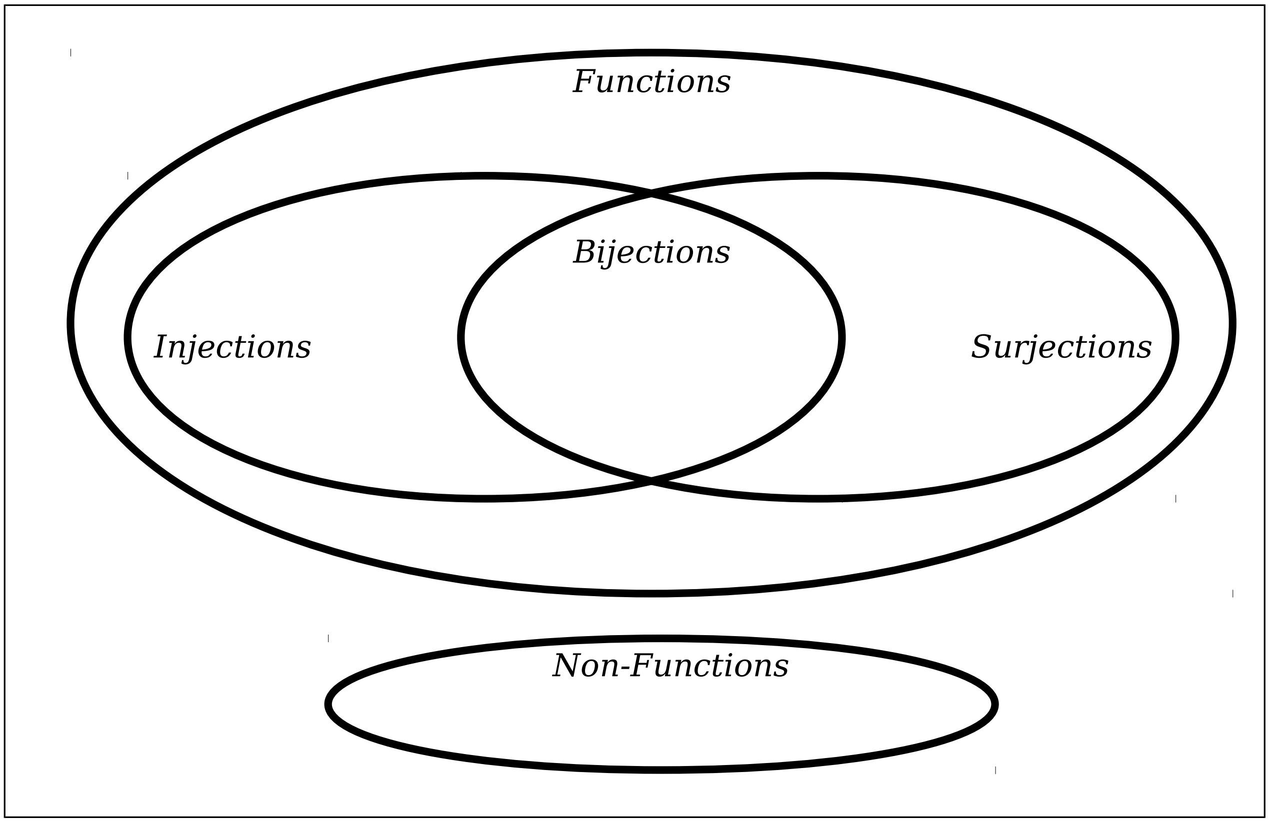
Intuitively, a chain is a series of values in ascending order according to the strict order $<_A$. The ***length*** of a chain is the number of elements in that chain. The ***height of*** $<_A$ is the length of the longest chain in $<_A$.

iv. There is a connection between the Hasse diagram of a strict order $<_A$ and the height of $<_A$. What is it? No formal proof is required.

v. Let $<_A$ be an arbitrary strict order over a set A. Prove that if $<_A$ has height two or less, then $\lessdot_A$ is a strict order. As a hint, you might want to draw some pictures and think about what your answers to parts (iii) and (iv) tell you.

Problem Four: Properties of Functions (4 Points)

Consider the following Venn diagram:



Below is a list of purported functions. For each of those purported functions, determine where in this Venn diagram that object goes. No justification is necessary.

1. $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = n^2$
2. $f : \mathbb{Z} \to \mathbb{N}$ defined as $f(n) = n^2$
3. $f : \mathbb{N} \to \mathbb{Z}$ defined as $f(n) = n^2$
4. $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(n) = n^2$
5. $f : \mathbb{R} \to \mathbb{N}$ defined as $f(n) = n^2$
6. $f : \mathbb{N} \to \mathbb{R}$ defined as $f(n) = n^2$
7. $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = \sqrt{n}$.
8. $f : \mathbb{R} \to \mathbb{R}$ defined as $f(n) = \sqrt{n}$.
9. $f : \mathbb{R} \to \{ x \in \mathbb{R} \mid x \geq 0 \}$ defined as $f(n) = \sqrt{n}$.
10. $f : \{ x \in \mathbb{R} \mid x \geq 0 \} \to \{ x \in \mathbb{R} \mid x \geq 0 \}$ defined as $f(n) = \sqrt{n}$.
11. $f : \mathbb{N} \to \wp(\mathbb{N})$, where f is some injective function.
12. $f : \{a, b, c\} \to \{x, y\}$, where f is some surjective function.
13. f : {breakfast, lunch, dinner} → {shakshuka, soondubu, maafe}, where f is an injection.

Problem Five: Left and Right Inverses (4 Points)

There's a weaker notion of an inverse function called a ***left inverse function***. Suppose that $f : A \to B$ is a function that isn't a bijection. It still might be possible to find a function $g : B \to A$ such that $g(f(a)) = a$ for all $a \in A$. A function meeting these requirements is called a ***left inverse*** of f.

i. Find examples of a function f and two *different* functions g and h such that both g and h are left inverses of f. This shows that left inverses don't have to be unique. (Two functions g and h are different if there is some x where $g(x) \neq h(x)$.) *(Hint: Define your functions through pictures.)*

ii. Prove that if f has a left inverse, then f is injective.

Another weaker notion of an inverse function is called a ***right inverse function***. Suppose that $f : A \to B$ is a function that isn't a bijection. It still might be possible to find a function $g : B \to A$ such that $f(g(b)) = b$ for all $b \in B$. A function meeting these requirements is called a ***right inverse*** of f.

iii. Find examples of a function f and two *different* functions g and h such that both g and h are right inverses of f. This shows that right inverses don't have to be unique.

iv. Prove that if f has a right inverse, then f is surjective.

Problem Six: Set Cardinalities (4 Points)

Let a and b be arbitrary objects such that $a \neq b$. Using the formal definition of equal cardinalities, prove that $|\mathbb{N} \times \{a, b\}| = |\mathbb{N}|$. Specifically, define a bijection $f : \mathbb{N} \times \{a, b\} \to \mathbb{N}$, then prove your function is a bijection. We are looking for a proof that calls back to the formal definition of bijections, so please be as rigorous and specific in your proof as possible.

We recommend that you draw out some pictures before trying to define a function so that you have an intuitive sense of how the bijection will pair the elements of $\mathbb{N} \times \{a, b\}$ and the elements of $\mathbb{N}$.

We will cover the material necessary to solve problems seven, eight, and nine in Monday's lecture.

Problem Seven: Understanding Diagonalization (2 Points)

Proofs by diagonalization are quite tricky and rely on nuanced arguments. In this problem, we'll ask you to review the diagonalization proof we covered in lecture to help you better understand how it works.

i. Consider the function $f : \mathbb{N} \to \wp(\mathbb{N})$ defined as $f(n) = \emptyset$. Trace through our proof of Cantor's theorem with this choice of f in mind. In the middle of the argument, the proof defines some set D in terms of f. Given that $f(n) = \emptyset$, what is that set D? Is it clear why $f(n) \neq D$ for any $n \in \mathbb{N}$?

ii. Repeat part (i) of this problem using the function $f : \mathbb{N} \to \wp(\mathbb{N})$ defined as

$$f(n) = \{\ m \in \mathbb{N} \mid m \geq n\ \}$$

Now what do you get for the set D? Is it clear why $f(n) \neq D$ for any $n \in \mathbb{N}$?

Problem Eight: Simplifying Cantor's Theorem? (2 Points)

In lecture, we proved Cantor's theorem, that if S is a set, then $|S| < |\wp(S)|$. Our proof used a diagonal argument, which is clever but tricky. Below is a purported proof that $|S| \neq |\wp(S)|$ that doesn't use a diagonal argument:

> *Theorem:* If S is a set, then $|S| \neq |\wp(S)|$.
>
> *Proof:* Let S be any set and consider the function $f : S \to \wp(S)$ defined as $f(x) = \{x\}$. To see that this is a valid function from S to $\wp(S)$, note that for any $x \in S$, we have $\{x\} \subseteq S$. Therefore, $\{x\} \in \wp(S)$ for any $x \in S$, so f is a legal function from S to $\wp(S)$.
>
> Let's now prove that f is injective. Consider any $x_1, x_2 \in S$ where $f(x_1) = f(x_2)$. We'll prove that $x_1 = x_2$. Because $f(x_1) = f(x_2)$, we have $\{x_1\} = \{x_2\}$. Since two sets are equal if and only if their elements are the same, this means that $x_1 = x_2$, as required.
>
> However, f is not surjective. Notice that $\emptyset \in \wp(S)$, since $\emptyset \subseteq S$ for any set S, but that there is no x such that $f(x) = \emptyset$; this is because $\emptyset$ contains no elements and $f(x)$ always contains one element. Since f is not surjective, it is not a bijection. Thus $|S| \neq |\wp(S)|$ ■

Unfortunately, this proof is incorrect. What's wrong with this proof? Justify your answer.

Problem Nine: Paradoxical Sets (5 Points)

What happens if we take *absolutely everything* and throw it into a set? If we do, we would get a set called the *universal set*, which we denote $\mathscr{U}$:

$$\mathscr{U} = \{ x \mid x \text{ exists } \}$$

Absolutely everything would belong to this set: $1 \in \mathscr{U}$, $\mathbb{N} \in \mathscr{U}$, philosophy $\in \mathscr{U}$, CS103 $\in \mathscr{U}$, etc. In fact, we'd even have $\mathscr{U} \in \mathscr{U}$, which is strange but not immediately a problem.

Unfortunately, the set $\mathscr{U}$ doesn't actually exist, as its existence would break mathematics.

i. Prove that if A and B are sets where $A \subseteq B$, then $|A| \leq |B|$. Although this probably makes intuitive sense, to formally prove this result, you need to find an injection $f : A \to B$ and prove that your function is injective.

ii. Using your result from (i), prove that if $\mathscr{U}$ exists at all, then $|\wp(\mathscr{U})| \leq |\mathscr{U}|$.

iii. Using your result from (ii) and Cantor's Theorem, prove that $\mathscr{U}$ does not exist.

The result you've proven shows that there is a collection of objects (namely, the collection of everything that exists) that cannot be put into a set. This goes against our intuition of what a set is. When this was discovered at the start of the twentieth century, it caused quite a lot of chaos in the math world and led to a reexamination of logical reasoning itself and a more formal definition of what objects can and cannot be gathered into a set. If you're curious to learn more about what sets can and cannot be created, take Math 161 (Set Theory).

Extra Credit Problem: Counting Functions (1 Point)

Let $\mathbb{N}^{\mathbb{N}}$ be the set of all functions whose domain and codomain are the natural numbers (that is, $\mathbb{N}^{\mathbb{N}}$ is the set $\{ f \mid f : \mathbb{N} \to \mathbb{N} \}$). Prove that $|\mathbb{N}| < |\mathbb{N}^{\mathbb{N}}|$.