Problem Set 4

This fourth problem set explores graph theory and the pigeonhole principle. Over the course of this problem set, you'll see lots of different families of graphs and will get a feel for some of the nifty properties of these classes of graphs. Plus, you'll develop a newfound appreciation for the pigeonhole principle.

As always, please feel free to drop by office hours, send us emails, or ask on Piazza if you have any questions. We'd be happy to help out.

Good luck, and have fun!

Checkpoint due Monday, October 19 at the start of lecture.

Remaining problems due Friday, October 23 at the start of lecture.

Write your solutions to the following problem and submit them online by Monday, October 19th at the start of class. This problem will be graded on a 0/1/2 scale based on whether you have attempted to solve all the problem, rather than on correctness. We will try to get these problems returned to you with feedback on your proof style this Wednesday, October 21st.

Checkpoint Problem: Graph Theory Party Tricks (2 Points)

Suppose that you have a party of $n \geq 2$ people. Each pair of people at the party either knows each other (are acquaintances) or does not know each other (are strangers).

Prove that there must be two people at the party who know exactly the same number of other people. *(Hint: How many options are there for the number of people each person knows?)*

The rest of these problems are due on Friday, October 23

Problem One: Graph Coloring (6 Points)

The ***degree*** of a node v in a graph G is the number of edges that have v as an endpoint. In other words, it's the number of edges v directly touches. Interestingly, there usually isn't much of a connection between the degree of the nodes in a graph and the number of colors necessary to color that graph.

i. Give an example of a 2-colorable graph where some node has degree seven. Briefly justify why your graph meets these criteria; no proof is necessary.

ii. Generalize your answer from part (i) by describing how, for any $n \geq 1$, you can build a 2-colorable graph where some node has degree at least n. This shows that there is no direct connection between the *maximum* degree of a node in a graph and the chromatic number of that graph.

iii. Give an example of a 2-colorable graph where every node has degree three. Briefly justify why your graph meets these criteria; no proof is necessary.

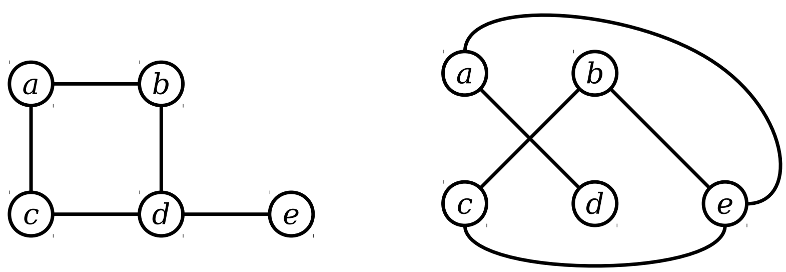
iv. Generalize your answer from part (iii) by describing how, for any $n \geq 1$, you can build a 2-colorable graph where every node has degree at least n. This shows that there is no direct connection between the *minimum* degree of a node in the graph and the chromatic number of that graph.

v. Give an example of a graph where every node has degree two but which is *not* 2-colorable. Briefly justify why your graph meets these criteria; no proof is necessary.

vi. Generalize your answer from part (v) by describing how, for any $n \geq 2$, you can build a connected graph with at least n nodes where every node has degree two but which is not 2-colorable. This shows that it's possible for each isolated part of a graph to look 2-colorable even though the graph as a whole is not.

Problem Two: Complements and Connectivity (4 Points)

If $G = (V, E)$ is an undirected graph, the ***complement of G***, denoted G^c, is a graph related to the original graph G. Intuitively, G^c has the same nodes as G, and its edges consist of all the edges missing from graph G. Formally speaking, G^c is the graph with the same nodes as G and with edges determined as follows: the edge $\{u, v\}$ is present in G^c if and only if $u \neq v$ and the edge $\{u, v\}$ is not present in G. As an example, here's a graph G and its complement graph G^c:



Graph G Graph G^c

Recall that a graph G is called ***connected*** if there is a path between any two nodes in G.

Prove that if G is an undirected graph, then G is connected or G^c is connected (or both). As a hint, look at Handout 13 and see the advice about how to prove a statement of the form $P \lor Q$.

Problem Three: Bipartite Graphs (5 Points)

The ***bipartite graphs*** are a special class of graphs with applications throughout computer science. An undirected graph $G = (V, E)$ is called ***bipartite*** if there is a way to partition the nodes V into two sets V_1 and V_2 so that every edge in E has one endpoint in V_1 and the other in V_2.

To help you get a better intuition for bipartite graphs, let's look at an example. Suppose that you have a group of people and a list of restaurants. You can illustrate which people like which restaurants by constructing a bipartite graph where V_1 is the set of people, V_2 is the set of restaurants, and there's an edge from a person p to a restaurant r if person p likes restaurant r.

Bipartite graphs have many interesting properties. One of the most fundamental is this one:

An undirected graph is bipartite if and only if it contains no cycles of odd length.

Intuitively, a bipartite graph contains no odd-length cycles because cycles alternate between the two groups V_1 and V_2, so any cycle has to have even length.

The trickier step is proving that if G contains no cycles of odd length, then G has to be bipartite. For now, assume that G has just one connected component; if G has multiple connected components, we can treat each one as a separate graph for the purposes of determining whether G is bipartite. (You don't need to prove this, but I'd recommend taking a minute to check why this is the case.)

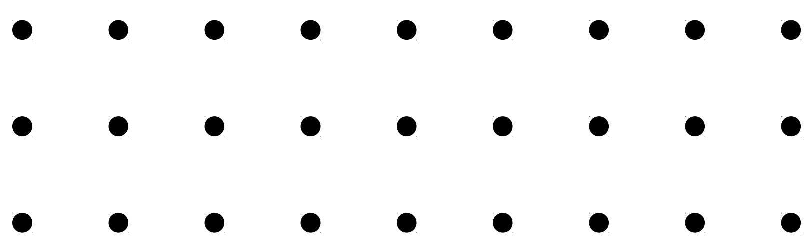
Suppose G is an undirected graph with no cycles of odd length. Choose any node $v \in V$. Let V_1 be the set of all nodes that are connected to v by a path of odd length and V_2 be the set of all nodes connected to v by a path of even length.

i. Prove that V_1 and V_2 have no nodes in common.

ii. Using your result from part (i), prove that if G has no cycles of odd length, then G is bipartite.

Problem Four: Coloring a Grid (4 Points)

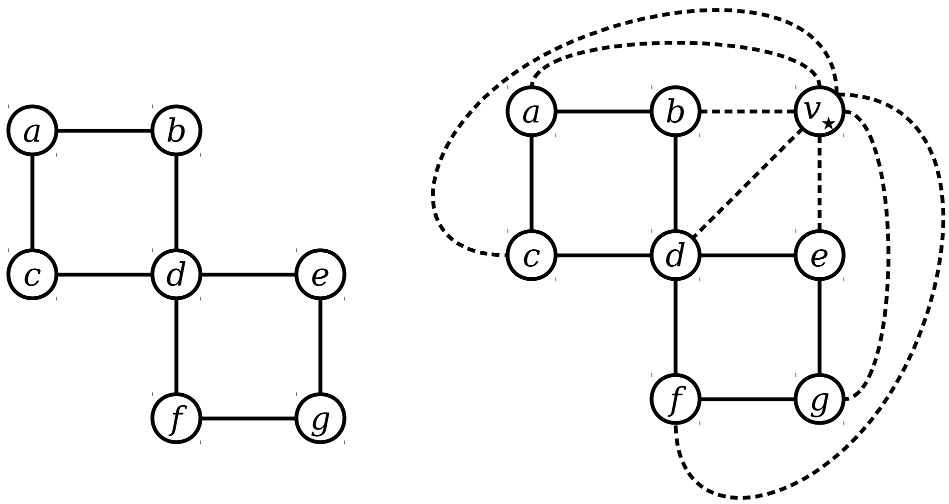
You are given a 3×9 grid of points, like the one shown below:



Suppose that you color each point in the grid either red or blue. Prove that no matter how you color those points, you can always find four points of the same color that form the corners of a rectangle.

Problem Five: Outerplanar Graphs (3 Points)

In this question, you'll see a class of graphs called the ***outerplanar graphs*** that are closely related to the planar graphs. Let's begin by introducing a new operation on graphs called ***augmentation***. If G is a graph, the augmentation of graph G, denoted Aug(G), is formed by adding a new node $v_\star$ to G, then adding edges from $v_\star$ to each other node in G. For example, below is a graph G and its augmentation Aug(G). To make it easier to see the changes between G and Aug(G), we've drawn the edges added in Aug(G) using dashed lines:



Graph G Aug(G)

Now, we can define the outerplanar graphs. An undirected graph G is called an ***outerplanar graph*** if Aug(G) is a planar graph. In other words, if Aug(G) is a *planar* graph, then the original graph G is an *outerplanar* graph.

Prove the ***three-color theorem:*** every outerplanar graph is 3-colorable.

Problem Six: Bipartite Complements (4 Points)

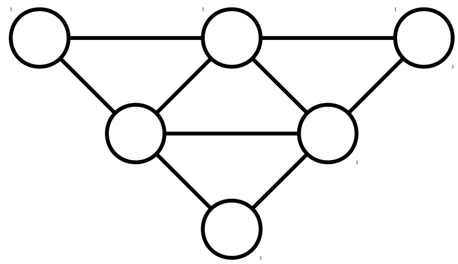
As a refresher, a ***k-coloring*** of a graph G is a way of assigning each node in G one of up to k different colors so that no two nodes in G linked by an edge are the same color. The ***chromatic number*** of a graph G, denoted $\chi(G)$, is the smallest value of k for which G is k-colorable.

Prove that if $G = (V, E)$ is a bipartite graph with n nodes, then $\chi(G^c) \geq \lceil n / 2 \rceil$.

Problem Seven: Chromatic and Independence Numbers (4 Points)

Let's introduce a new definition. An ***independent set*** in an undirected graph $G = (V, E)$ is a set $I \subseteq V$ such that if $x \in I$ and $y \in I$, then $\{x, y\} \notin E$. Intuitively, an independent set in G is a set of nodes where no two nodes in I are adjacent. The ***independence number*** of a graph G, denoted $\alpha(G)$, is the size of the largest independent set in G.

Consider the following graph G:



i. What is $\chi(G)$? What is $\alpha(G)$? No justification is required.

ii. Let r and s be arbitrary positive natural numbers. Prove that if G is an undirected graph with $rs+1$ nodes, then $\chi(G) \geq r+1$ or $\alpha(G) \geq s+1$ (or both).

Problem Eight: Friends and Strangers Revisited (5 Points)

A ***k-clique*** is an undirected graph consisting of k nodes, each of which is connected to each of the others.

In lecture, we proved the "Theorem on Friends and Strangers:" if you color all the edges of a 6-clique either red or blue, you can always find a 3-clique made purely of red edges or purely of blue edges.

It turns out that if you look at progressively larger and larger graphs, you can show that you can start finding progressively larger and larger cliques of a single color.

i. Prove that in any 10-clique where all edges are colored either red or blue, you can always find a red 3-clique or a blue 4-clique. *(Hint: Use a similar technique to the proof of the Theorem on Friends and Strangers. Also, use that result as a building block in your proof.)*

ii. Prove that in any 10-clique where all edges are colored either red or blue, you can always find a red 4-clique or a blue 3-clique.

iii. Prove that in any 20-clique where all edges are colored either red or blue, you can always find a red 4-clique or a blue 4-clique.

The sorts of results you've proven above are special cases of a result called *Ramsey's theorem* which says that for any numbers r and s, there is a number C_{rs} such that any C_{rs}-clique where each edge is colored either red or blue must contain a red r-clique or a blue s-clique. In some sense, this means that it's not possible to have complete and total disorder in large structures; any sufficiently large clique whose edges are colored necessarily must have some interesting substructure.

Extra Credit Problem: *k*-Regular Graphs (1 Point Extra Credit)

An undirected graph G is called ***k-regular*** if every node in G has degree exactly k. The ***girth*** of a graph is the length of the shortest simple cycle in G. If G has no cycles, its girth is infinite.

Prove that any k-regular graph with girth five has at least $k^2 + 1$ nodes.