Extra Practice Problems 1

Here's a set of practice problems you can work through to help prepare for the midterm. We'll release solutions to these problems on Wednesday. If you have any questions about them, please feel free to stop by office hours!

Problem One: First-Order Logic

Here's some more practice problems to help you get used to translating statements into first-order logic.

i. Given the predicates

   Person(p), which states that p is a person, and
   ParentOf(p_1, p_2), which states that p_1 is the parent of p_2,

   write a statement in first-order logic that says "someone is their own grandparent." (Paraphrased from an old novelty song.)

ii. Given the predicates

   Natural(n), which states that n is a natural number, and
   Integer(n), which states that n is an integer,

   along with the function symbol $f(n)$, which represents some particular function f, write a statement in first-order logic that says "$f : \mathbb{N} \to \mathbb{Z}$ is a bijection."

Problem Two: Functions

It's possible to find all sorts of weird functions from infinite sets into themselves. This question asks you to come up with functions with all sorts of properties from ℕ back to itself.

i. Find a function $f : \mathbb{N} \to \mathbb{N}$ that is both injective and surjective. You should briefly justify why your function has these properties, but no formal proof is necessary.

ii. Find a function $f : \mathbb{N} \to \mathbb{N}$ that is injective but not surjective. You should briefly justify why your function has these properties, but no formal proof is necessary.

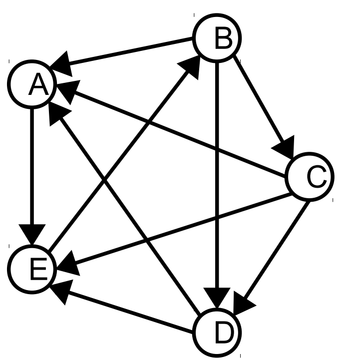
iii. Find a function $f : \mathbb{N} \to \mathbb{N}$ that is surjective but not injective. You should briefly justify why your function has these properties, but no formal proof is necessary.

iv. Find a function $f : \mathbb{N} \to \mathbb{N}$ that is neither injective nor surjective. You should briefly justify why your function has these properties, but no formal proof is necessary.

Problem Three: Binary Relations

This question explores the interaction between binary relations and tournaments.

Let's quickly refresh a definition. A ***tournament*** is a contest between some number of players in which each player plays each other player exactly once. We assume that no games end in a tie, so each game ends in a win for one of the players.



Here's a new definition to work with. If p is a player in tournament T, then we can define the set $W(p) = \{ x \mid x \text{ is a player in } T \text{ and } p \text{ beat } x \}$. Intuitively, $W(p)$ is the set of all the players that player p beat. For example, in the tournament on the left, $W(B) = \{A, C, D\}$.

Now, let's define a new binary relation. Let T be a tournament. We'll say that $p_1 \sqsubset_T p_2$ if $W(p_1) \subset W(p_2)$. Intuitively, $p_1 \sqsubset_T p_2$ means that p_2 beat every player that p_1 beat, plus some additional players.

For example, in the tournament to the left, we have that $D \sqsubset_T C$ because $W(D) = \{A, E\}$ and $W(C) = \{A, D, E\}$. Similarly, we know $A \sqsubset_T D$ since $W(A) = \{E\}$ and $W(D) = \{A, E\}$.

Prove that if T is any tournament, then $\sqsubset_T$ is a strict order over the players in T.