Extra Practice Problems 3

Here's one final set of practice problems for next week's midterm exam. We've released solutions to these problems along with this packet of problems, but we strongly recommend that you not look over them until you've completed these problems.

Problem One: First-Order Logic

Given the predicates

Set(S), which states that S is a set, and

$x \in S$, which states that x is an element of S,

write a statement in first-order logic that states "for any x and y, there is a set containing just the elements x and y." This is called the ***axiom of pairing***. Your formula can use any constructs of first-order logic (quantifiers, connectives, equality, etc.), but you should not use any functions or constants and should only use the predicates given above.

Problem Two: Propositional Logic

Below are a series of English descriptions of relations among propositional variables. For each description, write a propositional formula that precisely encodes that relation. Then, briefly explain the intuition behind your formula. You may find the online truth table tool useful here.

i. For the variables a, b, c, and d: the variables, written out in alphabetical order, alternate between true and false.

ii. For the variables a, b, c, and d: the variables, written out in alphabetical order, alternate between true and false, except that your formula cannot use the $\lor$ connective.

Problem Three: Binary Relations

Let's introduce a new definition. Let R and T be binary relations over the same set A. We'll say that R is ***no stronger than*** than T if the following statement is true:

$$\forall a \in A.\ \forall b \in A.\ (aRb \to aTb)$$

i. Let R and T be binary relations over the same set A where R is no stronger than T. Prove or disprove: if R is a strict order, then T is a strict order.

ii. Let R and T be binary relations over the same set A where R is no stronger than T. Prove or disprove: if T is a strict order, then R is a strict order.

iii. Let R and T be binary relations over the same set A where R is no stronger than T. Prove or disprove: if R is an equivalence relation, then T is an equivalence relation.

iv. Let R and T be binary relations over the same set A where R is no stronger than T. Prove or disprove: if T is an equivalence relation, then R is an equivalence relation.

Problem Four: Functions

Up to this point, most of our discussion of functions has involved functions from arbitrary domains to arbitrary codomains. If we restrict ourselves to functions with specific types of domains and codomains, then we can start exploring more nuanced and interesting classes of functions.

Let's suppose that we have a function $f : \mathbb{R} \to \mathbb{R}$. We'll say that f is an ***odd function*** if the following is true:

$$\forall x \in \mathbb{R}.\ f(-x) = -f(x)$$

This function explores properties of odd functions.

i. Prove that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are odd, then $g \circ f$ is also odd.

ii. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is odd and is a bijection, then f^{-1} is also odd.

We can define ***even functions*** as follows. A function $f : \mathbb{R} \to \mathbb{R}$ is called ***even*** if the following is true:

$$\forall x \in \mathbb{R}.\ f(-x) = f(x)$$

iii. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an even function, then f is ***not*** a bijection.

It turns out that every function $f : \mathbb{R} \to \mathbb{R}$ can be written as the sum of an odd function and an even function. The next few parts of this problem ask you to prove this.

iv. Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function. Prove that for any $r \in \mathbb{R}$, the function $r \cdot f : \mathbb{R} \to \mathbb{R}$ defined as $(r \cdot f)(x) = r \cdot f(x)$ is also odd.

v. Let $f : \mathbb{R} \to \mathbb{R}$ be an even function. Prove that for any $r \in \mathbb{R}$, the function $r \cdot f : \mathbb{R} \to \mathbb{R}$ defined as $(r \cdot f)(x) = r \cdot f(x)$ is also even.

vi. Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Prove that $g : \mathbb{R} \to \mathbb{R}$ defined as $g(x) = f(x) - f(-x)$ is odd.

vii. Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Prove that $h : \mathbb{R} \to \mathbb{R}$ defined as $h(x) = f(x) + f(-x)$ is even.

viii. Prove that any function $f : \mathbb{R} \to \mathbb{R}$ can be expressed as the sum of an odd function and an even function.