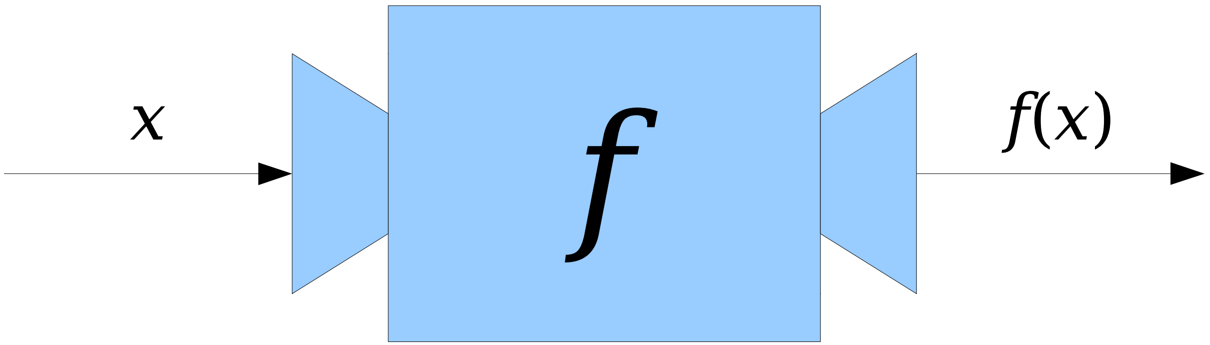
Functions

Part Two

Recap from Last Time

Rough Idea of a Function:

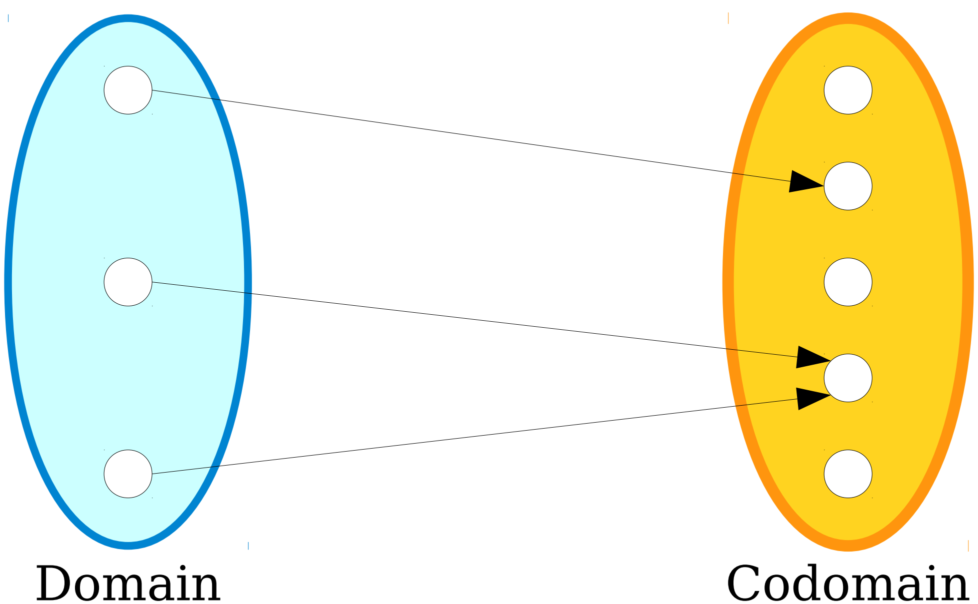
A function is an object f that takes in one input and produces exactly one output.



(This is not a complete definition – we'll revisit this in a bit.)

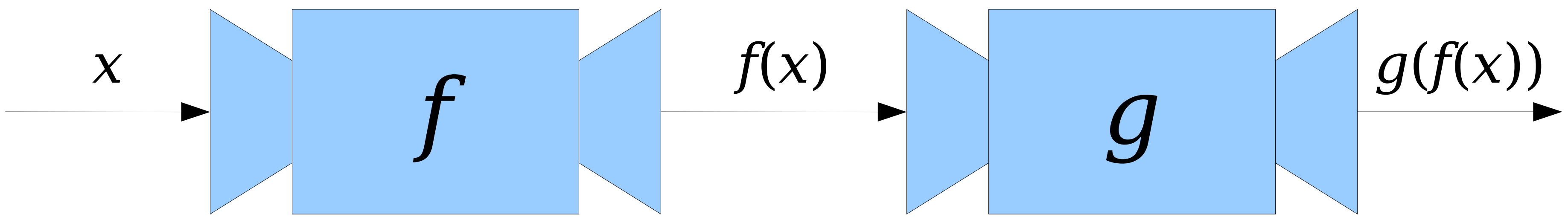
Domains and Codomains

- Every function f has two sets associated with it: its ***domain*** and its ***codomain***.
- A function f can only be applied to elements of its domain. For any x in the domain, $f(x)$ belongs to the codomain.
- If f has domain A and codomain B, we write $\boldsymbol{f : A \to B}$.



Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- Notice that the codomain of f is the domain of g. This means that we can use outputs from f as inputs to g.



Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- The ***composition of f and g***, denoted $\boldsymbol{g \circ f}$, is a function where
  - $g \circ f : A \to C$, and
  - $(g \circ f)(x) = g(f(x))$.

The name of the function is $g \circ f$. When we apply it to an input x, we write $(g \circ f)(x)$. I don't know why, but that's what we do.

- A few things to notice:
  - The domain of $g \circ f$ is the domain of f. Its codomain is the codomain of g.
  - Even though the composition is written $g \circ f$, when evaluating $(g \circ f)(x)$, the function f is evaluated first.

New Stuff!

A Topic for PS3

The Cartesian Product

- The ***Cartesian product*** of $A \times B$ of two sets is defined as

$$A \times B = \{\ (a, b) \mid a \in A \text{ and } b \in B\ \}$$

$$\underbrace{\{\, 0,\ 1,\ 2 \,\}}_{A} \times \underbrace{\{\, a,\ b,\ c \,\}}_{B} = \left\{ \begin{array}{l} (0, a),(0, b),(0, c), \\ (1, a),(1, b),(1, c), \\ (2, a),(2, b),(2, c) \end{array} \right\}$$

The Cartesian Product

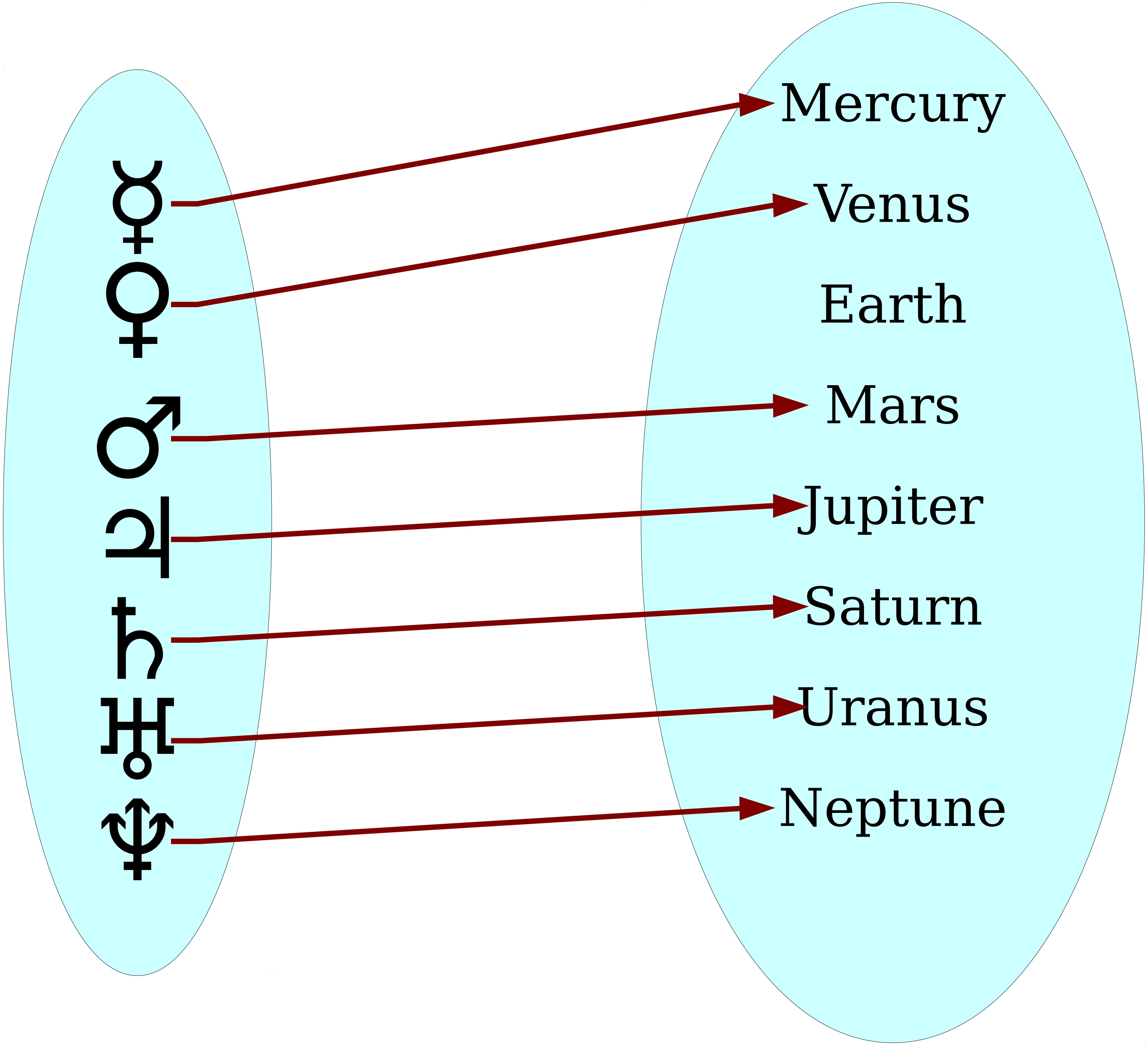
- The ***Cartesian product*** of $A \times B$ of two sets is defined as

$$A \times B = \{\ (a, b) \mid a \in A \text{ and } b \in B\ \}$$

- We denote $A^2 = A \times A$

$$\underbrace{\{0, 1, 2\}^2}_{A^2} = \left\{ \begin{matrix} (0, 0),(0, 1),(0, 2), \\ (1, 0),(1, 1),(1, 2), \\ (2, 0),(2, 1),(2, 2) \end{matrix} \right\}$$

Special Types of Functions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Mercury
Venus
Earth
Mars
Jupiter
Saturn
Uranus
Neptune

Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) if each element of the codomain has at most one element of the domain that maps to it.
  - A function with this property is called an ***injection***.
- Formally, $f : A \to B$ is an injection if this statement is true:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different"*)

- Equivalently:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same"*)

Injections and Composition

Injections and Composition

- ***Theorem:*** If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.
- Our goal will be to prove this result. To do so, we're going to have to call back to the formal definitions of injectivity and function composition.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof:

What's the high-level structure of this proof?

$$\forall f : A \to B.\ \forall g : B \to C.\ (\mathit{Inj}(f) \land \mathit{Inj}(g) \to \mathit{Inj}(g \circ f))$$

Therefore, we'll choose two arbitrary injective functions $f : A \to B$ and $g : B \to C$ and prove that $g \circ f$ is injective.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective.

What does it mean for $g \circ f : A \to C$ to be injective?

There are two equivalent definitions, actually!

$$\forall a_1 \in A.\ \forall a_2 \in A.\ ((g \circ f)(a_1) = (g \circ f)(a_2) \to a_1 = a_2)$$

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$$

Therefore, we'll choose arbitrary $a_1 \in A$ and $a_2 \in A$ where $(g \circ f)(a_1) = (g \circ f)(a_2)$ and prove that $a_1 = a_2$.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, we will prove for all $a_1 \in A$ and $a_2 \in A$ that if $(g \circ f)(a_1) = (g \circ f)(a_2)$, then $a_1 = a_2$.

Suppose that $(g \circ f)(a_1) = (g \circ f)(a_2)$.

How do you evaluate $(g \circ f)(a_1)$?

$$\boldsymbol{(g \circ f)(a_1) = g(f(a_1))}$$

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, we will prove for all $a_1 \in A$ and $a_2 \in A$ that if $(g \circ f)(a_1) = (g \circ f)(a_2)$, then $a_1 = a_2$.

Suppose that $(g \circ f)(a_1) = (g \circ f)(a_2)$. Expanding out the definition of $g \circ f$, this means that $g(f(a_1)) = g(f(a_2))$.

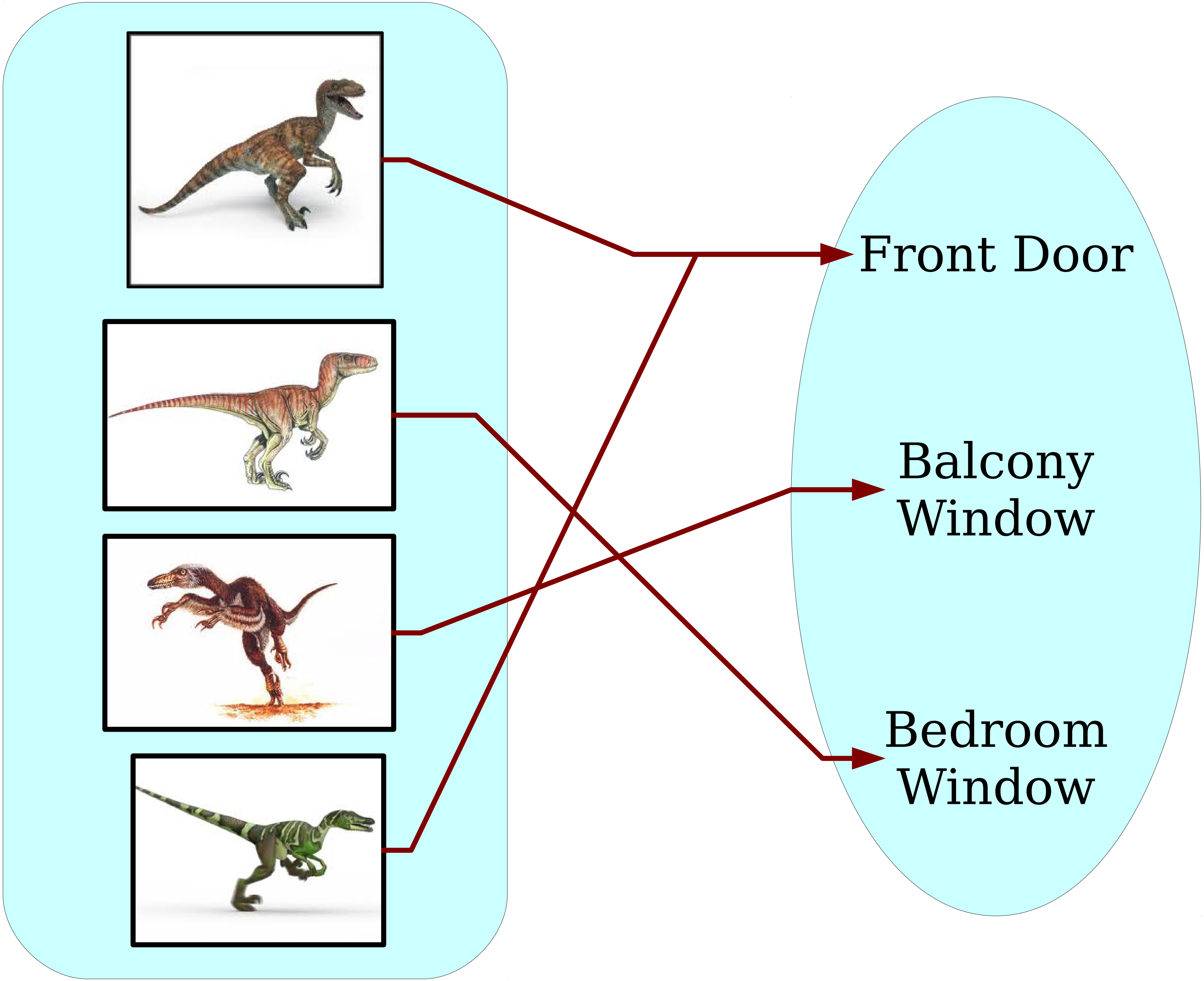
We know that g is injective. What does that mean?

$$\forall x \in A.\ \forall y \in A.\ (g(x) = g(y) \to x = y)$$

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, we will prove for all $a_1 \in A$ and $a_2 \in A$ that if $(g \circ f)(a_1) = (g \circ f)(a_2)$, then $a_1 = a_2$.

Suppose that $(g \circ f)(a_1) = (g \circ f)(a_2)$. Expanding out the definition of $g \circ f$, this means that $g(f(a_1)) = g(f(a_2))$. Since g is injective and $g(f(a_1)) = g(f(a_2))$, we know $f(a_1) = f(a_2)$. Similarly, since f is injective and $f(a_1) = f(a_2)$, we know that $a_1 = a_2$, as required. ■



Front Door
Balcony Window
Bedroom Window

Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) if each element of the codomain is "covered" by at least one element of the domain.
  - A function with this property is called a ***surjection***.
- Formally, $f : A \to B$ is a surjection if this statement is true:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every possible output, there's at least one possible input that produces it"*)

Composing Surjections

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

Proof:

What's the high-level structure of this proof?

$$\forall f : A \to B.\ \forall g : B \to C.\ (Sur(f) \land Sur(g) \to Sur(g \circ f))$$

Therefore, we'll choose two arbitrary surjective functions $f : A \to B$ and $g : B \to C$ and prove that $g \circ f$ is surjective.

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective.

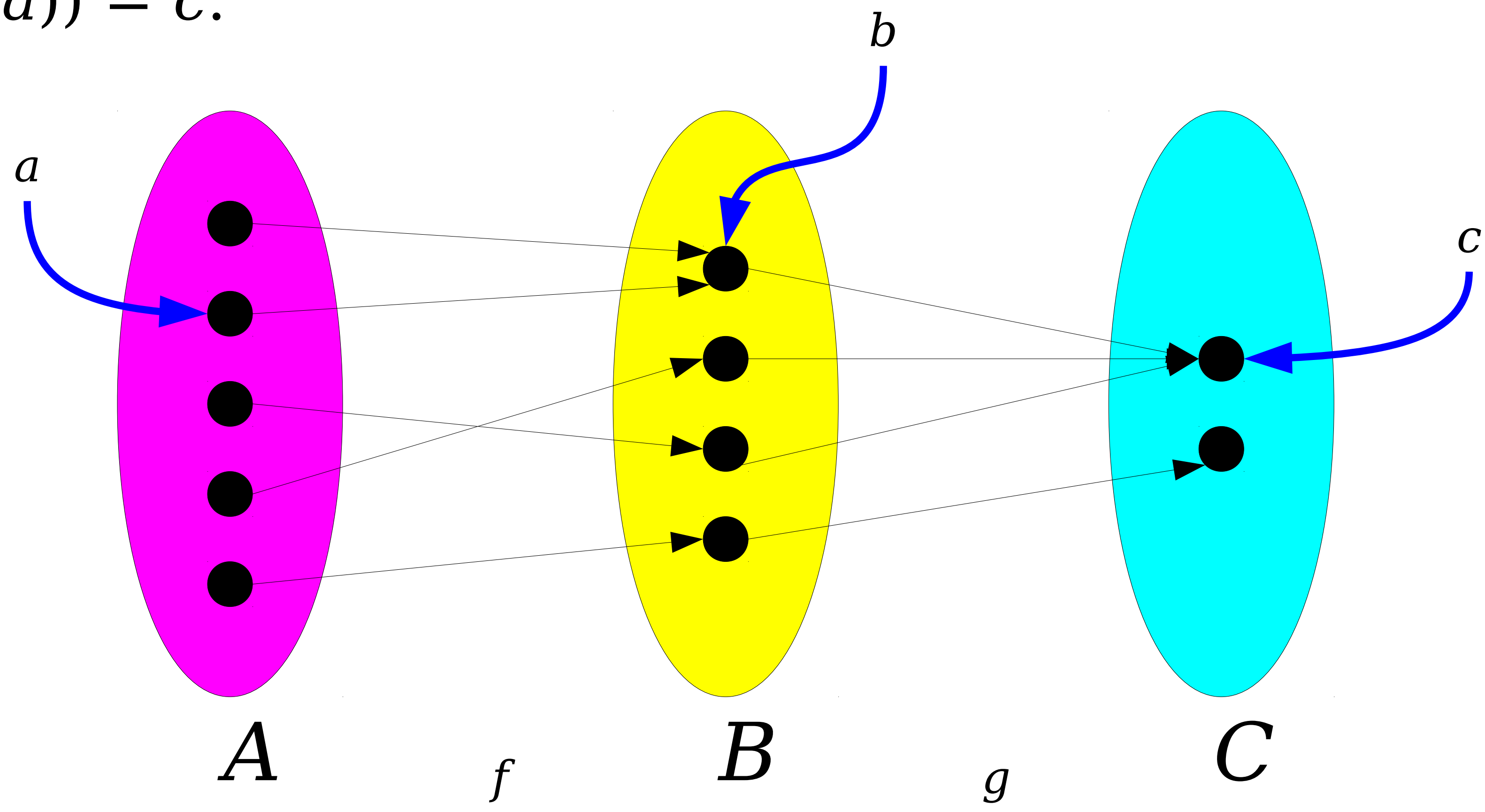
What does it mean for $g \circ f : A \to C$ to be surjective?

$$\forall c \in C.\ \exists a \in A.\ (g \circ f)(a) = c$$

Therefore, we'll choose arbitrary $c \in C$ and prove that there is some $a \in A$ such that $(g \circ f)(a) = c$.

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective. To do so, we will prove that for any $c \in C$, there is some $a \in A$ such that $(g \circ f)(a) = c$. Equivalently, we will prove that for any $c \in C$, there is some $a \in A$ such that $g(f(a)) = c$.



Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective. To do so, we will prove that for any $c \in C$, there is some $a \in A$ such that $(g \circ f)(a) = c$. Equivalently, we will prove that for any $c \in C$, there is some $a \in A$ such that $g(f(a)) = c$.

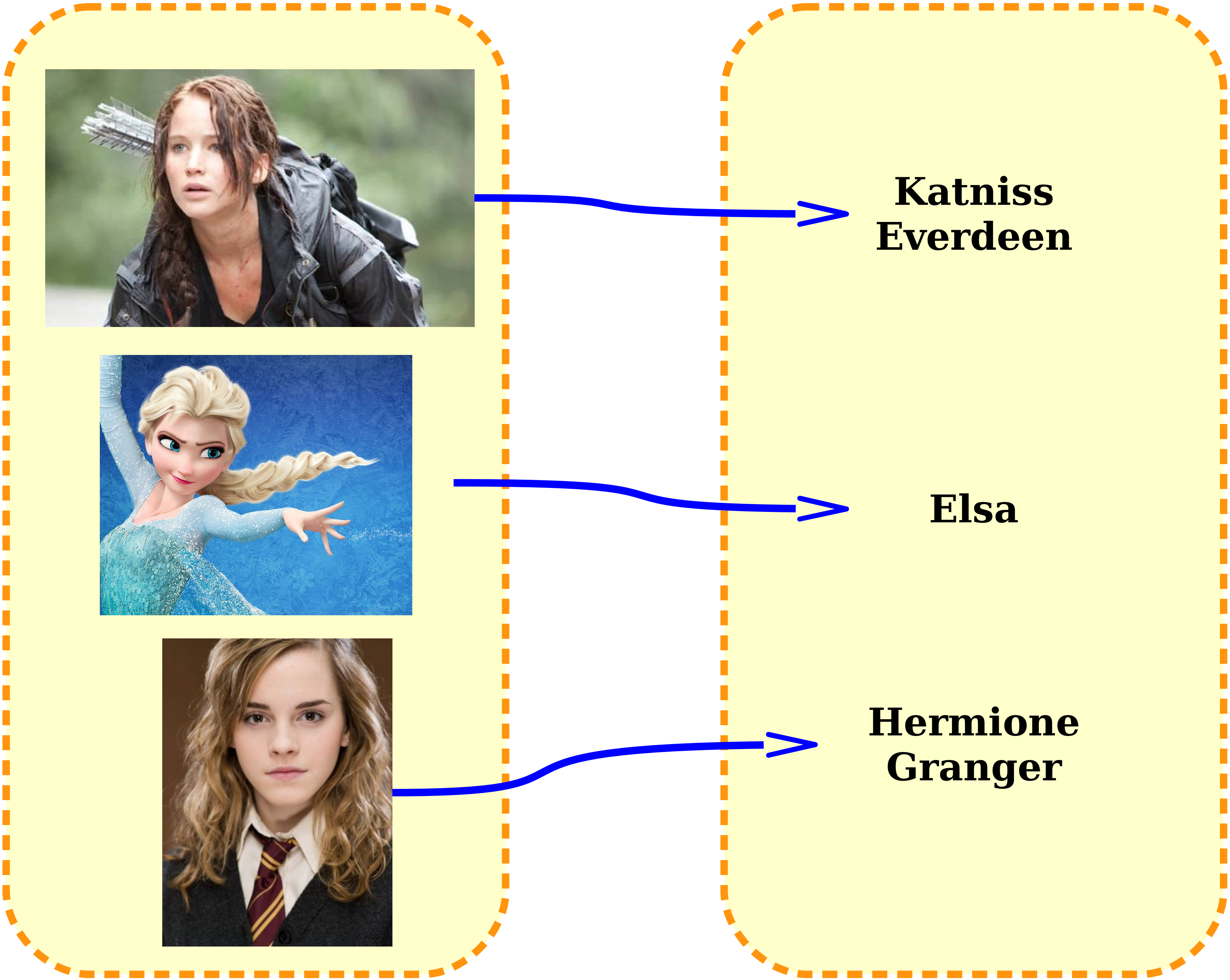
Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since $f : A \to B$ is surjective, there is some $a \in A$ such that $f(a) = b$. This means that there is some $a \in A$ such that

$$g(f(a)) = g(b) = c,$$

which is what we needed to show. ■

Injections and Surjections

- An injective function associates ***at most*** one element of the domain with each element of the codomain.
- A surjective function associates ***at least*** one element of the domain with each element of the codomain.
- What about functions that associate ***exactly one*** element of the domain with each element of the codomain?


Katniss Everdeen
Elsa
Hermione Granger

Bijections

- A function that associates each element of the codomain with a unique element of the domain is called ***bijective***.
  - Such a function is a ***bijection***.
- Formally, a bijection is a function that is both ***injective*** and ***surjective***.
- Bijections are sometimes called ***one-to-one correspondences***.
  - Not to be confused with “one-to-one functions.”

Bijections and Composition

- Suppose that $f : A \to B$ and $g : B \to C$ are bijections.
- Is $g \circ f$ necessarily a bijection?
- ***Yes!***
  - Since both f and g are injective, we know that $g \circ f$ is injective.
  - Since both f and g are surjective, we know that $g \circ f$ is surjective.
  - Therefore, $g \circ f$ is a bijection.

Time-Out for Announcements!

Problem Set Two

- Problem Set Two was due at 3:00PM today.
- Want to use late days? Submit up to 3:00PM on Monday of next week.
- We'll try to get everything graded and returned by Wednesday of next week.

Problem Set Three

- Problem Set Three goes out today.
  - Checkpoint is due on Monday.
  - Remaining problems due Friday.
- Play around with relations, functions, and cardinality!
  - (The last few problems require topics from Monday's lecture.)
- Need help? Ask questions on Piazza or stop by office hours!

Problem Set Three

- It seems like Q3 on PS3 printed one of the symbols wrong.
- The weird symbol that looks like this:

$$\dot{<}$$

- should look like this:

$$\lessdot$$

Problem Set Three

- It seems like Q7 on PS3 printed one of the symbols wrong as well (sorry!)
- The function $f(n)$ should be

$$f(n) = \{ m \in \mathbb{N} \mid m \geq n \}$$

Your Questions

"What were the grade distributions for the first PSet?"

"What is the average grade for Problem Set 1, and for problem sets in general?"

The median grade was 31 / 34, including the checkpoint.

Generally speaking, assignment scores tend to be pretty high because you have lots of time to work on them. Remember that they're only 1/3 of your grade and that most of the variance comes from exams. Think of this as a "strategic point reserve" you can spend if one of the exams doesn't go well.

"Do you think grades often get in the way of learning? Or, are they useful evaluations?"

Yes. And yes.

Grades are useful as a feedback mechanism and as an incentive to keep people motivated. Things go wrong when people start obsessing over every point and think that their GPA defines them. (I'm speaking from personal experience on both sides of the desk.)

“If you could spend 1 year devoted to solving any problem, what problem would you choose?”

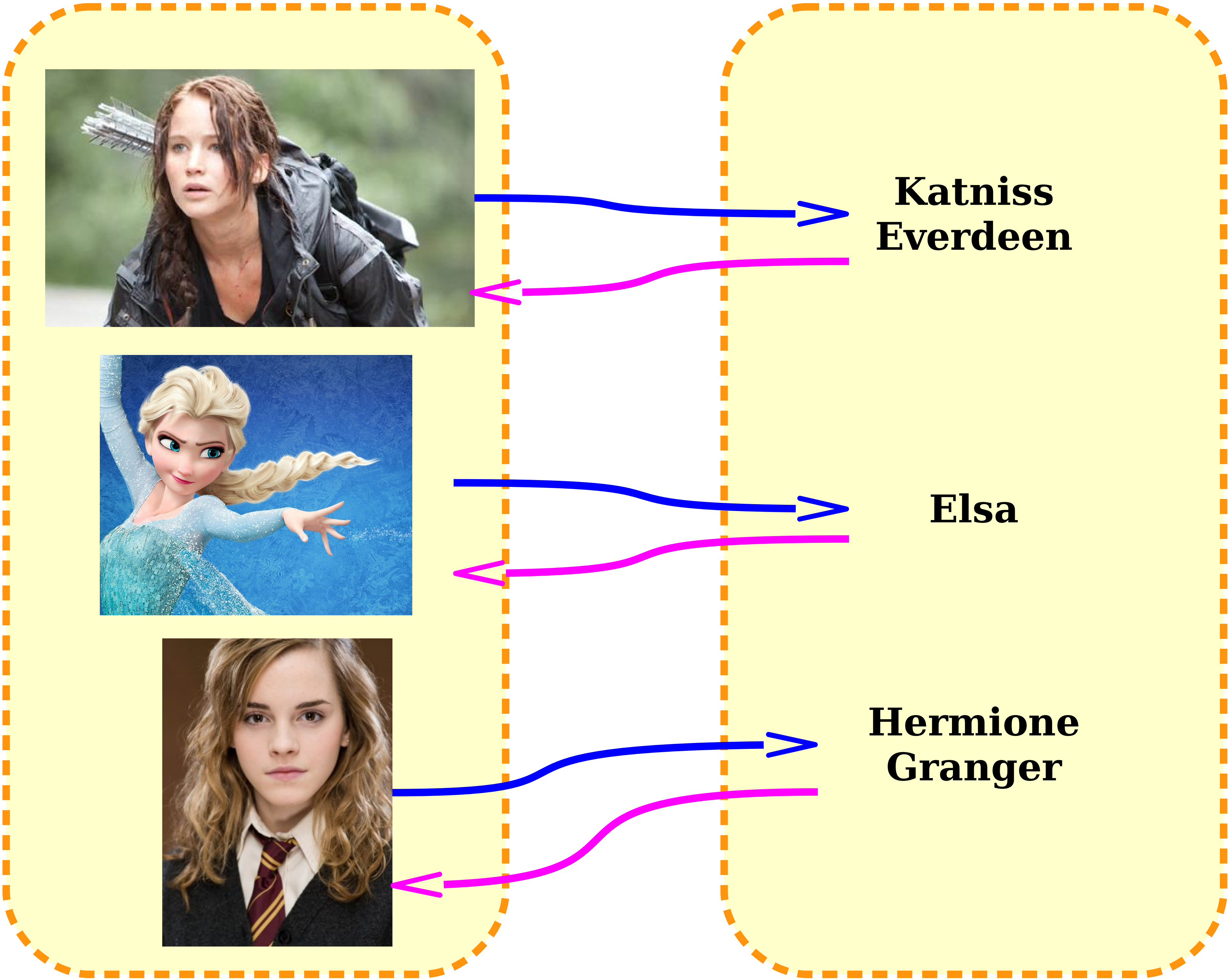
I had to really think about this one. I think I'd probably work trying to help refugees who have arrived in the US resettle. It's something where just writing a check doesn't do too much and where the human connection is critical.

“Why is the lecture hall so cold? :(”

This room can get miserably hot if the AC is off. I assume they're just keeping it on because the alternative is so much worse.

Back to CS103!

Inverse Functions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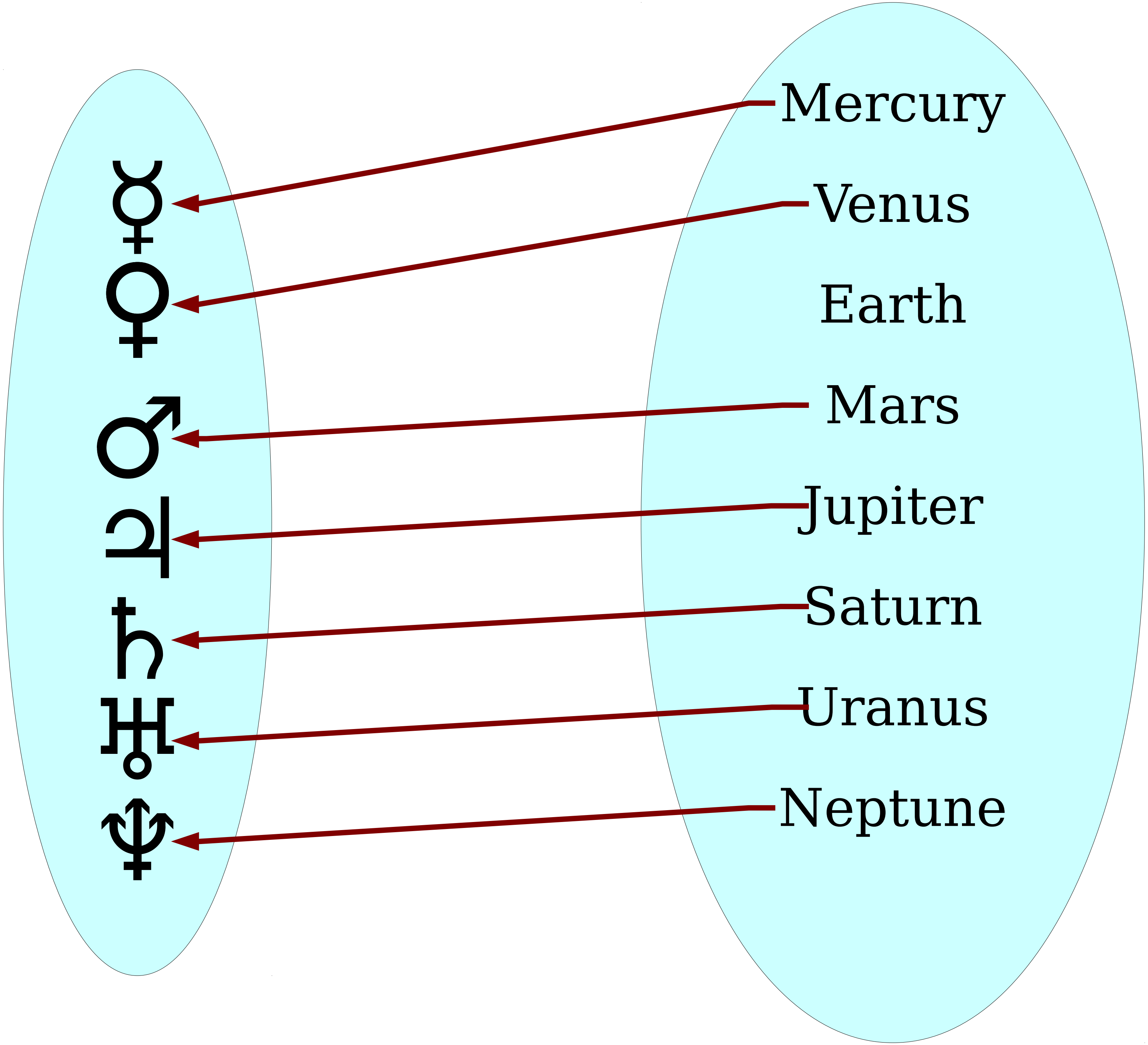
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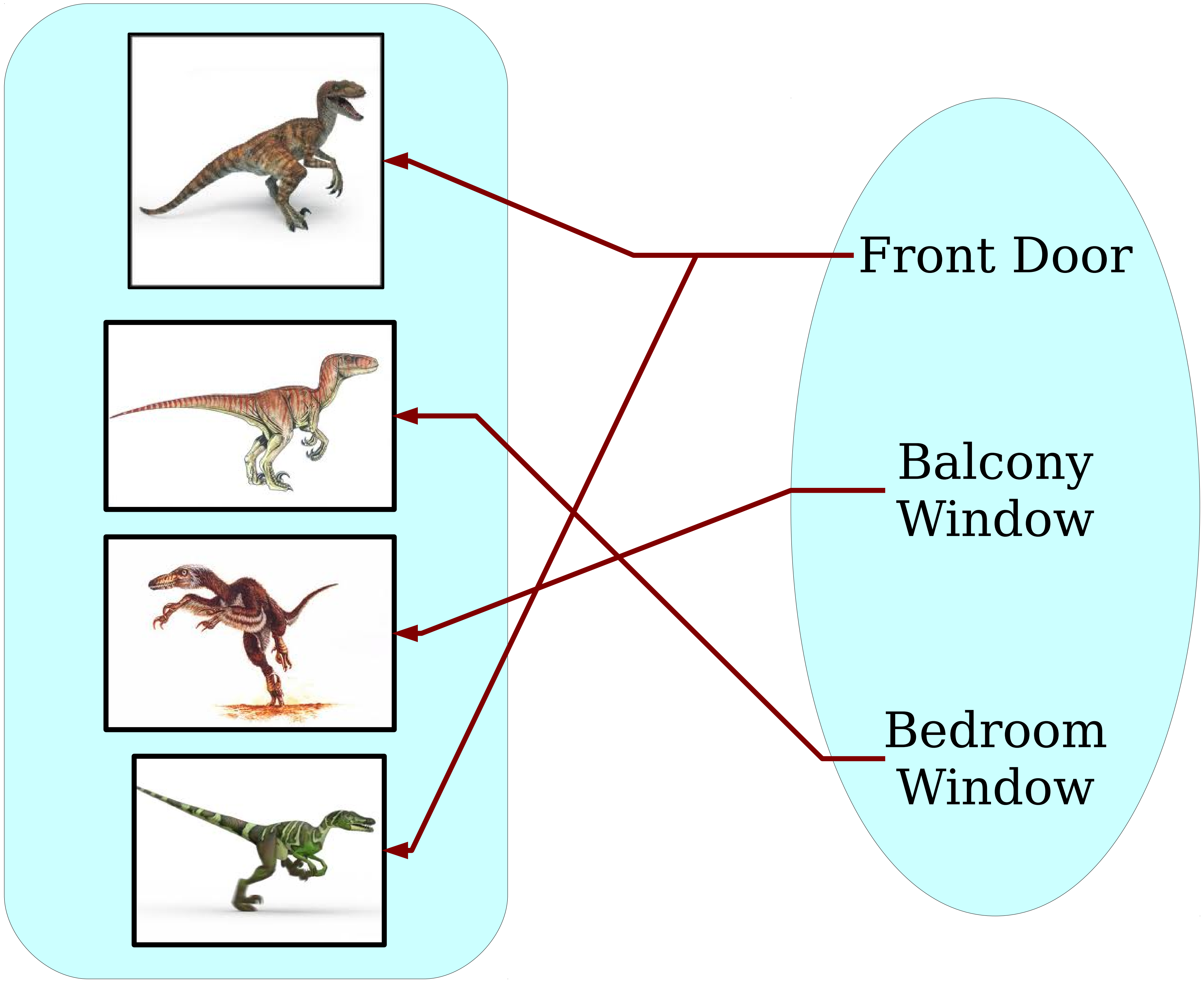
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Inverse Functions

- In some cases, it's possible to "turn a function around."
- Let $f : A \to B$ be a function. A function $f^{-1} : B \to A$ is called the ***inverse of f*** if the following is true:

$$\forall a \in A.\ \forall b \in B.\ (f(a) = b \leftrightarrow f^{-1}(b) = a)$$

- In other words, if f maps a to b, then f^{-1} maps b back to a.
- Not all functions have inverses (we just saw a few).
- If f is a function that has an inverse, then we say that f is ***invertible***.

A Useful Fact about Inverses

Lemma: Let $f : A \to B$ be invertible and let $f^{-1} : B \to A$ be its inverse. Then for any $a \in A$ and for any $b \in B$, we have $f^{-1}(f(a)) = a$ and $f(f^{-1}(b)) = b$.

Proof: First, consider any $a \in A$. We will prove that $f^{-1}(f(a)) = a$. To see this, let $b = f(a)$.

What does it mean that f^{-1} is the inverse of f?

$$\forall a \in A.\ \forall b \in B.\ f(a) = b \leftrightarrow f^{-1}(b) = a$$

Lemma: Let $f : A \to B$ be invertible and let $f^{-1} : B \to A$ be its inverse. Then for any $a \in A$ and for any $b \in B$, we have $f^{-1}(f(a)) = a$ and $f(f^{-1}(b)) = b$.

Proof: First, consider any $a \in A$. We will prove that $f^{-1}(f(a)) = a$. To see this, let $b = f(a)$. Since $f(a) = b$ and f^{-1} is the inverse of a, we see that

$$f^{-1}(b) = a. \qquad (1)$$

Substituting $b = f(a)$ into equation (1) tells us $f^{-1}(f(a)) = a$, as required.

Next, consider any $b \in B$. We will prove that $f(f^{-1}(b)) = b$. To see this, let $a = f^{-1}(b)$. Since f^{-1} is the inverse of f, this means that

$$f(a) = b. \qquad (2)$$

Plugging $a = f^{-1}(b)$ into equation (2) tells us $f(f^{-1}(b)) = b$, as required. ■

Which functions have inverses?

Inverse Functions

- ***Theorem:*** Let $f : A \to B$. Then f is invertible if and only if f is a bijection.
- To prove this result, we need to prove that
  - if $f : A \to B$ is invertible, then f is a bijection, and
  - if $f : A \to B$ is a bijection, then f is invertible.
- These are separate steps, so we'll do each one individually.

Theorem: If $f : A \to B$ is invertible, then f is a bijection.

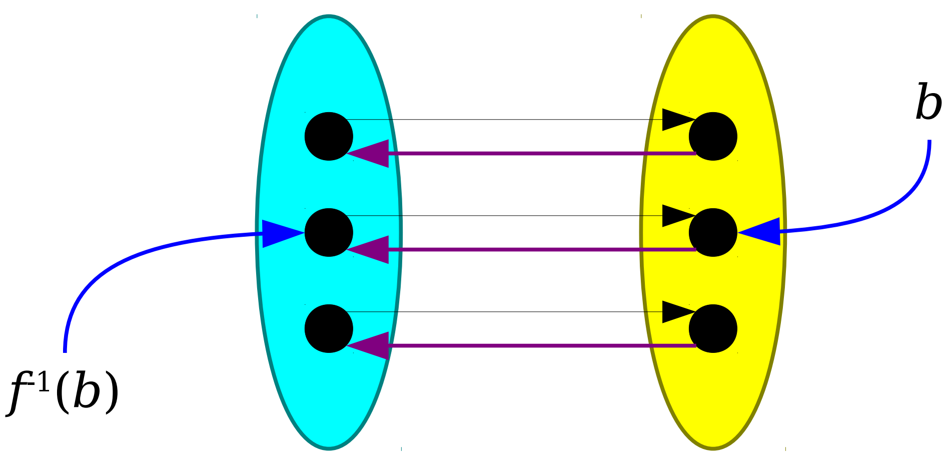
Proof: Let $f : A \to B$ be an invertible function and let f^{-1} be its inverse. We need to prove that f is a bijection, so we will show that f is injective and surjective.

First, we'll prove that f is injective. To do so, consider any $a_1, a_2 \in A$ where $f(a_1) = f(a_2)$. We need to show that $a_1 = a_2$. Applying f^{-1} to both sides of $f(a_1) = f(a_2)$ tells us that

$$f^{-1}(f(a_1)) = f^{-1}(f(a_2)).$$

Using our lemma, we then see that $a_1 = a_2$, as required.

Next, we will prove that f is surjective. To do so, consider any $b \in B$. We need to show that there is an $a \in A$ such that $f(a) = b$.



Theorem: If $f : A \to B$ is invertible, then f is a bijection.

Proof: Let $f : A \to B$ be an invertible function and let f^{-1} be its inverse. We need to prove that f is a bijection, so we will show that f is injective and surjective.

First, we'll prove that f is injective. To do so, consider any $a_1, a_2 \in A$ where $f(a_1) = f(a_2)$. We need to show that $a_1 = a_2$. Applying f^{-1} to both sides of $f(a_1) = f(a_2)$ tells us that

$$f^{-1}(f(a_1)) = f^{-1}(f(a_2)).$$

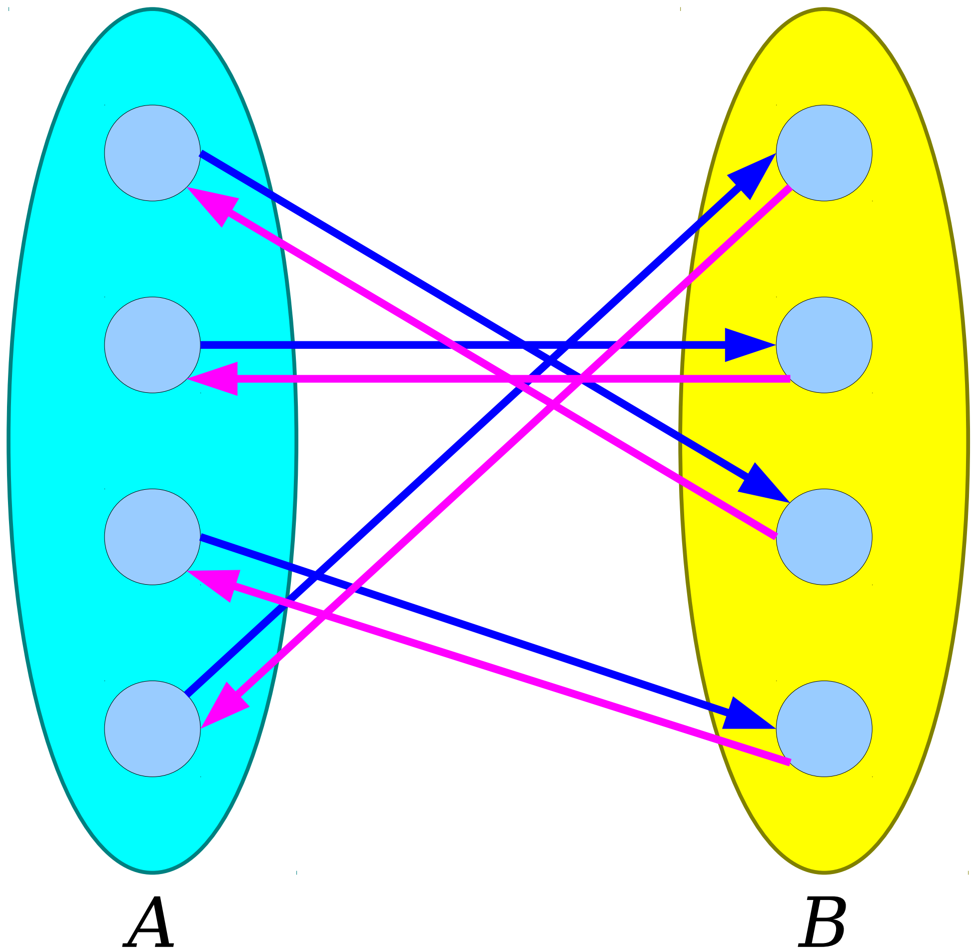
Using our lemma, we then see that $a_1 = a_2$, as required.

Next, we will prove that f is surjective. To do so, consider any $b \in B$. We need to show that there is an $a \in A$ such that $f(a) = b$. Let $a = f^{-1}(b)$. Then

$$f(a) = f(f^{-1}(b)) = b.$$

So there is an $a \in A$ (namely, $f^{-1}(b)$) such that $f(a) = b$, as required. ■

Bijections are Invertible



Define $f^{1}(b)$ to be "the unique choice of a where $f(a) = b$"

$$\forall a \in A.\ \forall b \in B.\ f(a) = b \leftrightarrow f^{-1}(b) = a$$

Theorem: If $f : A \to B$ is a bijection, then f is invertible.

Proof: Let $f : A \to B$ be a bijection. We need to show that there is a function $f^{1} : B \to A$ such that $f(a) = b$ if and only if $f^{1}(b) = a$.

Let's begin by proving the following, helpful fact: for any $b \in B$, there is exactly one $a \in A$ such that $f(a) = b$.

How do you express uniqueness in first-order logic?

$$
\begin{aligned}
&\exists a \in A.\ (f(a) = b\ \land \\
&\qquad \forall a' \in A.\ (a' \neq a \to f(a') \neq b)) \\
&)
\end{aligned}
$$

We'll therefore prove that there is ***at least*** one a where $f(a) = b$, then will prove that if you pick any other $a' \in A$, then $f(a') \neq b$.

Theorem: If $f : A \to B$ is a bijection, then f is invertible.

Proof: Let $f : A \to B$ be a bijection. We need to show that there is a function $f^{1} : B \to A$ such that $f(a) = b$ if and only if $f^{1}(b) = a$.

Let's begin by proving the following, helpful fact: for any $b \in B$, there is exactly one $a \in A$ such that $f(a) = b$.

First, we will prove that there is *at least* one $a \in A$ such that $f(a) = b$. Because f is a bijection, we know it's surjective, and therefore that for any $b \in B$ there is at least one $a \in A$ such that $f(a) = b$.

Next, we will prove that there is *at most* one $a \in A$ such that $f(a) = b$.

$$\begin{aligned}&\exists a \in A.\ (f(a) = b\ \land \\ &\qquad\qquad \forall a' \in A.\ (a' \neq a \to f(a') \neq b)) \\ &)\end{aligned}$$

Theorem: If $f : A \to B$ is a bijection, then f is invertible.

Proof: Let $f : A \to B$ be a bijection. We need to show that there is a function $f^1 : B \to A$ such that $f(a) = b$ if and only if $f^1(b) = a$.

Let's begin by proving the following, helpful fact: for any $b \in B$, there is exactly one $a \in A$ such that $f(a) = b$.

First, we will prove that there is *at least* one $a \in A$ such that $f(a) = b$. Because f is a bijection, we know it's surjective, and therefore that for any $b \in B$ there is at least one $a \in A$ such that $f(a) = b$.

Next, we will prove that there is *at most* one $a \in A$ such that $f(a) = b$. Suppose that $f(a) = b$ and consider any $a' \in A$ where $a' \neq a$. Because f is injective and $a' \neq a$, we know $f(a') \neq f(a)$. Since $f(a) = b$, this means that $f(a') \neq b$. Thus there is at most one $a \in A$ where $f(a) = b$.

Now consider the function $f^1 : B \to A$ defined as follows: let $f^1(b)$ be the unique choice of $a \in A$ for which $f(a) = b$. This function is well-defined because, as we proved above, for any $b \in B$ there is a unique $a \in A$ where $f(a) = b$. Notice further that this definition of f^1 ensures that f^1 is inverse of f; by definition $f(a) = b$ if and only if $f^1(b) = a$.

Overall, we have shown that if f is a bijection, then there exists a function f^1 which is the inverse of f. Therefore, if f is a bijection, then f is invertible. ■

Where We Are

- Phew! That was a lot of stuff!
- We now know
  - what an injection, surjection, and bijection are;
  - that injections, surjections, and bijections are closed under composition; and
  - that bijections are invertible and invertible functions are bijections.
- You might wonder why this all matters. Well, there's a good reason...

Cardinality Revisited

Cardinality

- Recall *(from our first lecture!)* that the ***cardinality*** of a set is the number of elements it contains.
- If S is a set, we denote its cardinality by $|S|$.
- For finite sets, cardinalities are natural numbers:
  - $|\{1, 2, 3\}| = 3$
  - $|\{100, 200\}| = 2$
- For infinite sets, we introduced ***infinite cardinals*** to denote the size of sets:

$$|\mathbb{N}| = \aleph_0$$

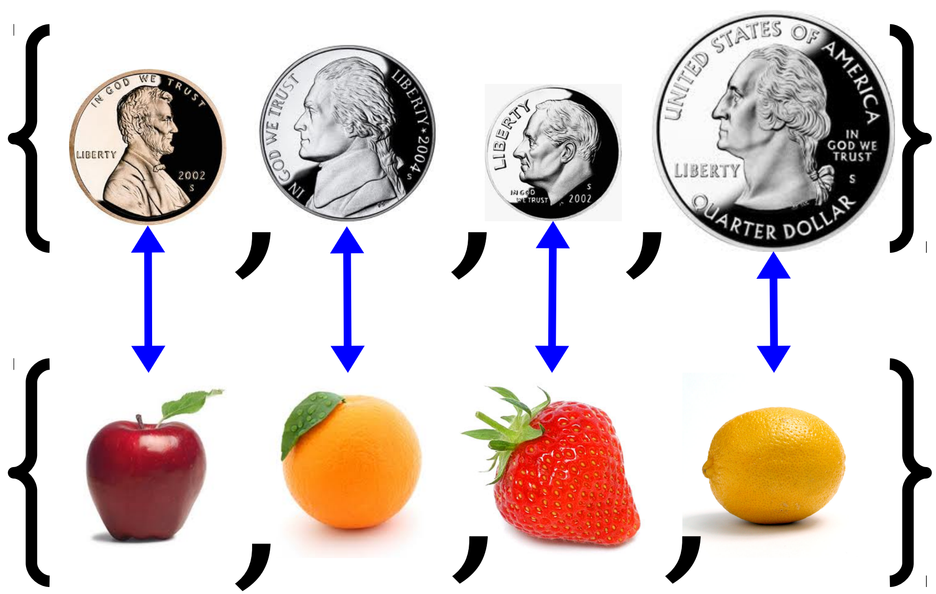
Defining Cardinality

- It is difficult to give a rigorous definition of what cardinalities actually are.
  - What is 4? What is $\aleph_0$?
- ***Idea:*** Define cardinality as a *relation* between two sets rather than as an absolute quantity.

Comparing Cardinalities

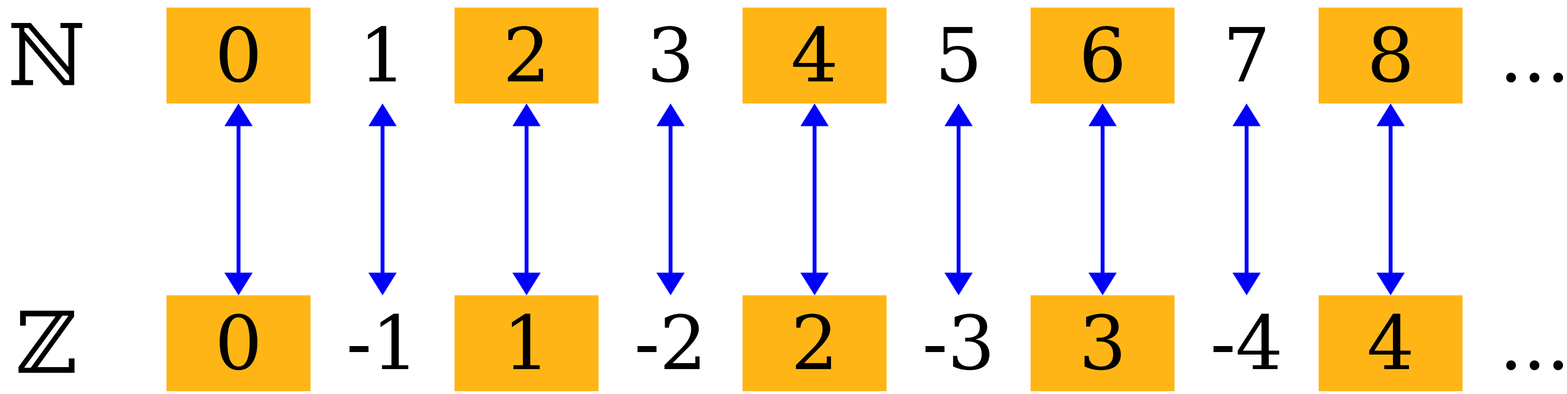
- The relationships between set cardinalities are defined in terms of functions between those sets.
- Here is the formal definition of what it means for two sets to have the same cardinality:

$|S| = |T|$ if there exists a *bijection* $f : S \to T$



Infinity is Weird…

Infinite Cardinalities



$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ & \end{cases}$$

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
$\mathbb{Z}$	0	-1	1	-2	2	-3	3	-4	4	...

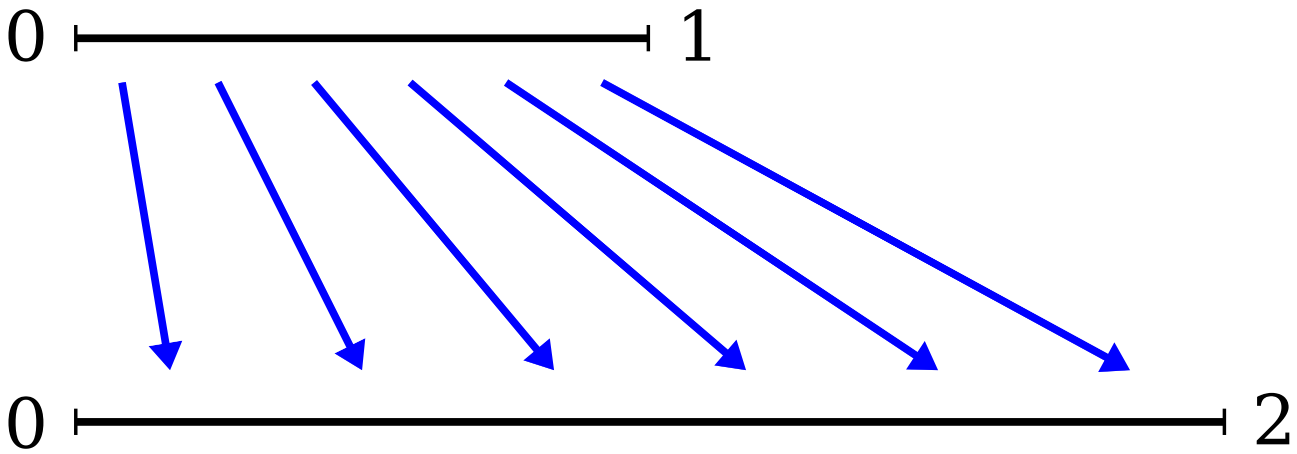
$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ & \end{cases}$$

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
$\mathbb{Z}$	0	-1	1	-2	2	-3	3	-4	4	...

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -(n+1)/2 & \text{otherwise} \end{cases}$$

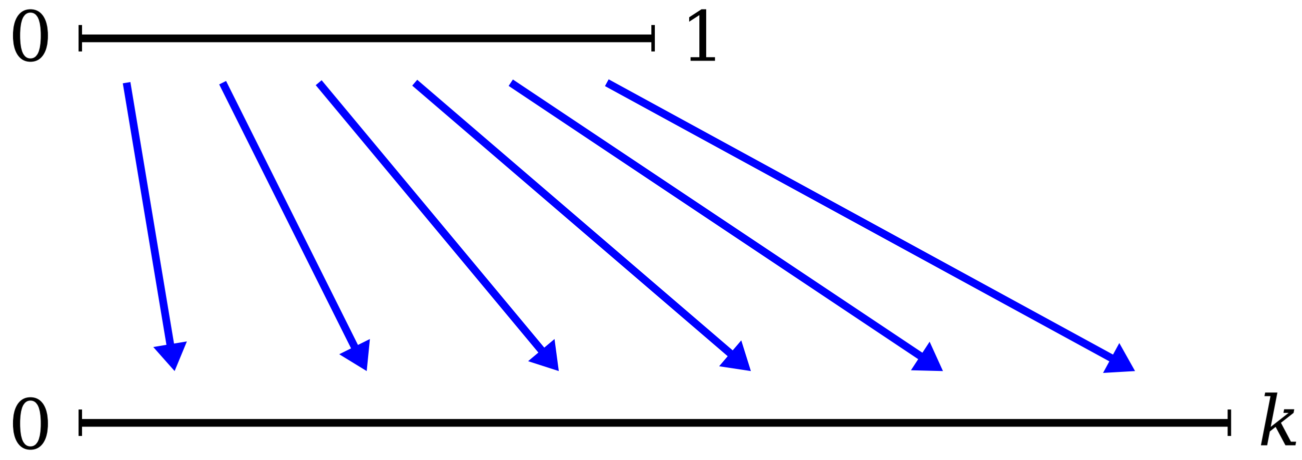
Home on the Range



$$f : [0, 1] \to [0, 2]$$
$$f(x) = 2x$$

$$|[0, 1]| = |[0, 2]|$$

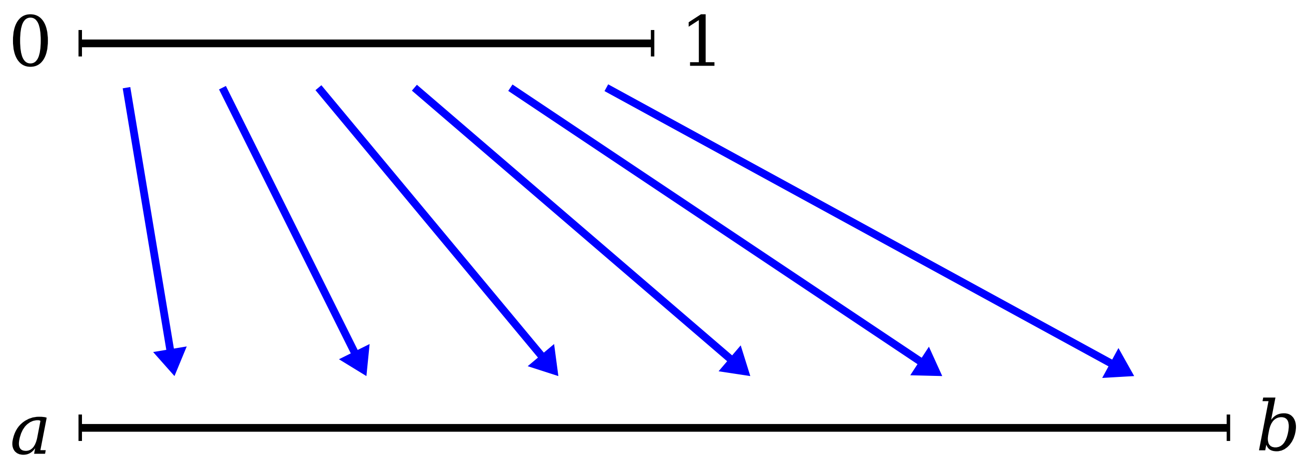
Home on the Range



$$f : [0, 1] \to [0, k]$$
$$f(x) = kx$$

$$|[0, 1]| = |[0, k]|$$

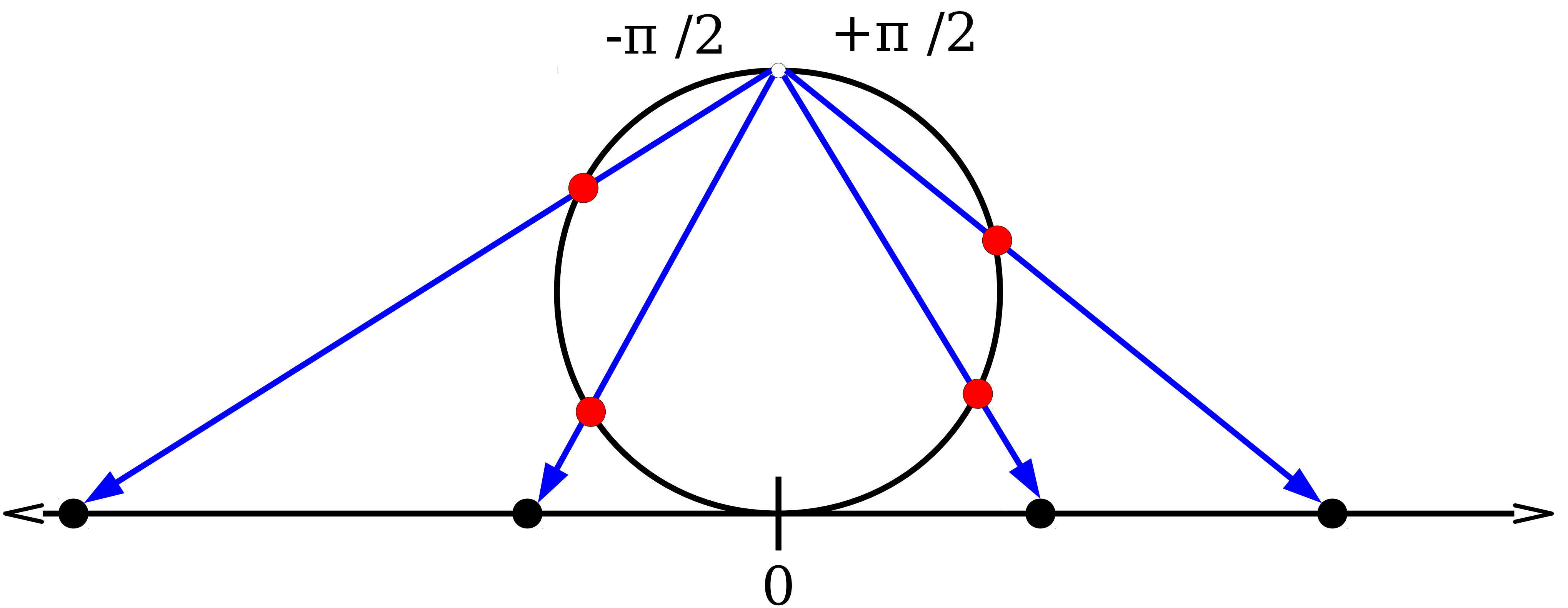
Home on the Range



$$f : [0, 1] \to [a, b]$$
$$f(x) = (b - a)x + a$$

$$|[0, 1]| = |[a, b]|$$

Put a Ring On It



$f : (-\pi/2, \pi/2) \to \mathbb{R}$

$f(x) = \tan x$

$|(-\pi/2, \pi/2)| = |\mathbb{R}|$

Properties of Cardinality

- For any sets R, S, and T, the following are true:
  - $|S| = |S|$.
    - Define $f : S \to S$ as $f(x) = x$.
  - If $|S| = |T|$, then $|T| = |S|$.
    - If $f : S \to T$ is a bijection, then $f^{-1} : T \to S$ is a bijection.
  - If $|R| = |S|$ and $|S| = |T|$, then $|R| = |T|$.
    - If $f : R \to S$ and $g : S \to T$ are bijections, then $g \circ f : R \to T$ is a bijection.