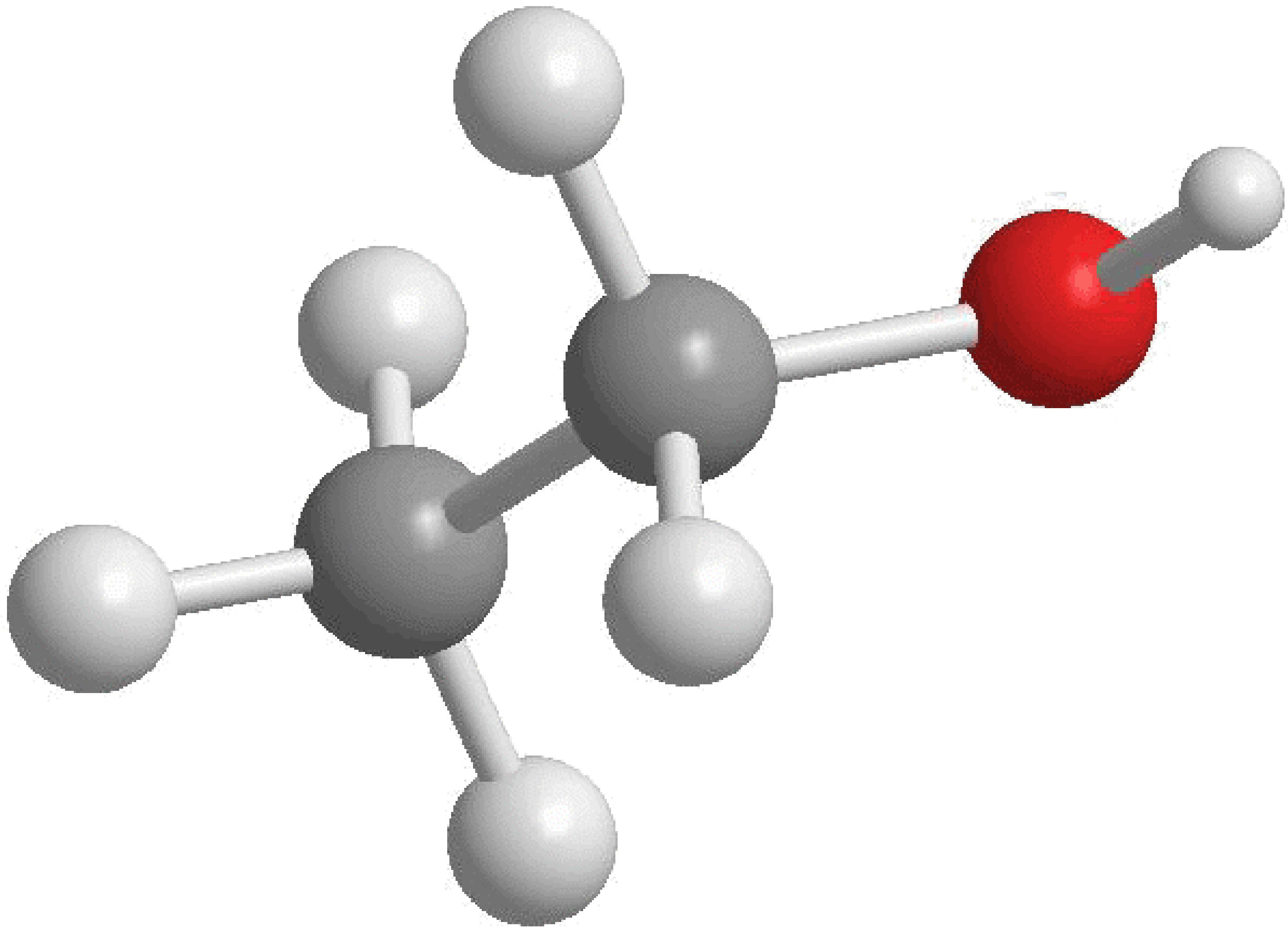
Graphs

Chemical Bonds



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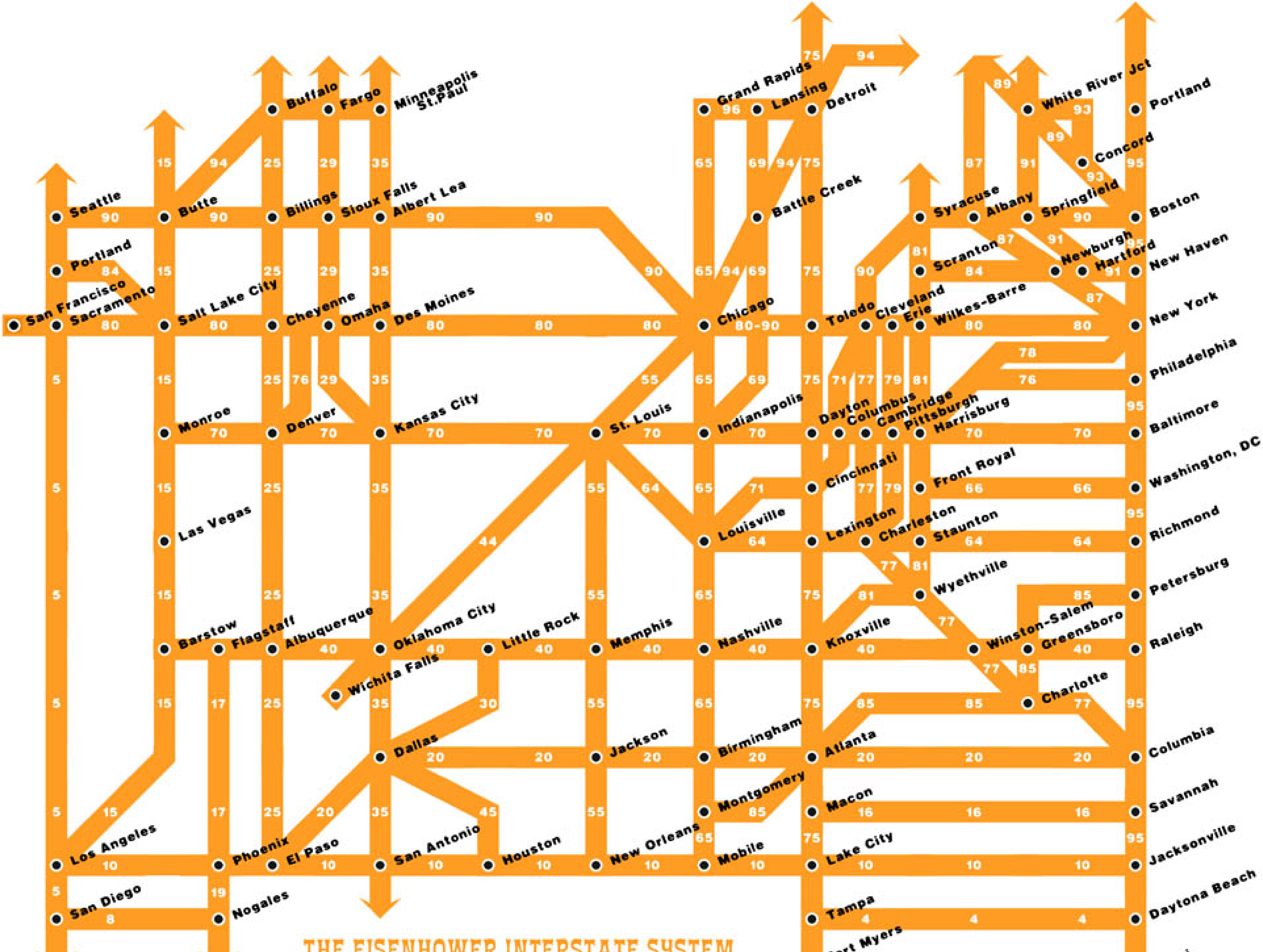
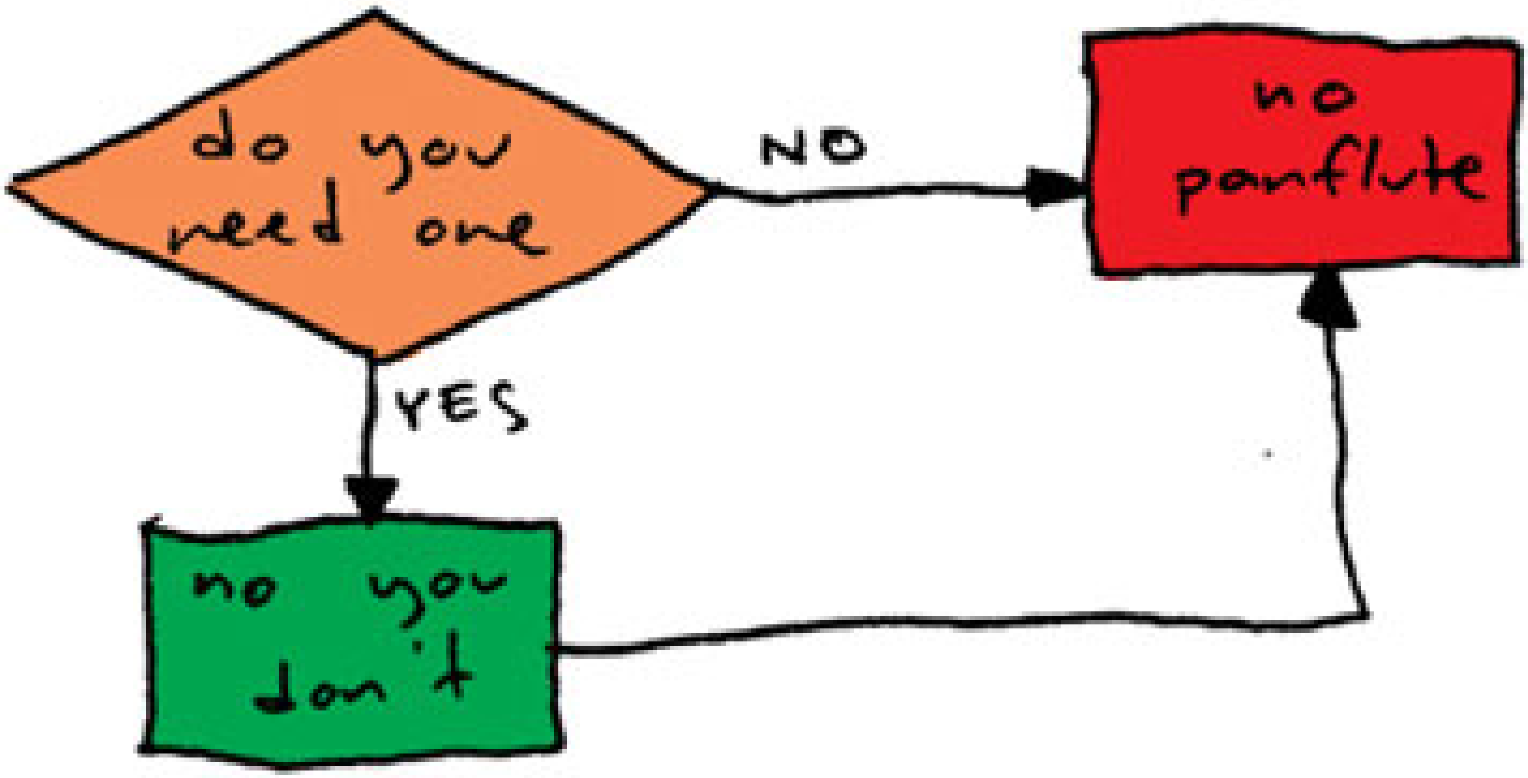


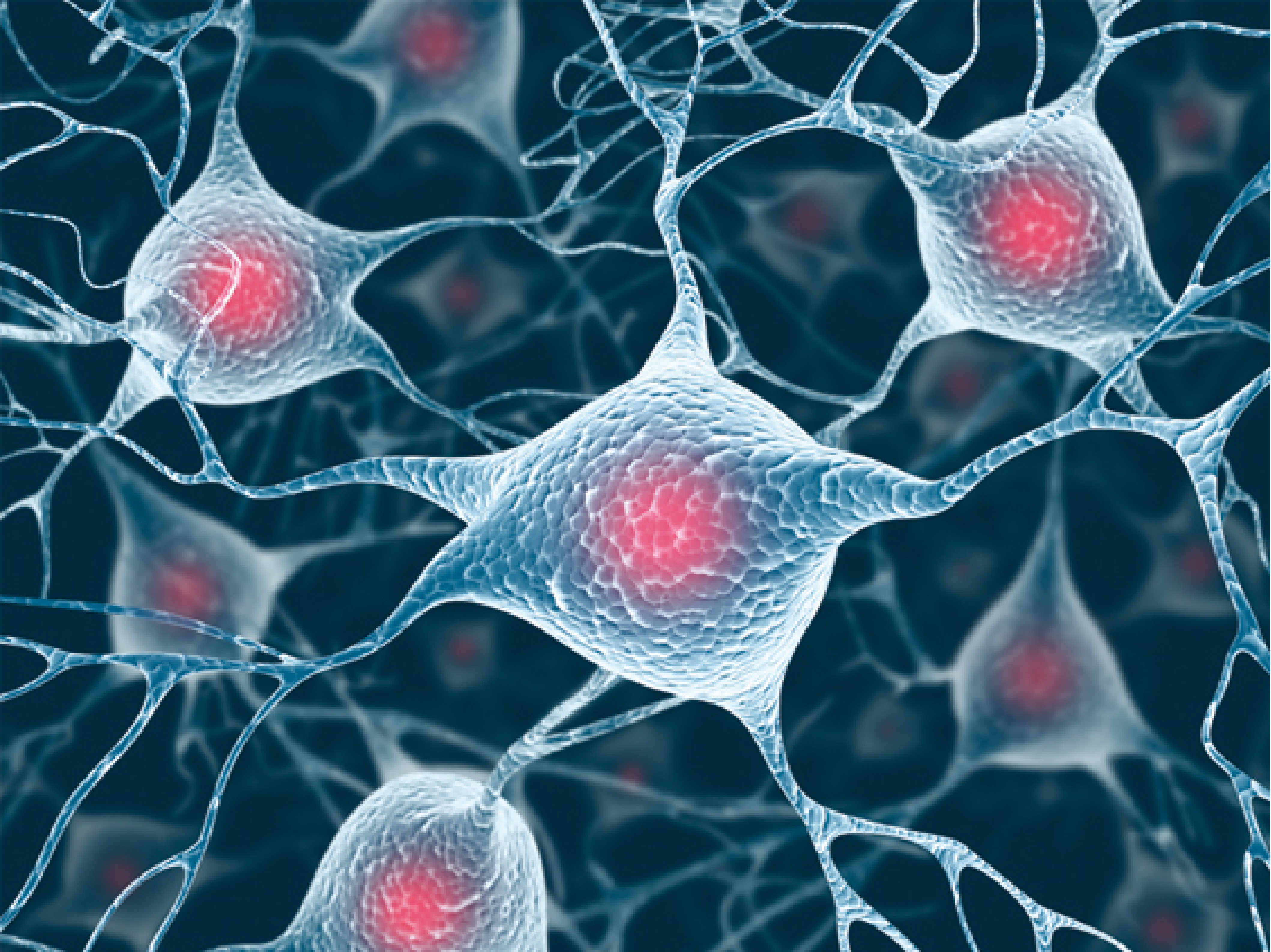
THE EISENHOWER INTERSTATE SYSTEM

(simplified)

CHRIS YATES 2007

PANFLUTE FLOWCHART



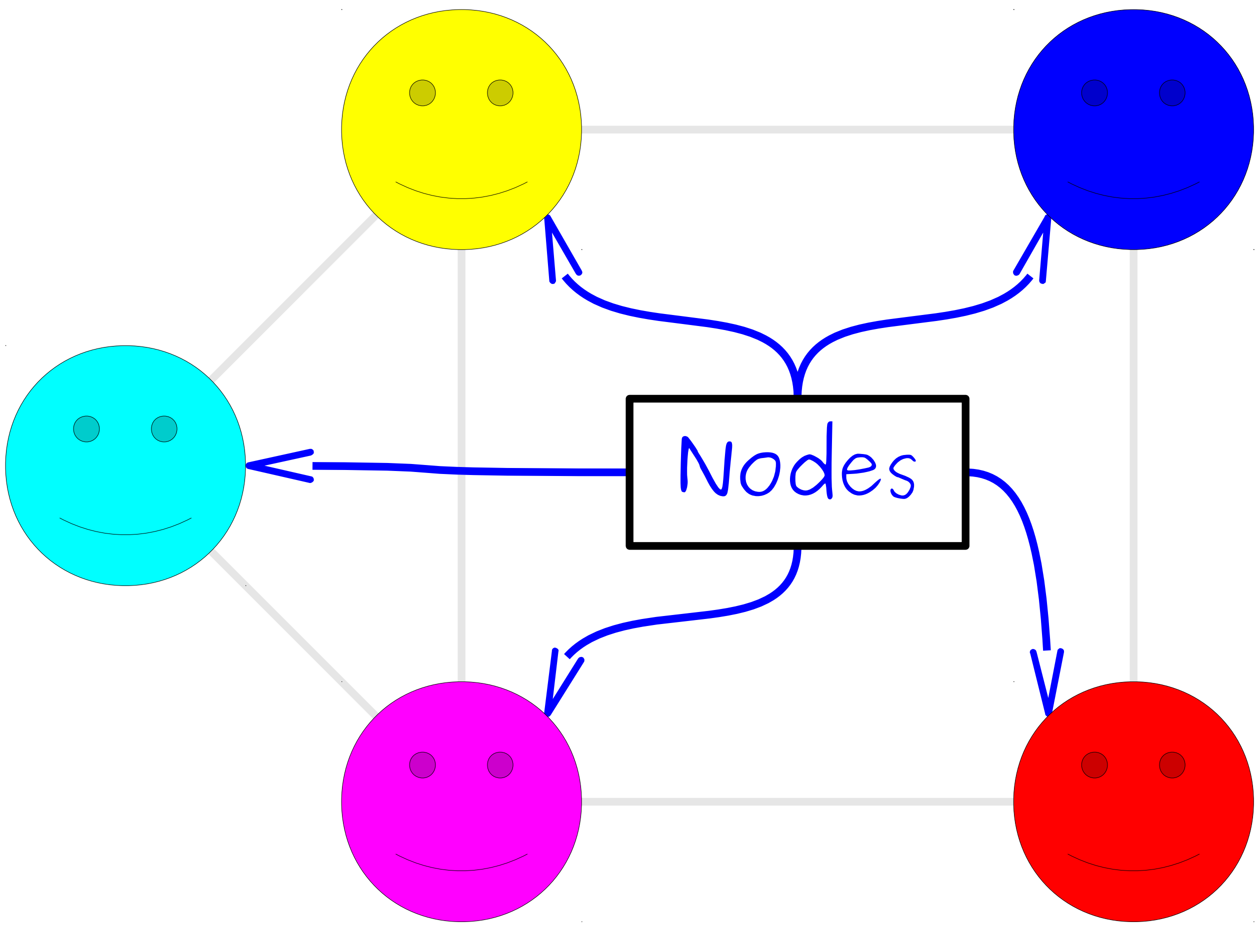


Me too!

What's in Common

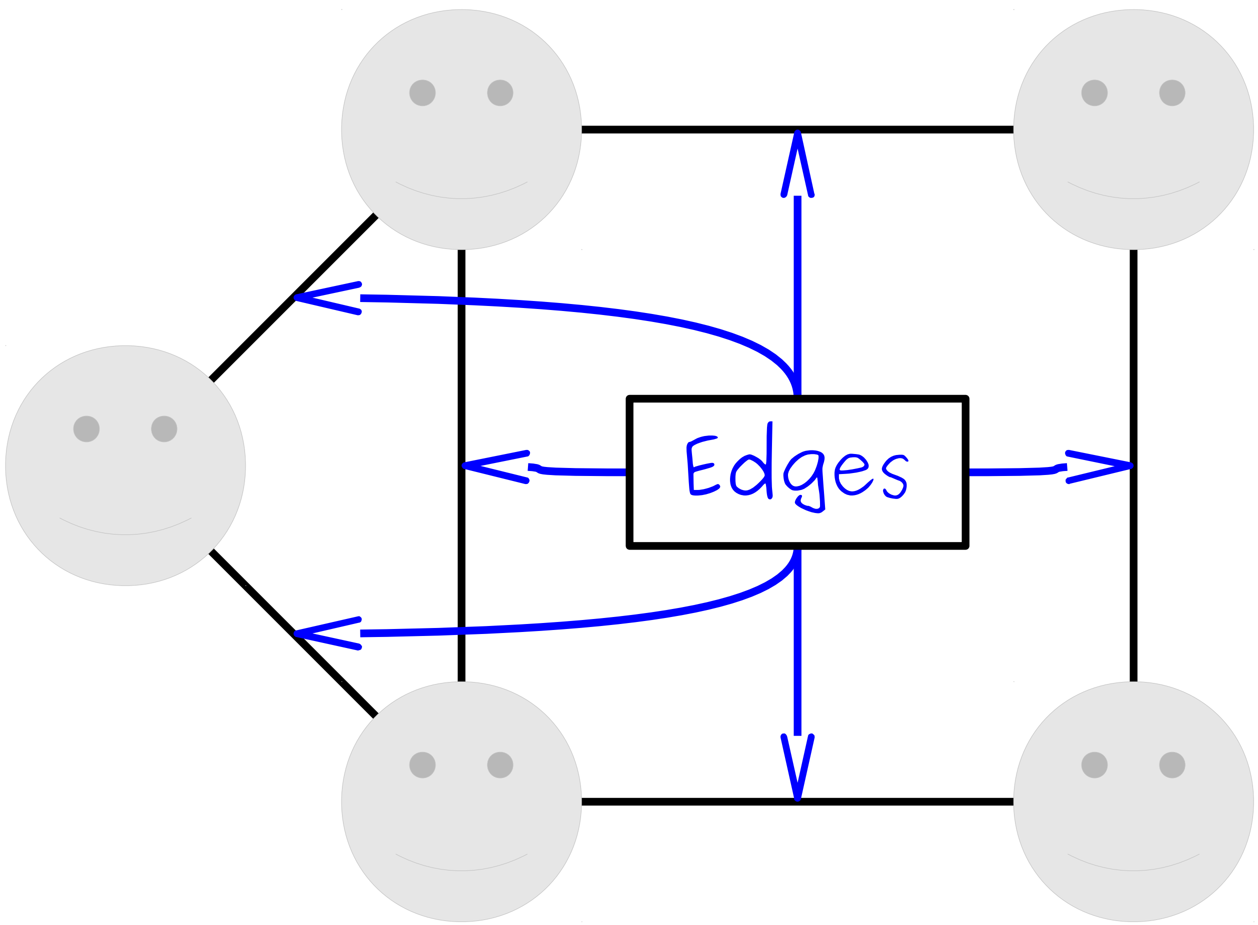
- Each of these structures consists of
  - Individual objects and
  - Links between those objects.
- Goal: find a general framework for describing these objects and their properties.

A ***graph*** is a mathematical structure for representing relationships.



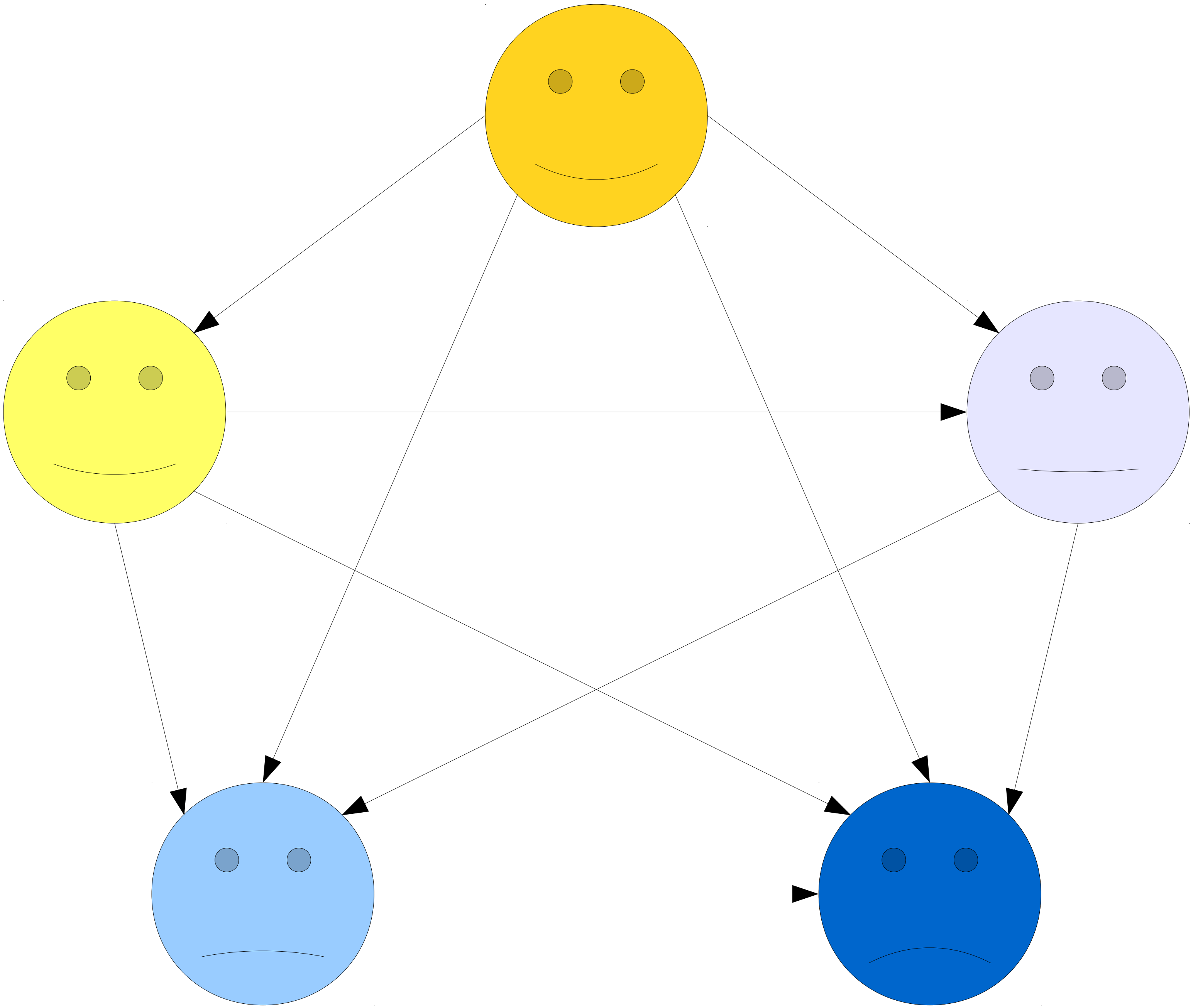
A graph consists of a set of ***nodes*** (or ***vertices***) connected by ***edges*** (or ***arcs***)

A ***graph*** is a mathematical structure for representing relationships.

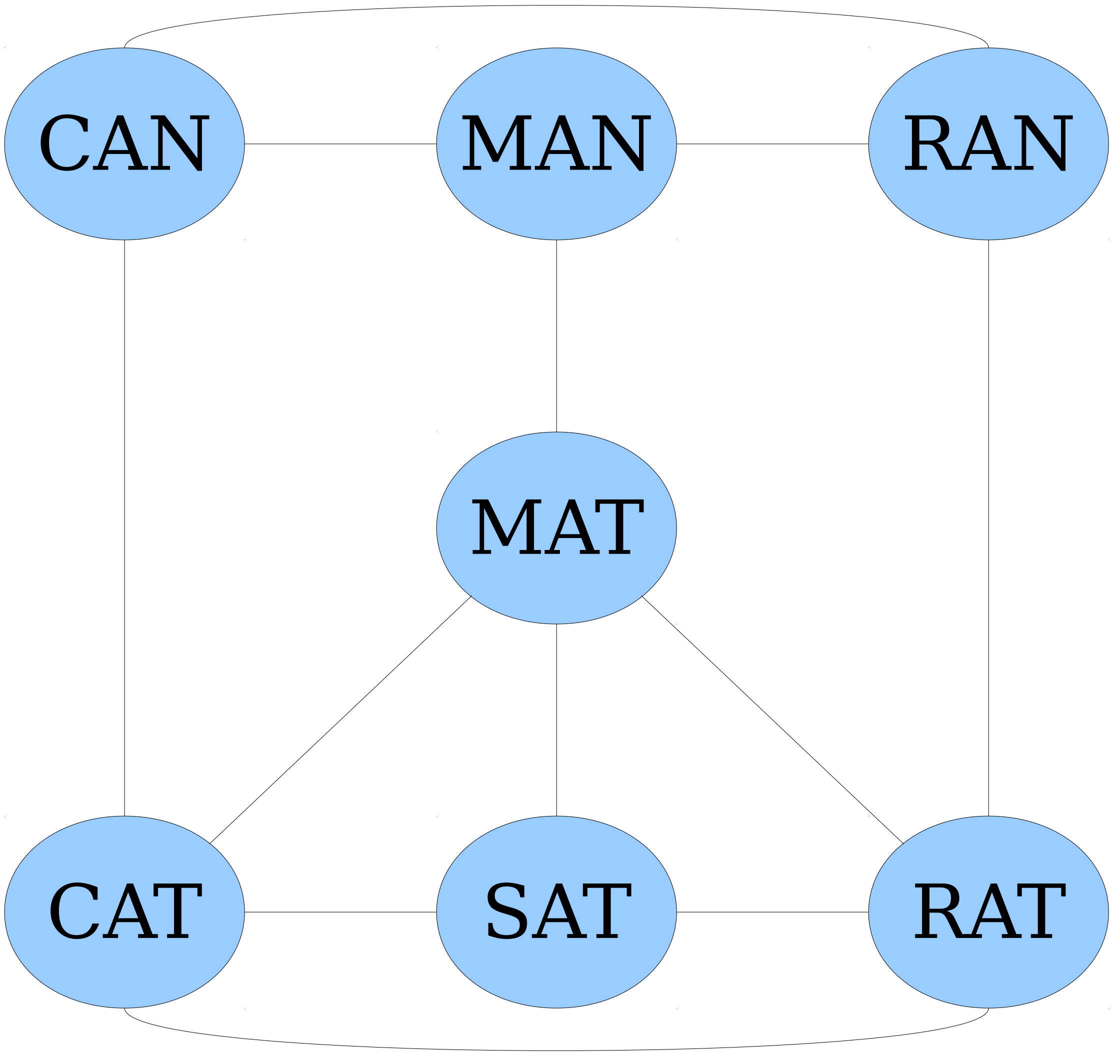


A graph consists of a set of ***nodes*** (or ***vertices***) connected by ***edges*** (or ***arcs***)

Some graphs are ***directed***.



Some graphs are ***undirected***.



Going forward, we're primarily going to focus on undirected graphs.

The term “graph” generally refers to undirected graphs unless specified otherwise.

Formalizing Graphs

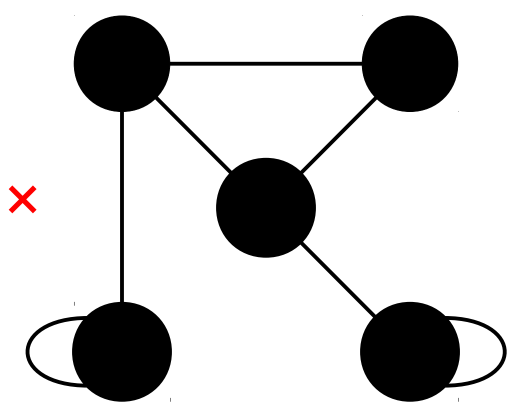
- How might we define a graph mathematically?
- Need to specify
  - What the nodes in the graph are, and
  - What the edges are in the graph.
- The nodes can be pretty much anything.
- What about the edges?

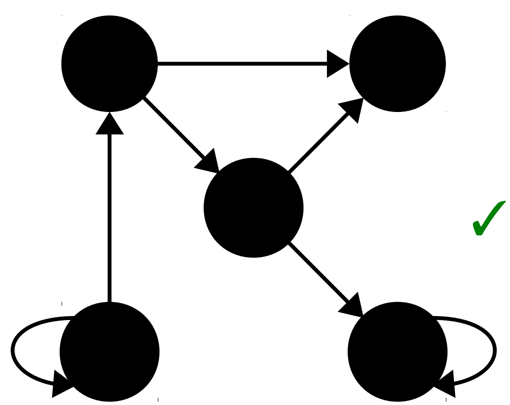
Formalizing Graphs

- An ***unordered pair*** is a set $\{a, b\}$ of two elements (remember that sets are unordered).
  - $\{0, 1\} = \{1, 0\}$
- Formally, an undirected graph is an ordered pair $G = (V, E)$, where
  - V is a set of nodes, which can be anything, and
  - E is a set of edges, which are unordered pairs of nodes.
- Formally, a directed graph is an ordered pair $G = (V, E)$, where
  - V is a set of nodes, which can be anything, and
  - E is a set of edges, which are *ordered* pairs of nodes.

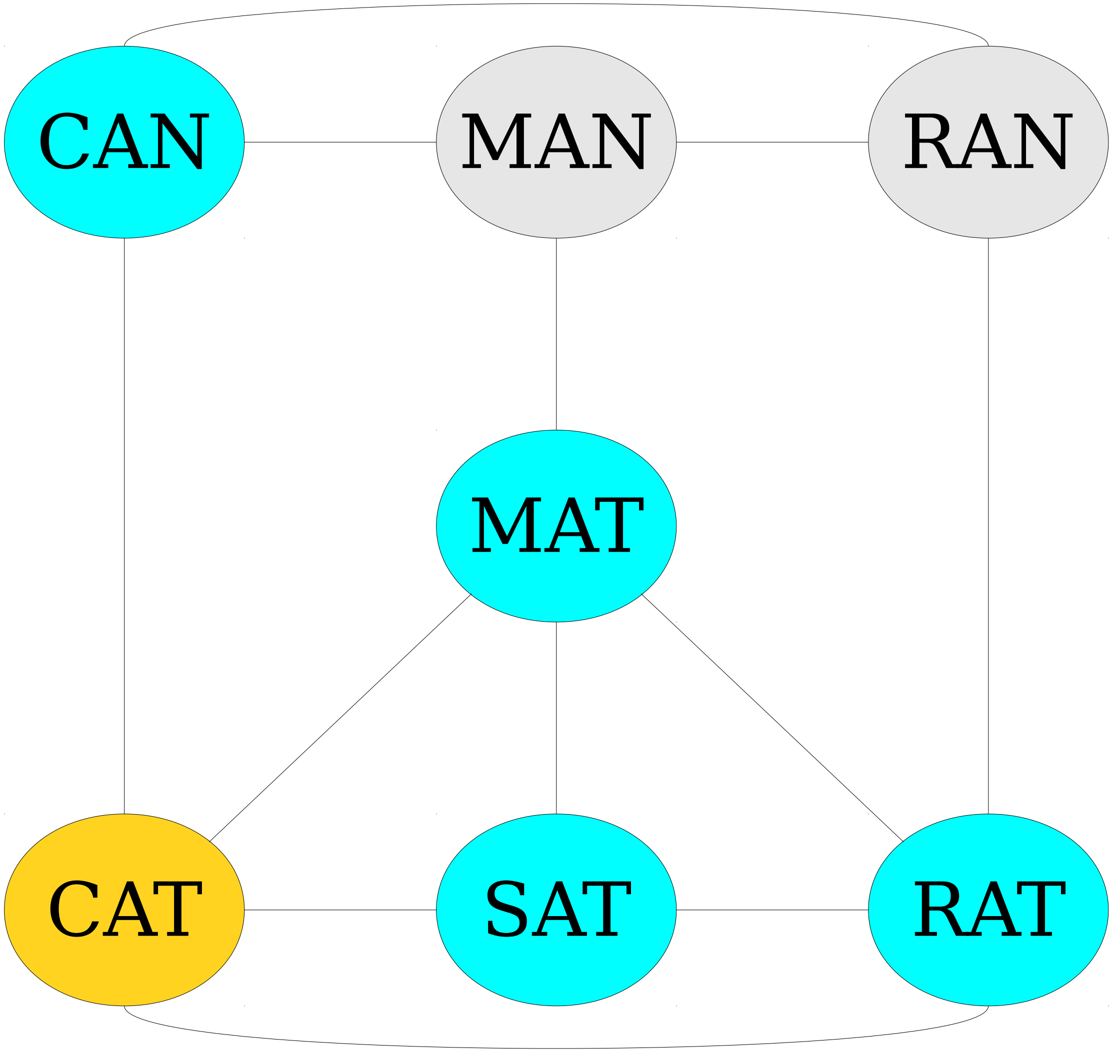
Self-Loops

- An edge from a node to itself is called a ***self-loop***.
- In undirected graphs, self-loops are generally not allowed unless specified otherwise.
  - This is mostly to keep the math easier. If you allow self-loops, a lot of results get messier and harder to state.
- In directed graphs, self-loops are generally allowed unless specified otherwise.





Standard Graph Terminology

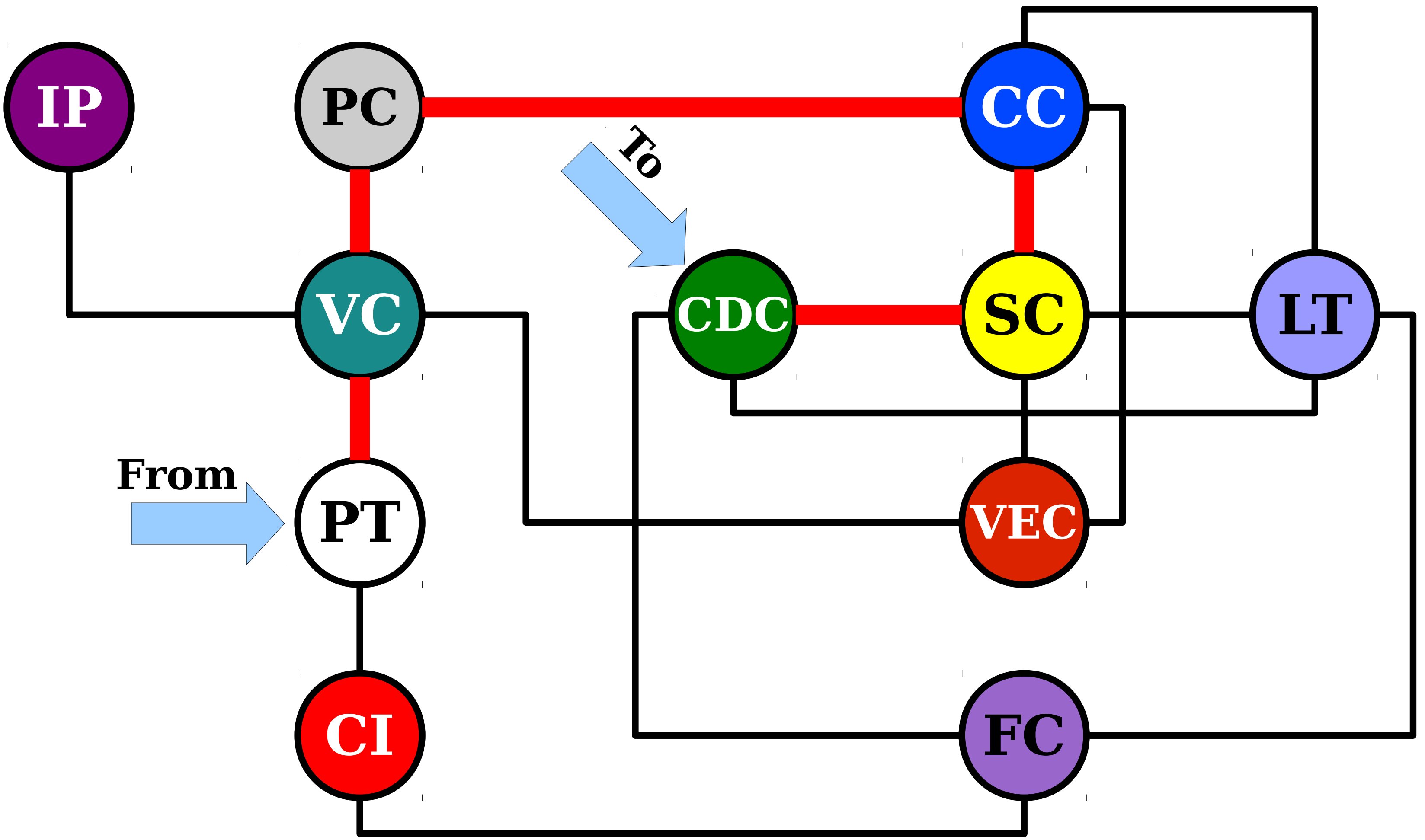


Two nodes are called ***adjacent*** if there is an edge between them.

Using our Formalisms

- Let $G = (V, E)$ be a graph.
- Intuitively, two nodes are adjacent if they're linked by an edge.
- Formally speaking, we say that two nodes $u, v \in V$ are ***adjacent*** if $\{u, v\} \in E$.

Navigating a Graph



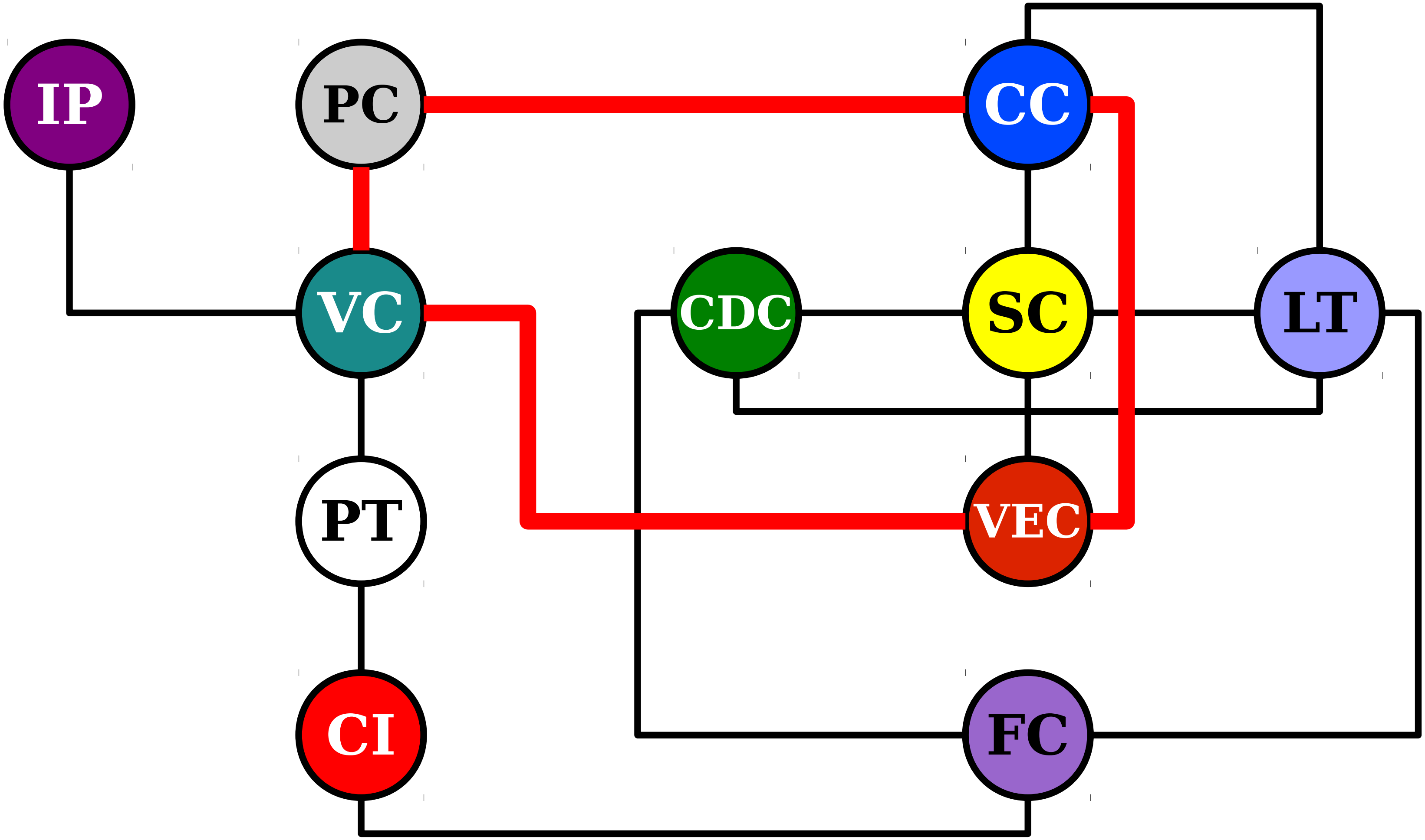
PT → VC → PC → CC → SC → CDC

A ***path*** from v_1 to v_n is a sequence of nodes $v_1, v_2, \ldots, v_n$ where $\{v_k, v_{k+1}\} \in E$ for all natural numbers in the range $1 \leq k \leq n - 1$.

The ***length*** of a path is the number of edges it contains, which is one less than the number of nodes in the path.

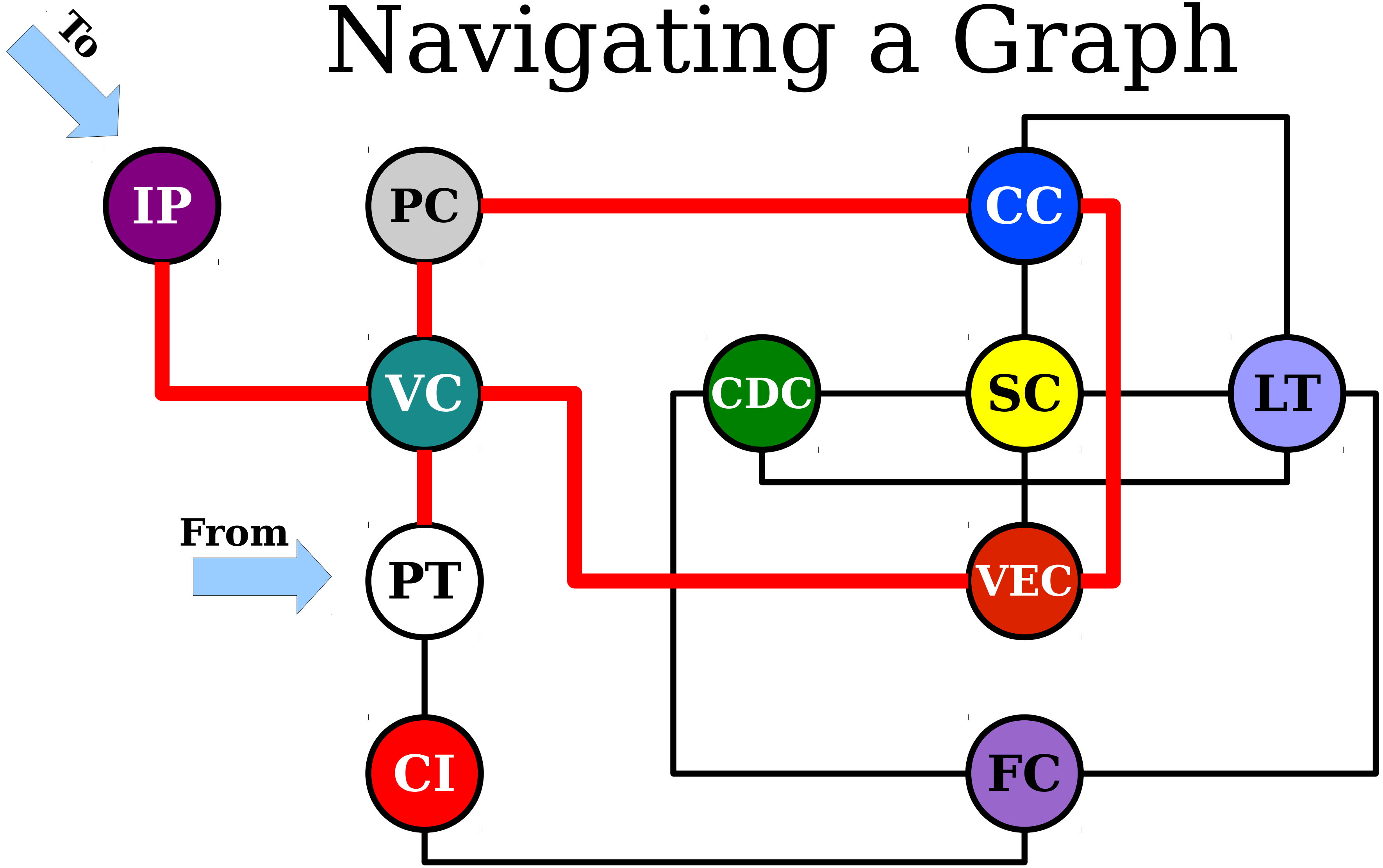
As an edge case, paths of length zero are allowed.

Navigating a Graph



PC → CC → VEC → VC → PC

Navigating a Graph



PT → VC → PC → CC → VEC → VC → IP

A ***cycle*** in a graph is a path from a node to itself.

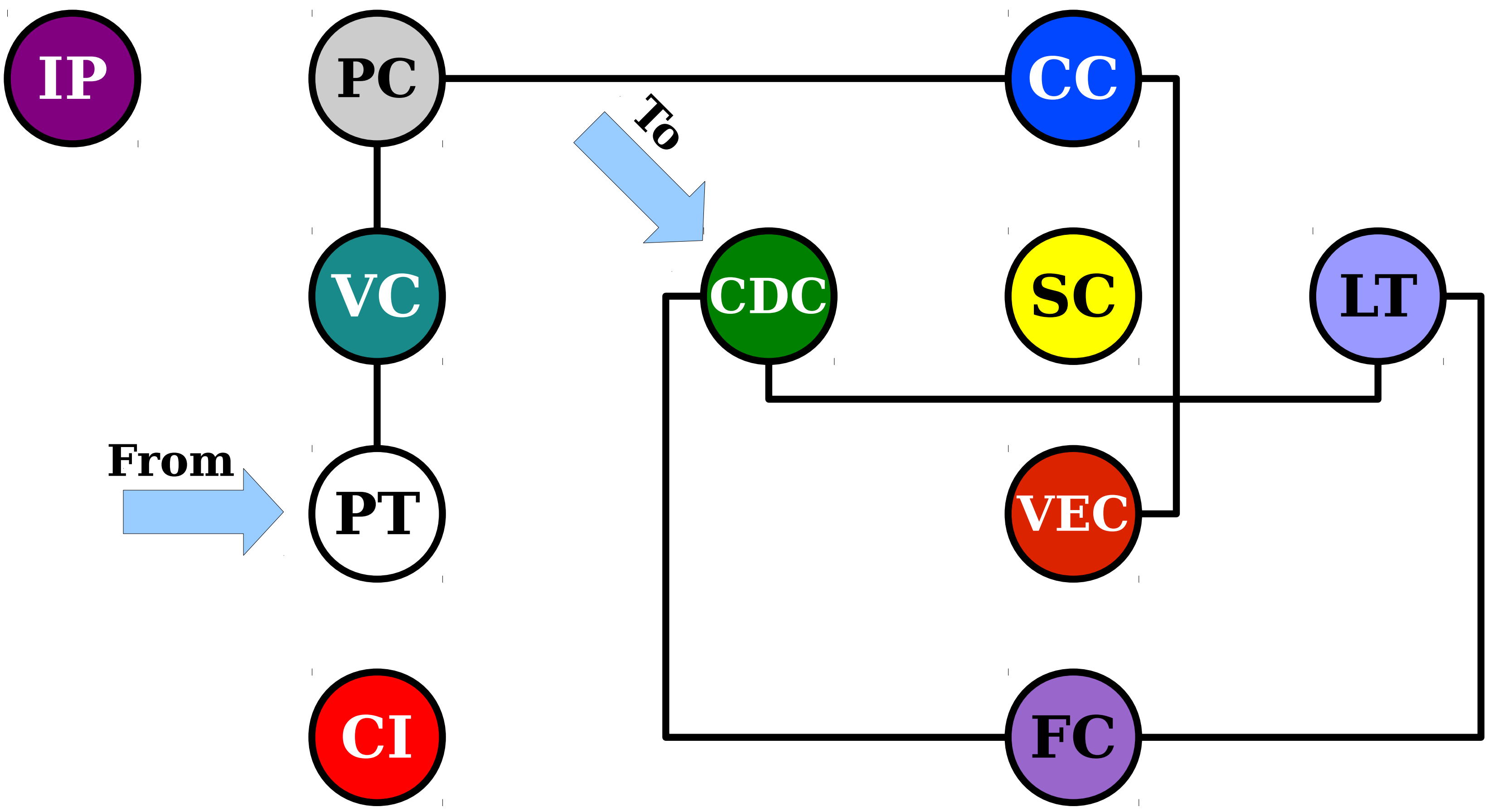
The ***length*** of a cycle is the number of edges in that cycle.

A ***simple path*** in a graph is a path that does not revisit any nodes or edges.

A ***simple cycle*** in a graph is a cycle that does not revisit any nodes or edges (except the start/end node).

Usually, the ***empty path*** starting and ending at a node and containing no edges is considered a simple path.

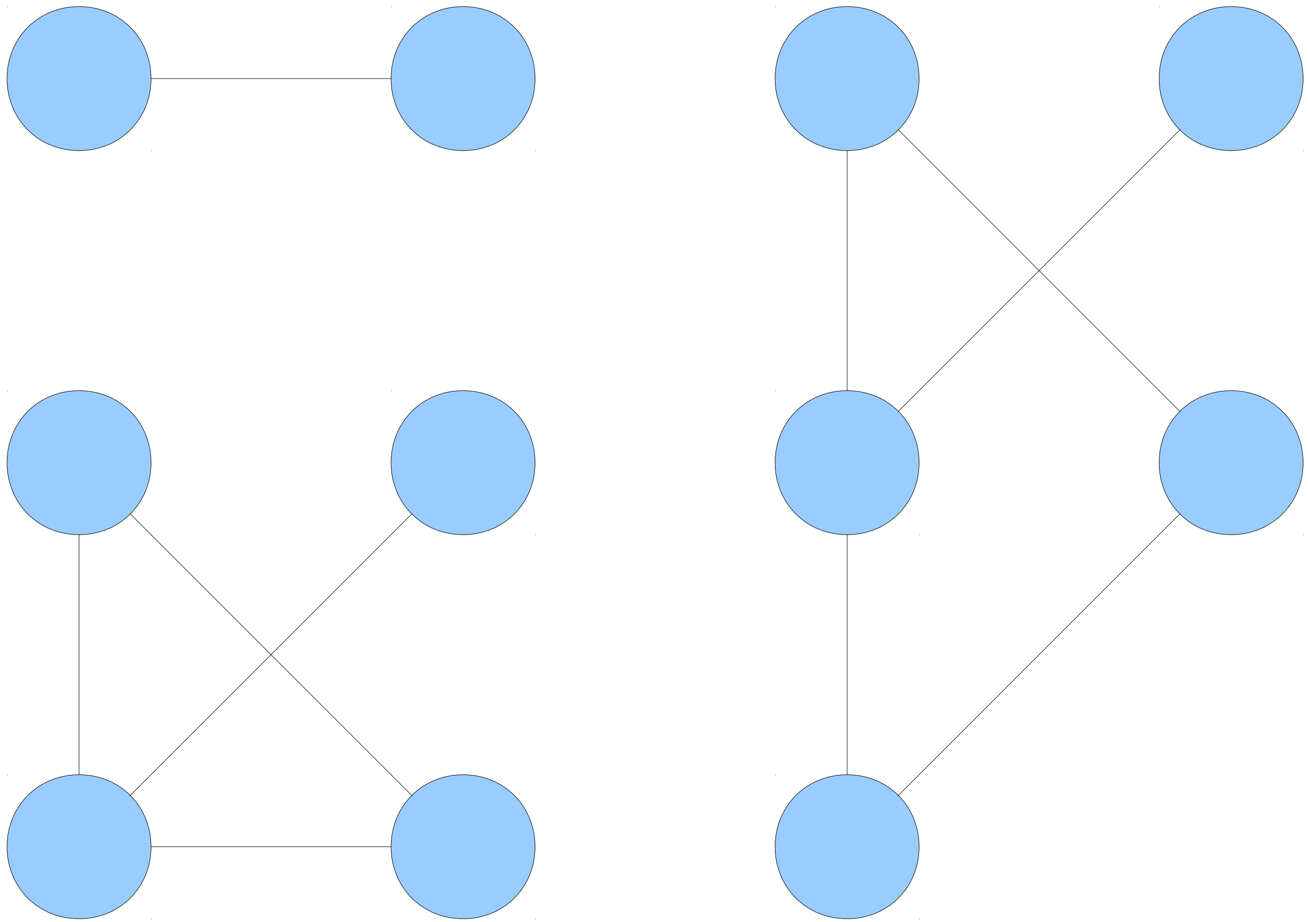
Navigating a Graph

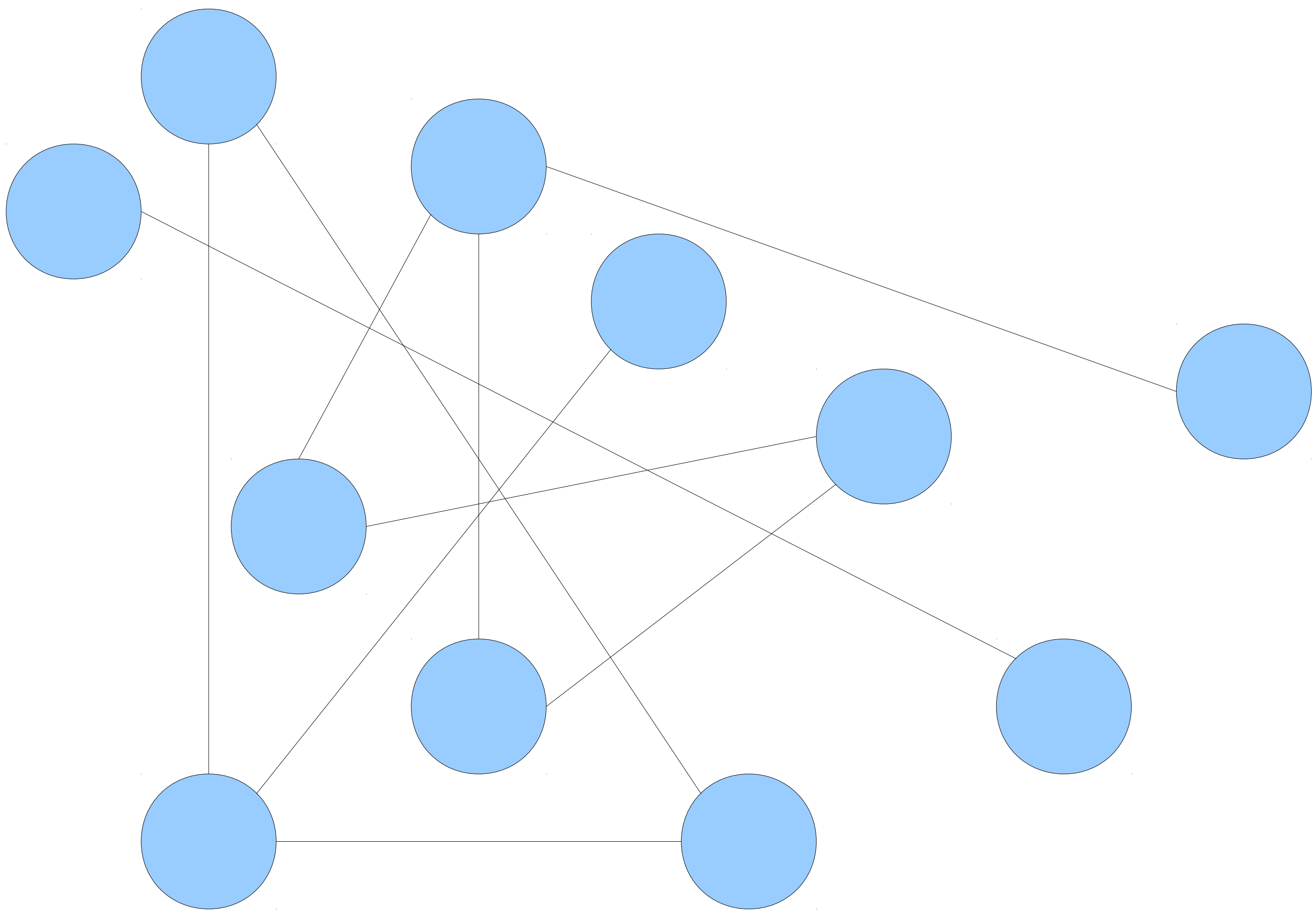


Two nodes u and v are called ***connected*** if there is a path from u to v.

A graph as a whole is called ***connected*** if all pairs of nodes in the graph are connected.

Connected Components





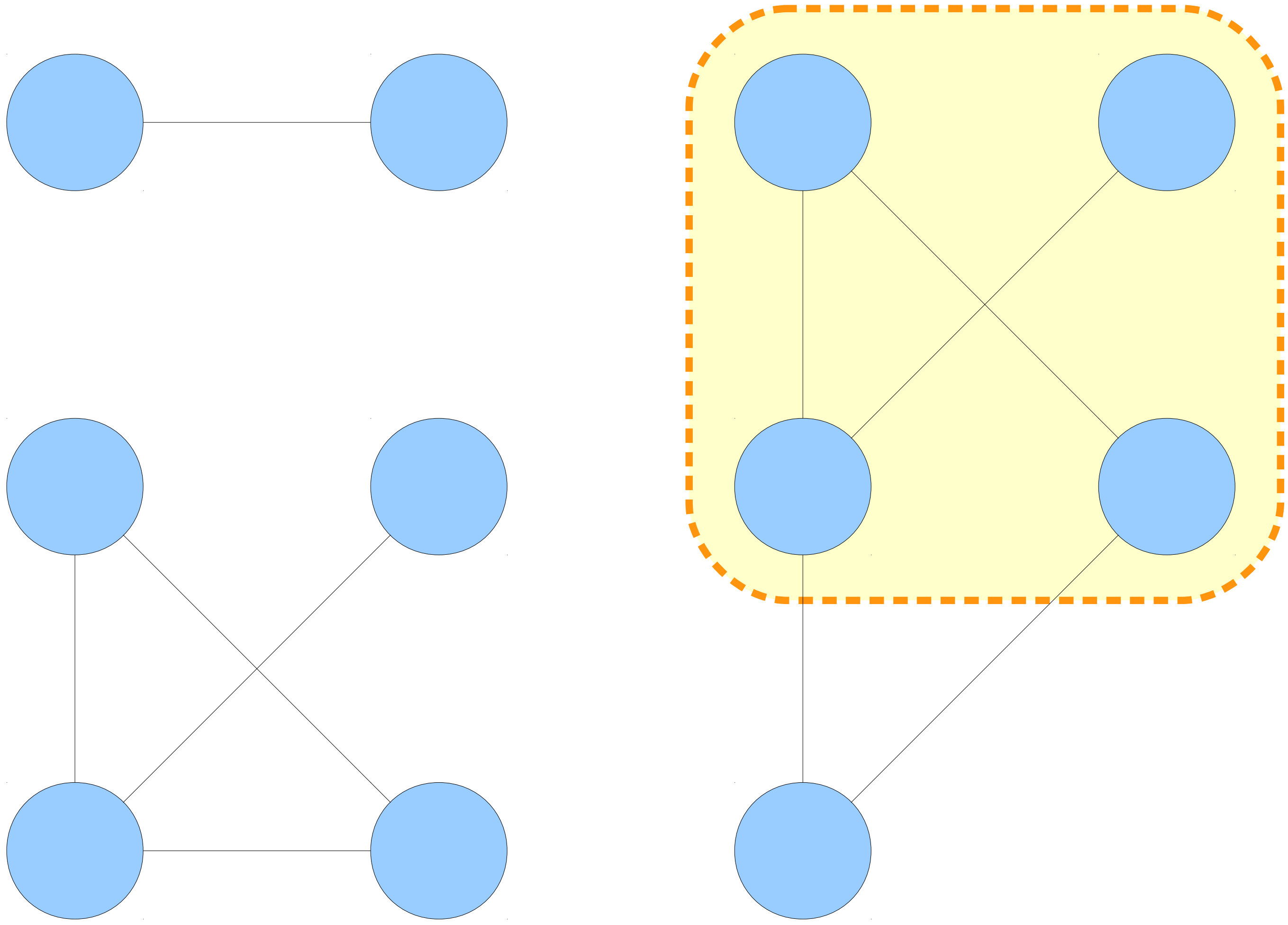
An Initial Definition

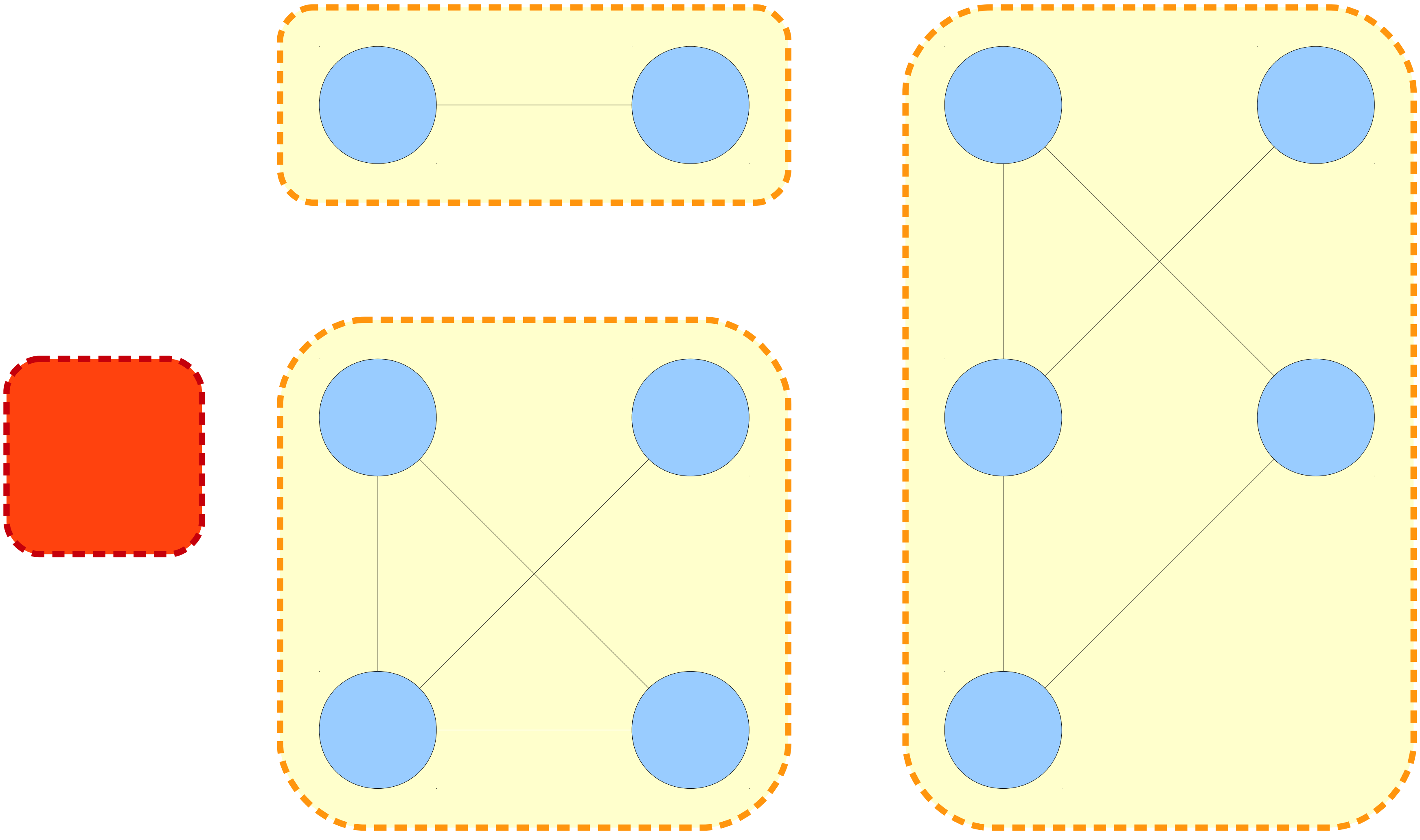
- **Attempted Definition #1:** A *piece* of an undirected graph $G = (V, E)$ is a set $C \subseteq V$ where

$$\forall u \in C.\ \forall v \in C.\ Connected(u, v)$$

- Intuition: a piece of a graph is a set of nodes that are all connected to one another.

⚠ *This definition has some problems;* ⚠ *please don't use it as a reference.*

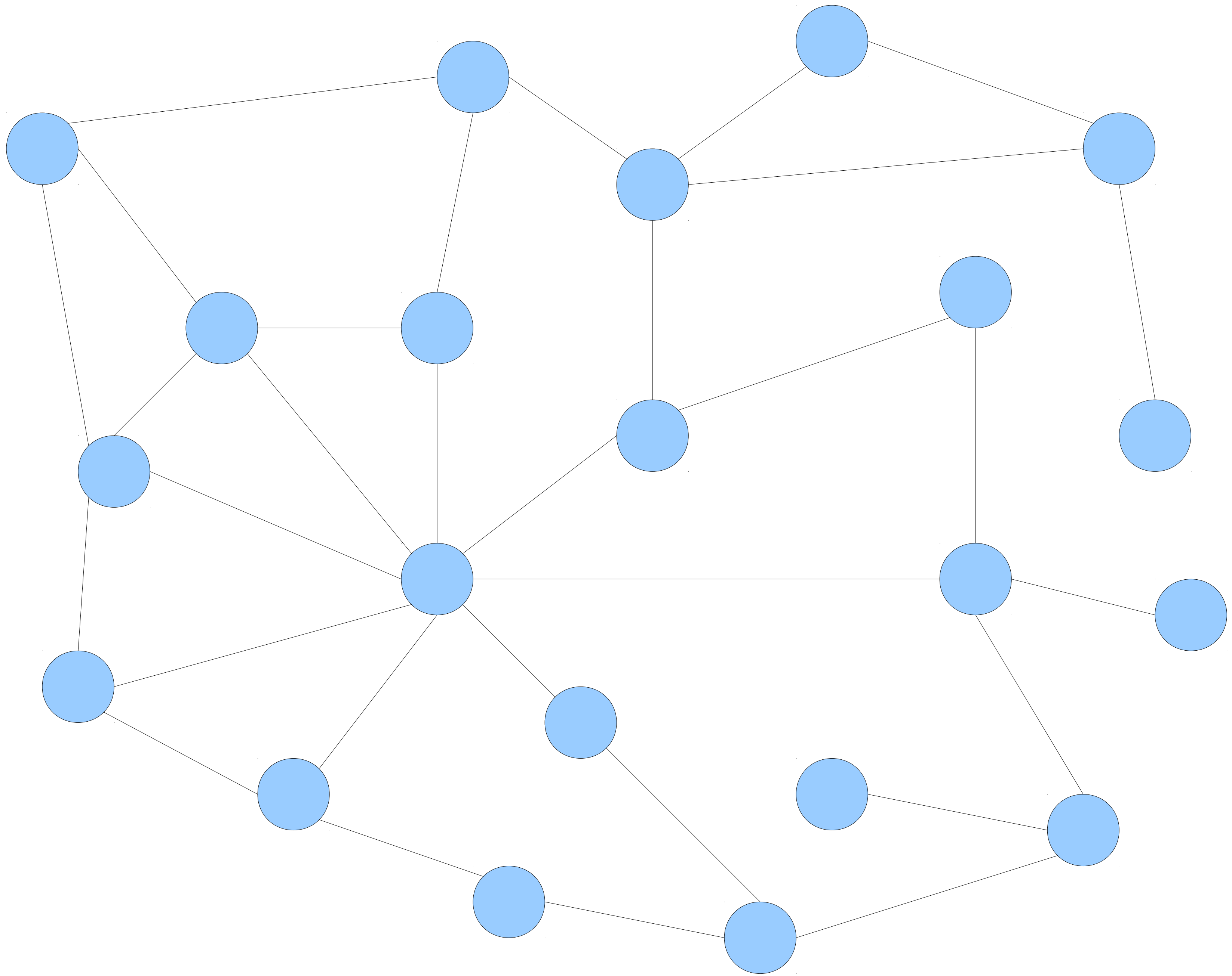


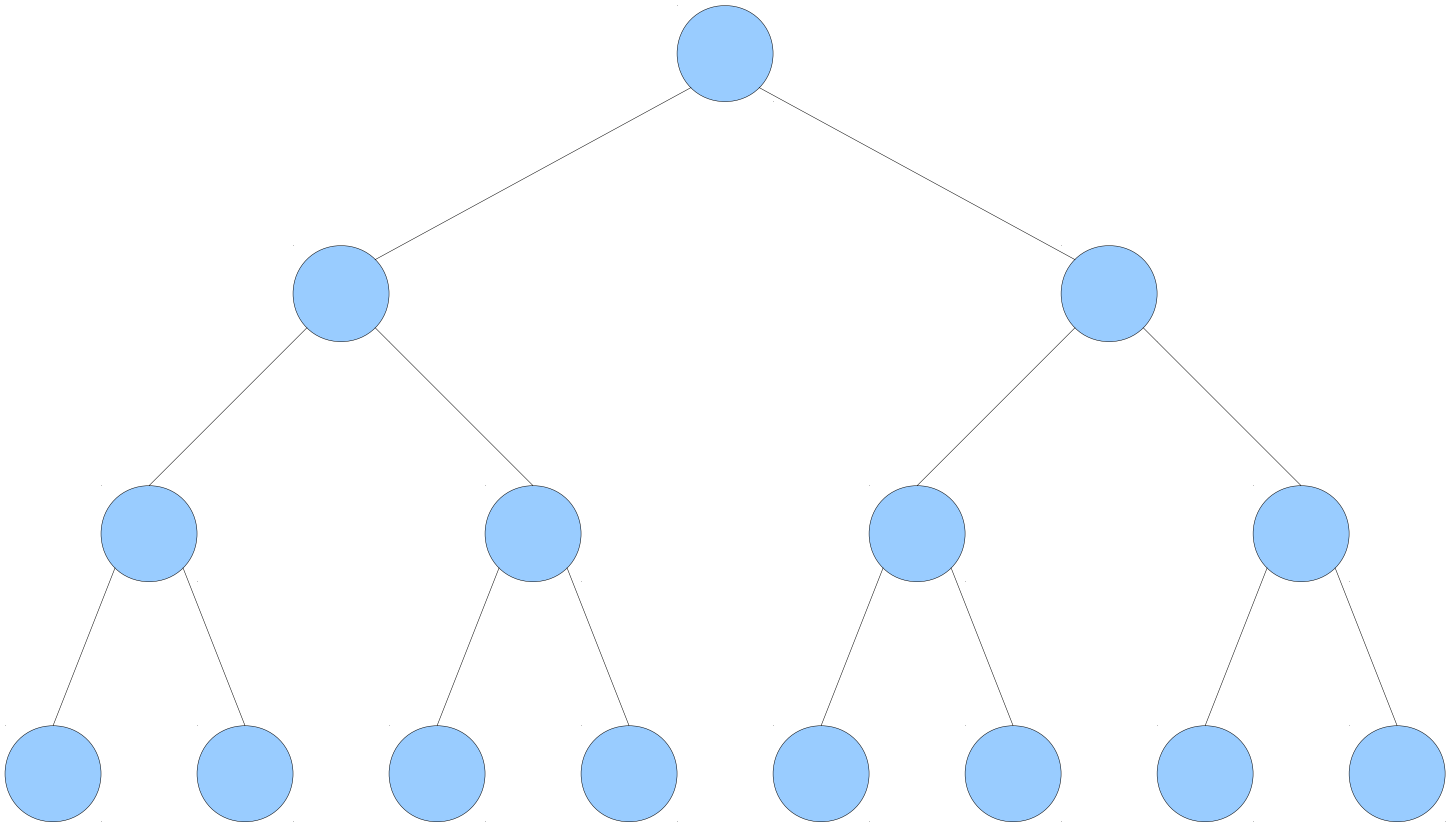


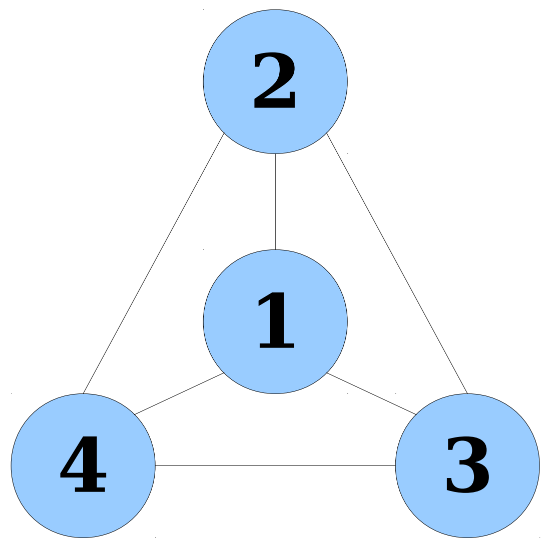
A Final Definition

- **Definition:** A ***connected component*** of an undirected graph $G = (V, E)$ is a *nonempty* set $C \subseteq V$ where
  - $\forall u \in C. \forall v \in C. Connected(u, v)$
  - $\forall u \in C. \forall v \in V - C. \neg Connected(u, v)$
- Intuition: a connected component is a nonempty set of nodes that are all connected to one another that includes as many nodes as possible.

Planar Graphs



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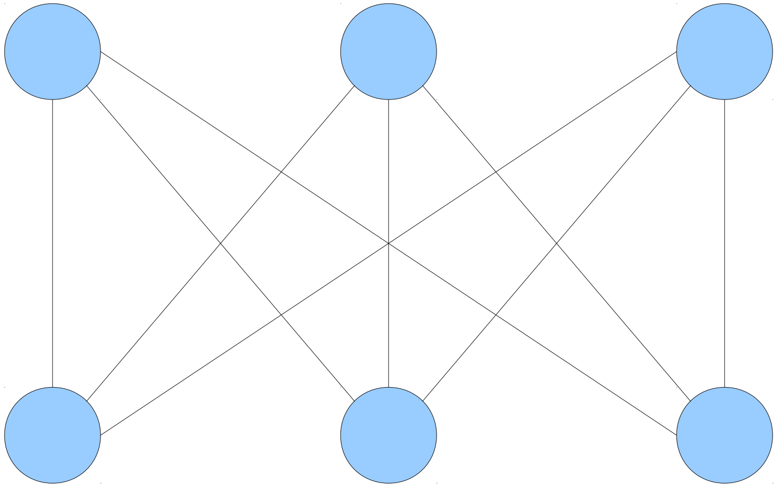
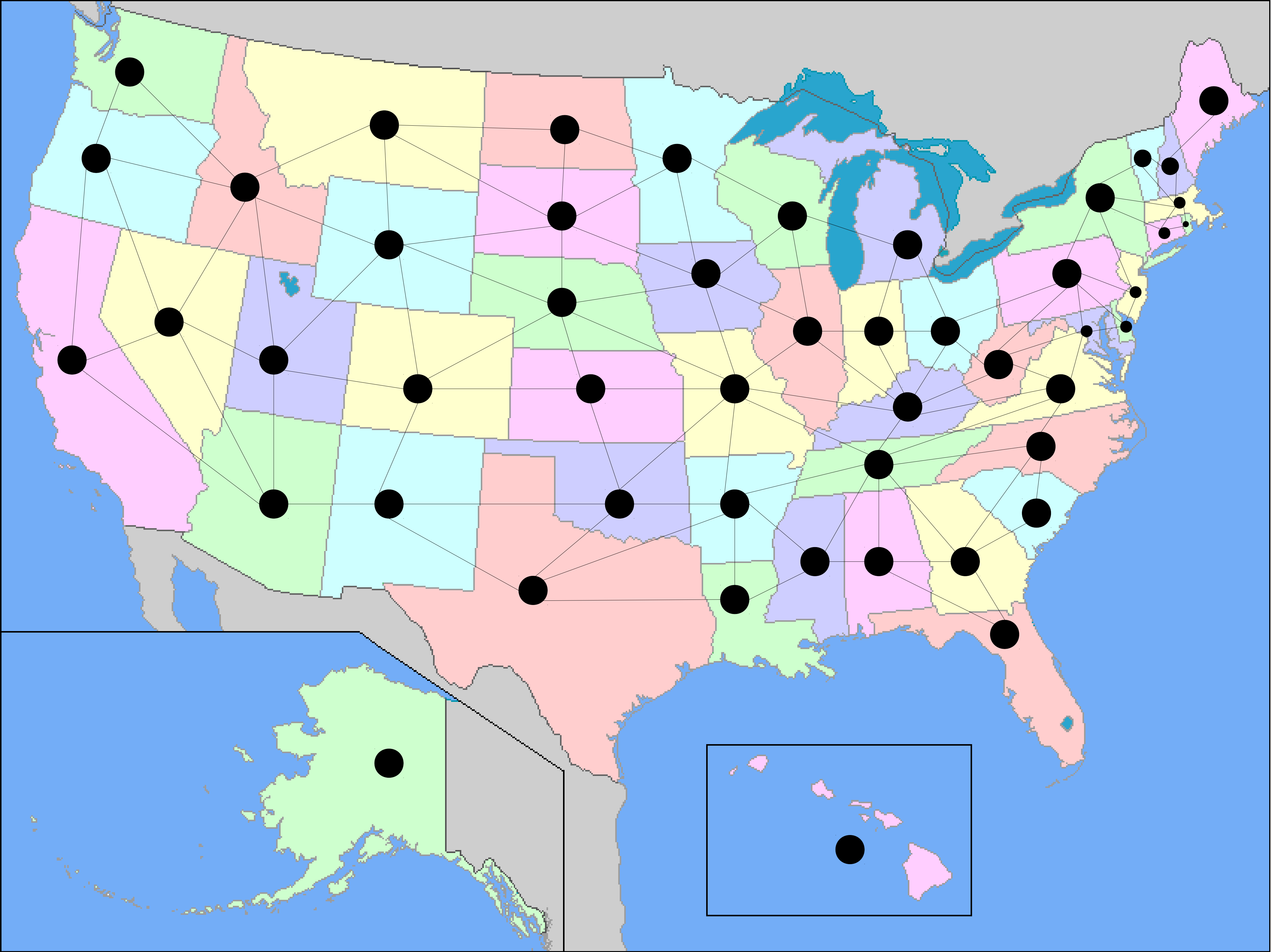


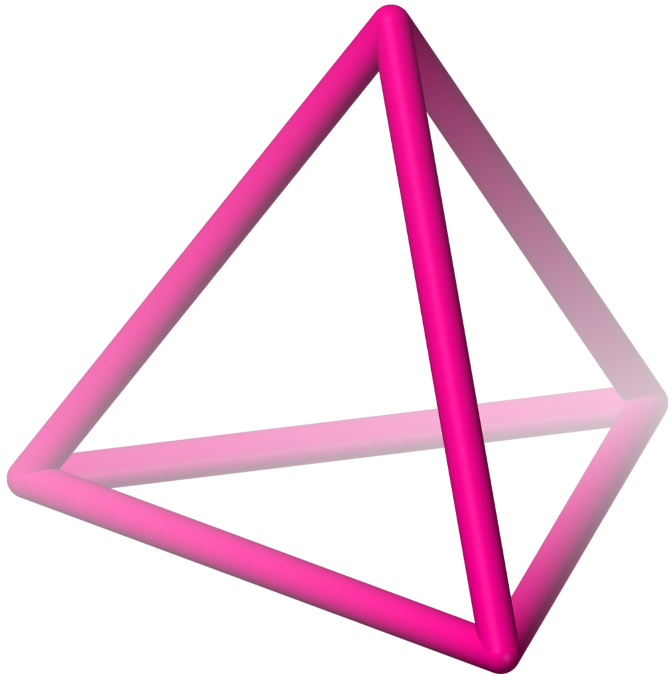
This graph is called the ***utility graph***. There is no way to draw it in the plane without edges crossing.

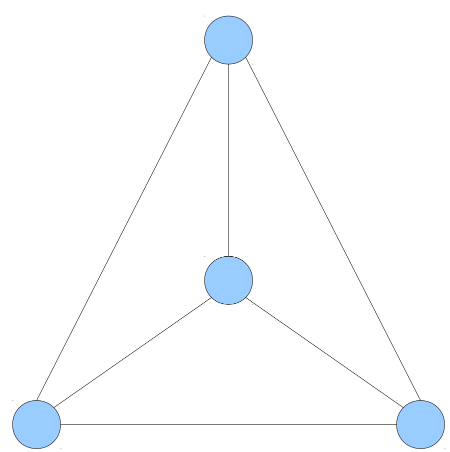
A graph is called a ***planar graph*** if there is some way to draw it in a 2D plane without any of the edges crossing.

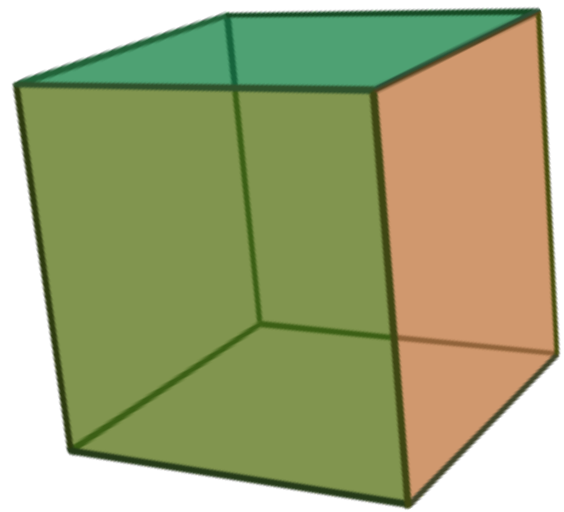
A Fun (And Strangely Addicting) Game:

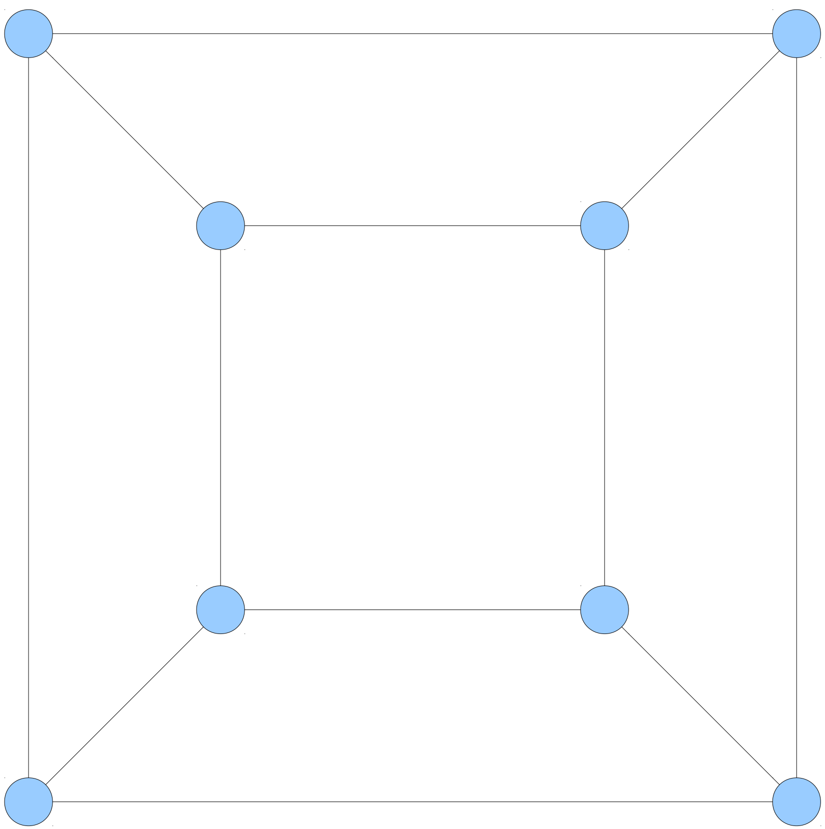
http://planarity.net/

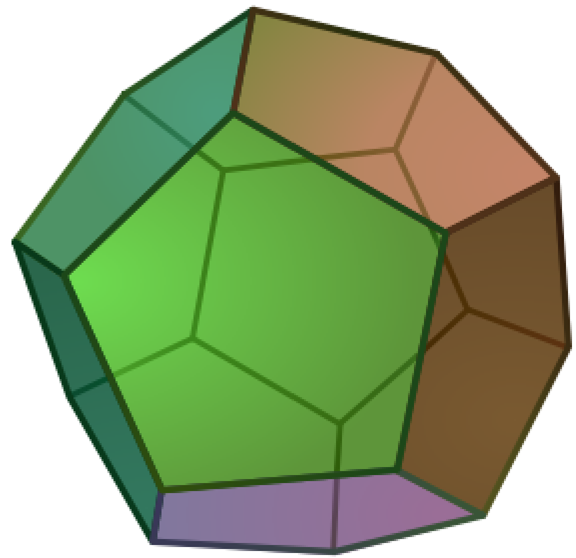


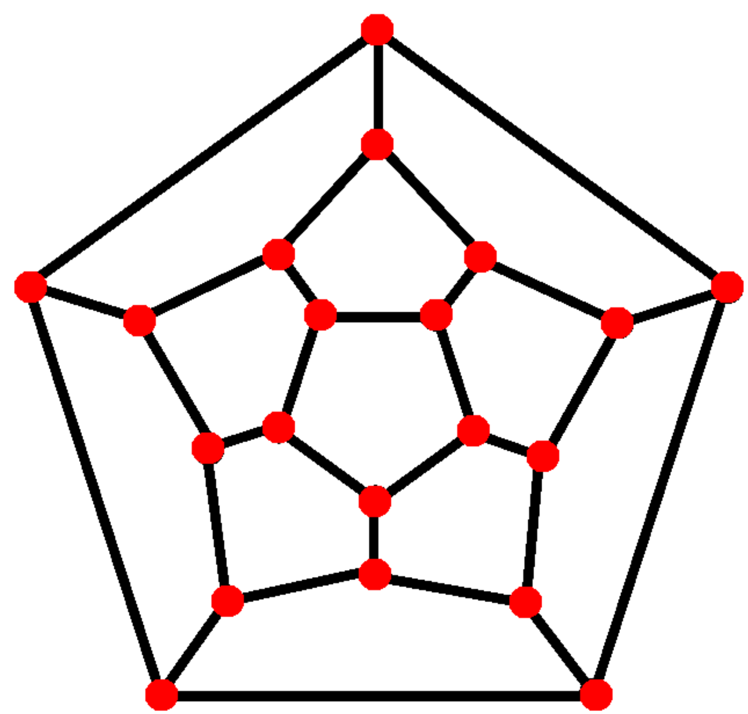




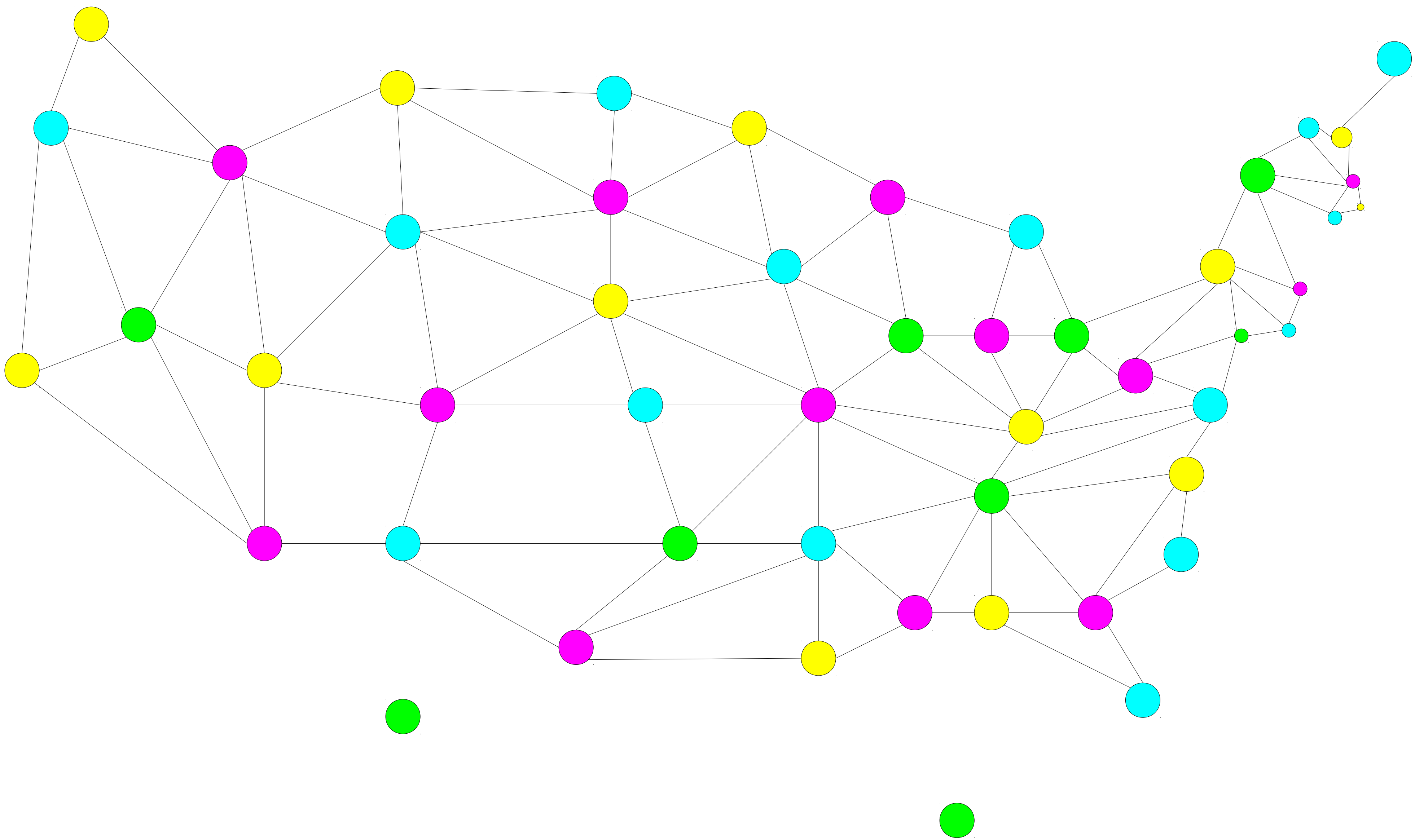








Graph Coloring



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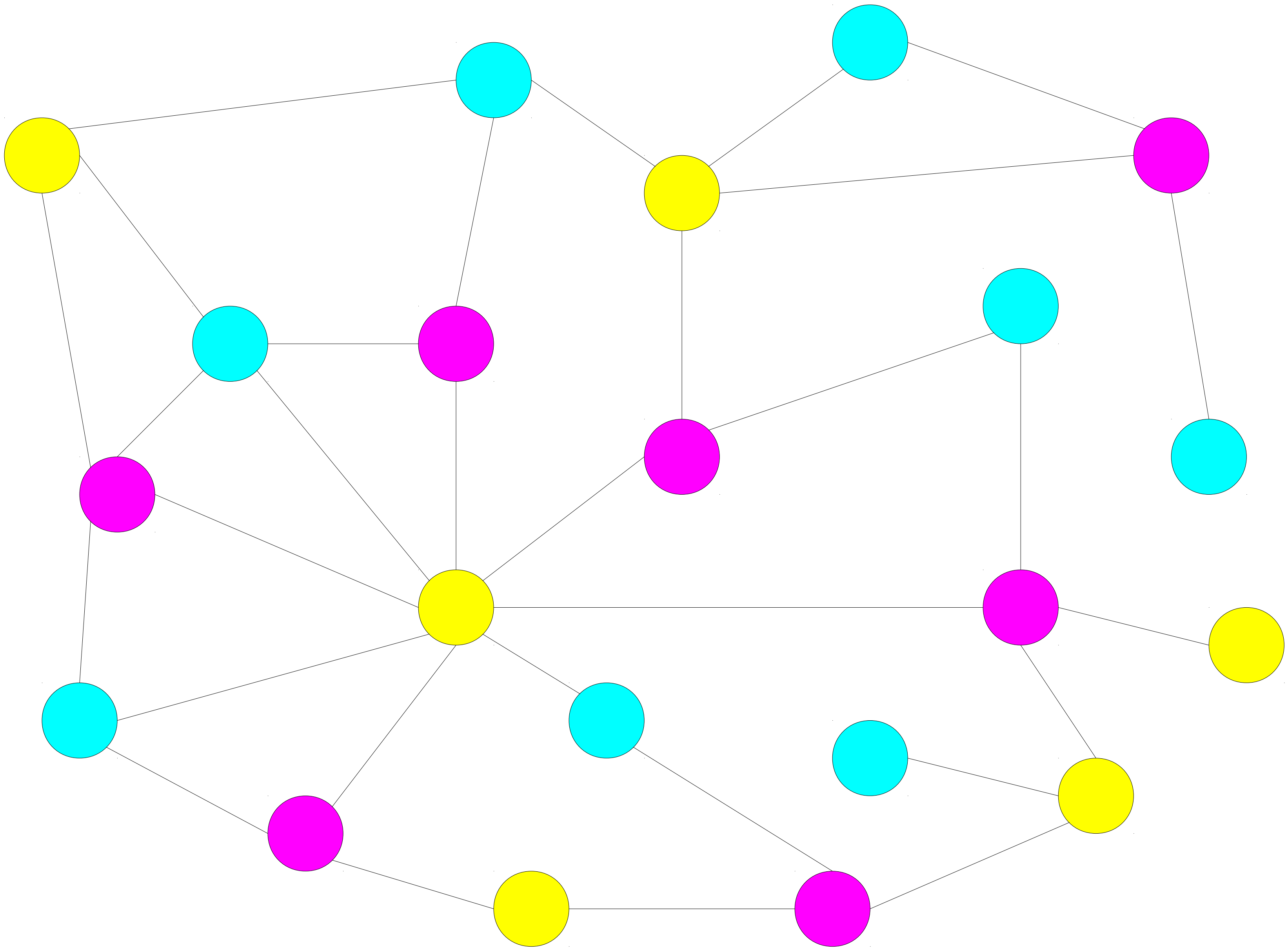

Graph Coloring

- An undirected graph $G = (V, E)$ with no self-loops (edges from a node to itself) is called ***k-colorable*** if the nodes in V can be assigned one of k different colors such that no two nodes of the same color are joined by an edge.
- The minimum number of colors needed to color a graph is called that graph's ***chromatic number***.
  - The chromatic number of a graph G is usually denoted $\boldsymbol{\chi(G)}$, from the Greek *χρώμα* ("color").

Theorem (Four-Color Theorem): Every planar graph is 4-colorable.

- **1850s:** Four-Color Conjecture posed.
- **1879:** Kempe proves the Four-Color Theorem.
- **1890:** Heawood finds a flaw in Kempe's proof.
- **1976:** Appel and Haken design a computer program that proves the Four-Color Theorem. The program checked 1,936 specific cases that are “minimal counterexamples;” any counterexample to the theorem must contain one of the 1,936 specific cases.
- **1980s:** Doubts rise about the validity of the proof due to errors in the software.
- **1989:** Appel and Haken revise their proof and show it is indeed correct. They publish a book including a 400-page appendix of all the cases to check.
- **1996:** Roberts, Sanders, Seymour, and Thomas reduce the number of cases to check down to 633.
- **2005:** Werner and Gonthier repeat the proof using an established automatic theorem prover (Coq), improving confidence in the truth of the theorem.