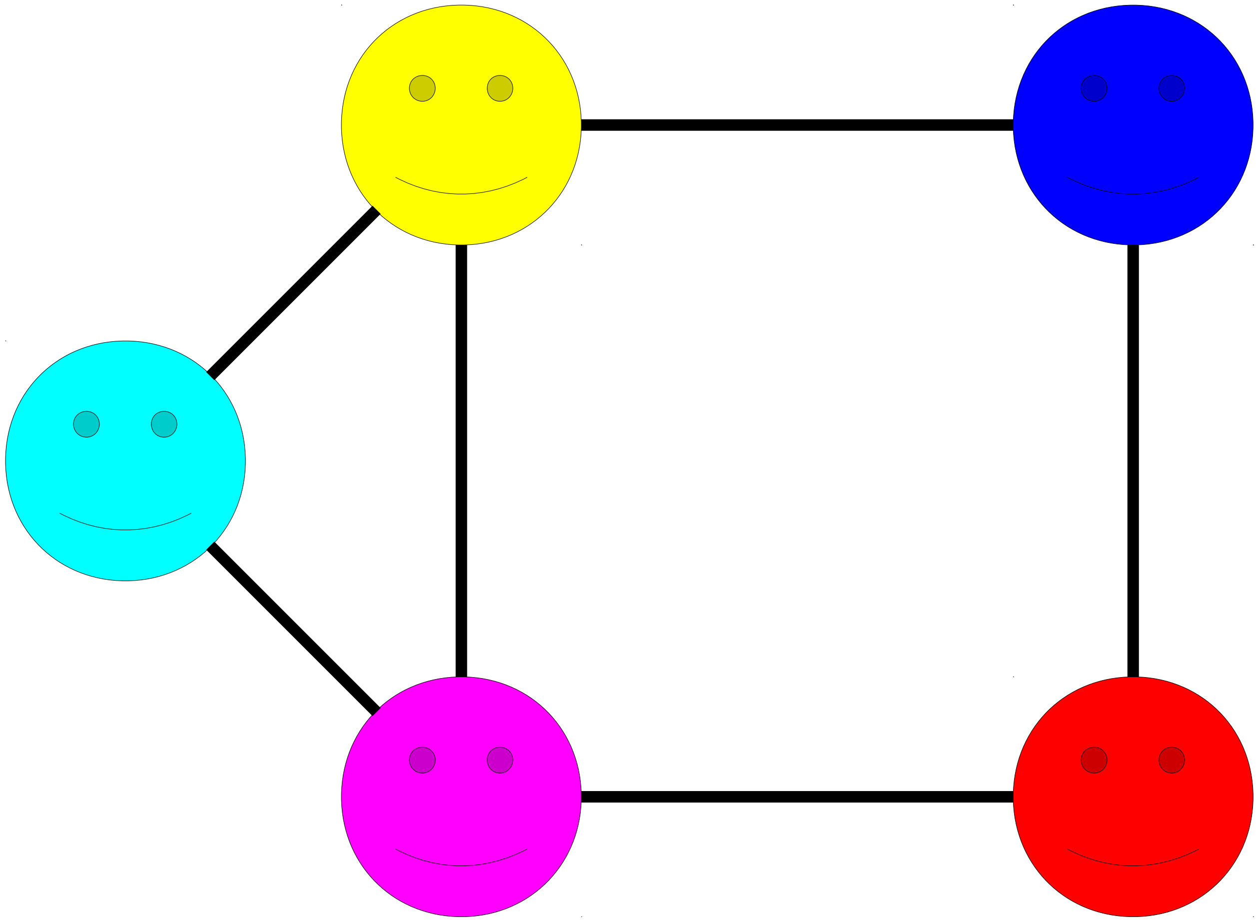
Graphs

Part Two

A ***graph*** is a mathematical structure for representing relationships.



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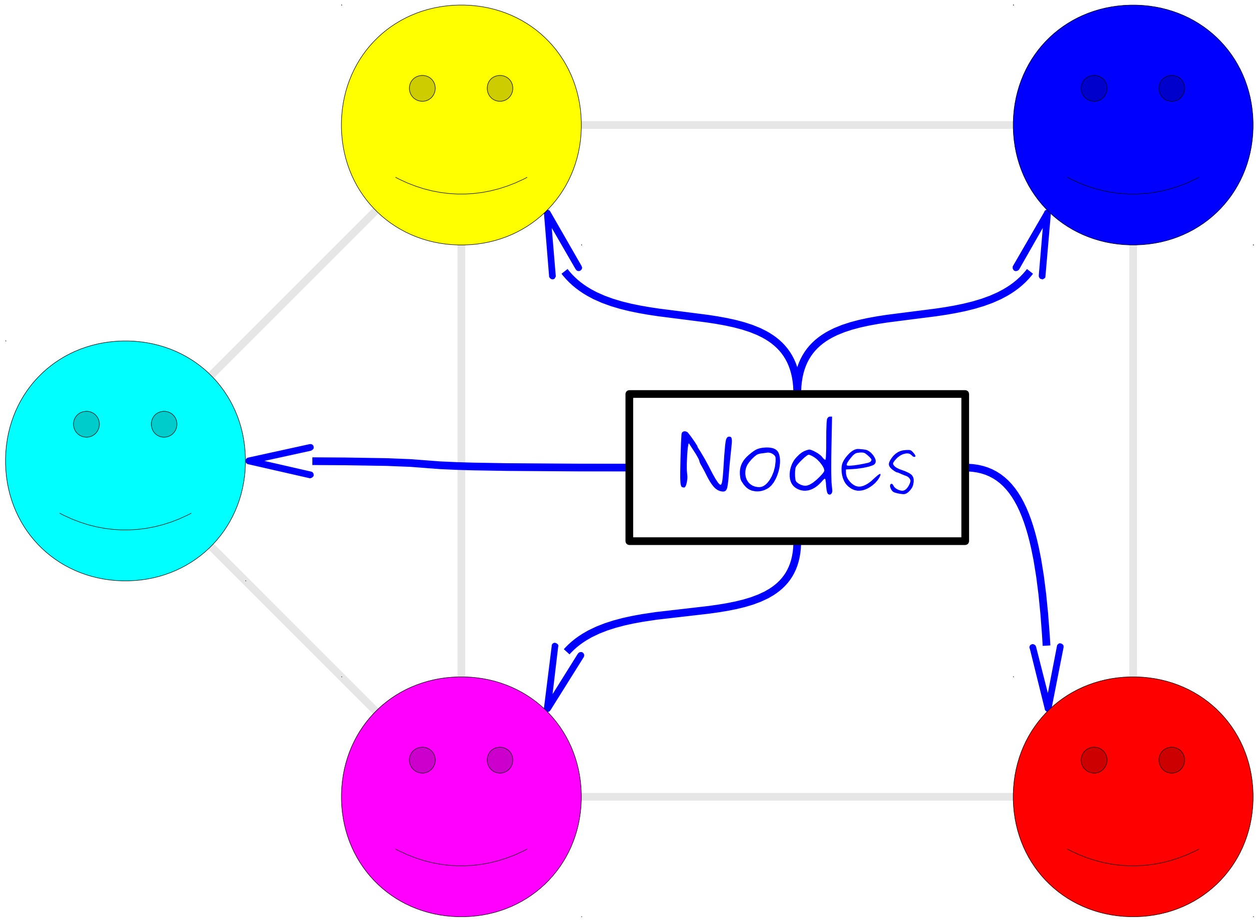

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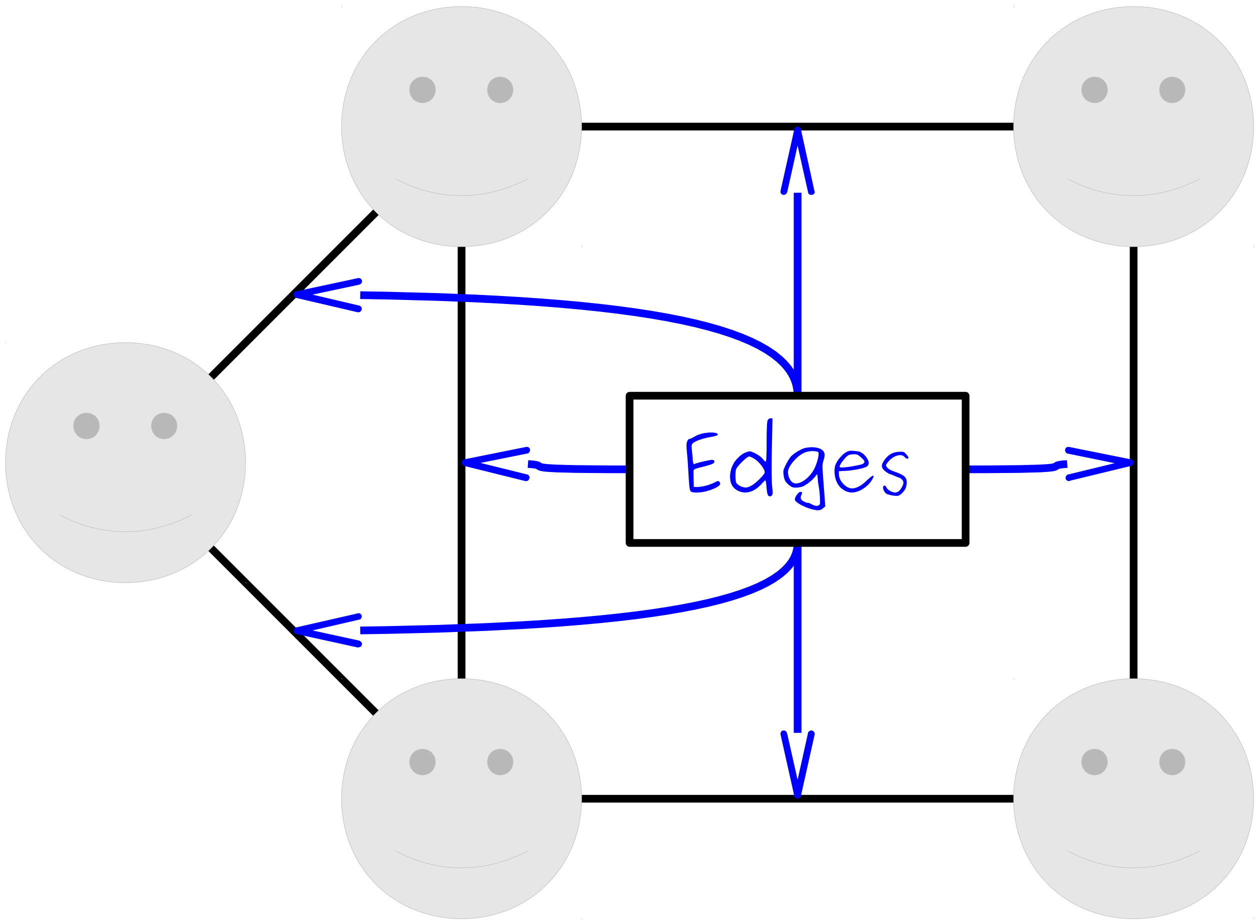

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Graph Coloring

- An undirected graph $G = (V, E)$ with no self-loops (edges from a node to itself) is called ***k-colorable*** if the nodes in V can be assigned one of k different colors such that no two nodes of the same color are joined by an edge.
- The minimum number of colors needed to color a graph is called that graph's ***chromatic number***.
  - The chromatic number of a graph G is usually denoted $\boldsymbol{\chi(G)}$, from the Greek *χρώμα* ("color").

The Pigeonhole Principle and Graphs

The ***pigeonhole principle*** is the following:

If m objects are placed into n bins, where $m > n$, then some bin contains at least two objects.

(We sketched a proof in Lecture #02)

A Surprising Application

Theorem: Suppose that every point in the real plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.

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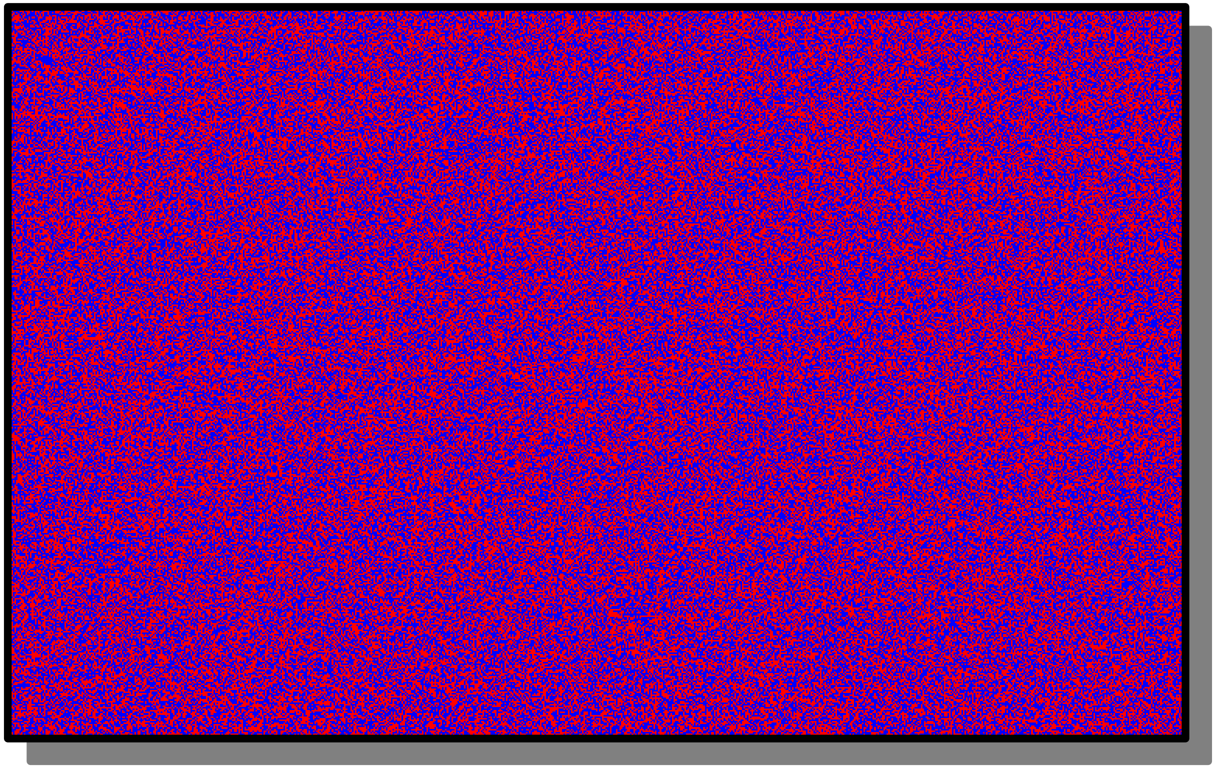

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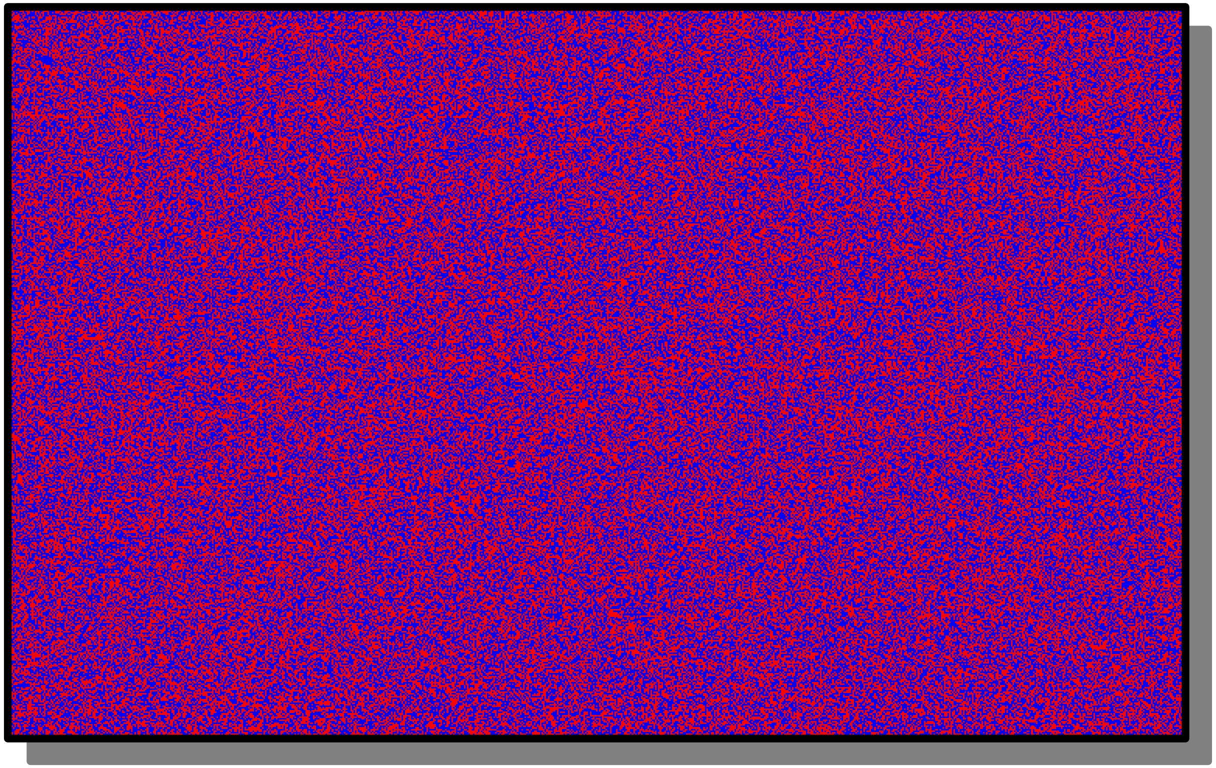

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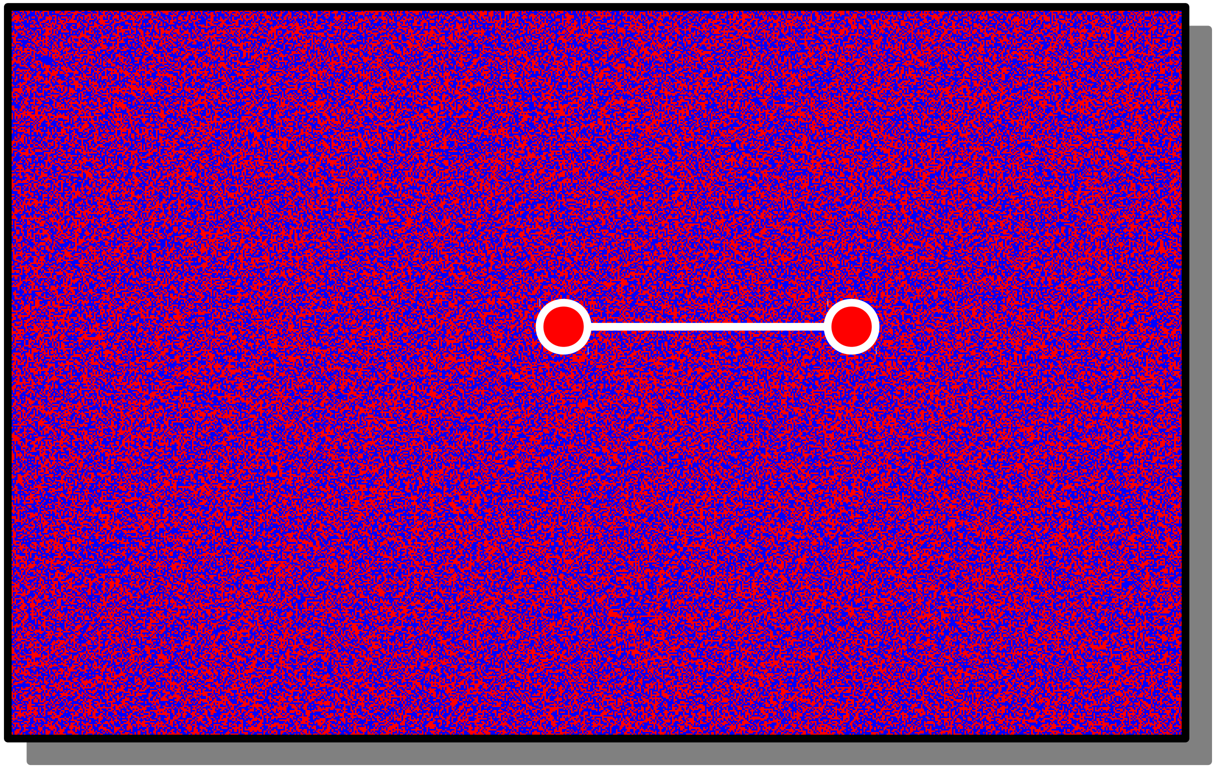

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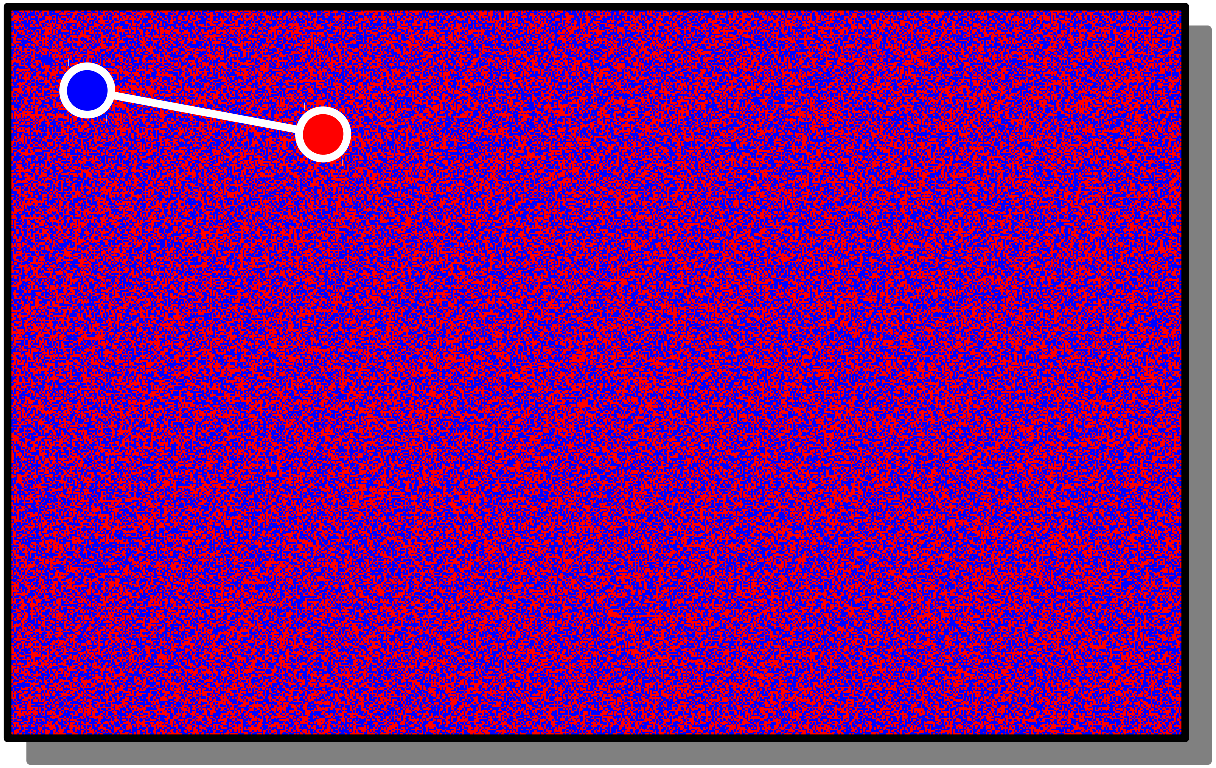

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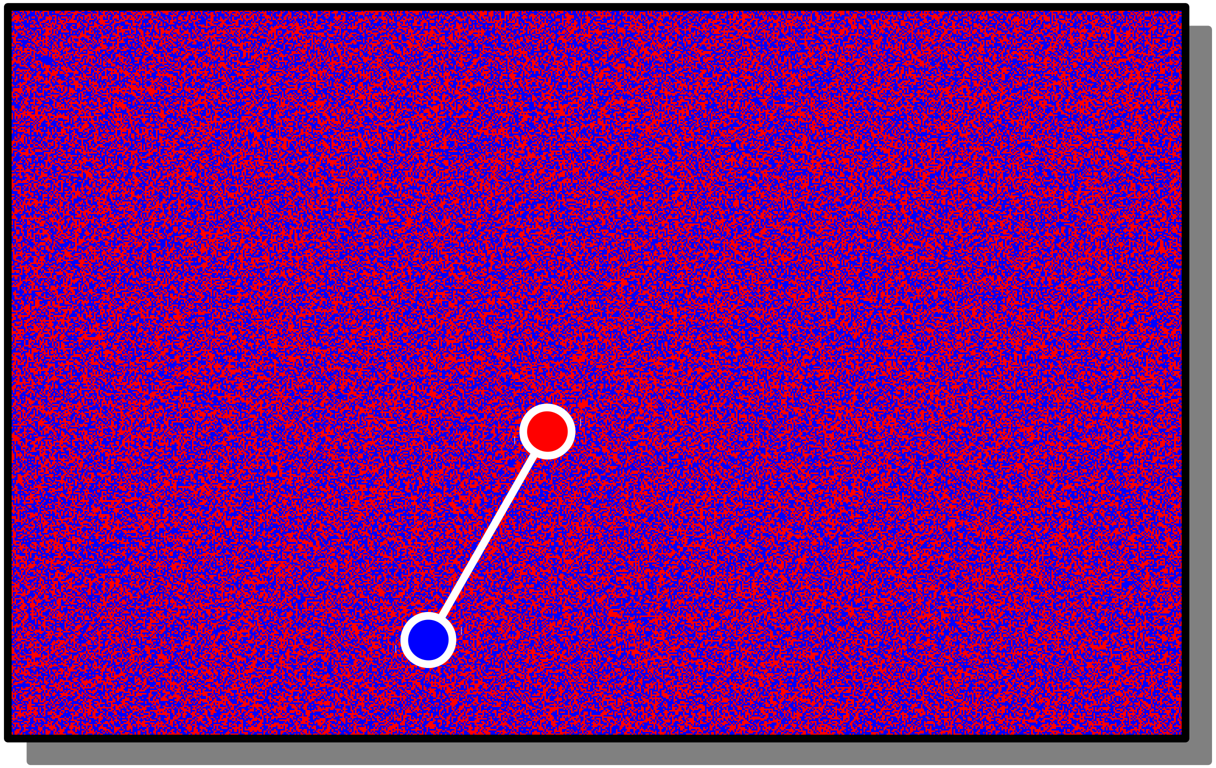

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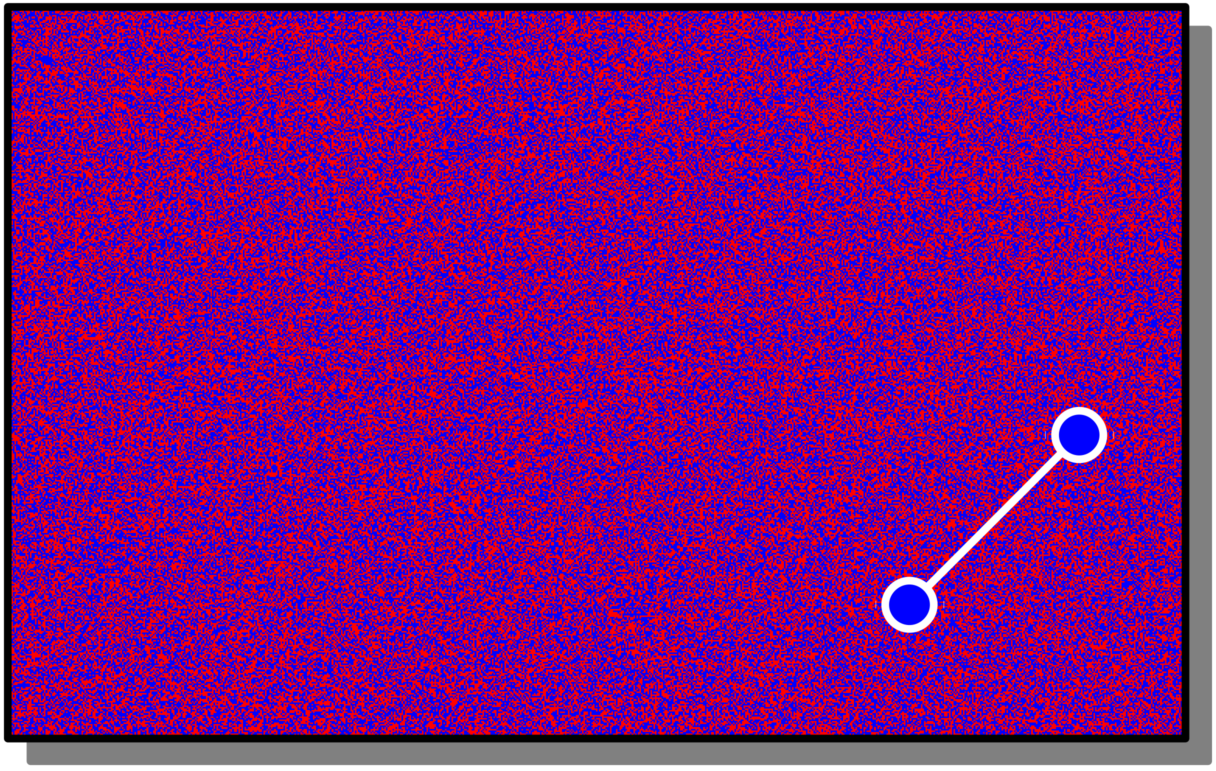

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Suppose we pick k points in the plane. What is the minimum choice of k we can pick before we're guaranteed to get at least two points of the same color?

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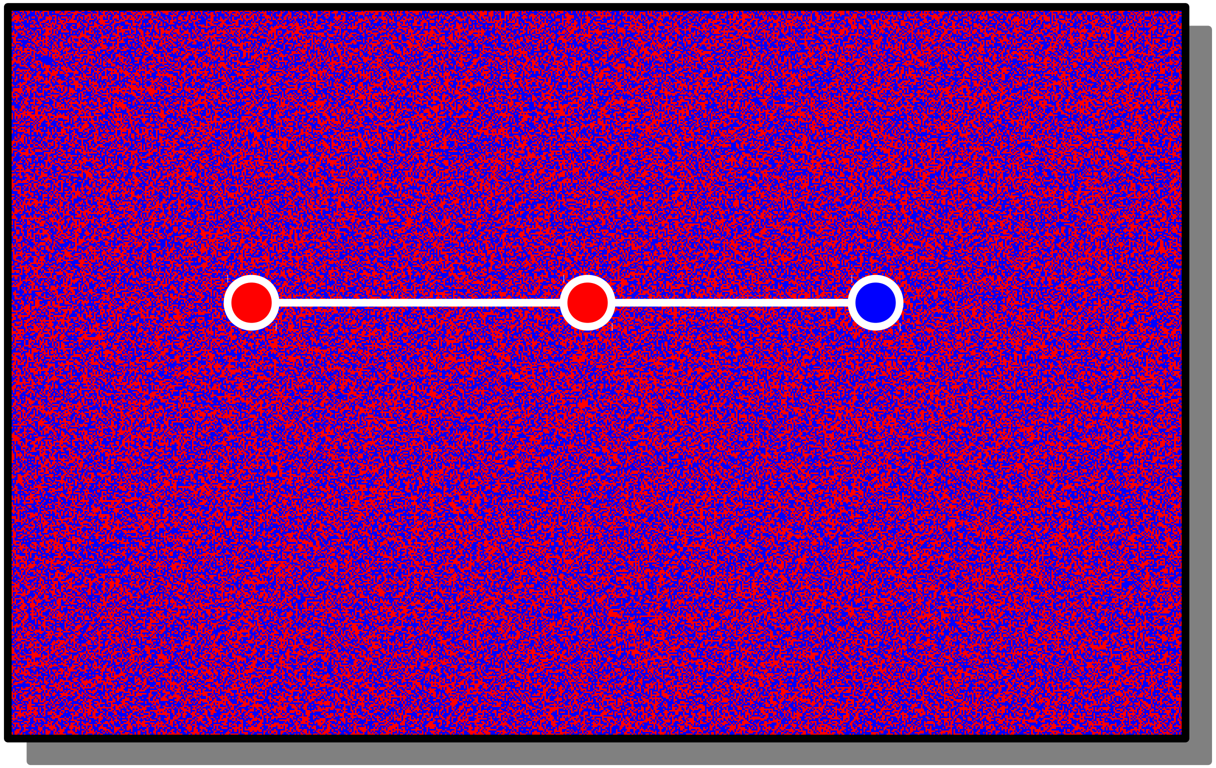

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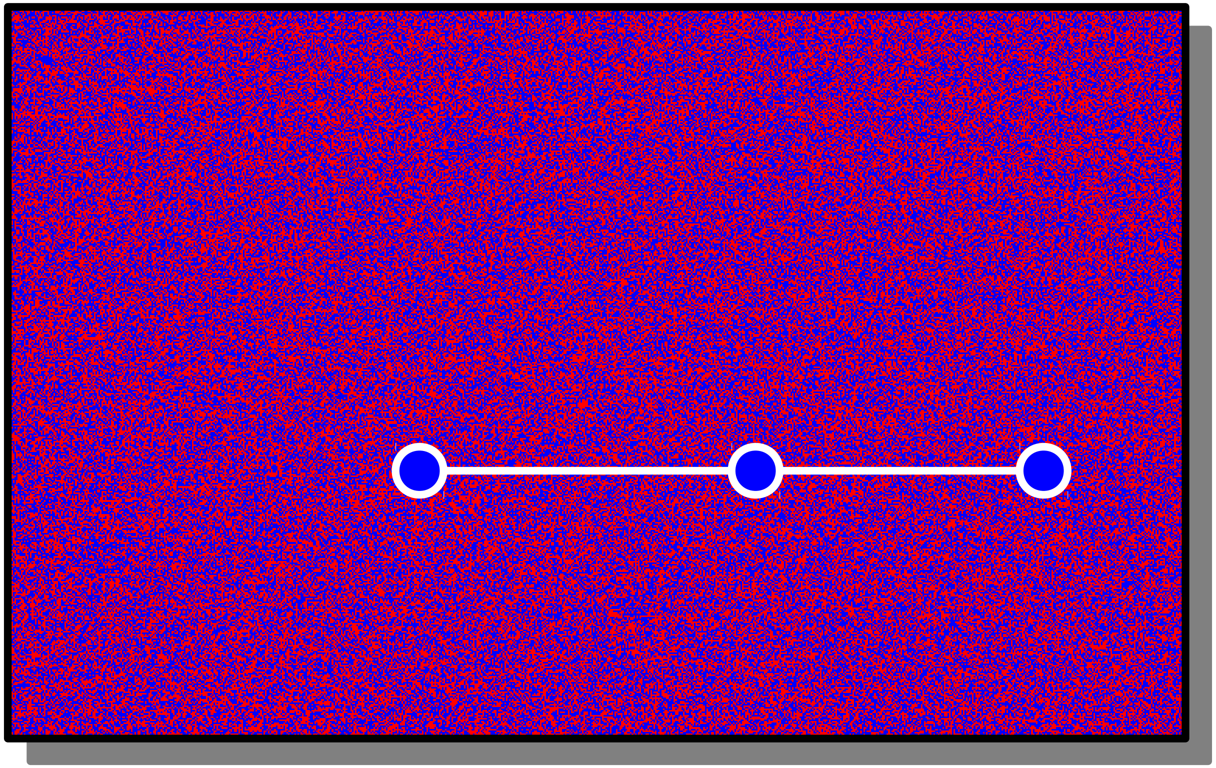

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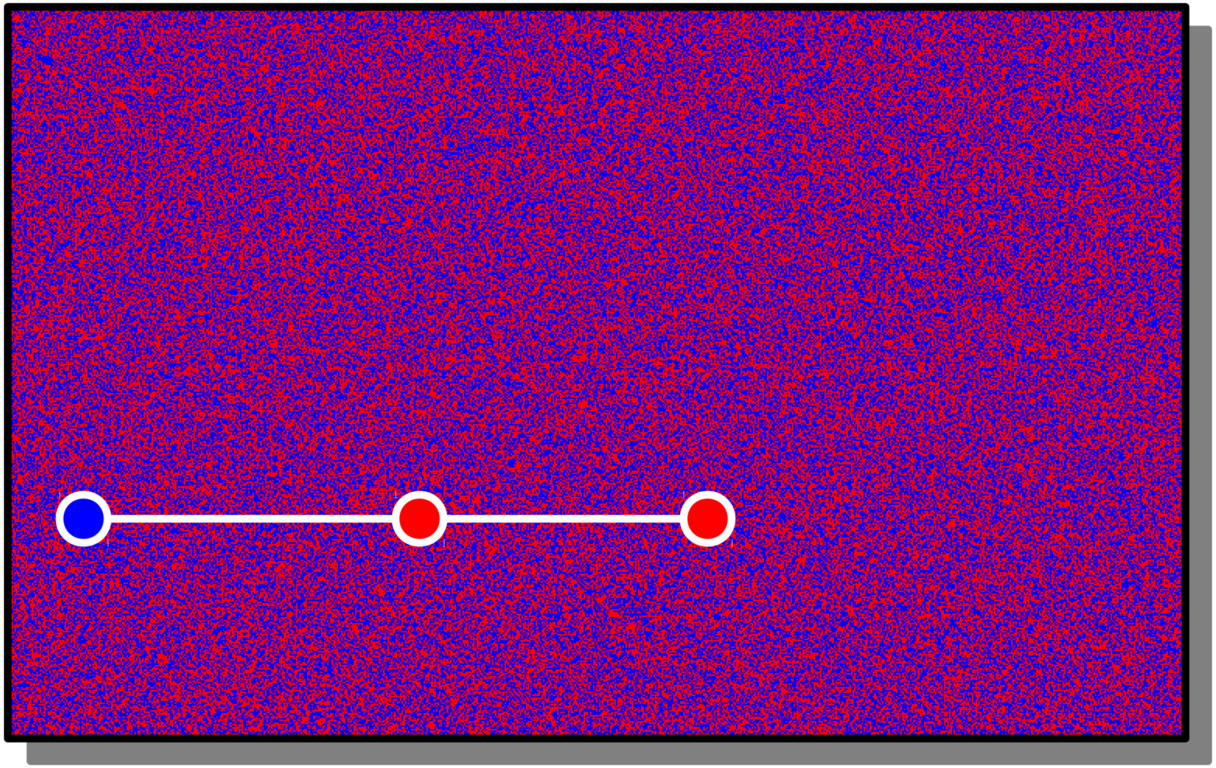

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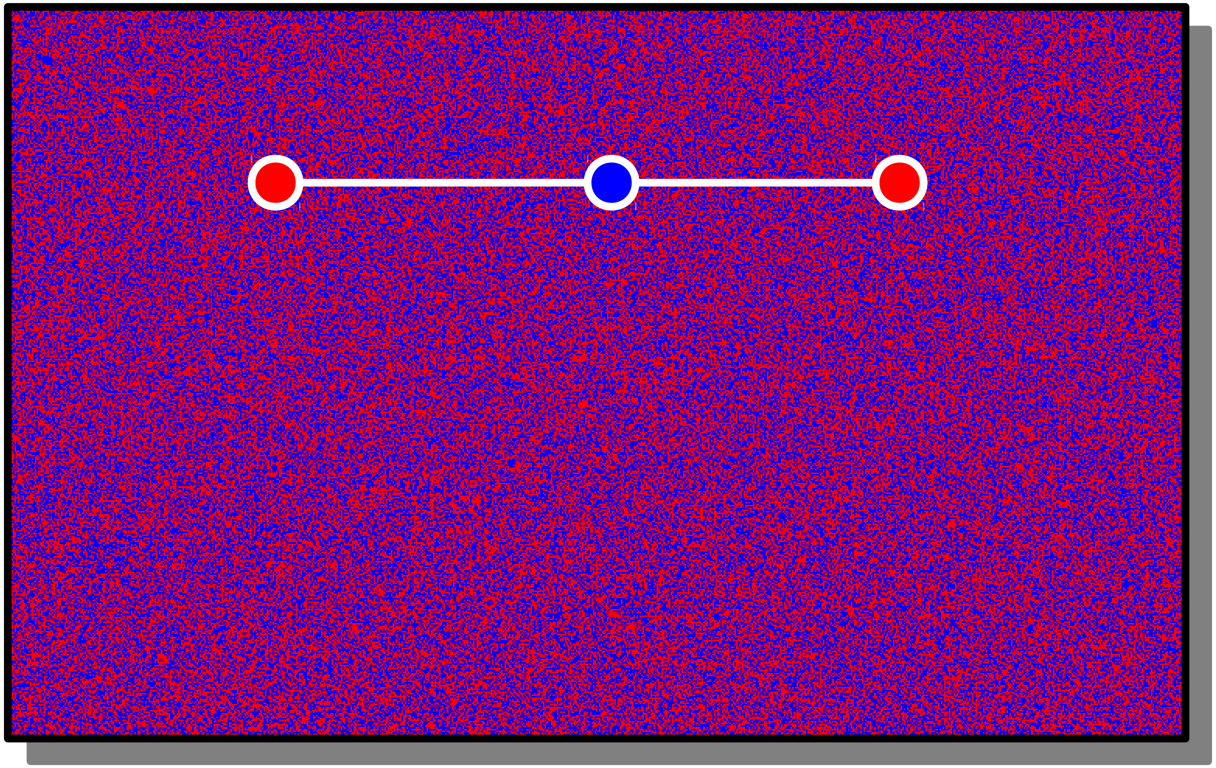

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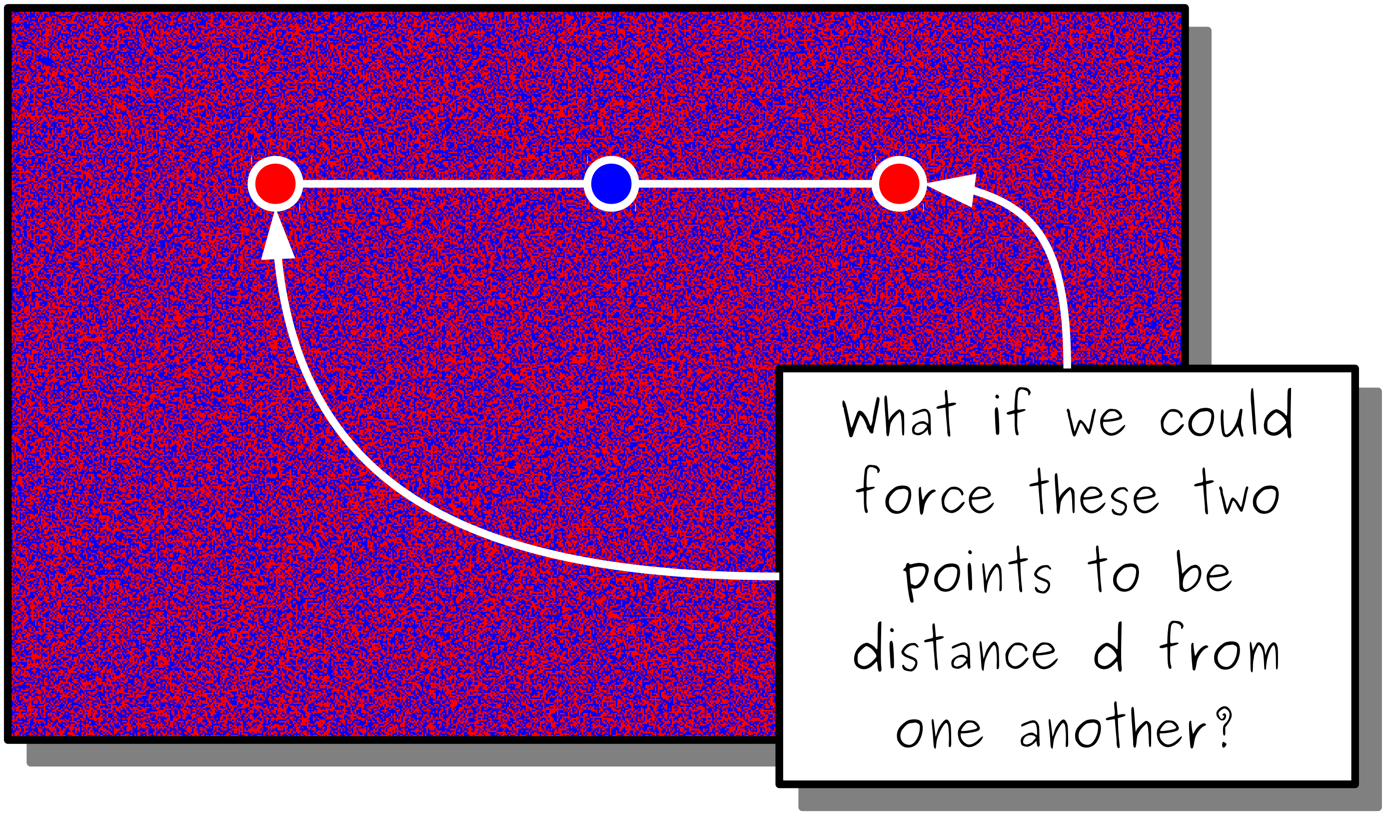

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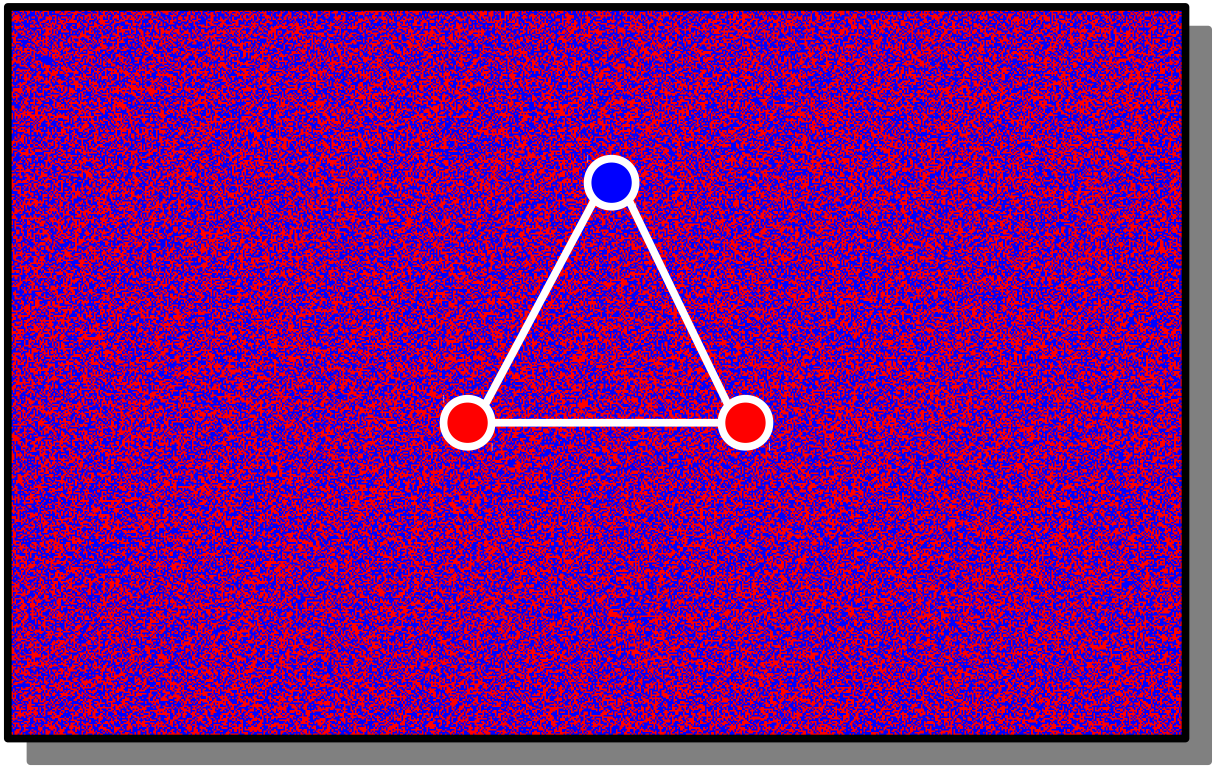

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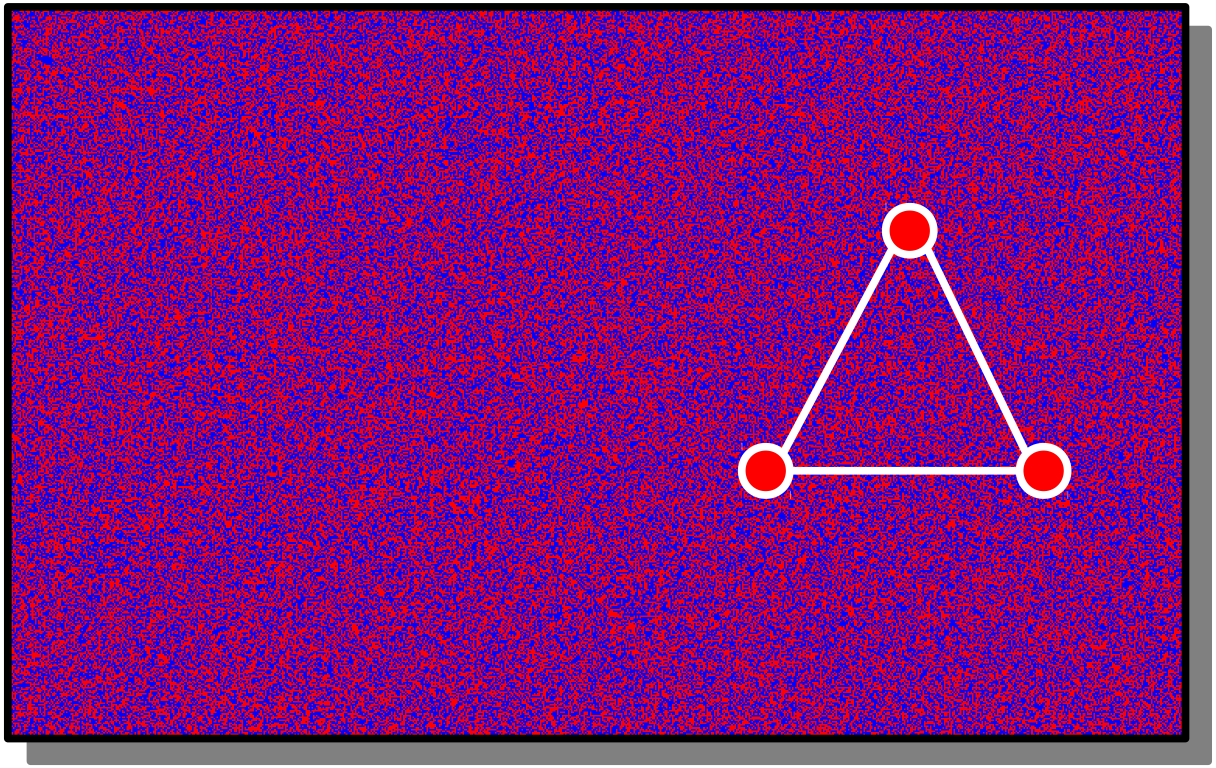

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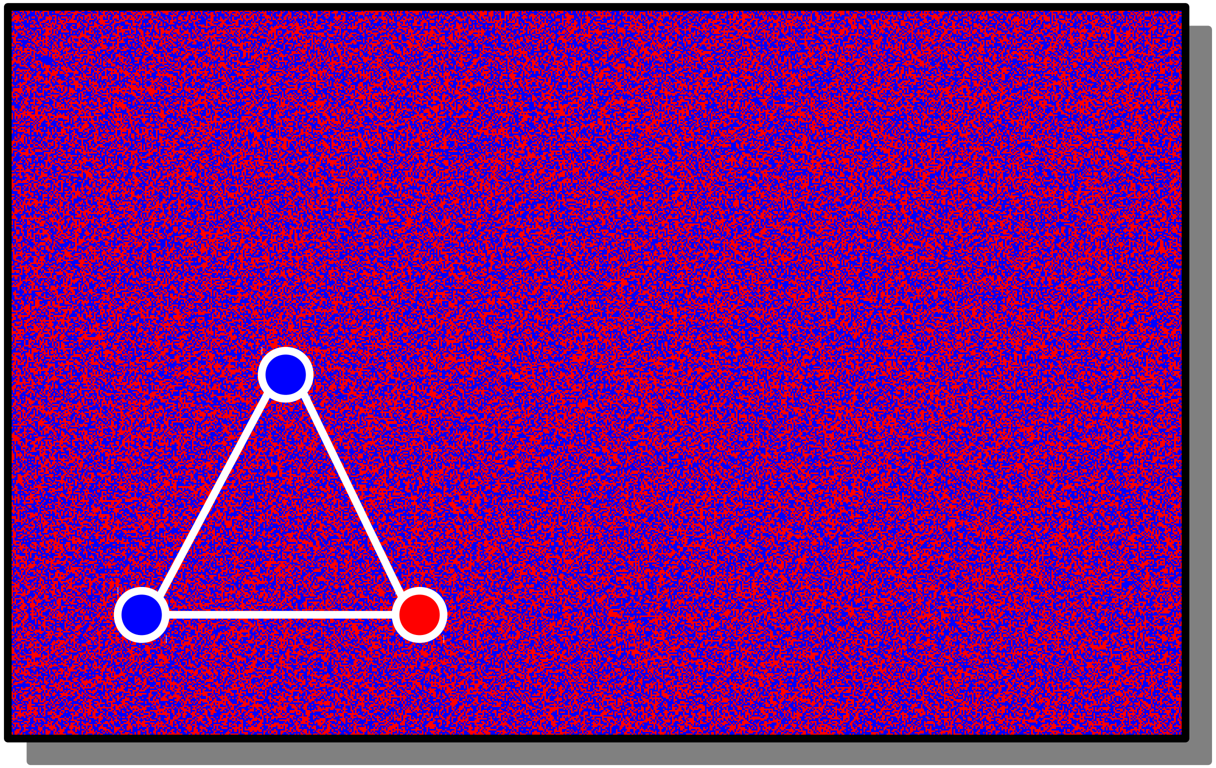

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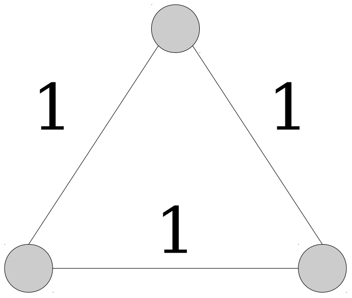
Proof: Consider any equilateral triangle whose sides are length 1. Put this triangle anywhere in the plane. Because the triangle has three vertices and each point in the plane is only one of two different colors, by the pigeonhole principle at least two of the vertices must have the same color. These vertices are at distance 1 from each other, as required. ■

The Hadwiger-Nelson Problem

- No matter how you color the points of the plane, there will always be two points at distance 1 that are the same color.
- Relation to graph coloring:
  - Every point in the real plane is a node.
  - There's an edge between two points that are at distance exactly one.
- ***Question:*** What is the chromatic number of this graph? (That is, how many colors do you need to ensure no points at distance one are the same color?)
- This is the ***Hadwiger-Nelson problem***.

Unit-Distance Graphs

- A ***unit-distance graph*** is a graph that can be drawn in a way where each edge has length one.
- Here's the unit-distance graph we used to show that the solution to Hadwiger-Nelson isn't two.



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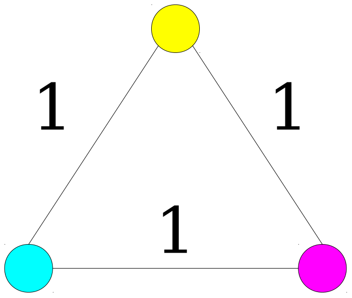

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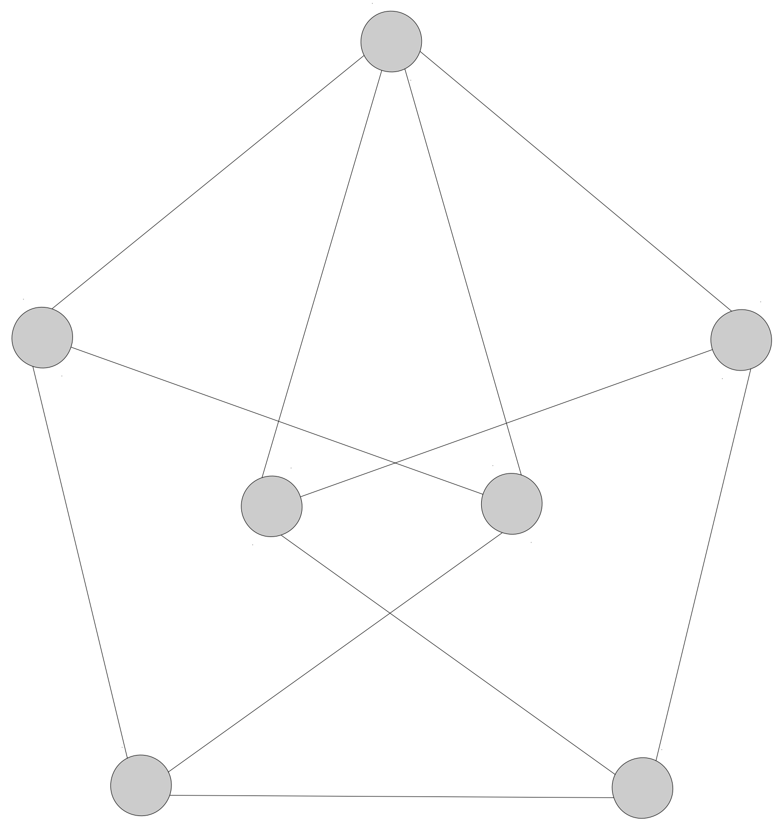
Theorem (Moser): The answer to the Hadwiger-Nelson problem is not three.

Proof: Find a unit-distance graph G where $\chi(G) > 3$.

The Moser Spindle



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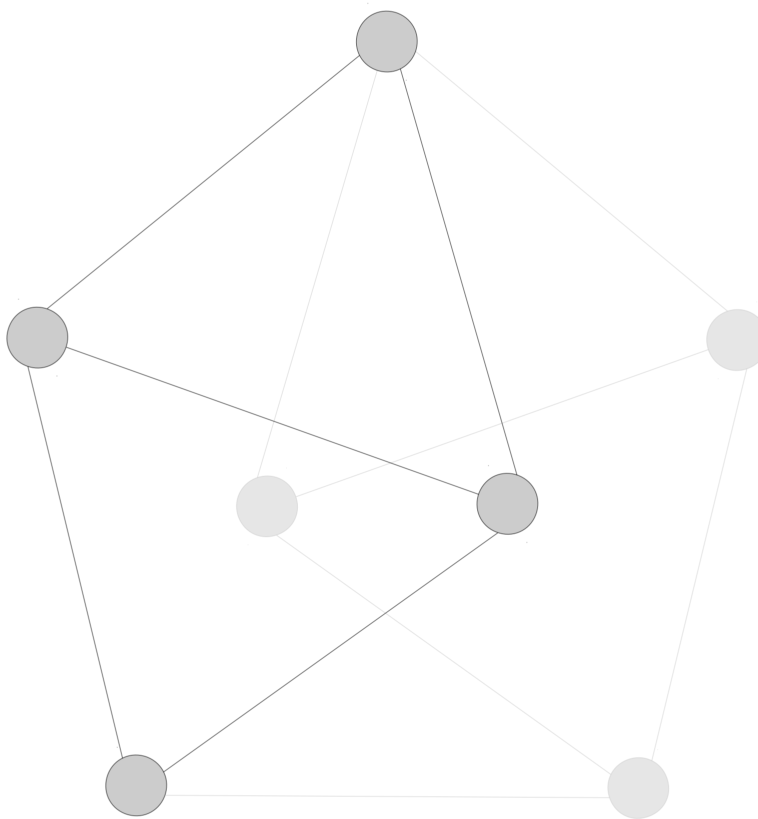

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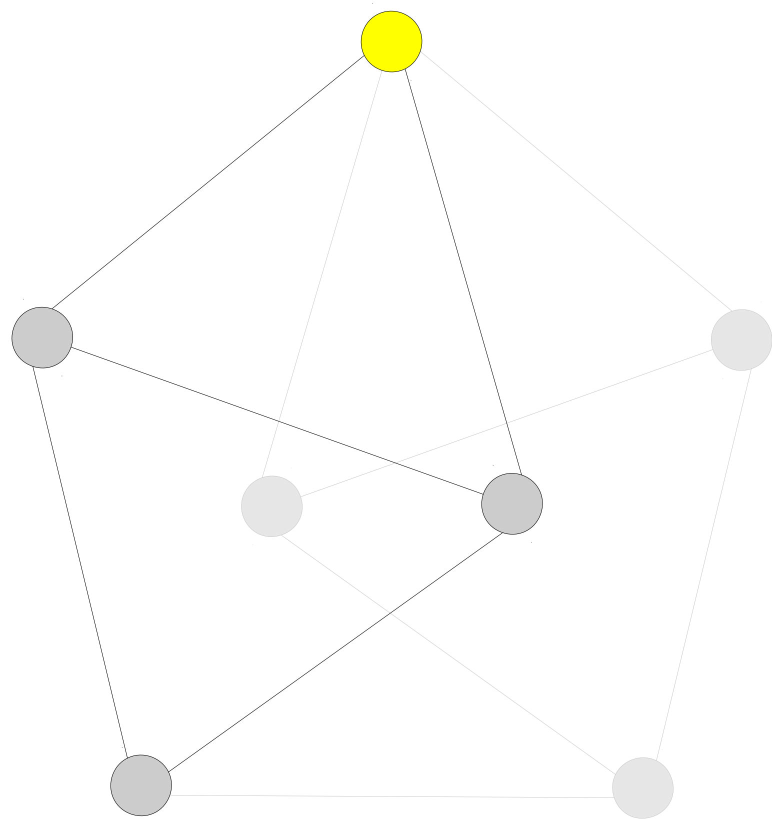

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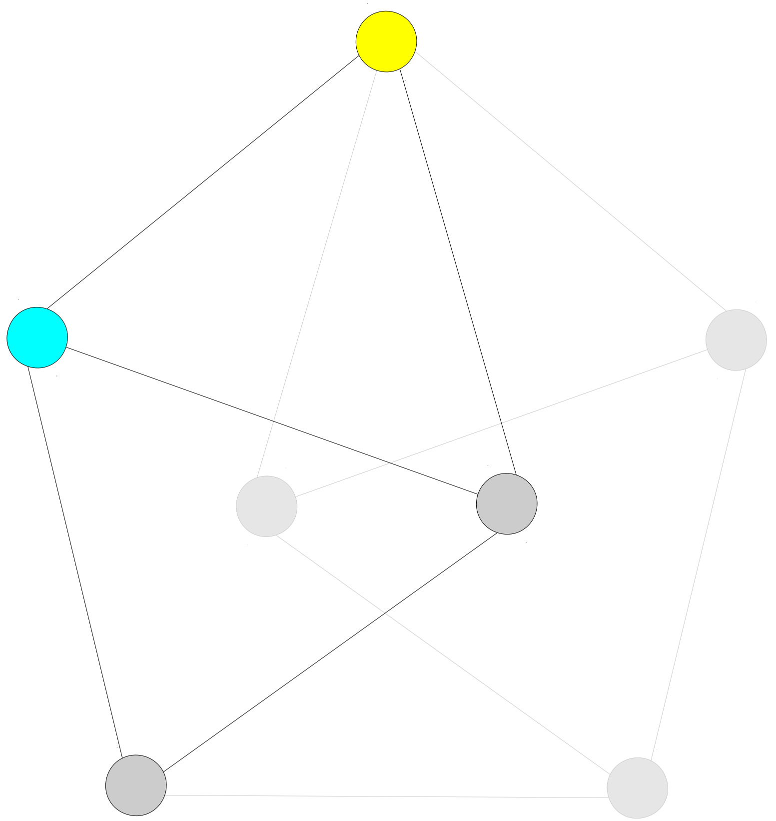

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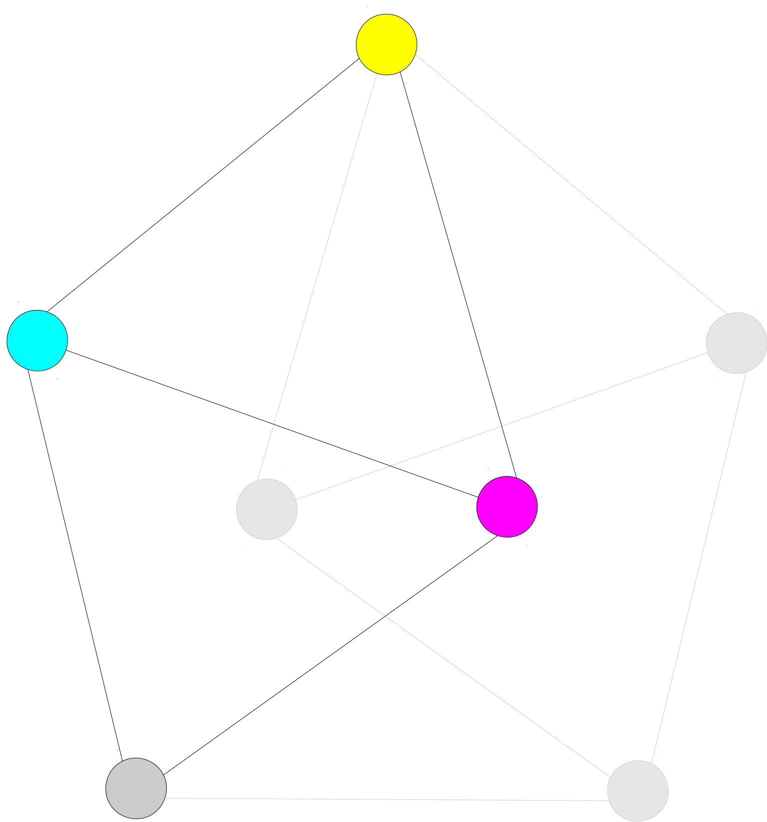

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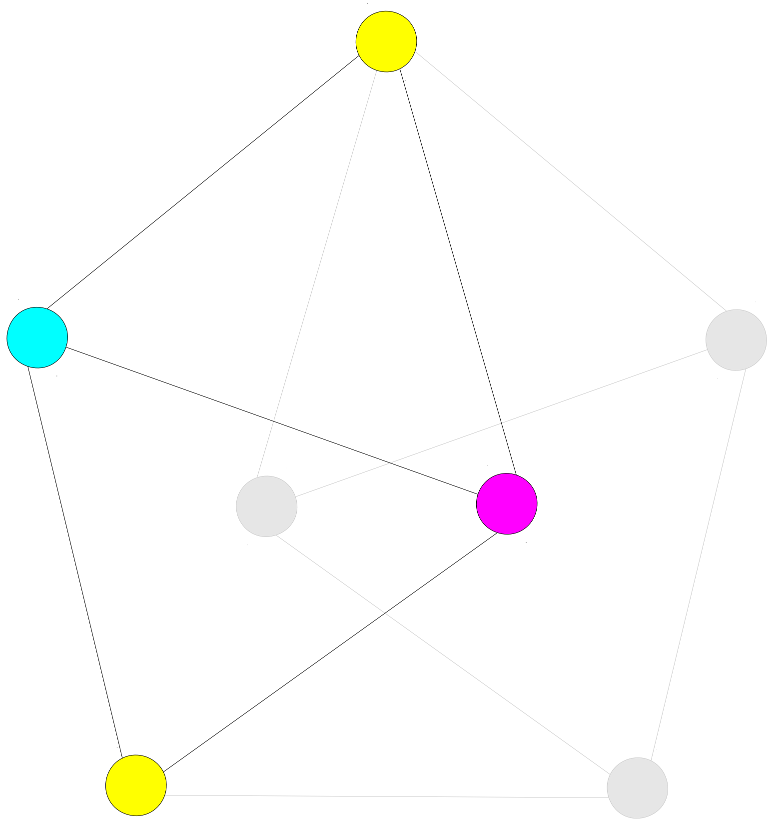

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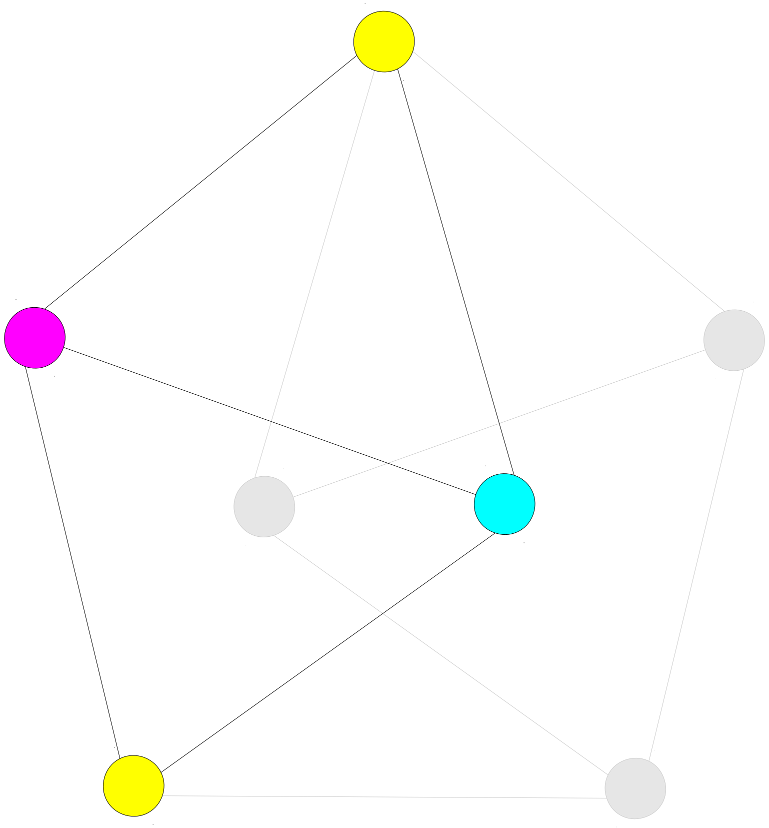

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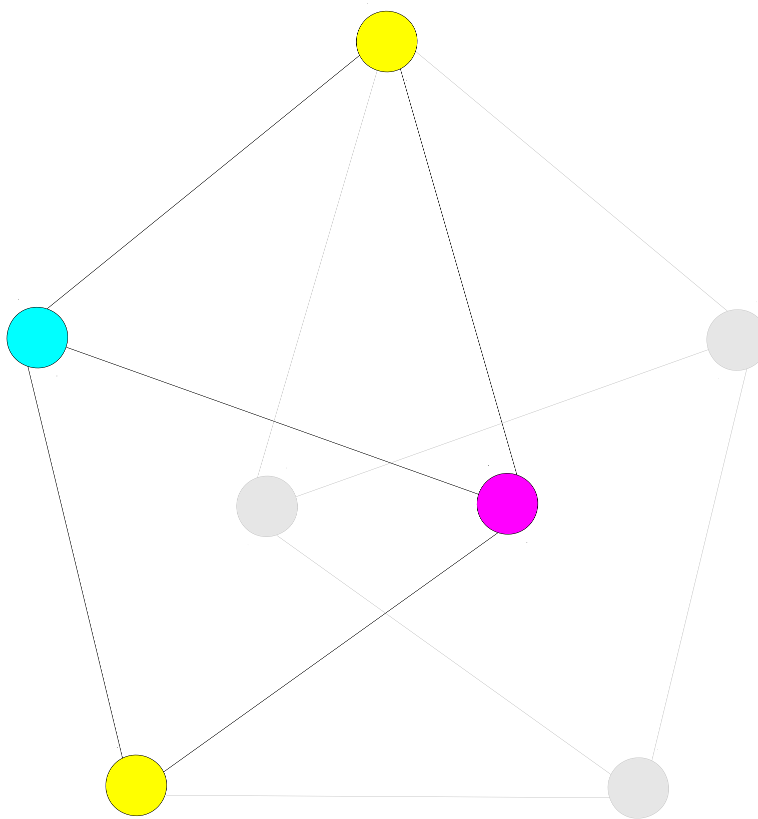

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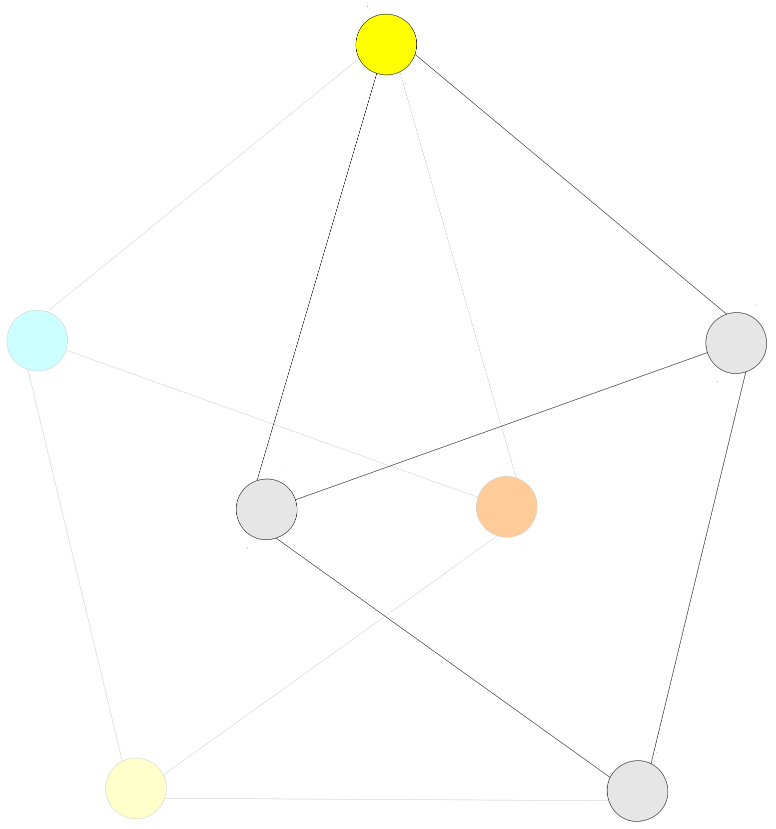

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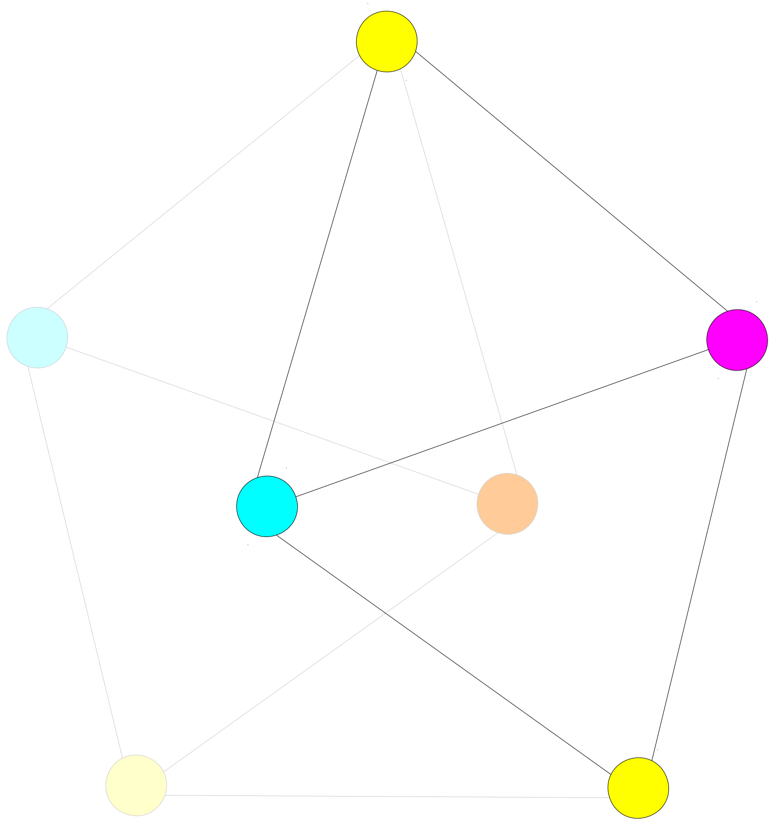

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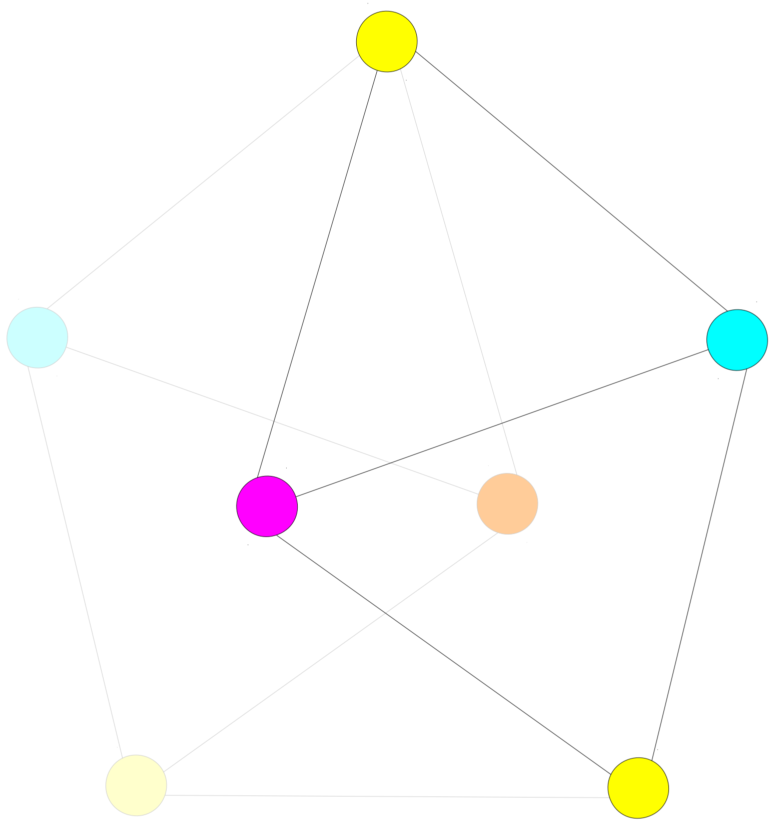

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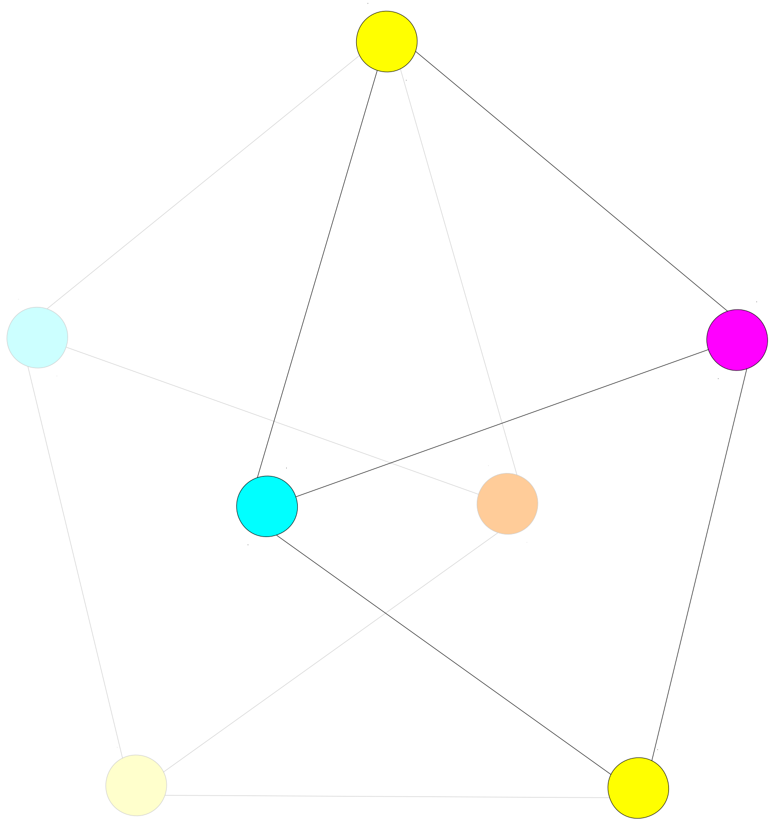

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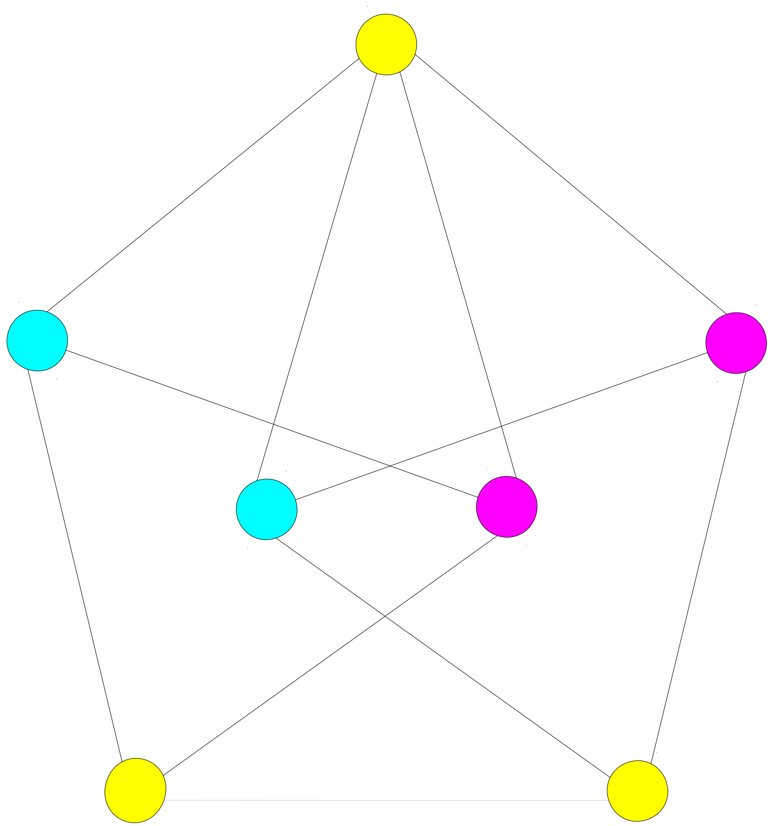

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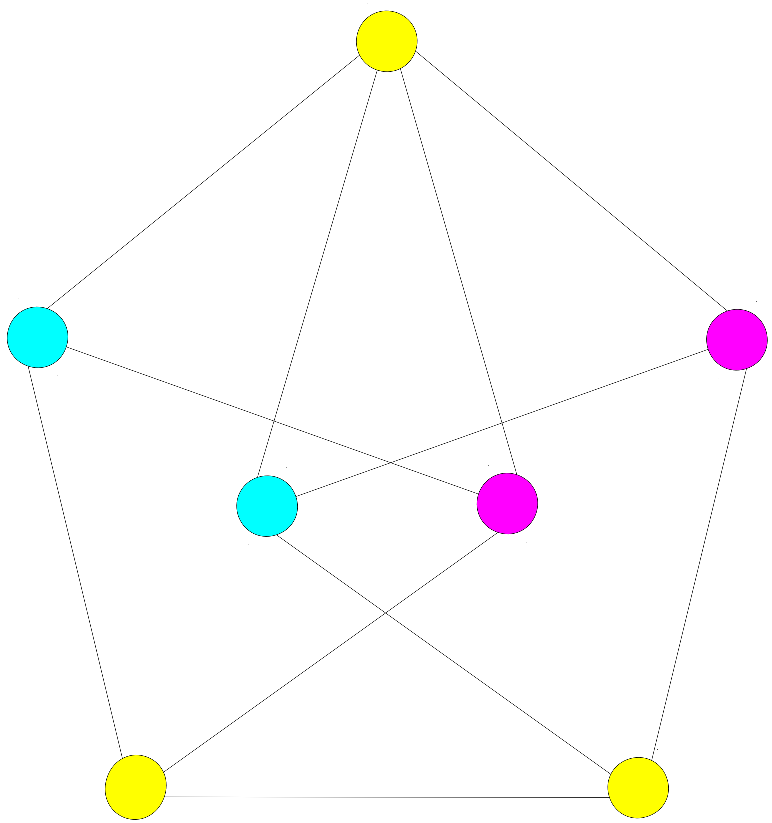
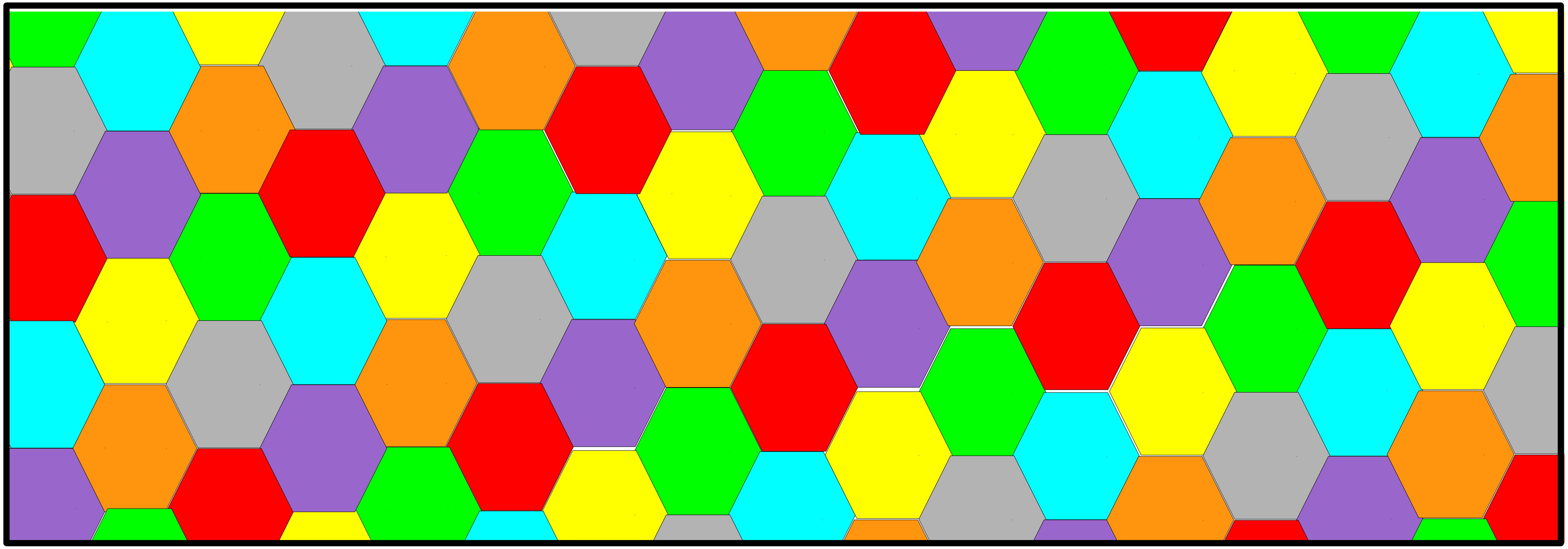


What We Know

- We know that the solution to Hadwiger-Nelson is at least four, since the Moser spindle is a unit-distance graph with chromatic number four.
- It turns out that it's possible to color the plane using seven colors so that no two points at distance 1 are the same color by tiling the plane with hexagons of diameter ¾.



The Hadwiger-Nelson Problem

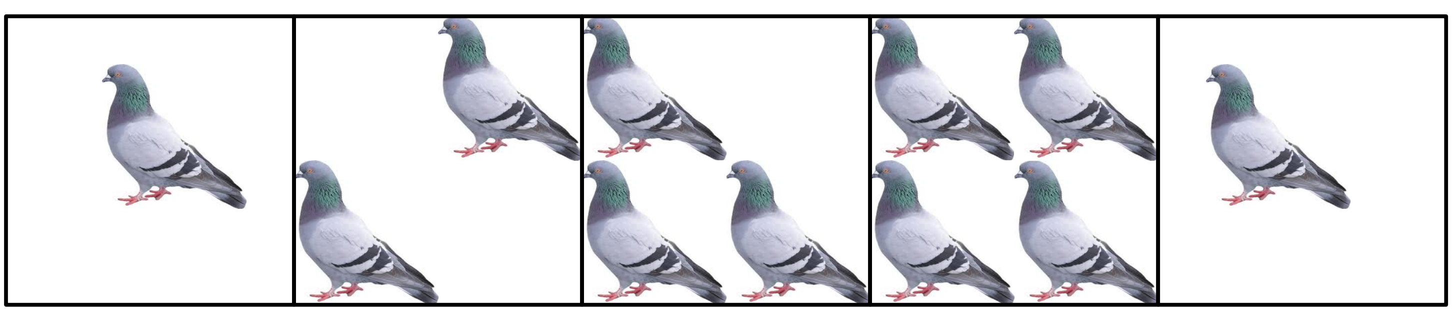
- We know that the answer to Hadwiger-Nelson problem can't be 1, 2, or 3.
- We also know the solution to the Hadwiger-Nelson problem must be at most 7.
- This means that the answer to the Hadwiger-Nelson problem must be 4, 5, 6, or 7.
- ***Amazing fact:*** No one knows which of these numbers is correct!
- ***This is an open problem in mathematics!***

The Generalized Pigeonhole Principle

The ***generalized pigeonhole principle*** is the following:

If m objects are placed into n bins, then some bin contains at least $\lceil m/n \rceil$ objects and some bin contains at most $\lfloor m/n \rfloor$ objects.

(Here, $\lceil x \rceil$ is the ***ceiling function*** and denotes the smallest integer greater than or equal to x, and $\lfloor x \rfloor$ is the ***floor function*** and denotes the largest integer less than or equal to x.)



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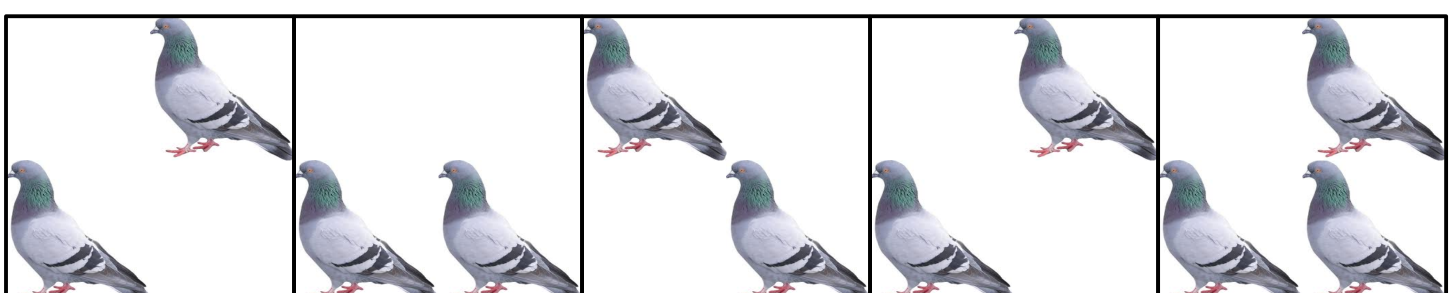

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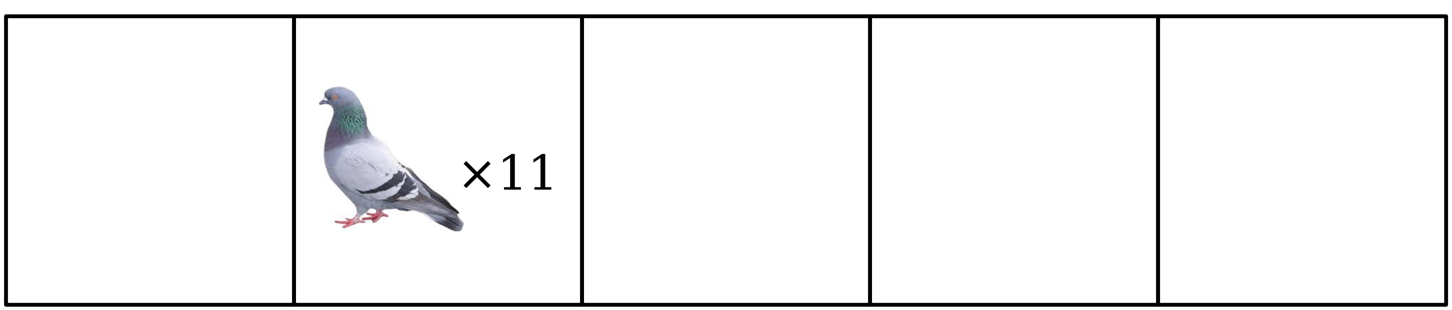

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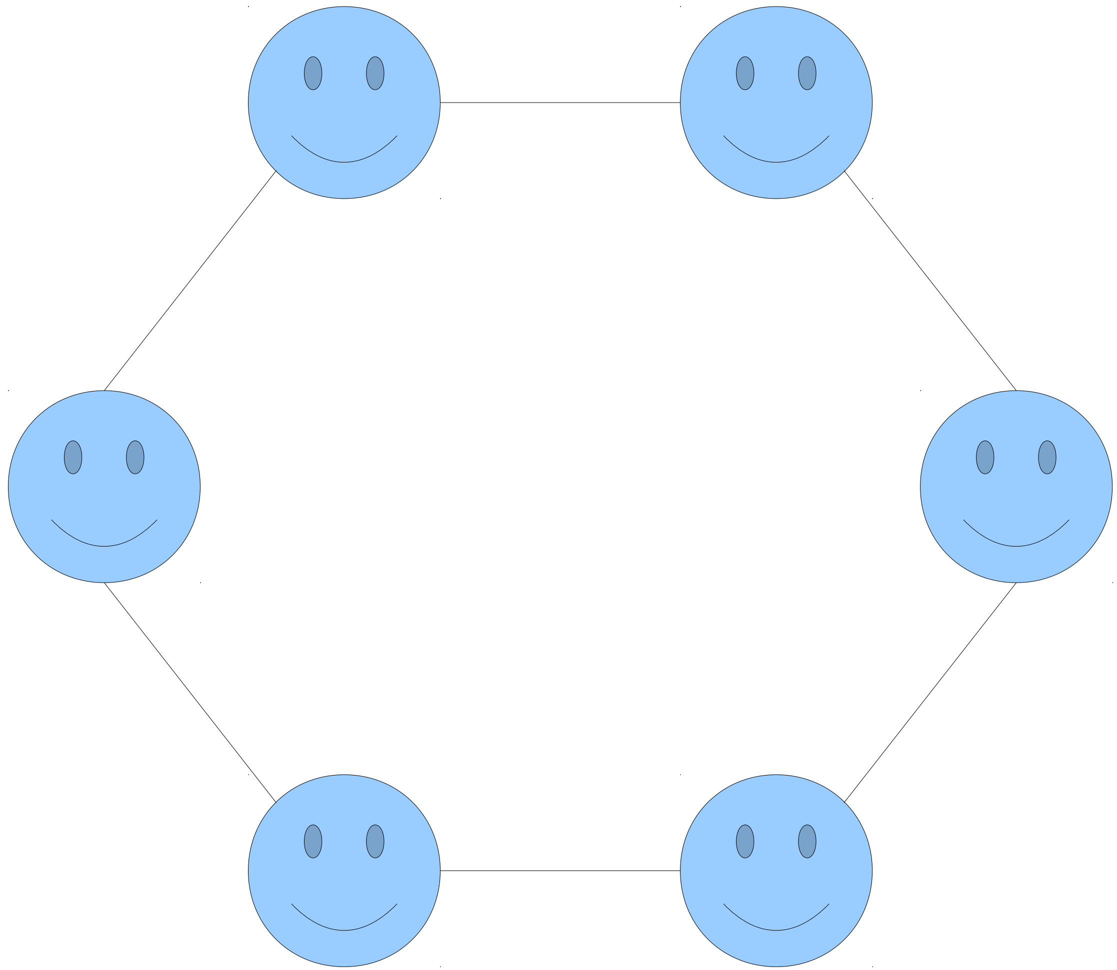
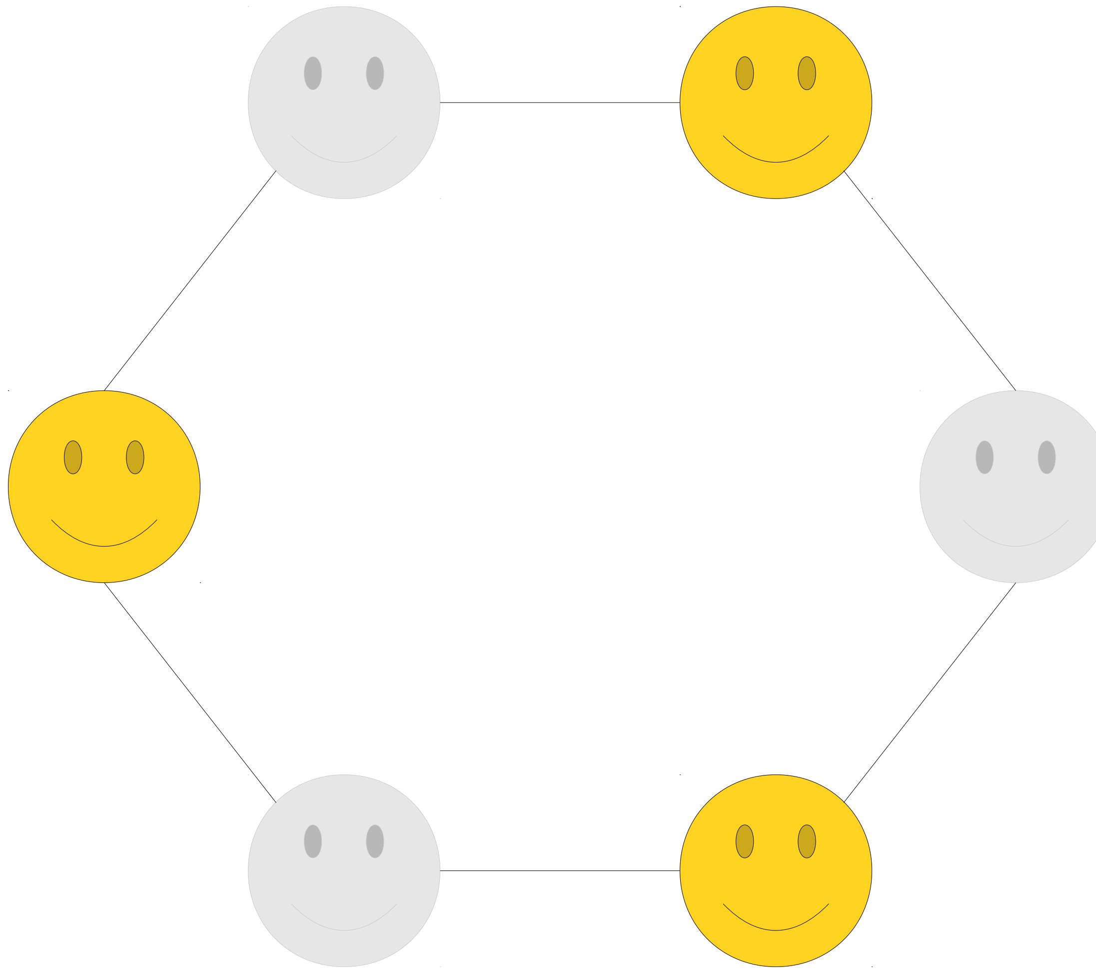
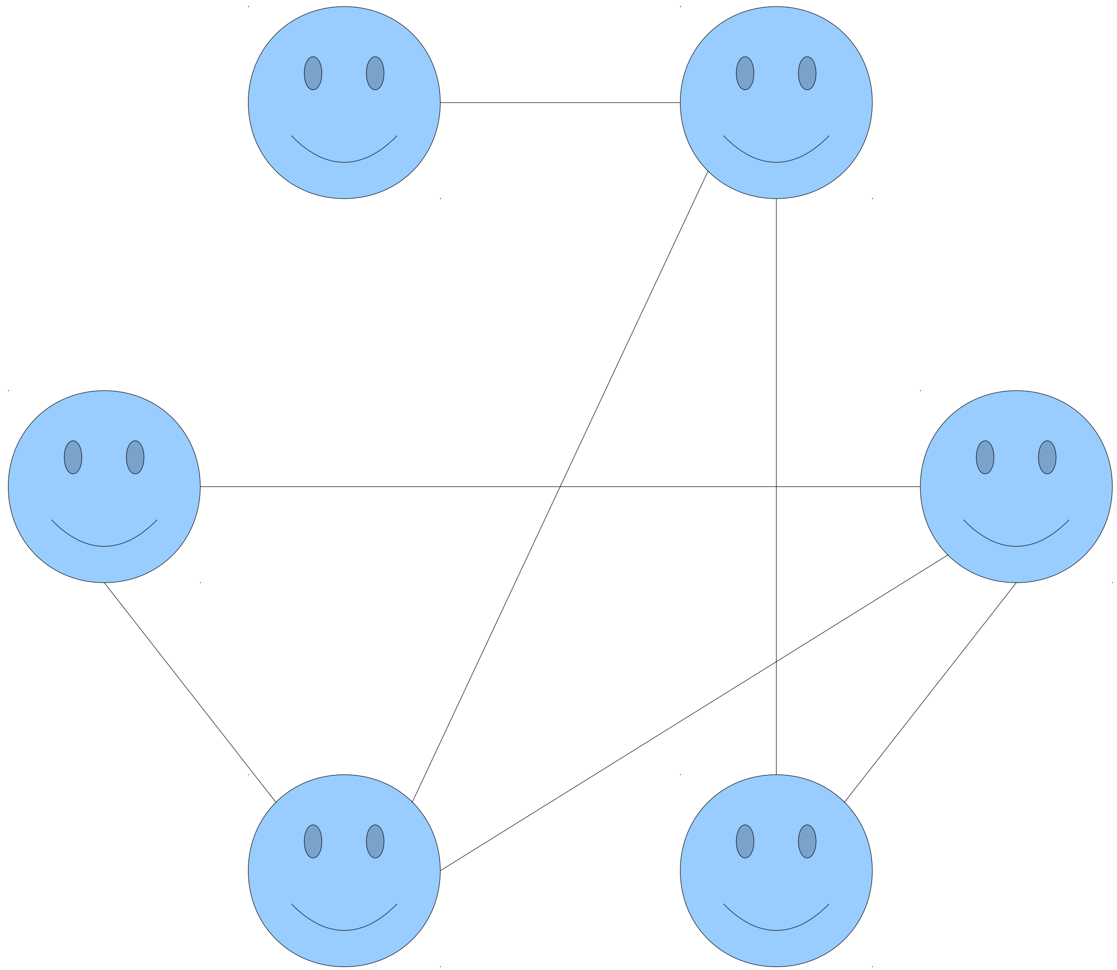
An Application: Friends and Strangers

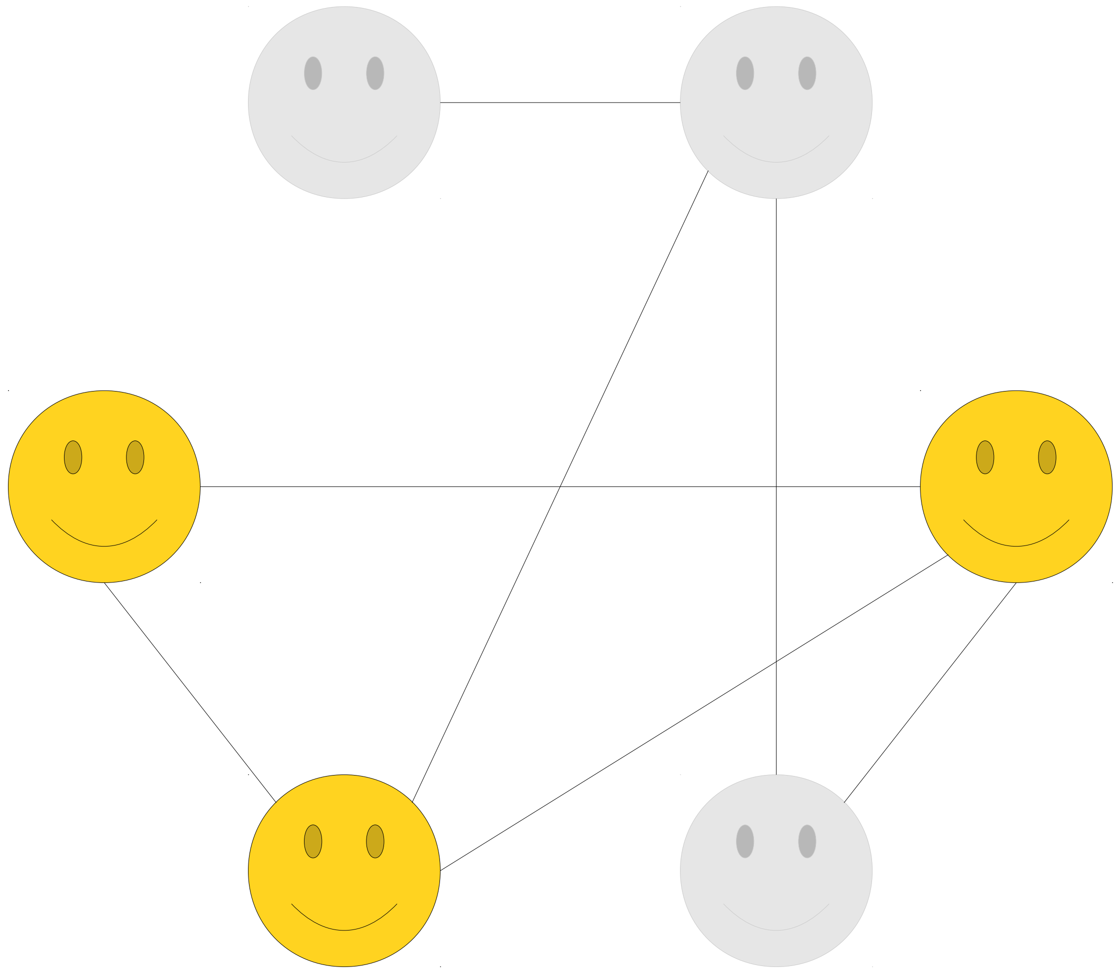
Friends and Strangers

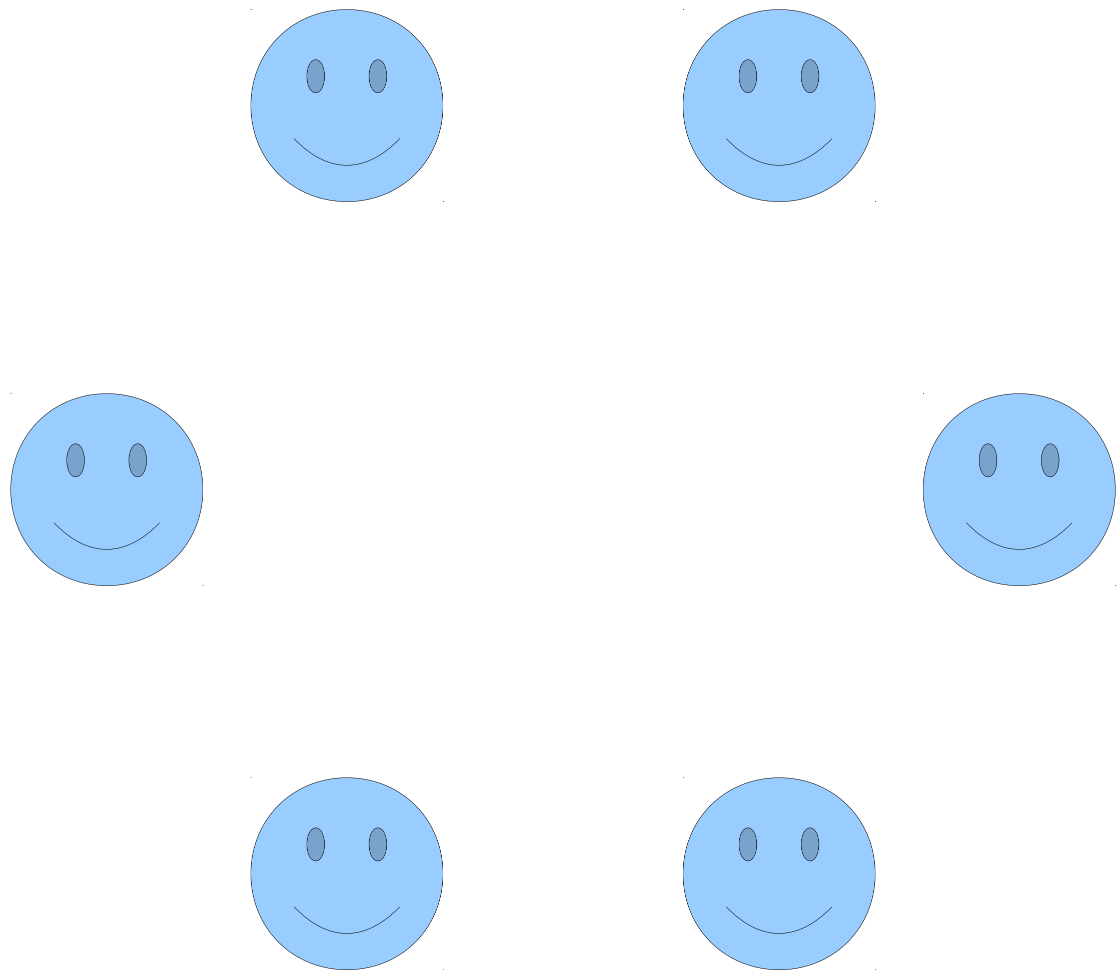
- Suppose you have a party of six people. Each pair of people are either friends (they know each other) or strangers (they do not).
- ***Theorem:*** Any such party must have a group of three mutual friends (three people who all know one another) or three mutual strangers (three people where no one knows anyone else).

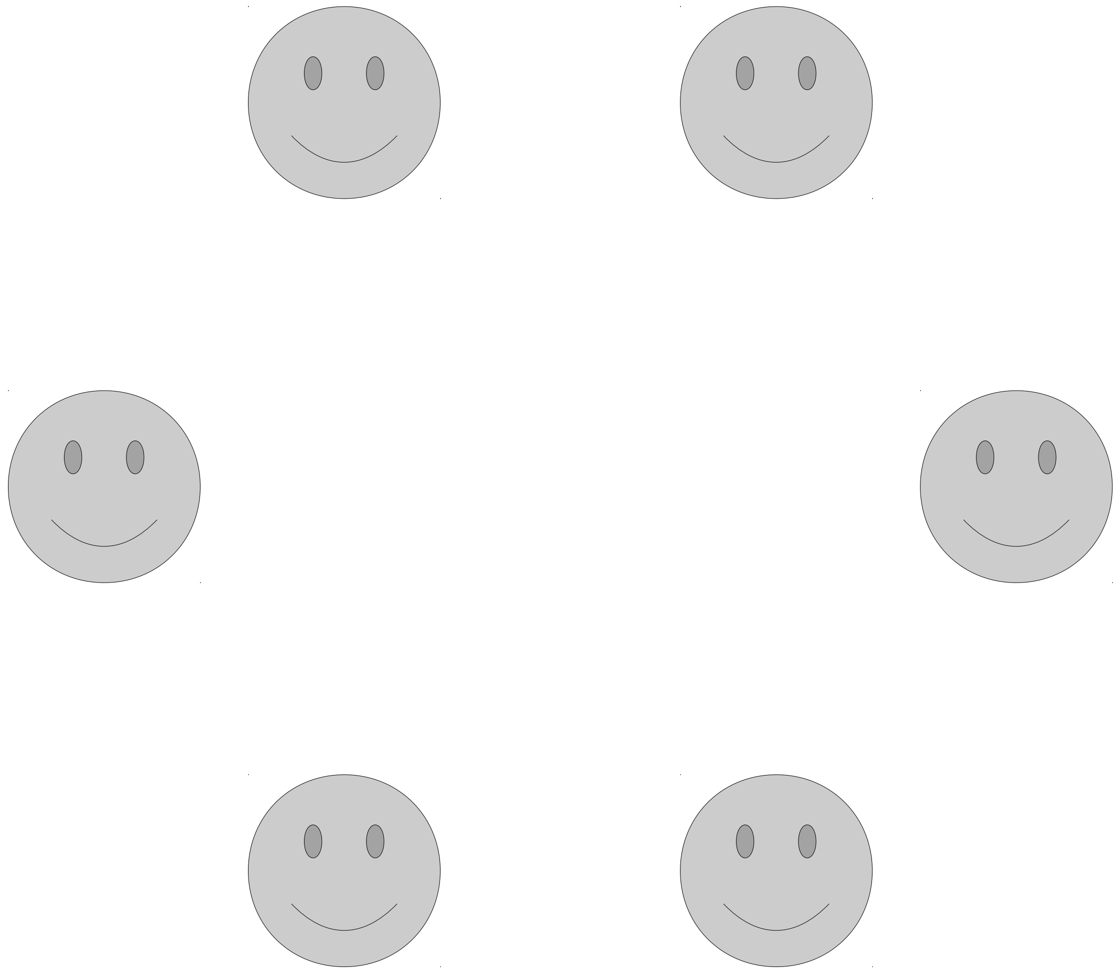


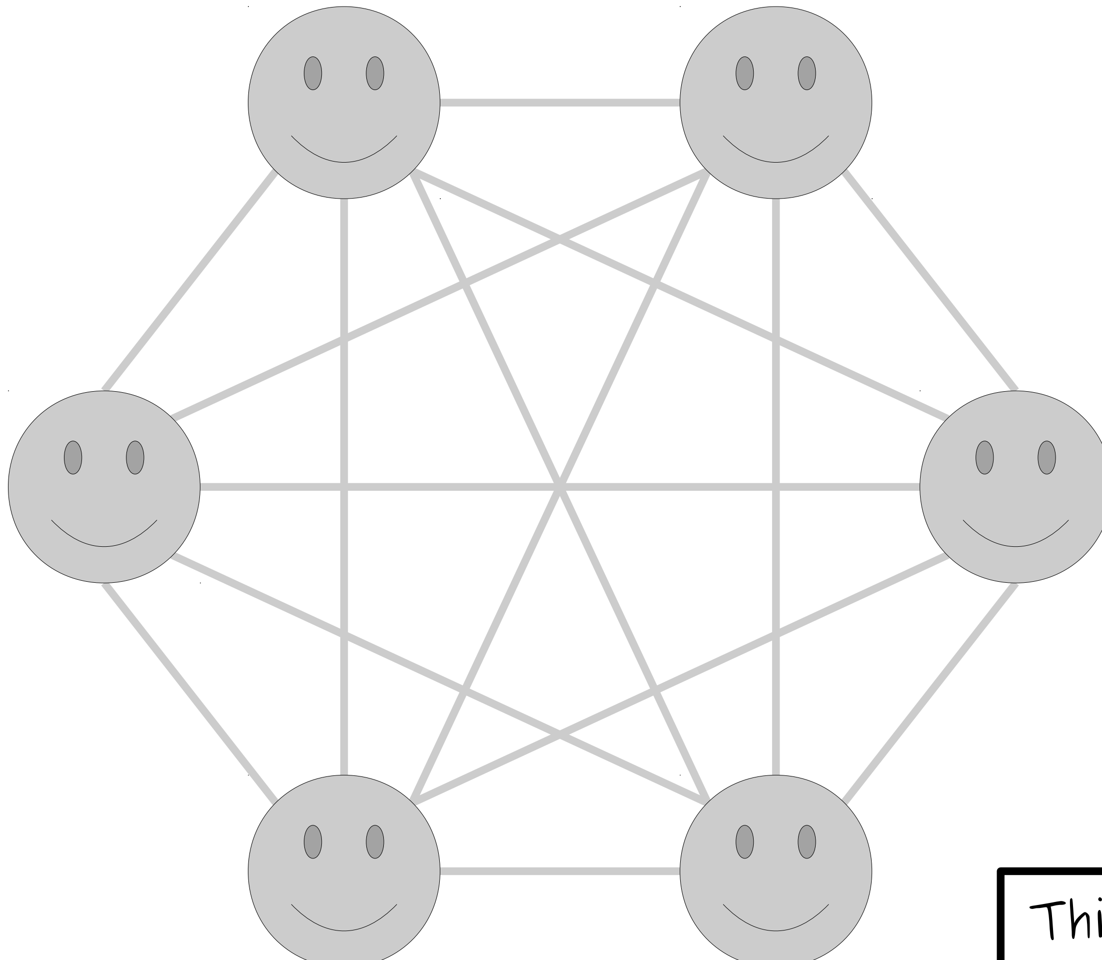




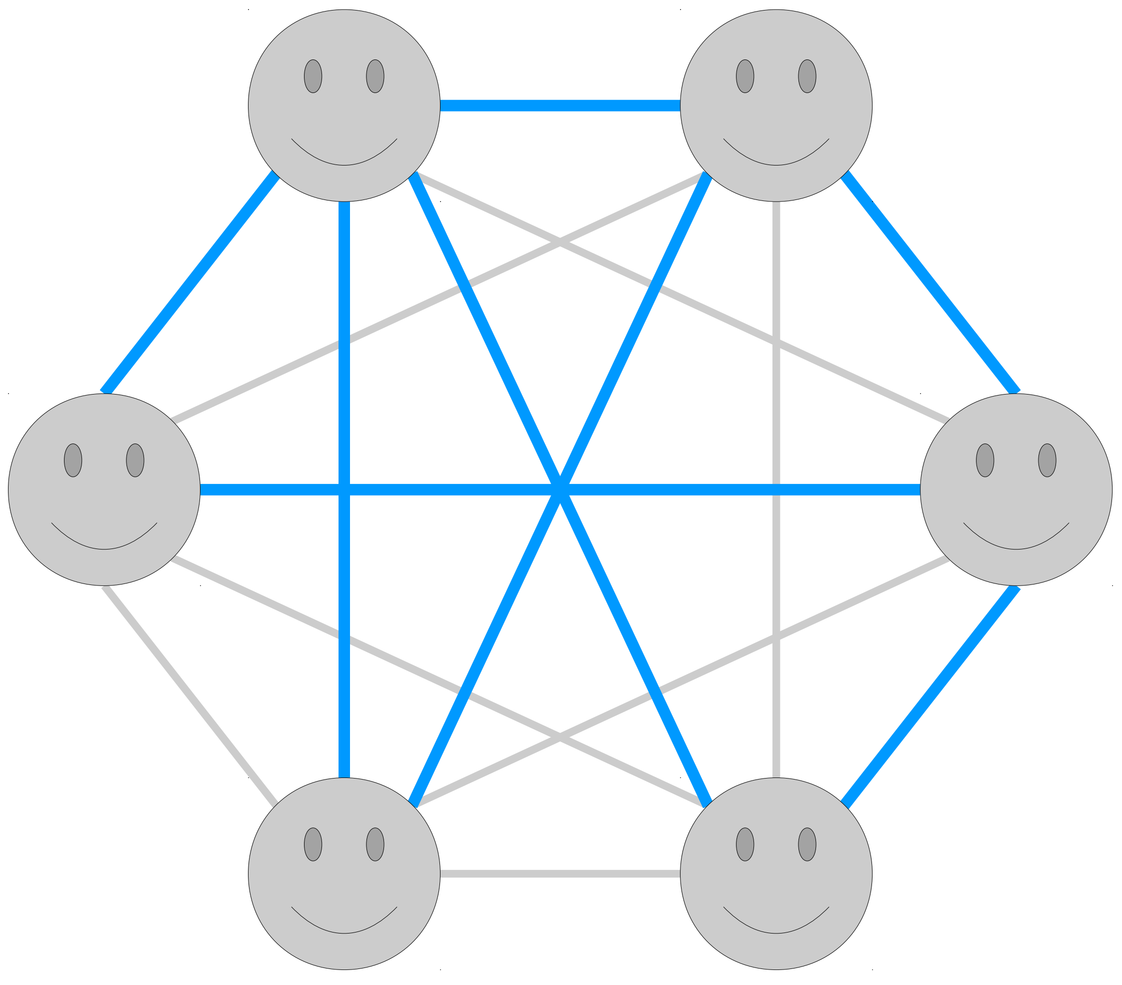


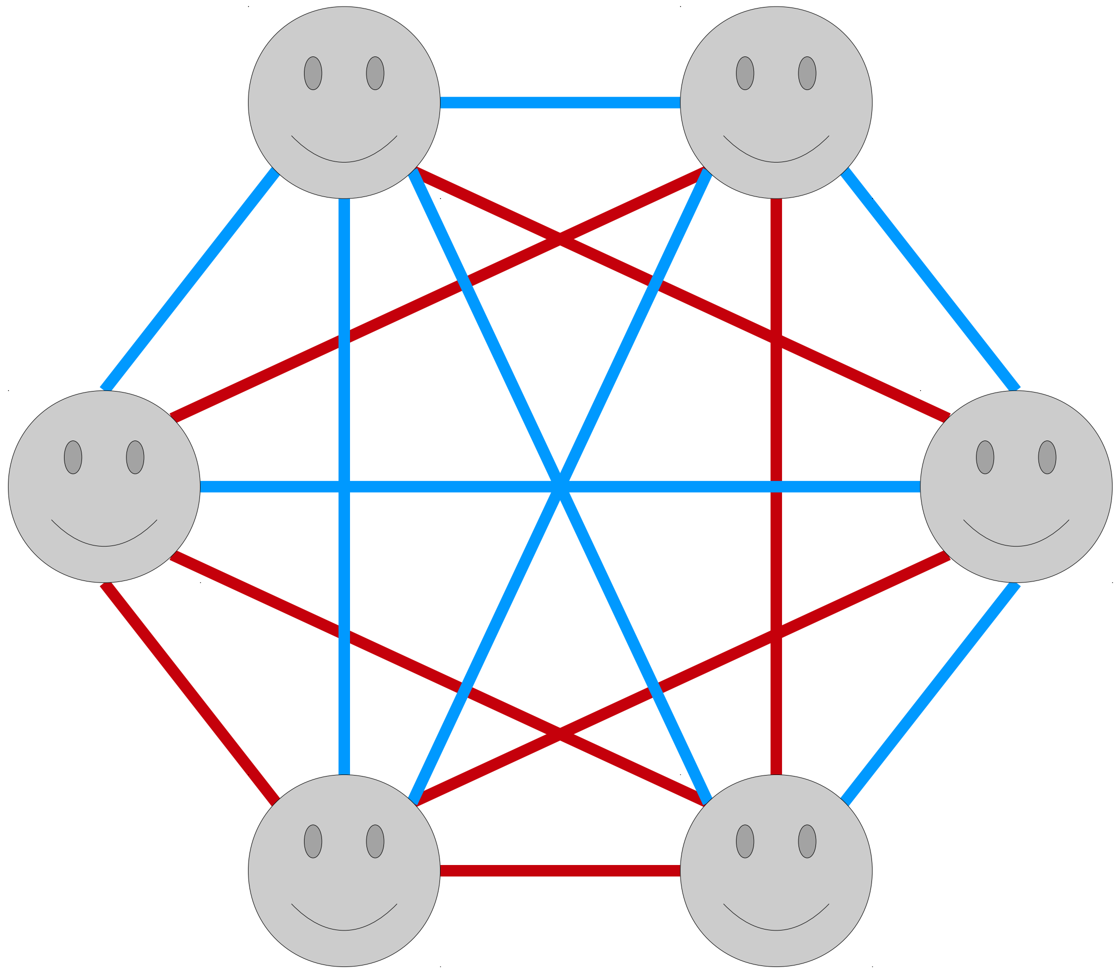


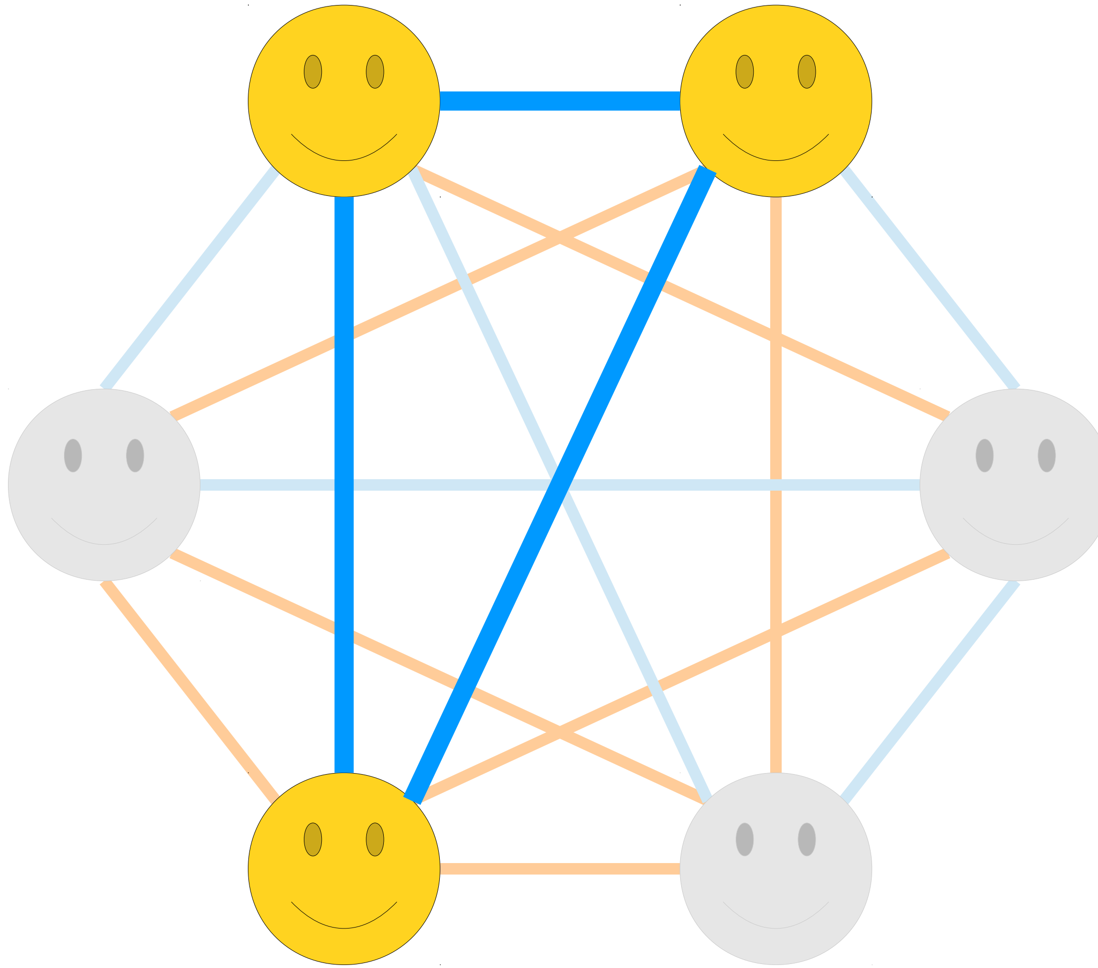


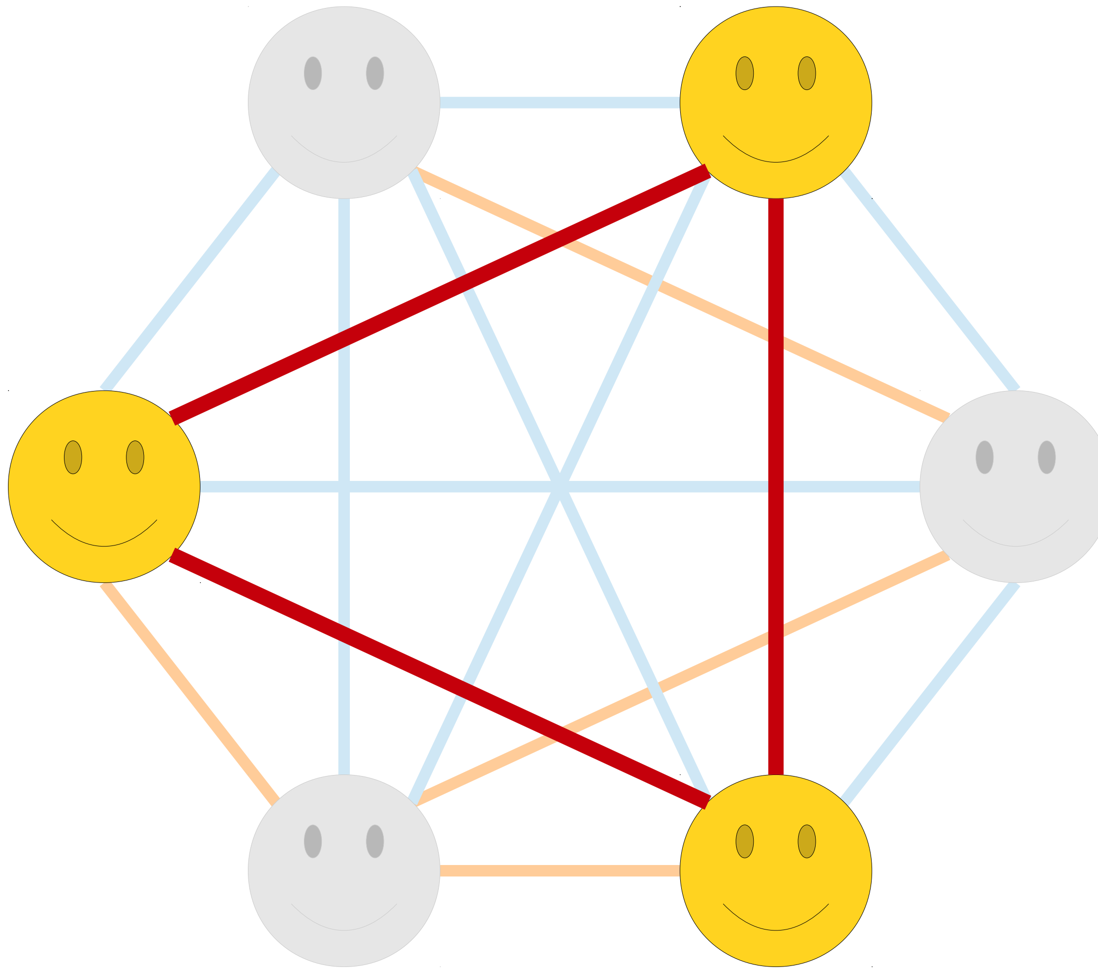


This graph is called a *6-clique*, by the way.



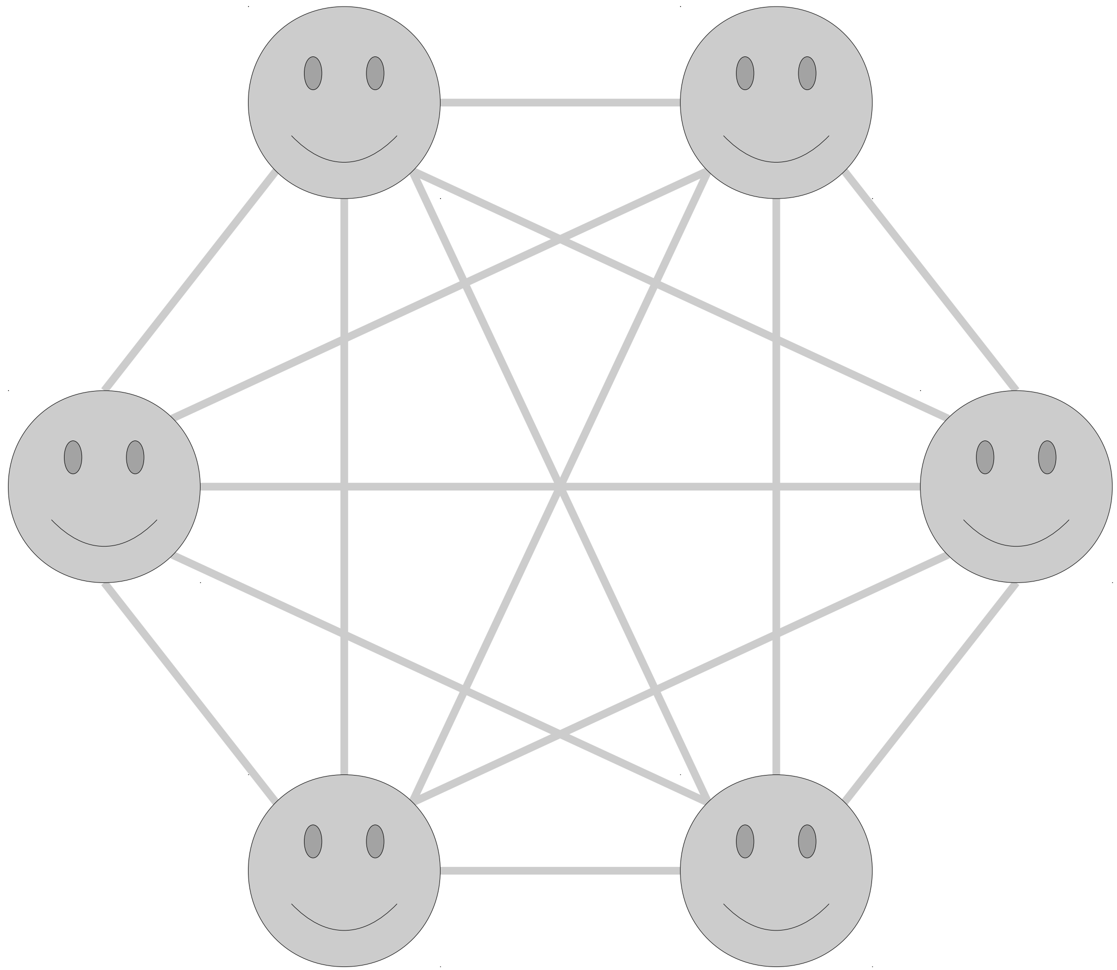


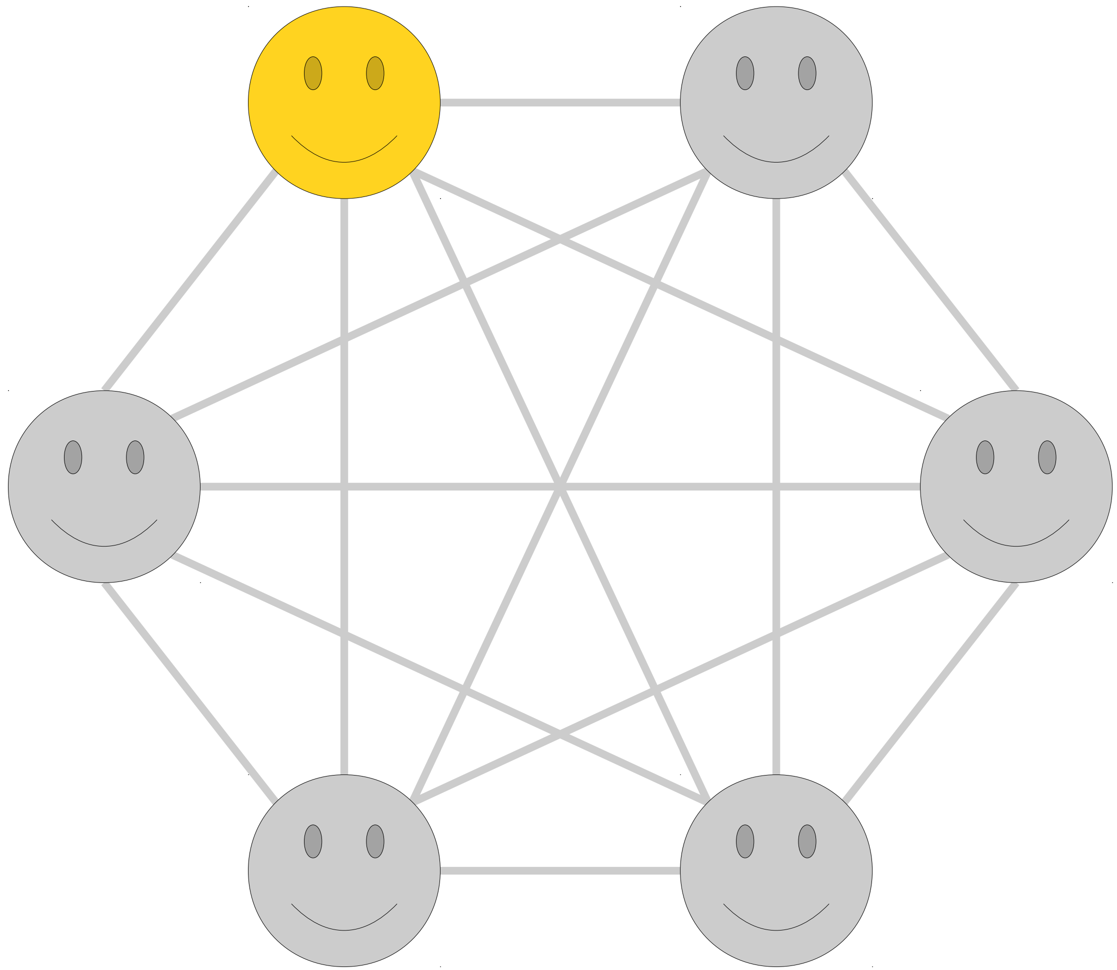


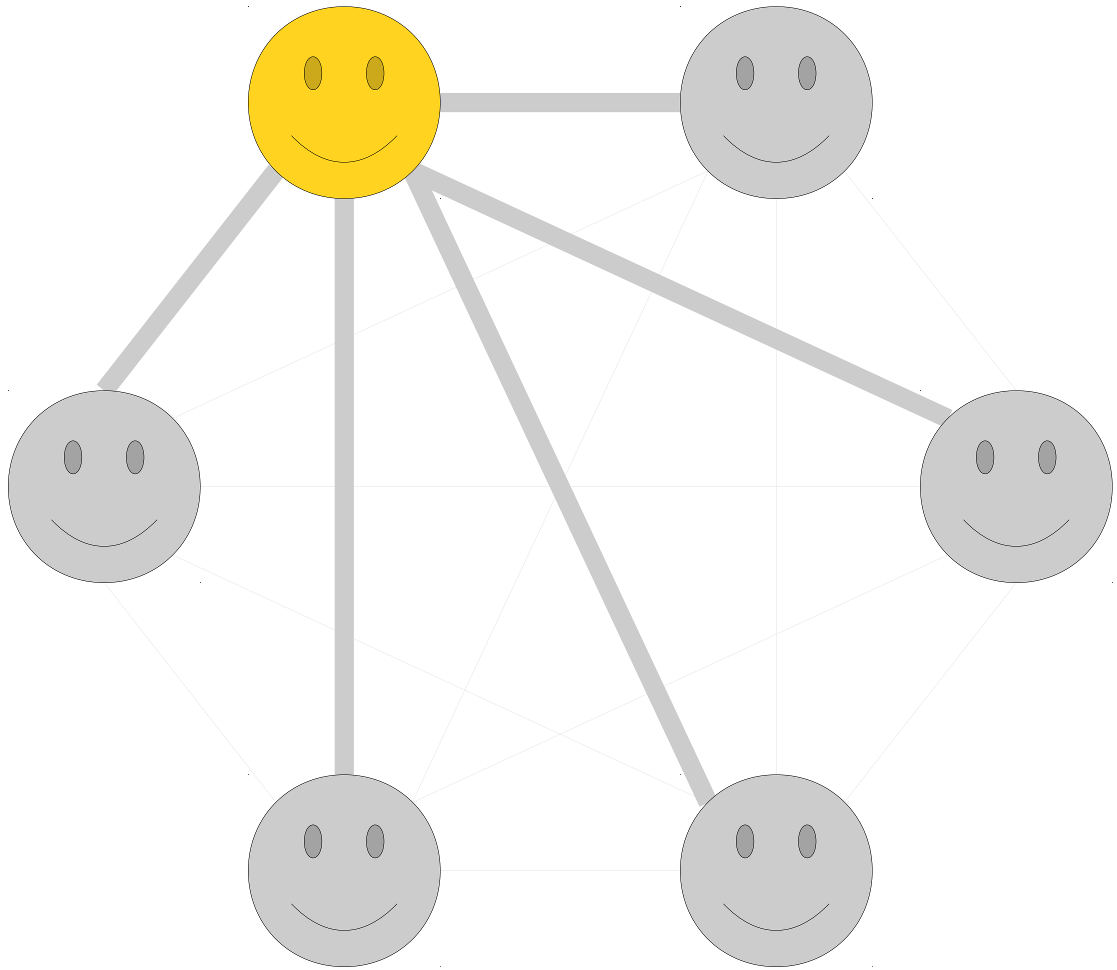


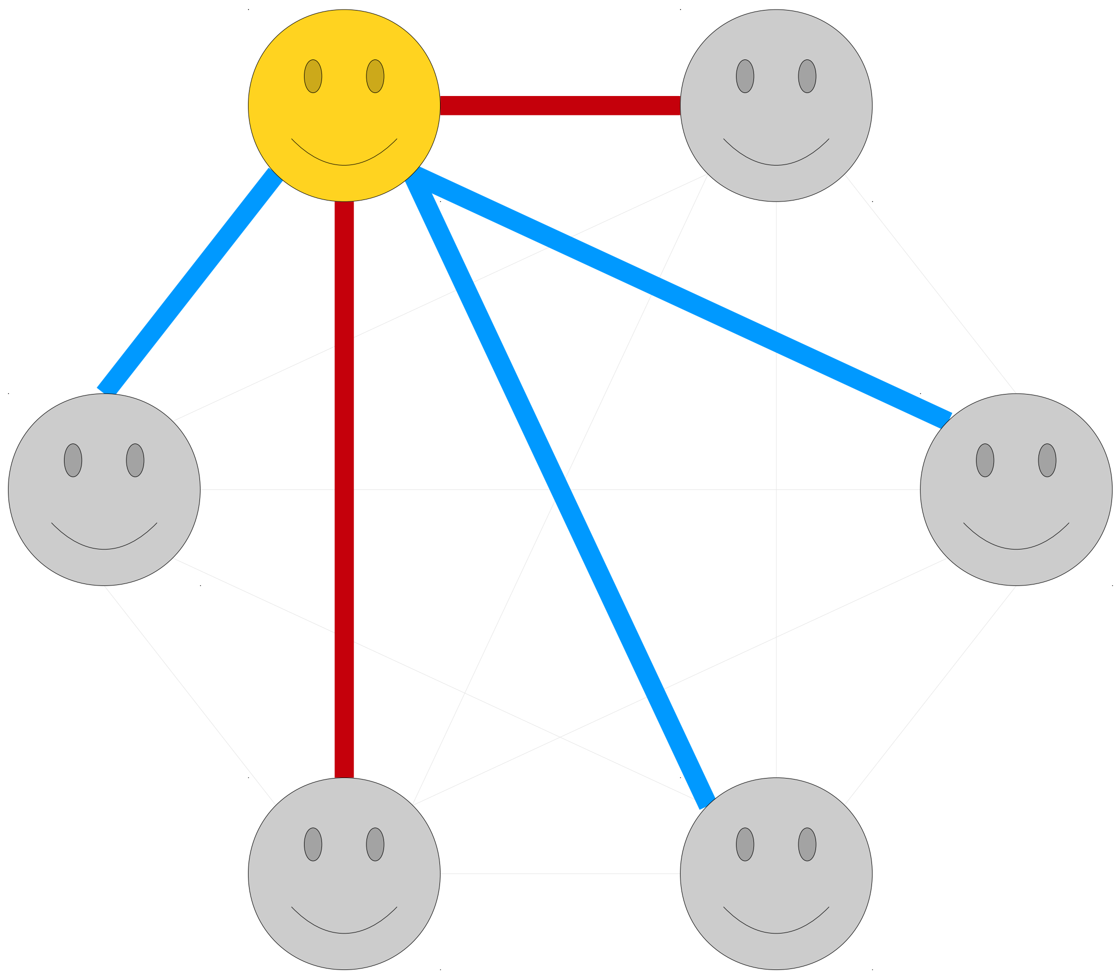
Friends and Strangers Restated

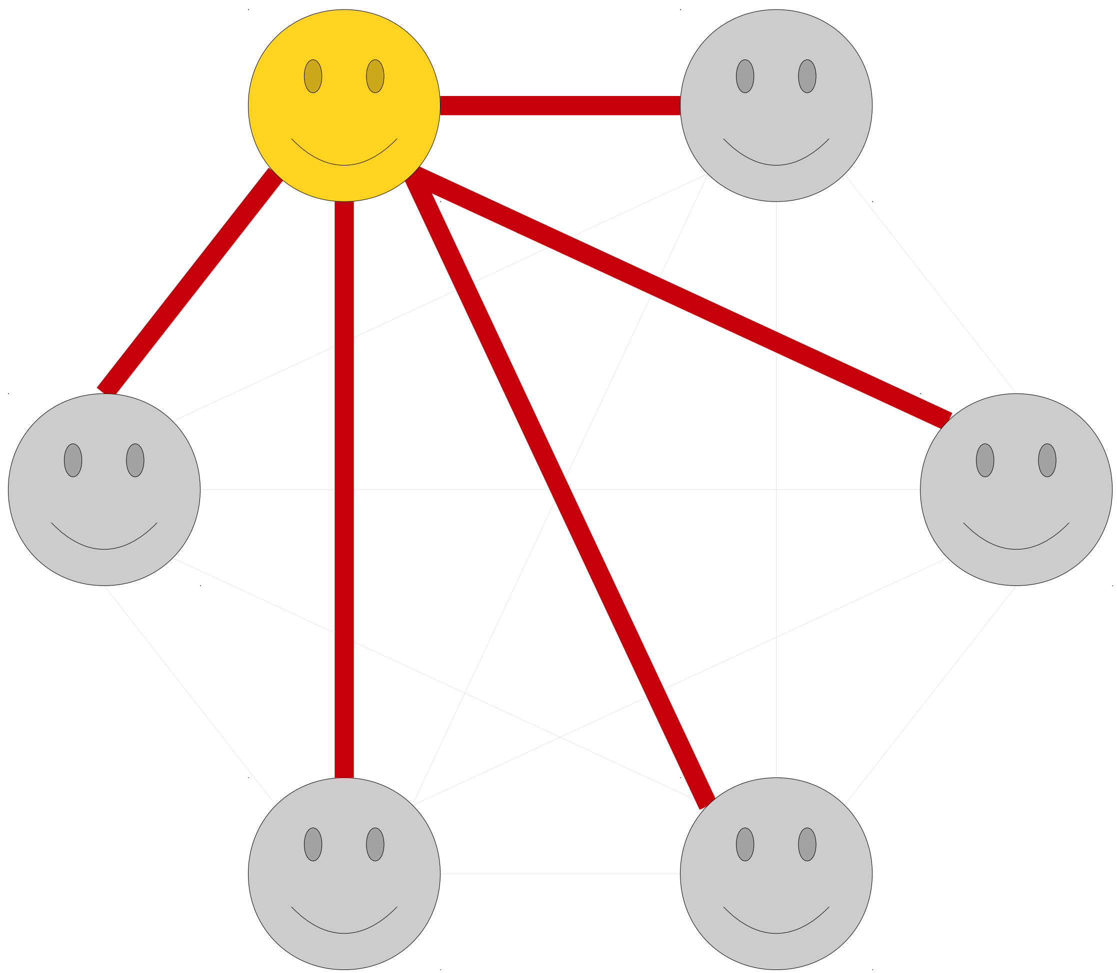
- From a graph-theoretic perspective, the Theorem on Friends and Strangers can be restated as follows:
- ***Theorem:*** Consider a 6-clique where every edge is colored red or blue. The the graph contains a red triangle, a blue triangle, or both.
- How can we prove this?

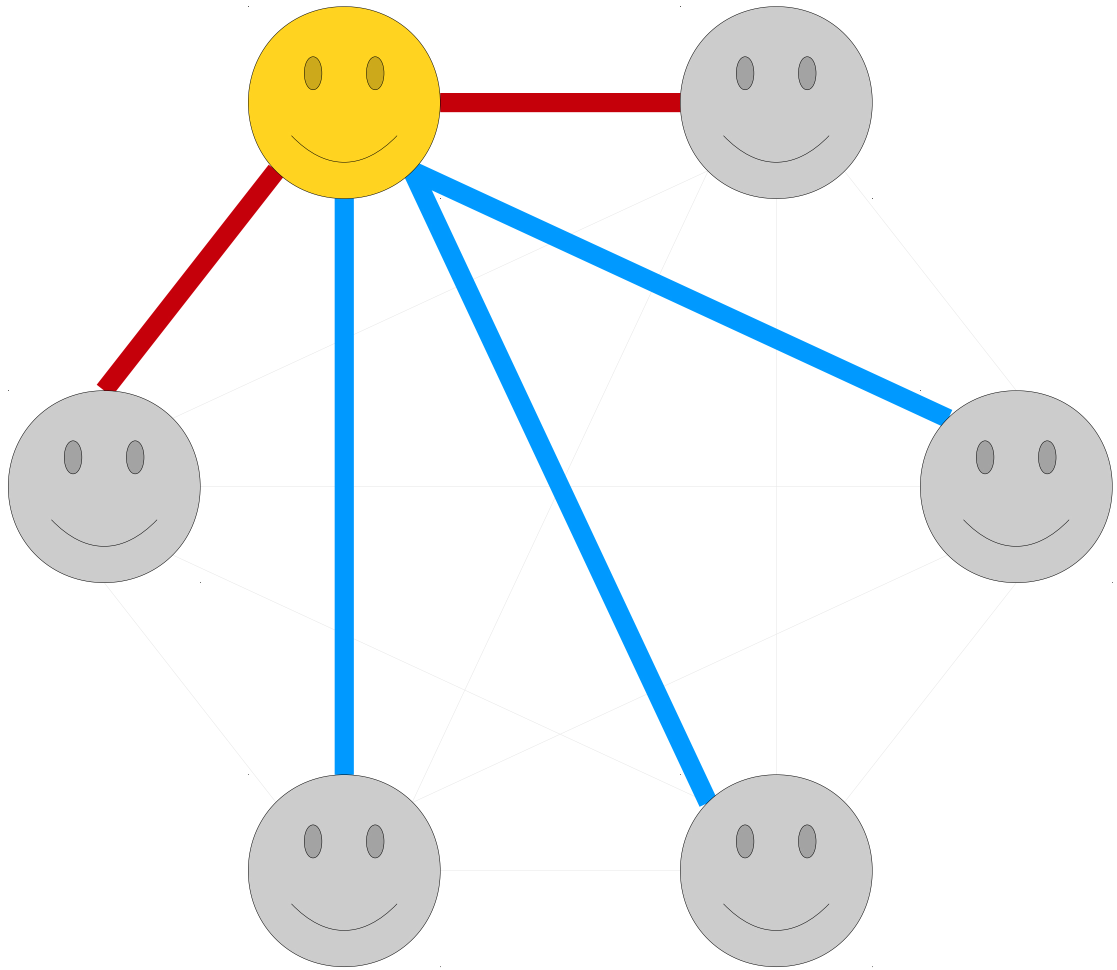


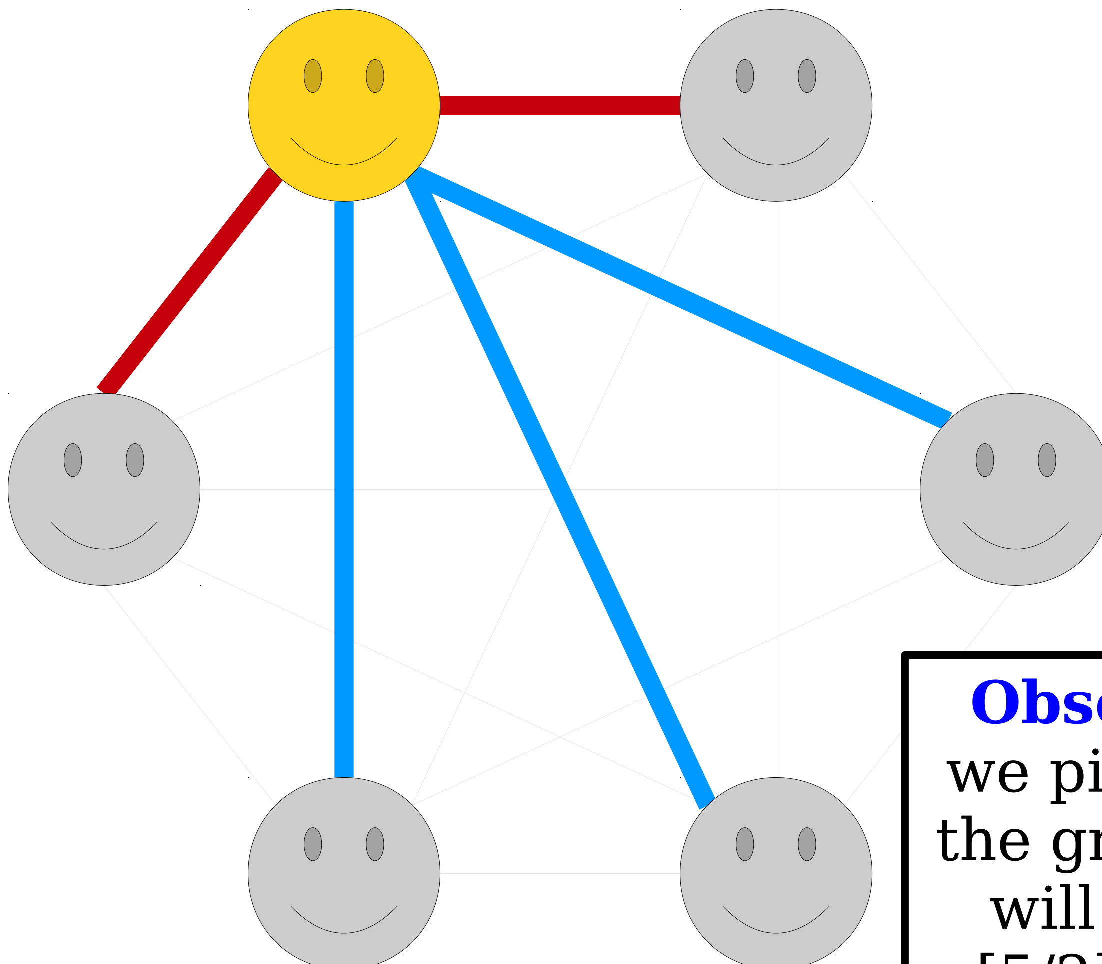




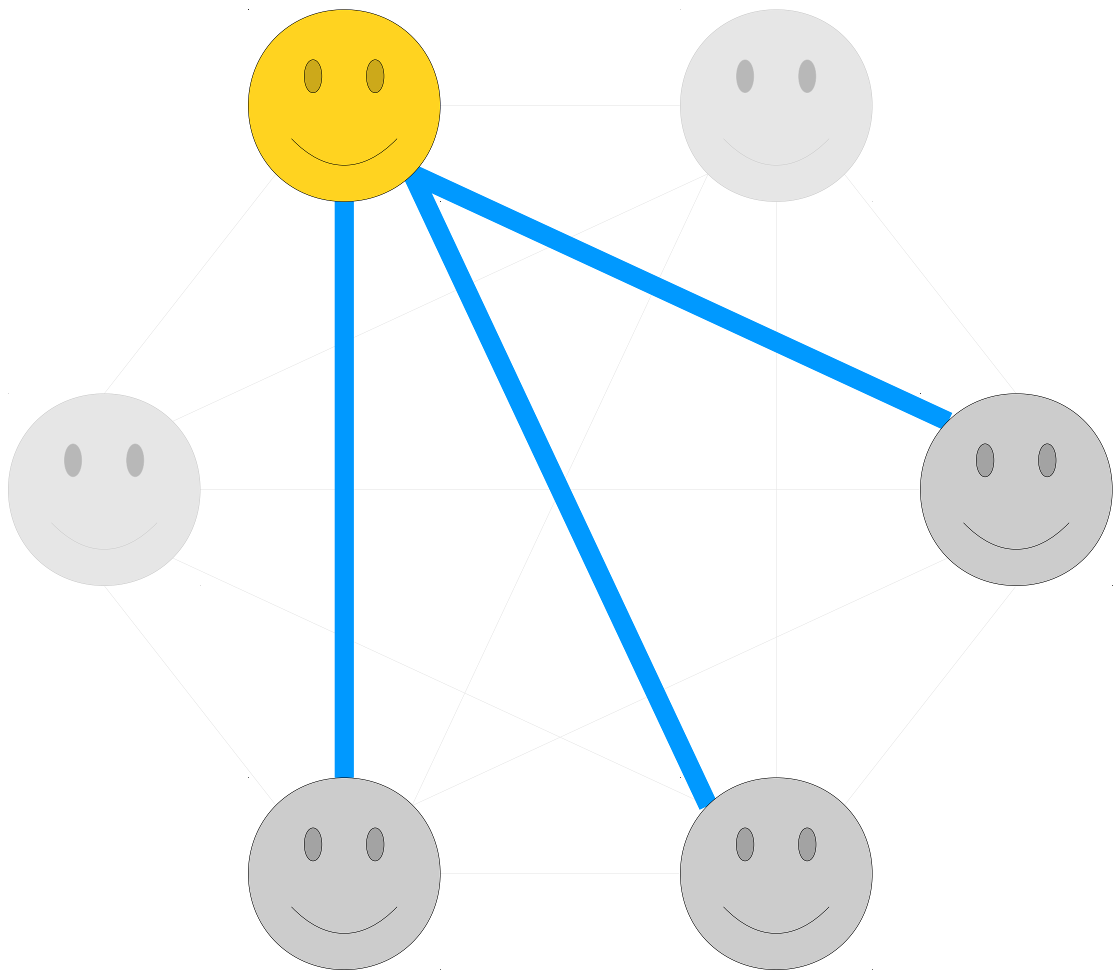


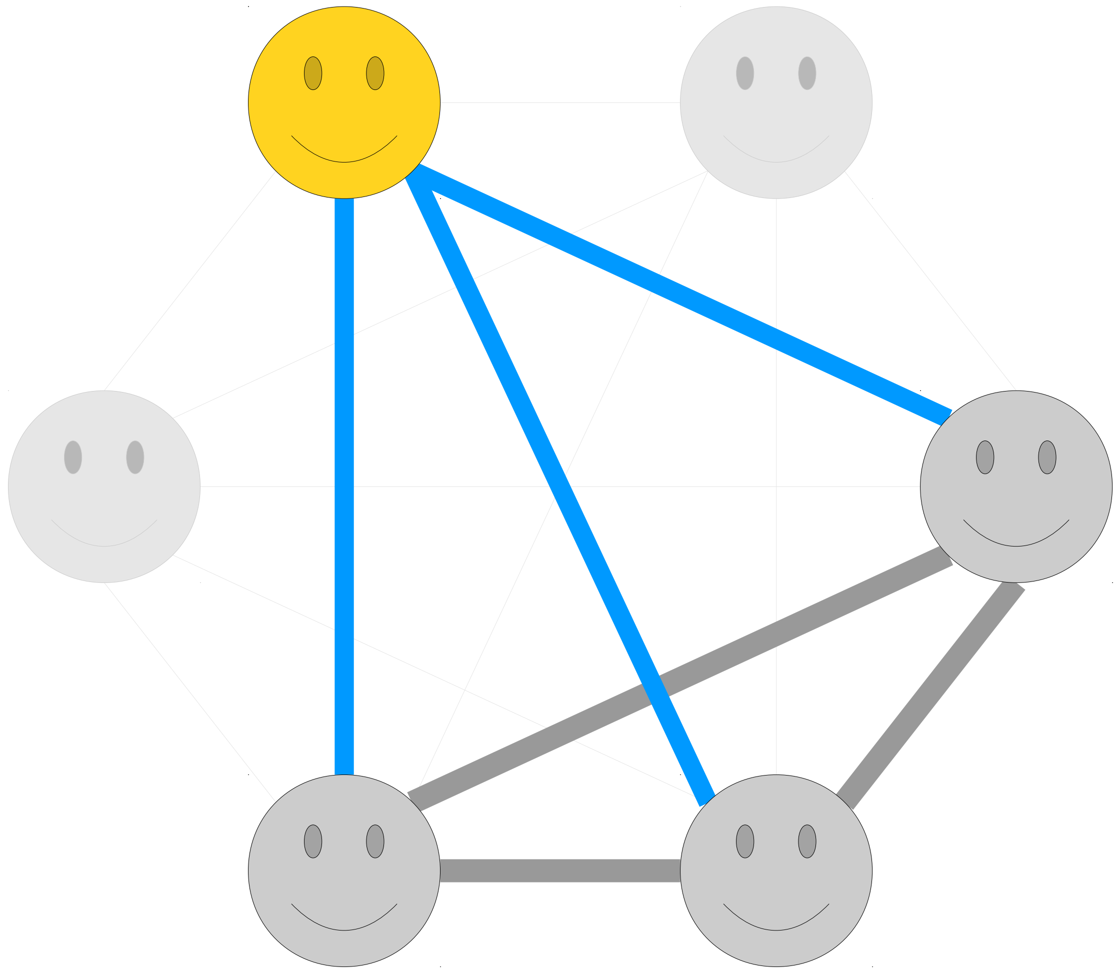


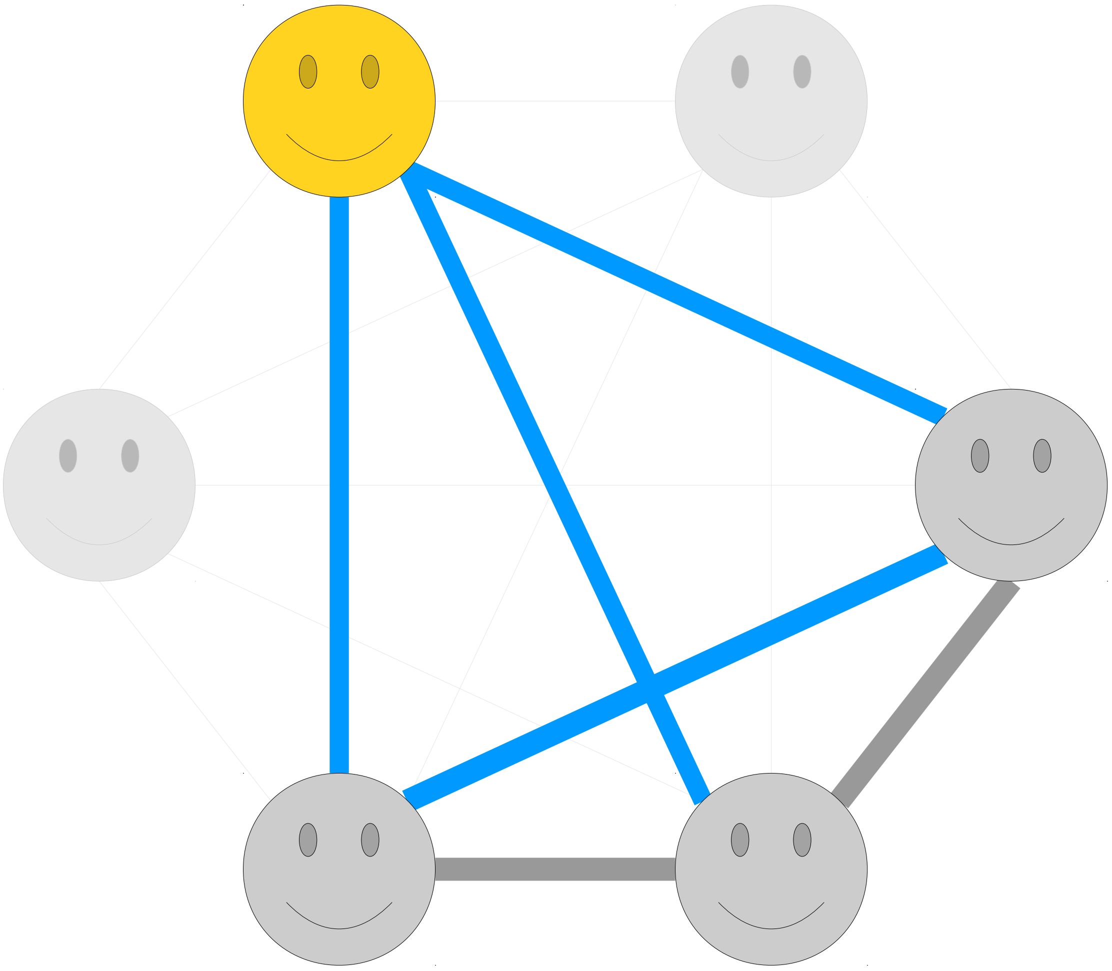


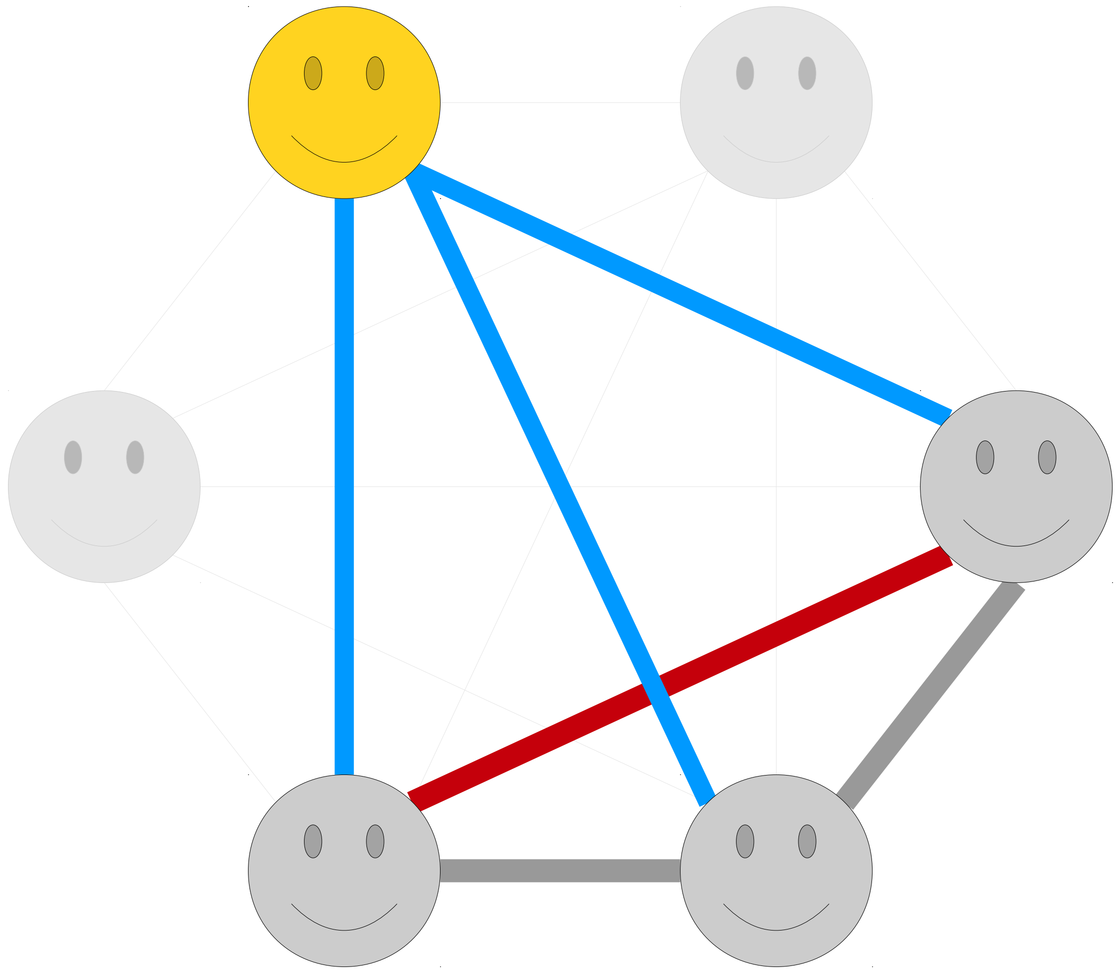


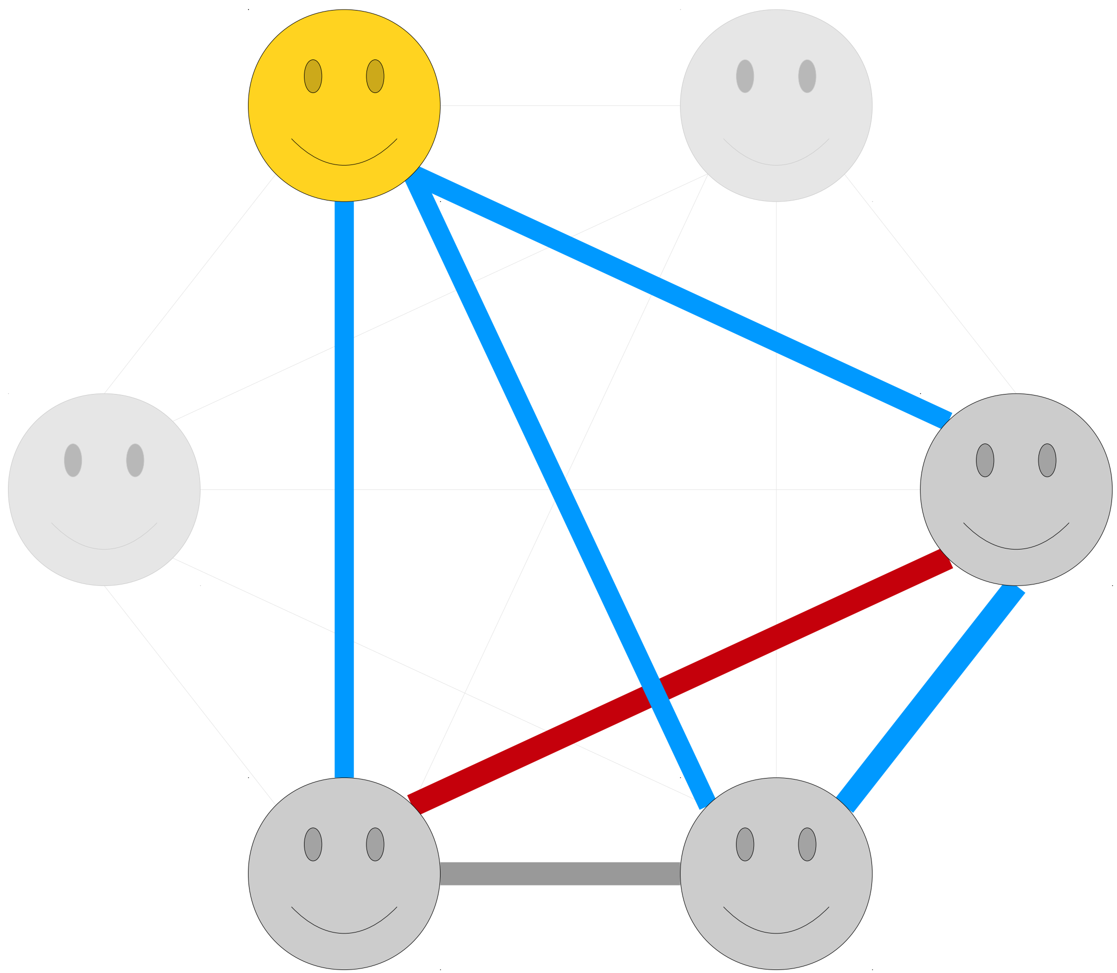
Observation 1: If we pick any node in the graph, that node will have at least ⌈5/2⌉ = 3 edges of the same color incident to it.

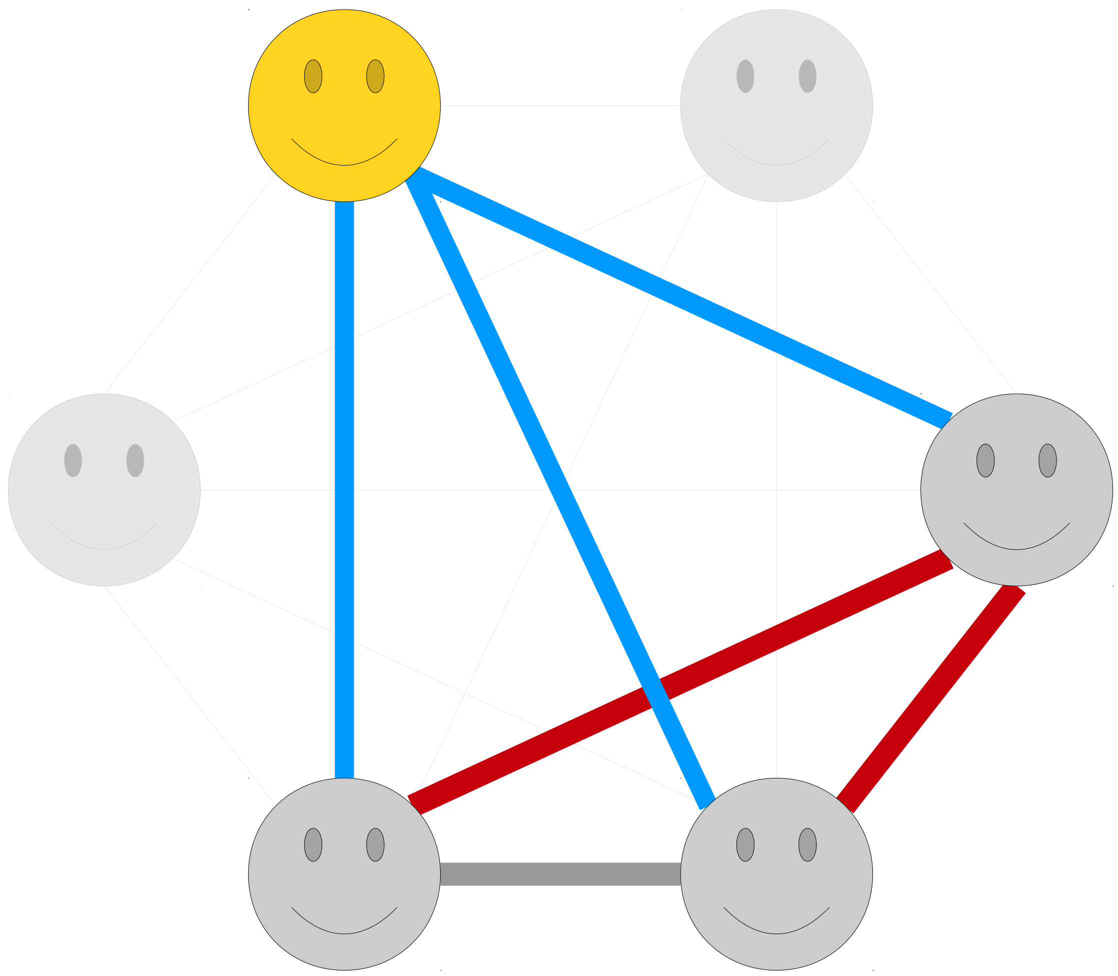


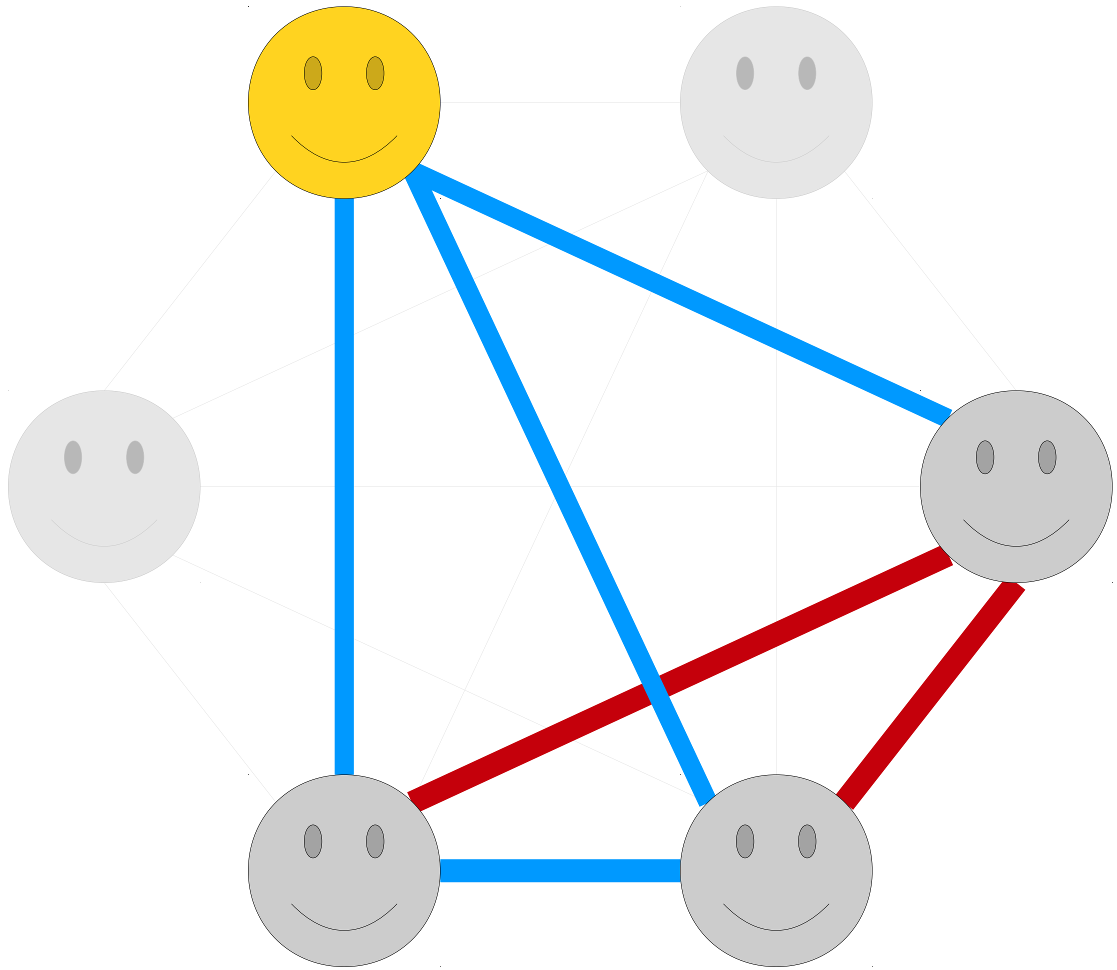


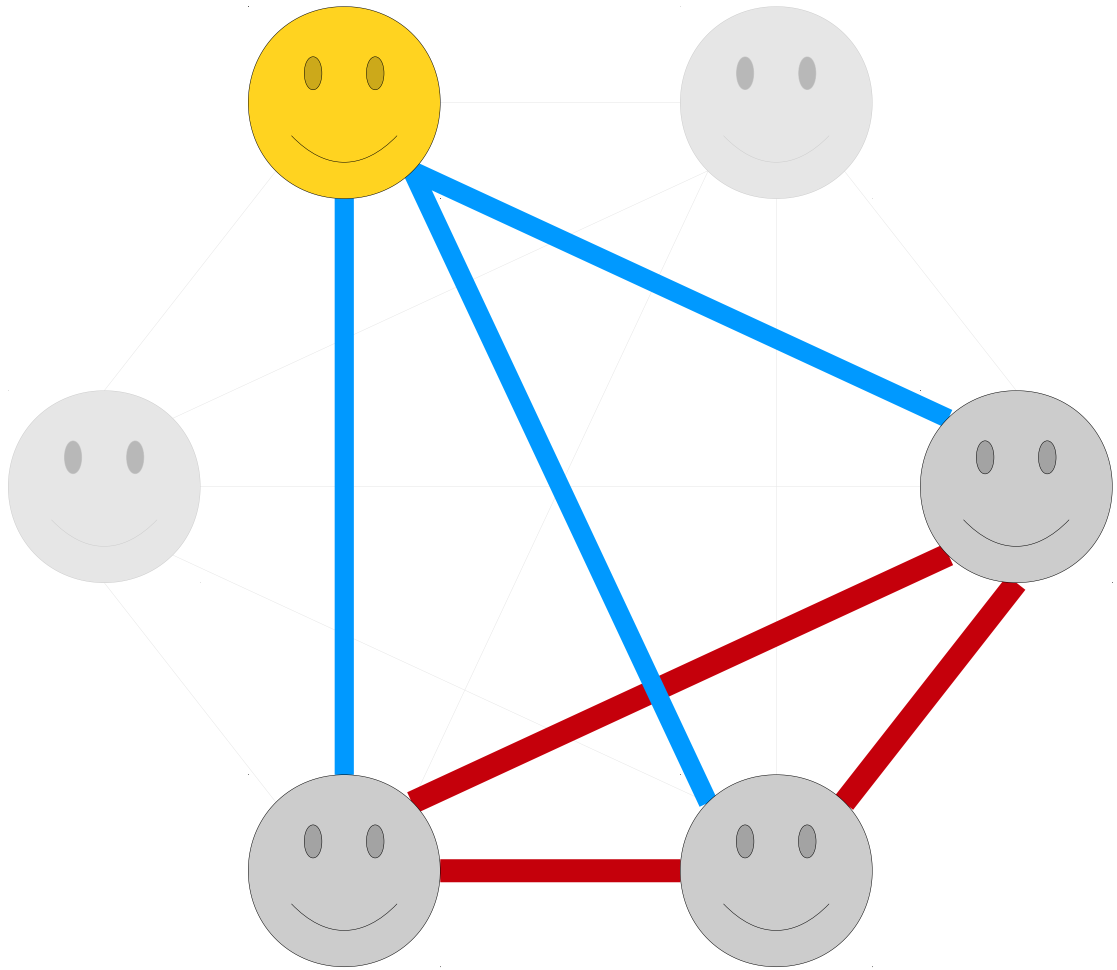












Theorem: Consider a 6-clique in which every edge is colored either red or blue. Then there must be a triangle of red edges, a triangle of blue edges, or both.

Proof: Consider any node x in the 6-clique. It is incident to five edges and there are two possible colors for those edges. Therefore, by the generalized pigeonhole principle, at least $\lceil 5 / 2 \rceil = 3$ of those edges must be the same color. Call that color c_1 and let the other color be c_2.

Let r, s, and t be three of the nodes connected to node x by an edge of color c_1. If any of the edges $\{r, s\}$, $\{r, t\}$, or $\{s, t\}$ are of color c_1, then one of those edges plus the two edges connecting back to node x form a triangle of color c_1. Otherwise, all three of those edges are of color c_2, and they form a triangle of color c_2. Overall, this gives a red triangle or a blue triangle, as required. ■

What This Means

- The proof we just did was along the following lines:

If you choose a sufficiently large object, you are guaranteed to find a large subobject of type A or a large subobject of type B.

- Intuitively, it's not possible to find gigantic objects that have absolutely no patterns or structure in them – there is no way to avoid having some interesting structure.
- There are numerous theorems of this sort. The mathematical field of ***Ramsey theory*** explicitly studies problems of this type.