Mathematical Induction

Part Two

Let P be some property. The ***principle of mathematical induction*** states that if

If it starts true…

$P(0)$ is true

…and it stays true…

and

$$\forall k \in \mathbb{N}.\ (P(k) \to P(k+1))$$

then

$$\forall n \in \mathbb{N}.\ P(n)$$

…then it's always true.

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: Let $P(n)$ be the statement "the sum of the first n powers of two is $2^n - 1$." We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

For our base case, we need to show $P(0)$ is true, meaning that the sum of the first zero powers of two is $2^0 - 1$. Since the sum of the first zero powers of two is zero and $2^0 - 1$ is zero as well, we see that $P(0)$ is true.

For the inductive step, assume that for some $k \in \mathbb{N}$ that $P(k)$ holds, meaning that

$$2^0 + 2^1 + \ldots + 2^{k-1} = 2^k - 1. \qquad (1)$$

We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers of two is $2^{k+1} - 1$. To see this, notice that

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \ldots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \qquad \textit{(via (1))} \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Therefore, $P(k + 1)$ is true, completing the induction. ■

Induction in Practice

- Typically, a proof by induction will not explicitly state $P(n)$.
- Rather, the proof will describe $P(n)$ implicitly and leave it to the reader to fill in the details.
- Provided that there is sufficient detail to determine
  - what $P(n)$ is;
  - that $P(0)$ is true; and that
  - whenever $P(k)$ is true, $P(k+1)$ is true,

  the proof is usually valid.

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: By induction.

For our base case, we'll prove the theorem is true when $n = 0$. The sum of the first zero powers of two is zero, and $2^0 - 1 = 0$, so the theorem is true in this case.

For the inductive step, assume the theorem holds when $n = k$ for some arbitrary $k \in \mathbb{N}$. Then

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \ldots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

So the theorem is true when $n = k+1$, completing the induction. ■

Variations on Induction: **Starting Later**

Induction Starting at 0

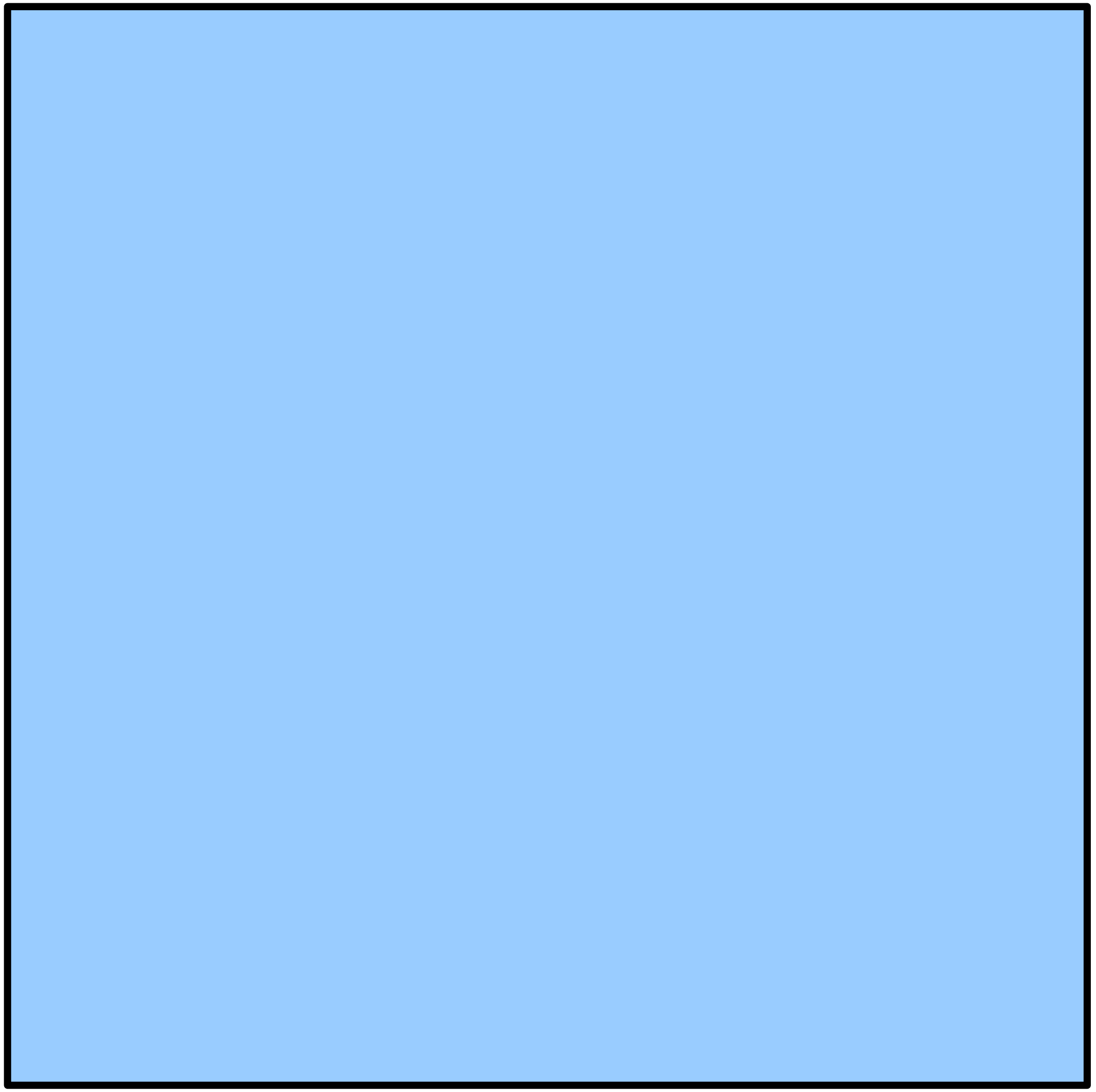
- To prove that $P(n)$ is true for all natural numbers greater than or equal to 0:
  - Show that $P(0)$ is true.
  - Show that for any $k \geq 0$, that if $P(k)$ is true, then $P(k+1)$ is true.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to 0.

Induction Starting at $\boldsymbol{m}$

- To prove that $P(n)$ is true for all natural numbers greater than or equal to $\boldsymbol{m}$:
  - Show that $P(\boldsymbol{m})$ is true.
  - Show that for any $k \geq \boldsymbol{m}$, that if $P(k)$ is true, then $P(k+1)$ is true.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to $\boldsymbol{m}$.

Variations on Induction: **Bigger Steps**

Subdividing a Square



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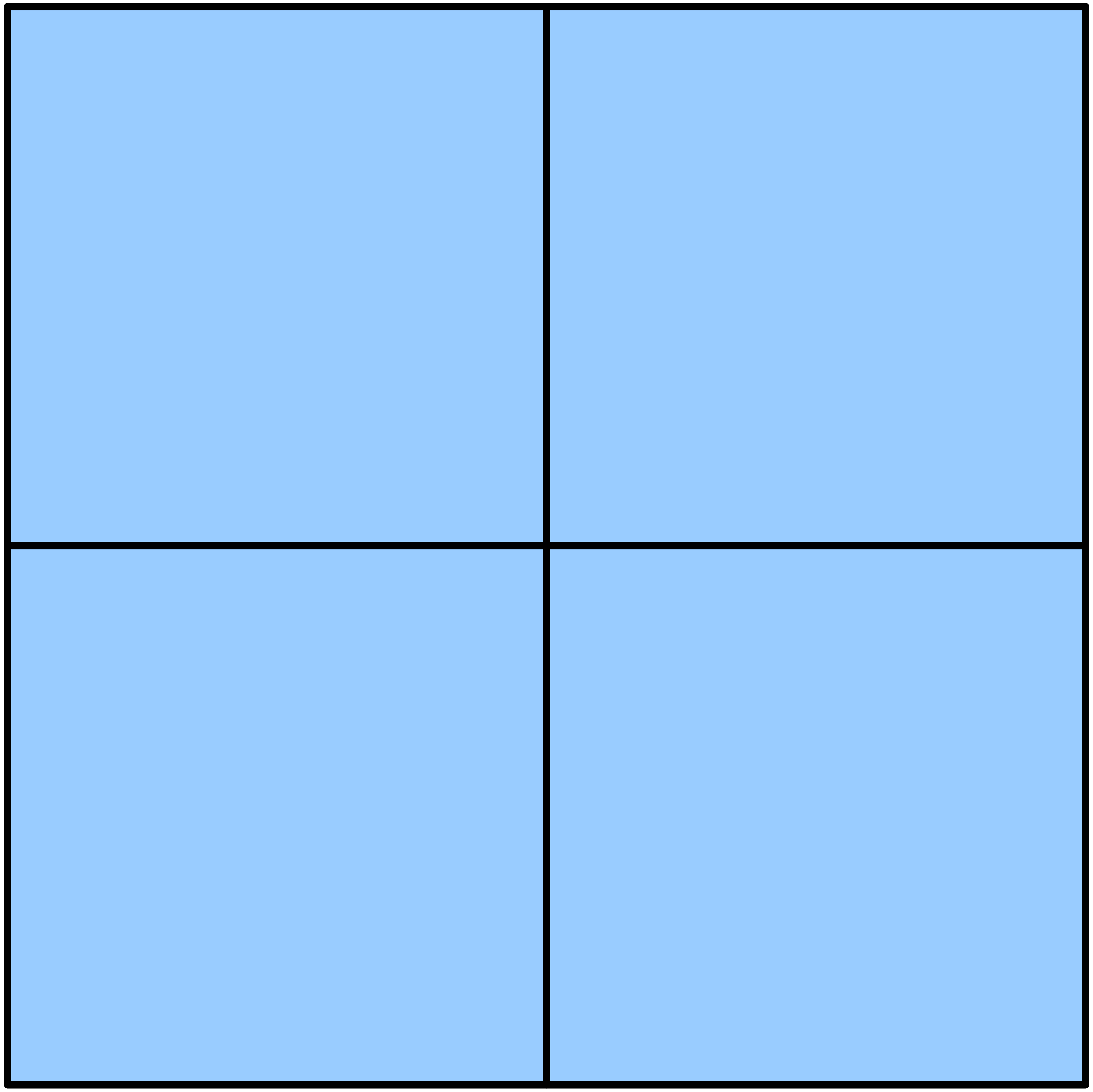

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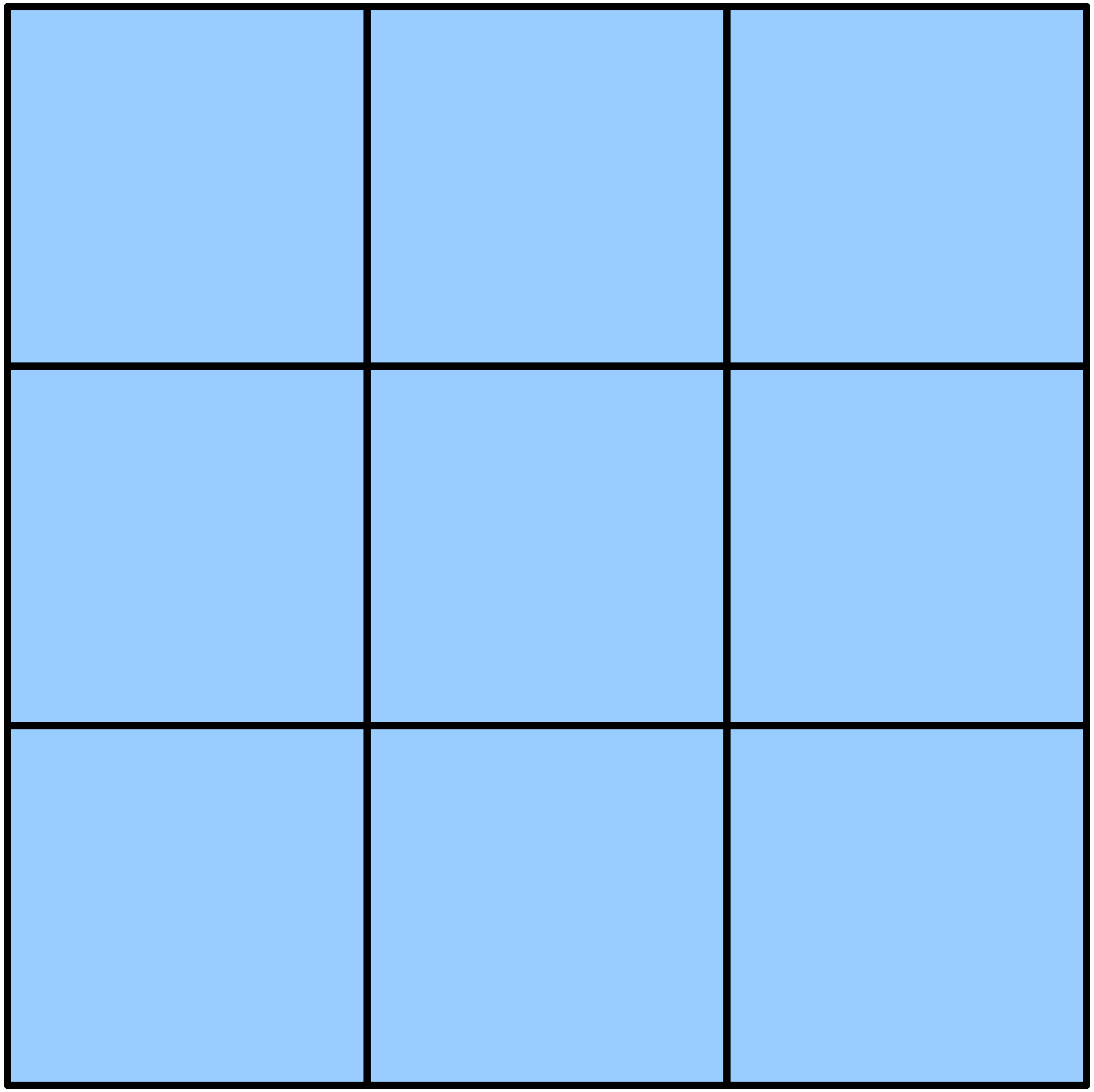

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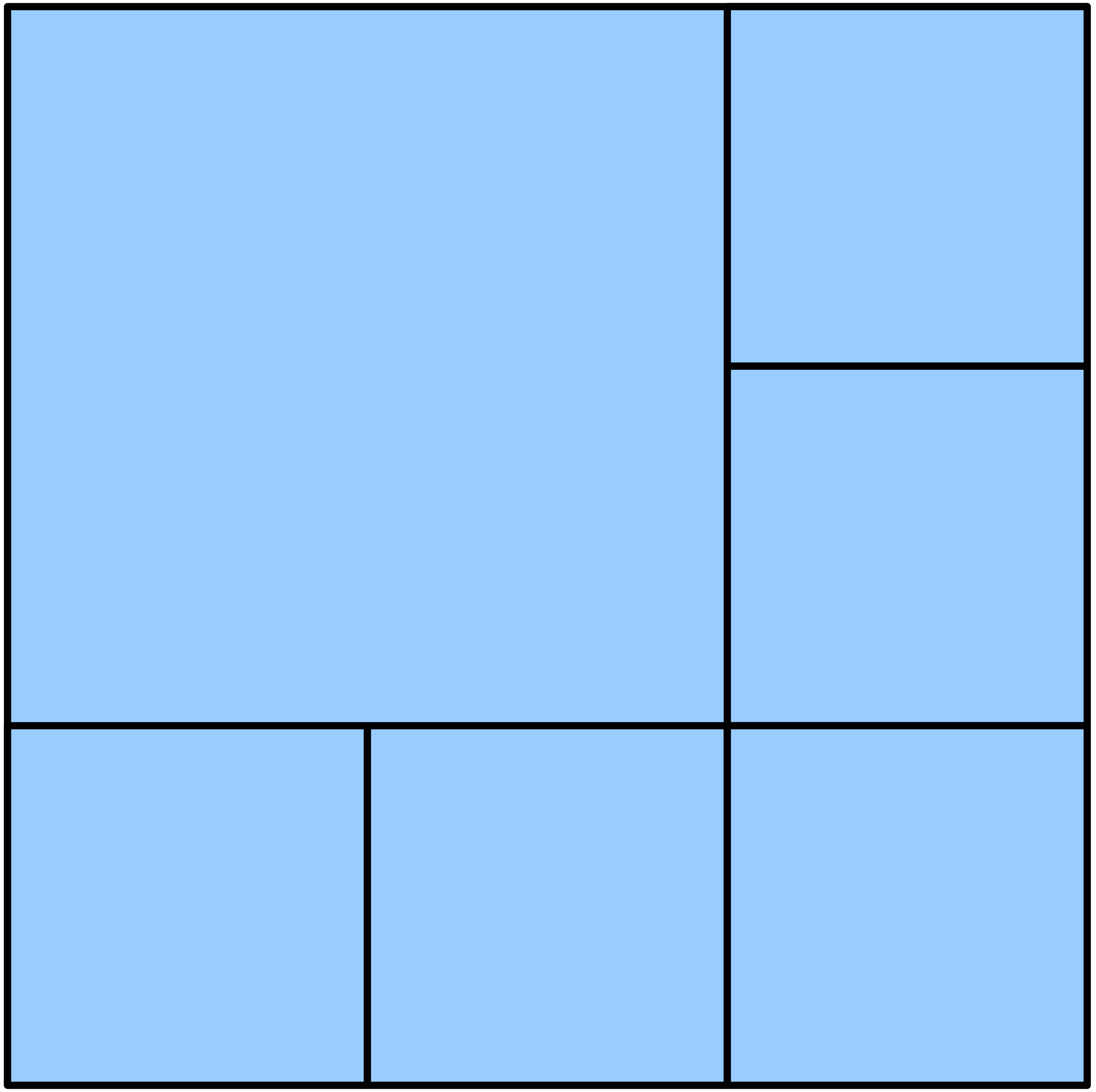

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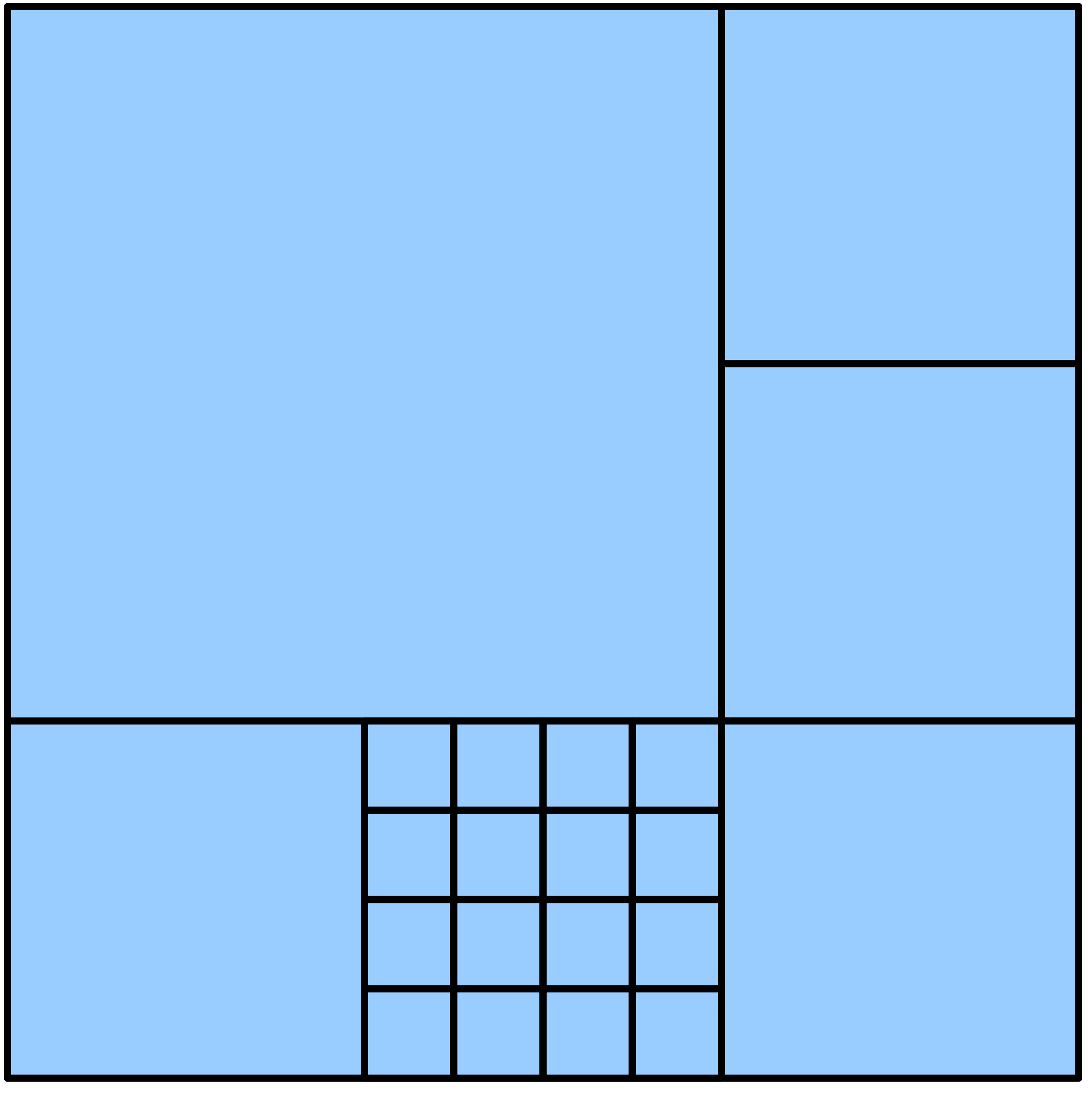


For what values of n can a square be subdivided into n squares?

1 2 3 4 5 6 7 8 9 10 11 12

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

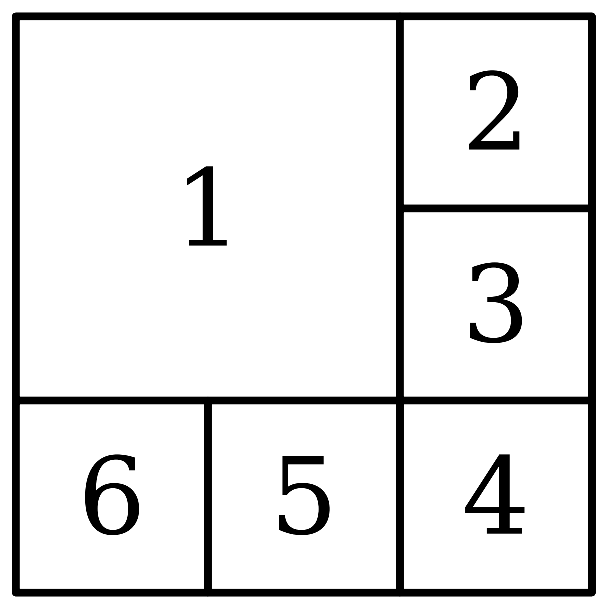
1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

1

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

1	2
4	3

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12



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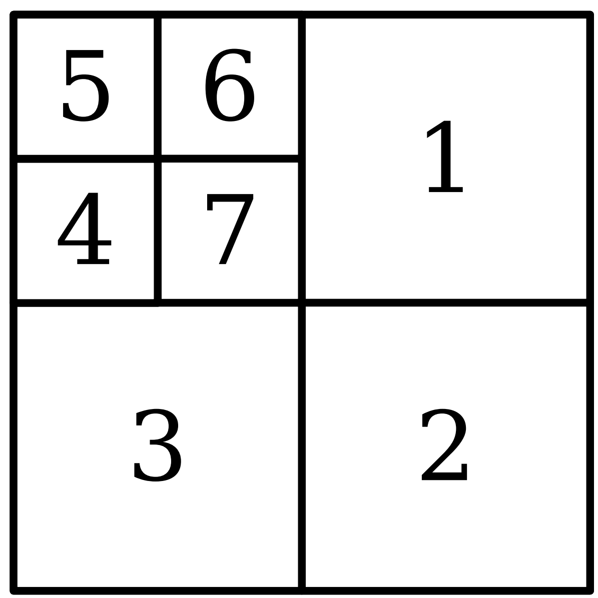

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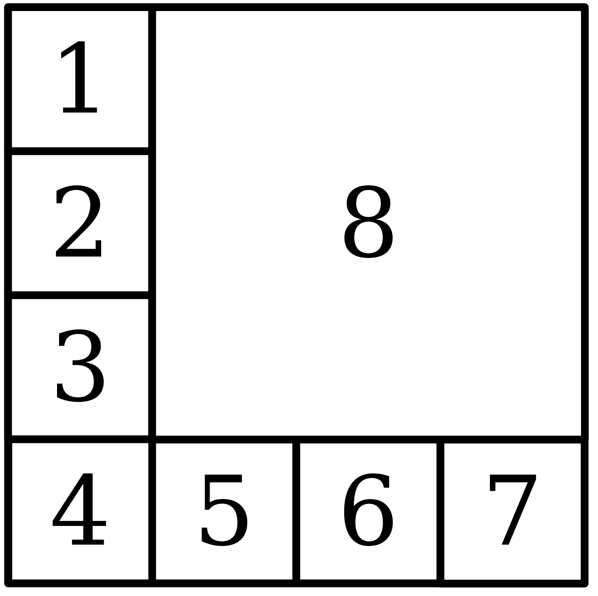

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1	2	3
8	9	4
7	6	5

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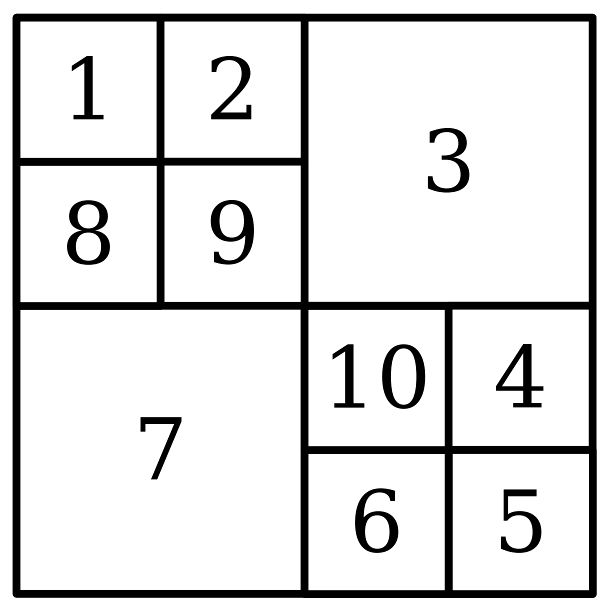

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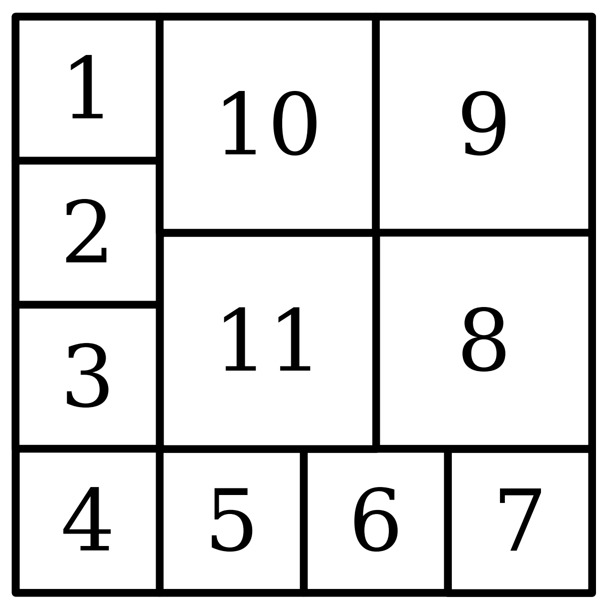

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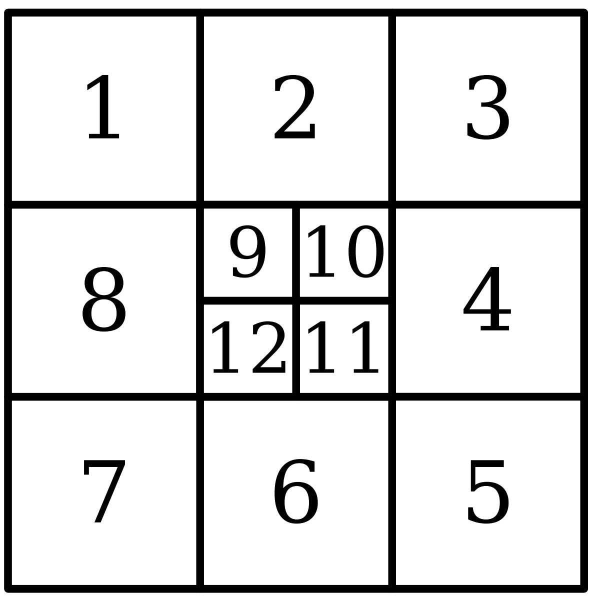

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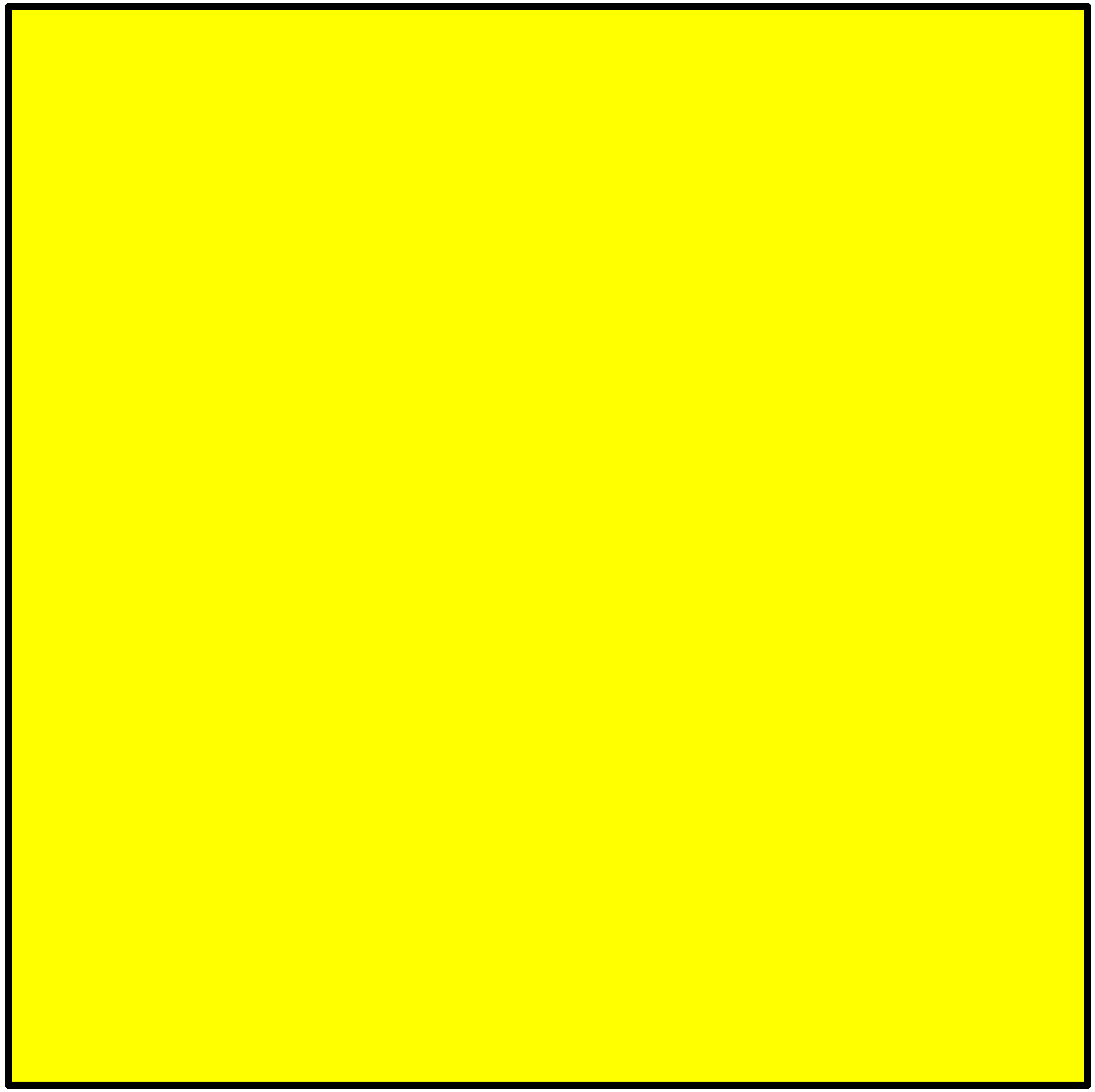

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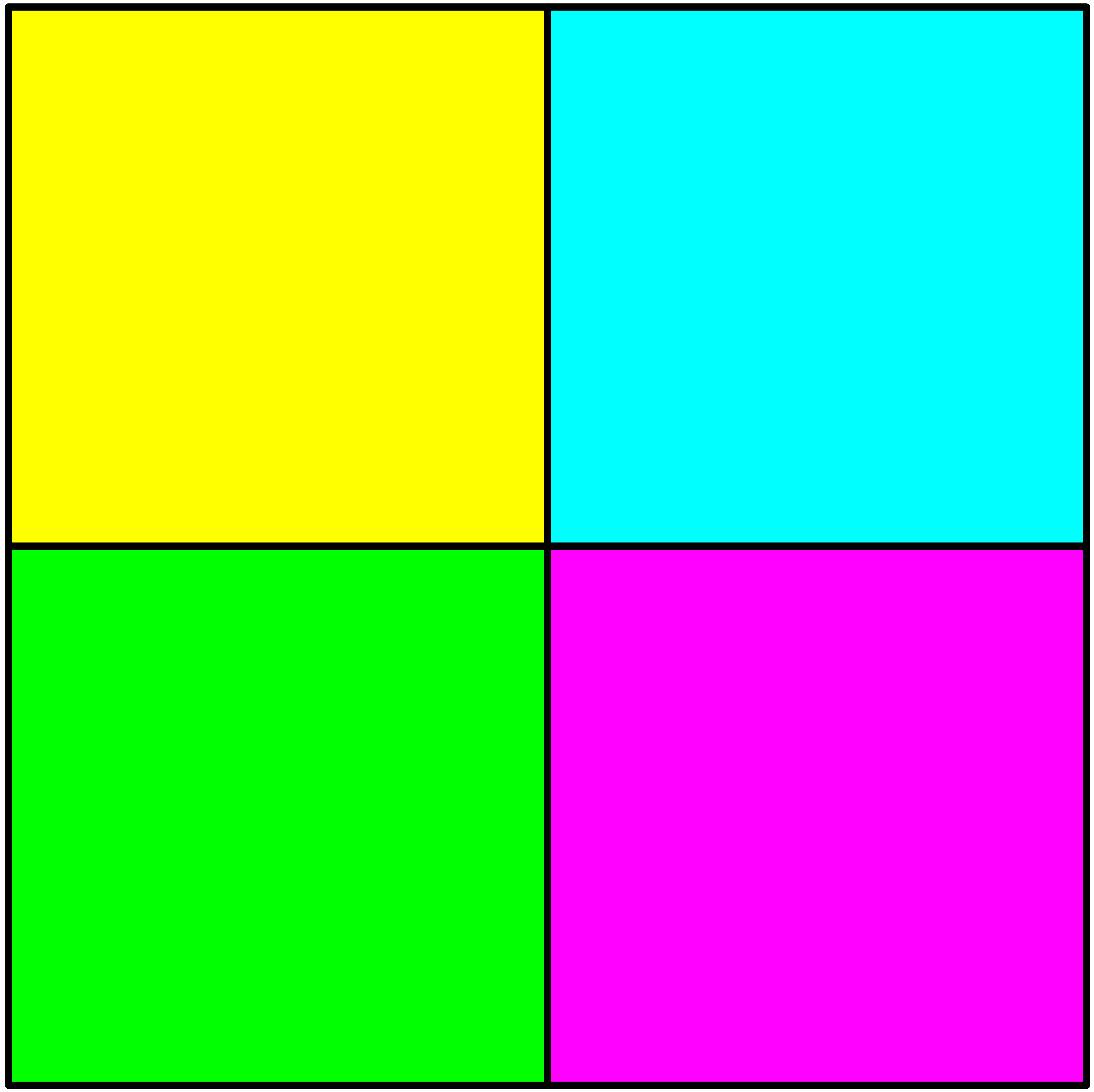

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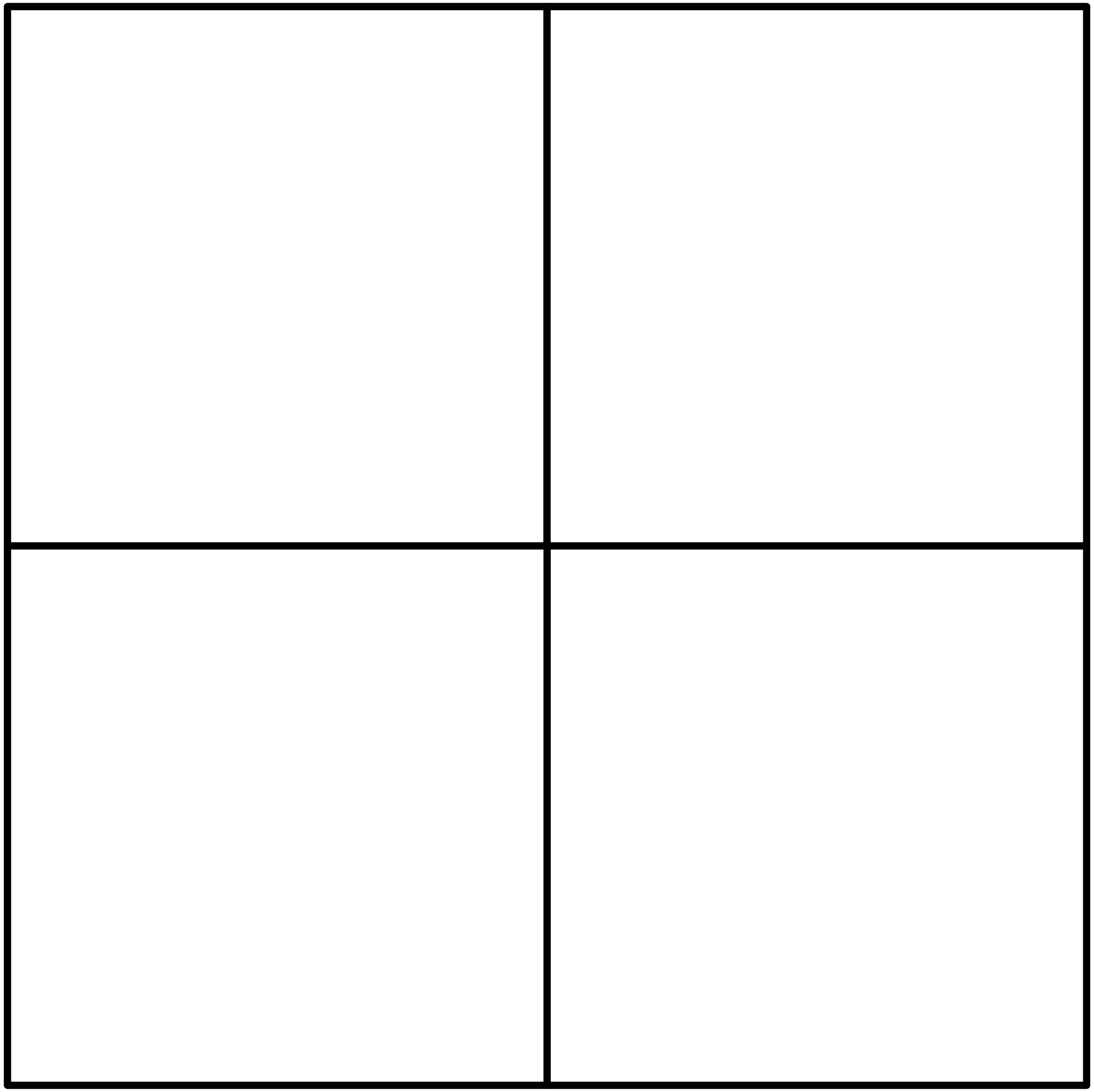

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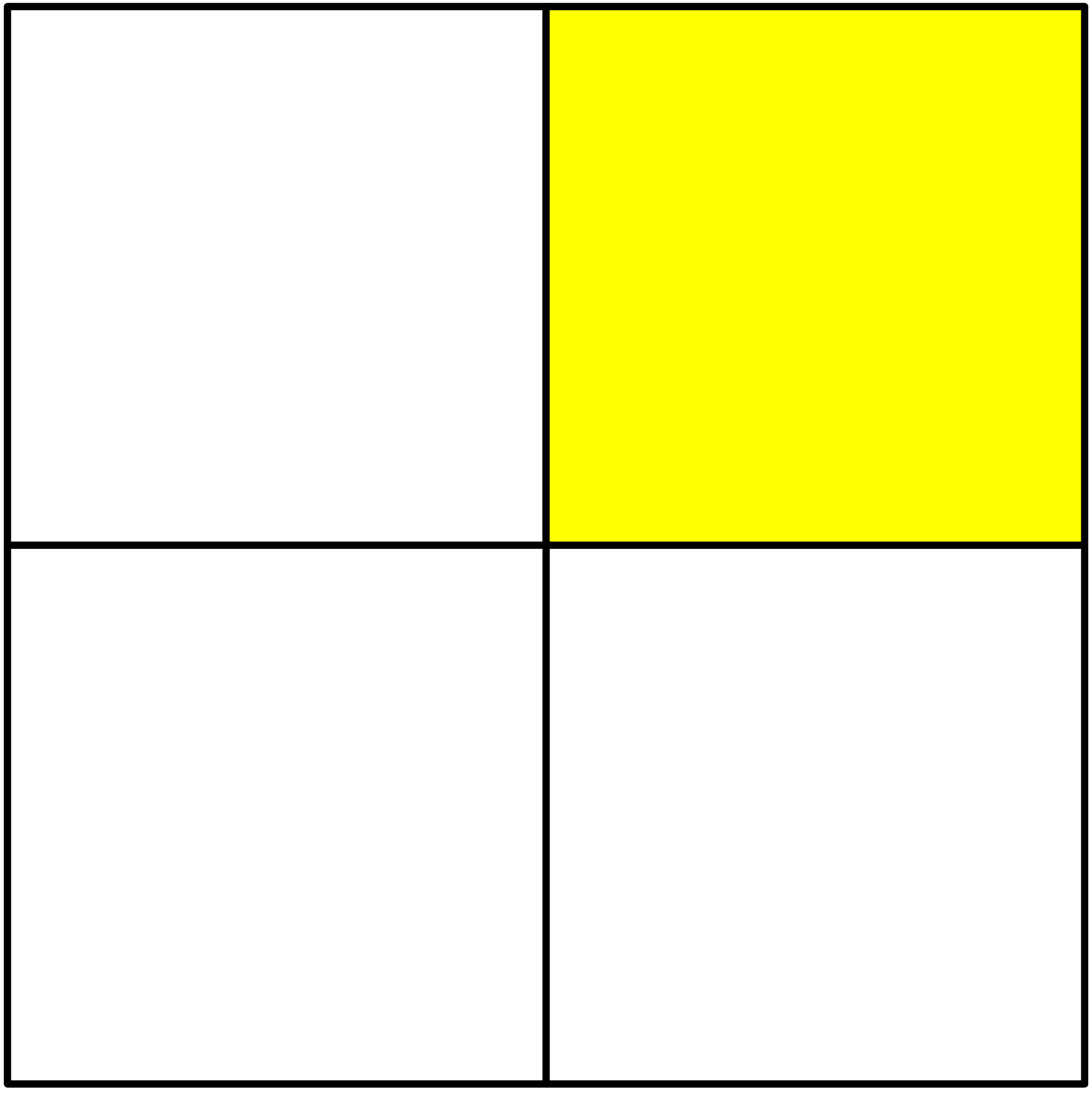

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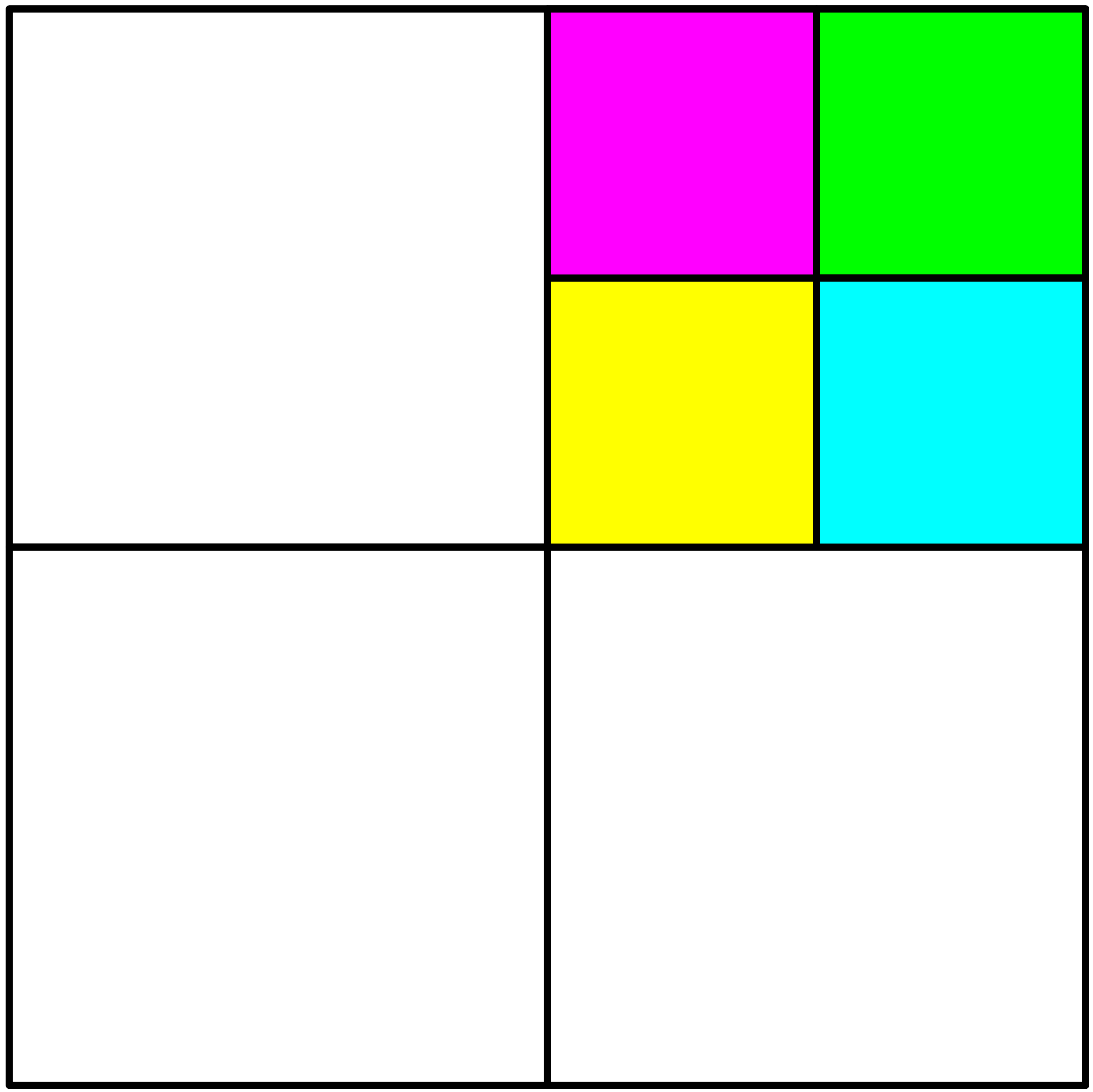

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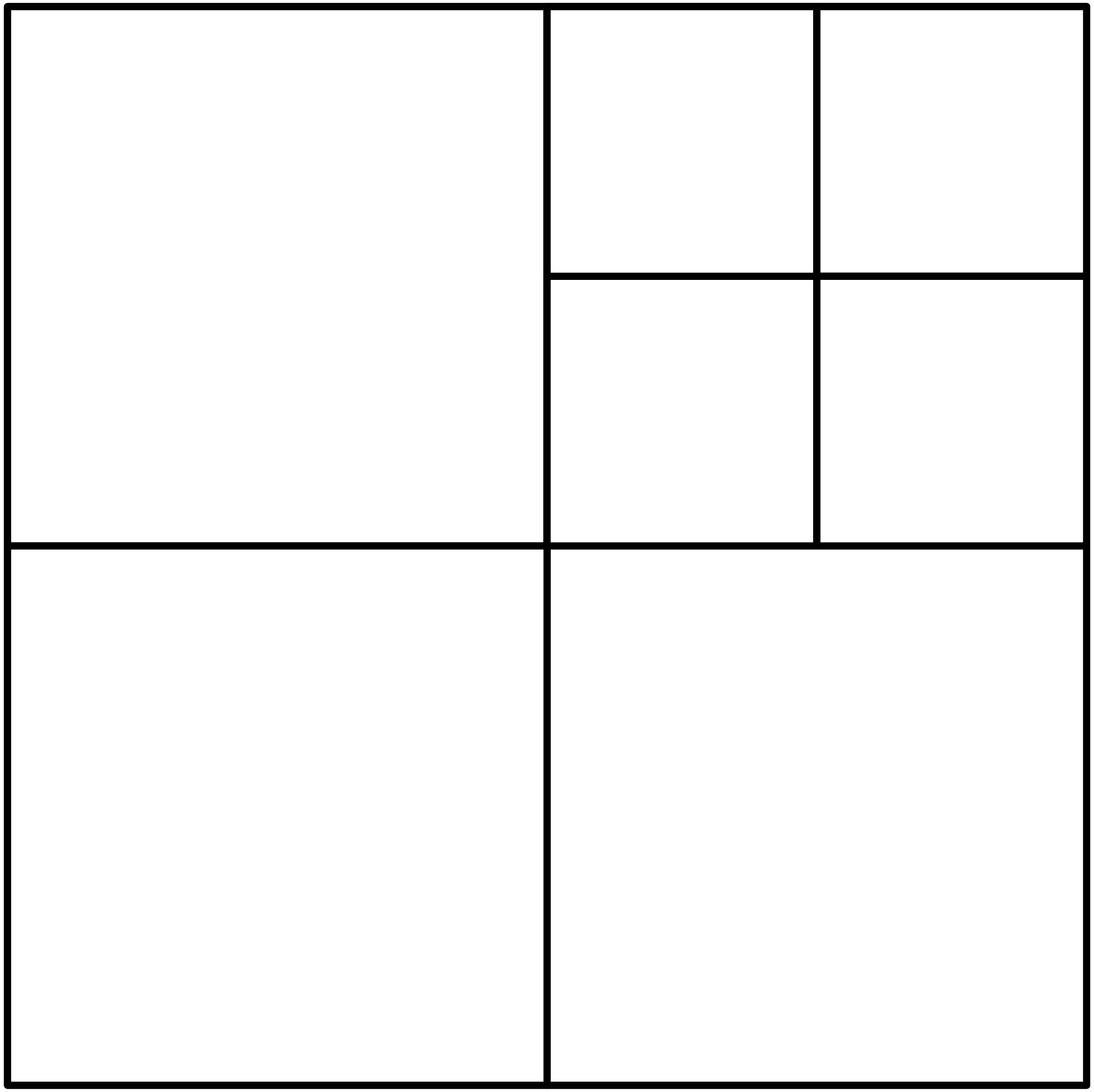

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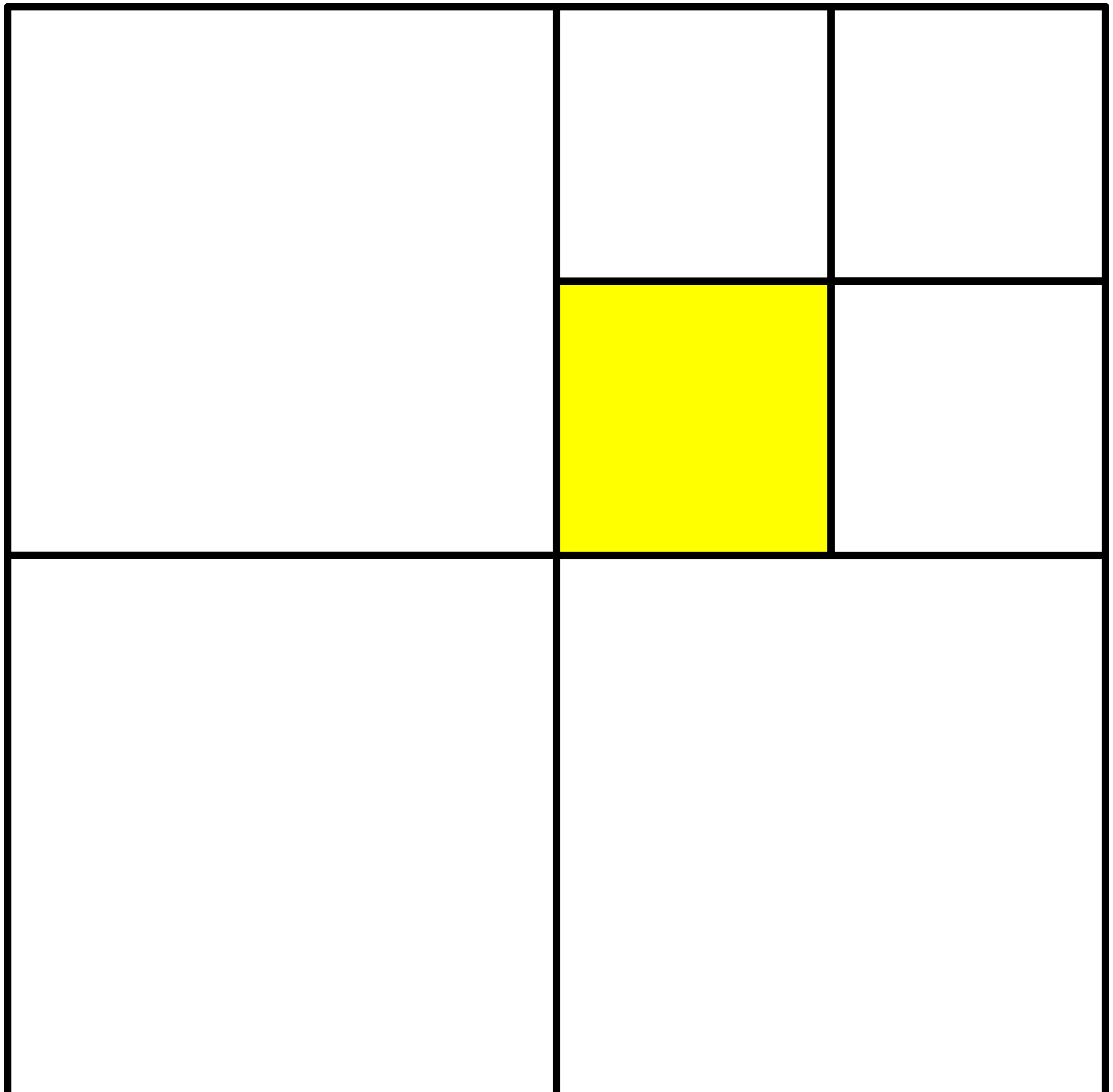

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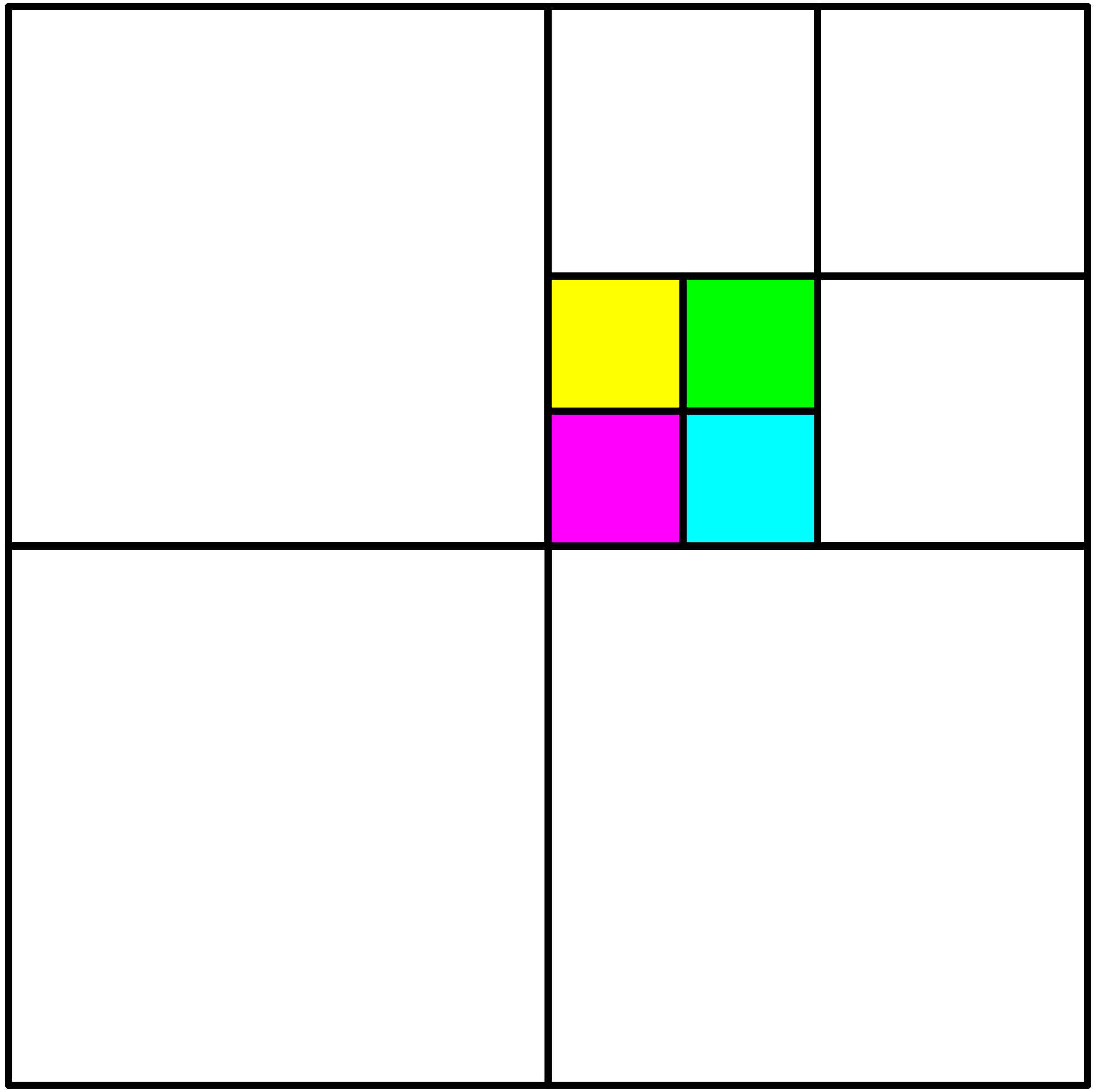

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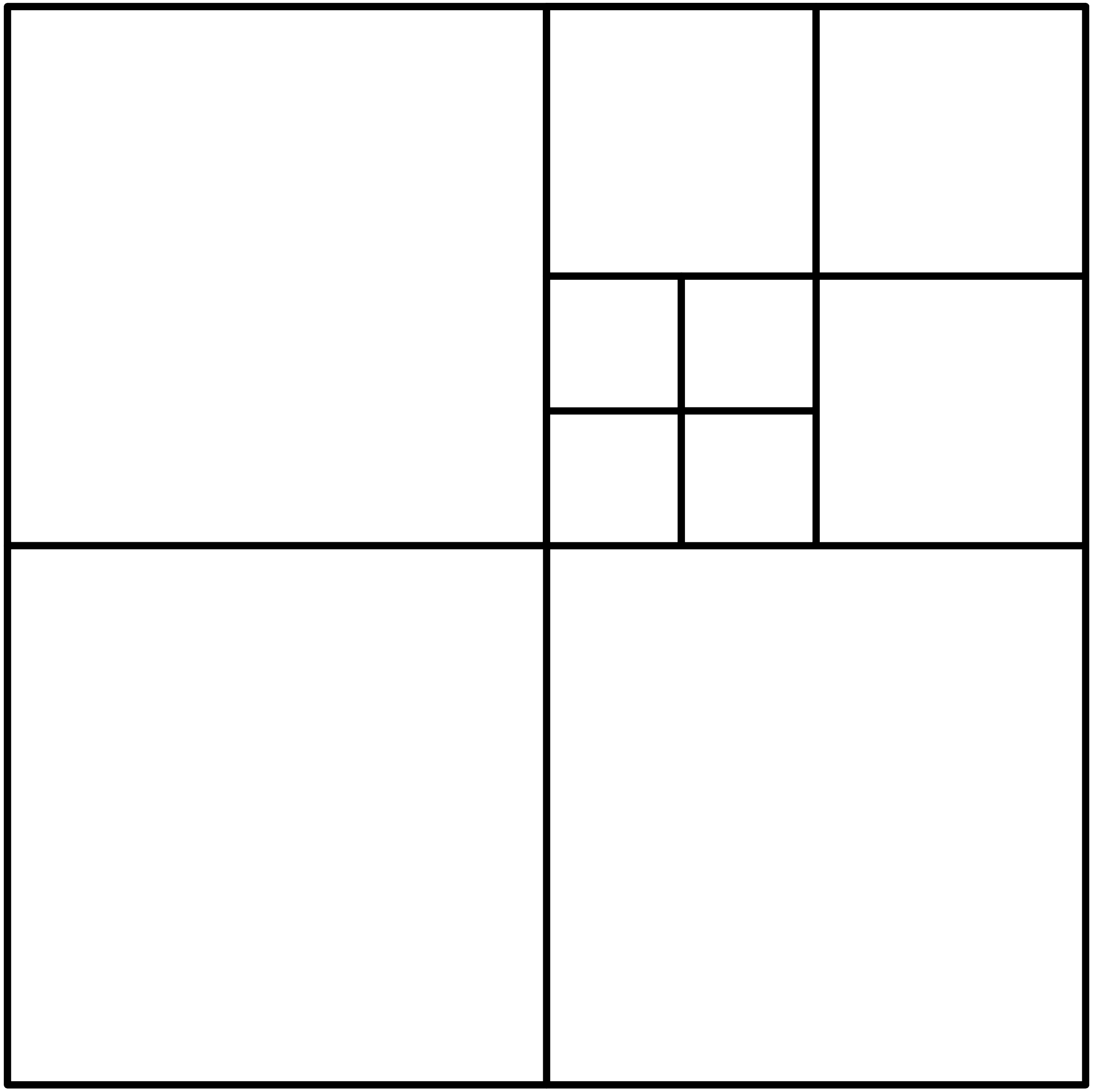

The Key Insight

- If we can subdivide a square into n squares, we can also subdivide it into $n + 3$ squares.
- Since we can subdivide a bigger square into 6, 7, and 8 squares, we can subdivide a square into n squares for any $n \geq 6$:
  - For multiples of three, start with 6 and keep adding three squares until n is reached.
  - For numbers congruent to one modulo three, start with 7 and keep adding three squares until n is reached.
  - For numbers congruent to two modulo three, start with 8 and keep adding three squares until n is reached.

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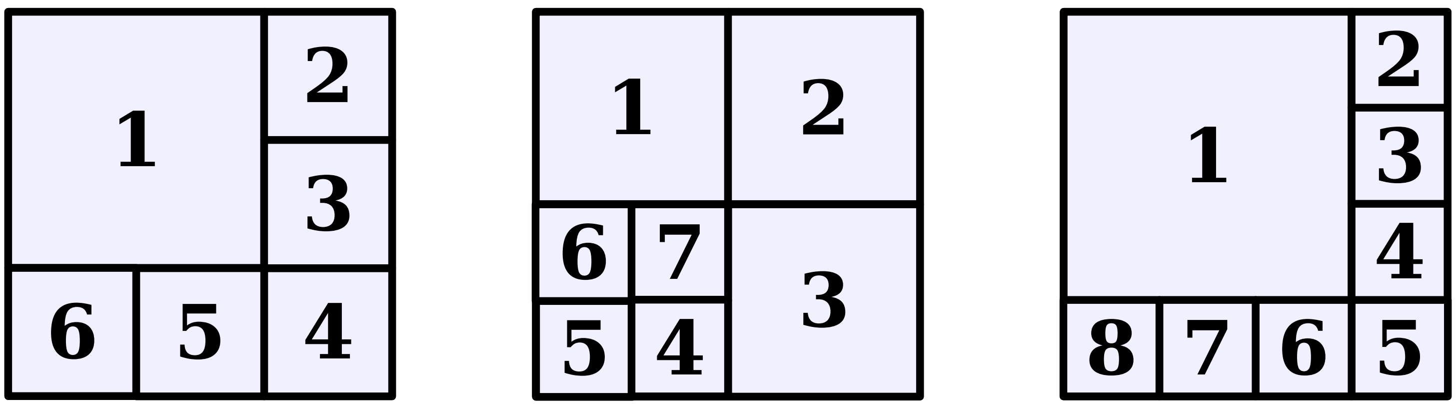

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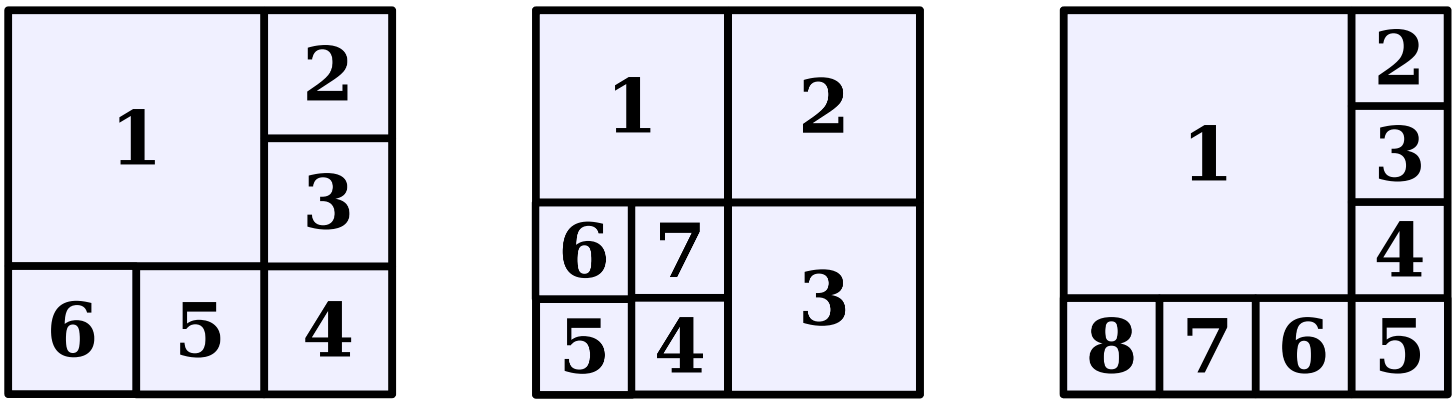

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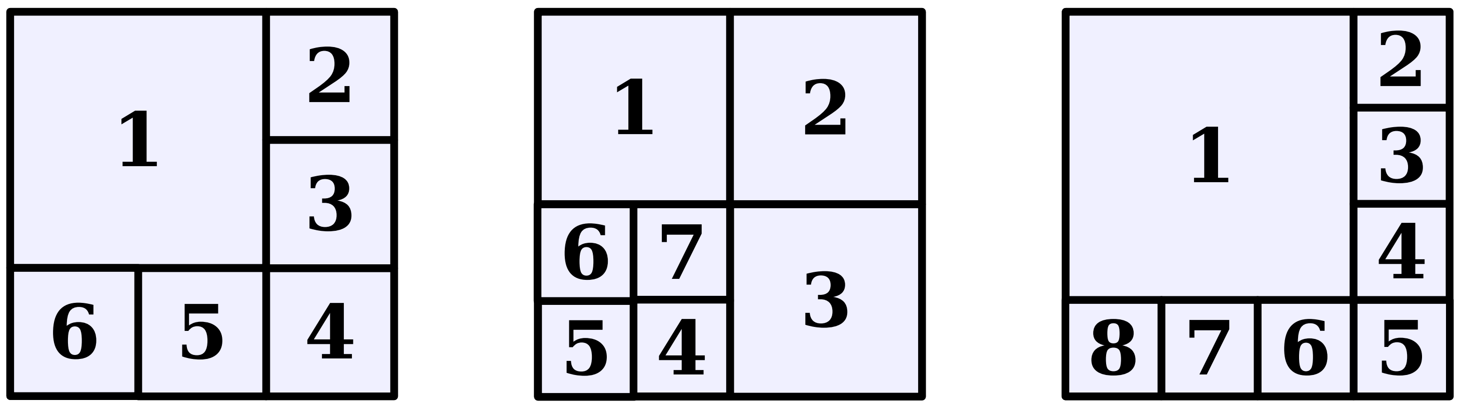

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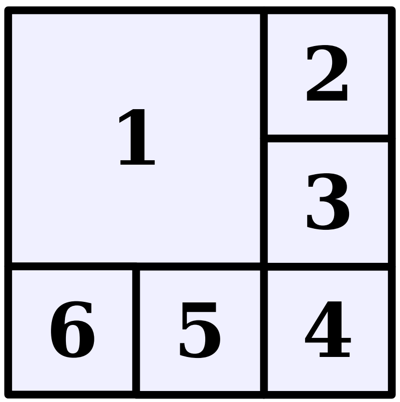

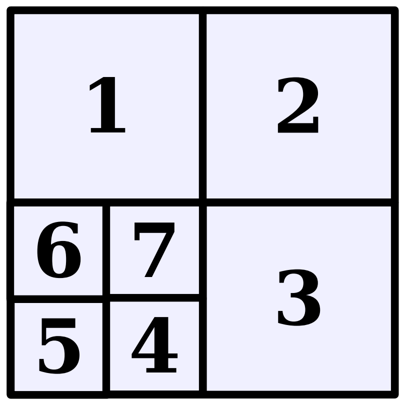

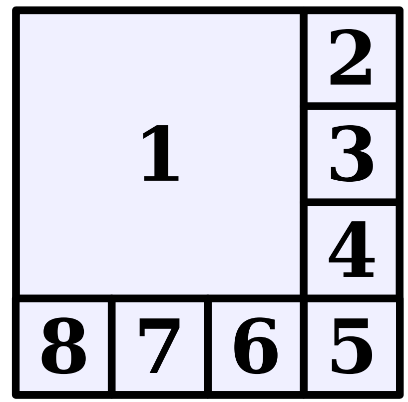

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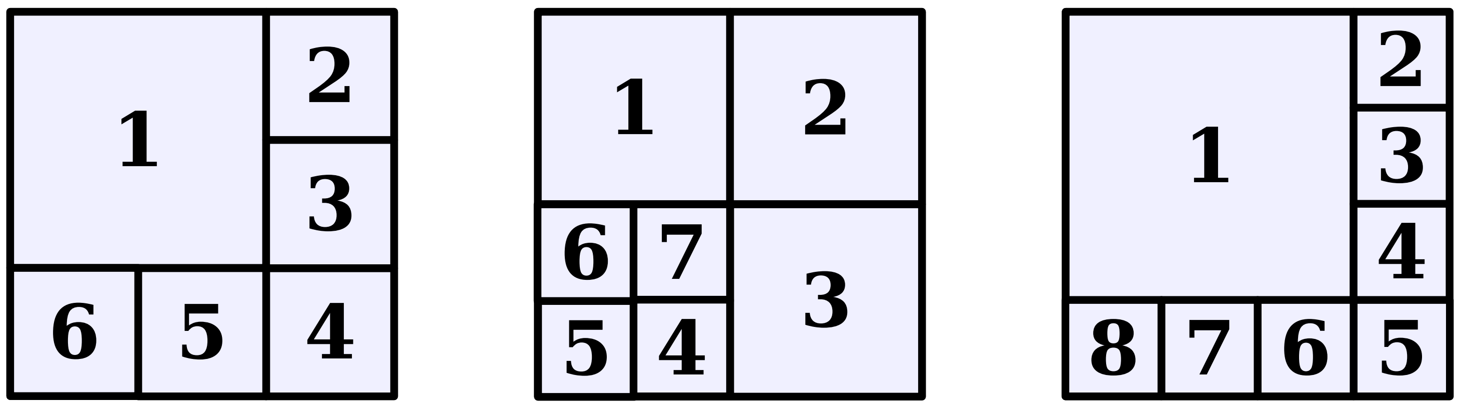

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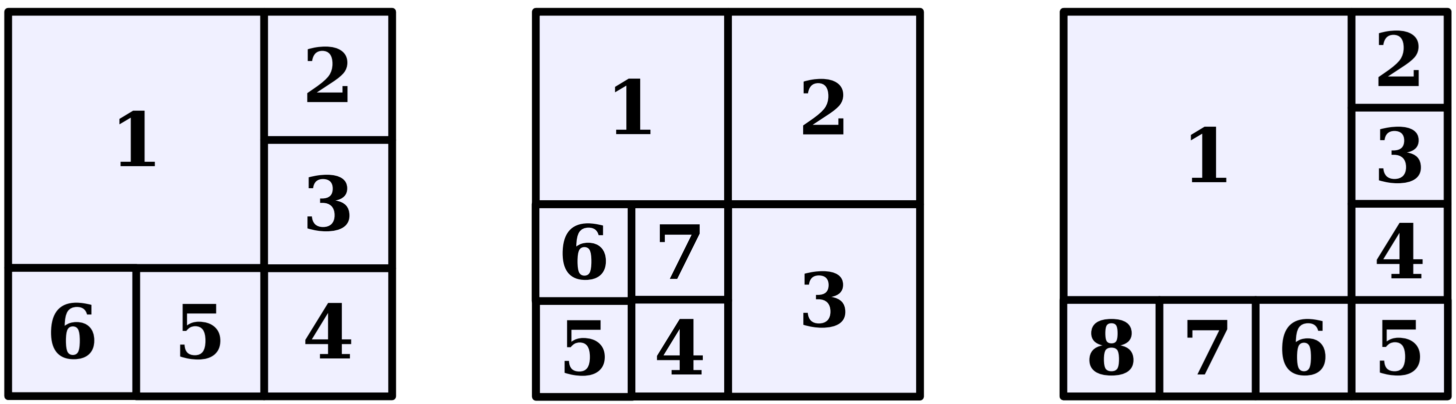

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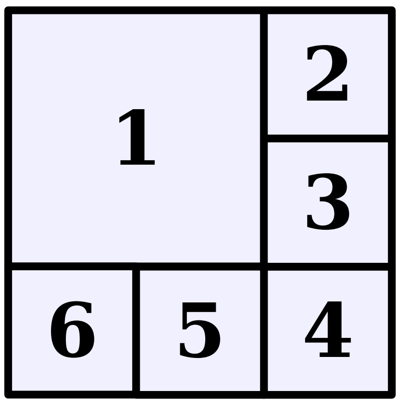

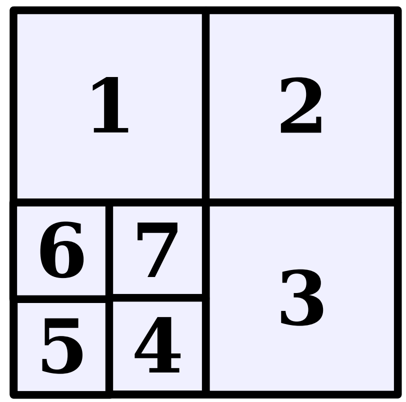

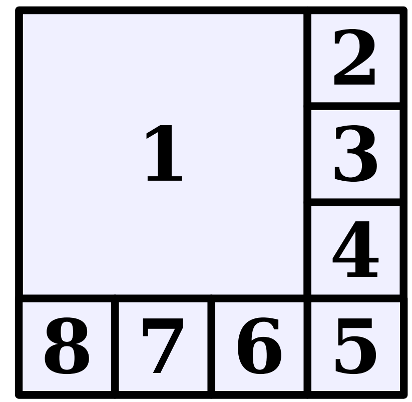

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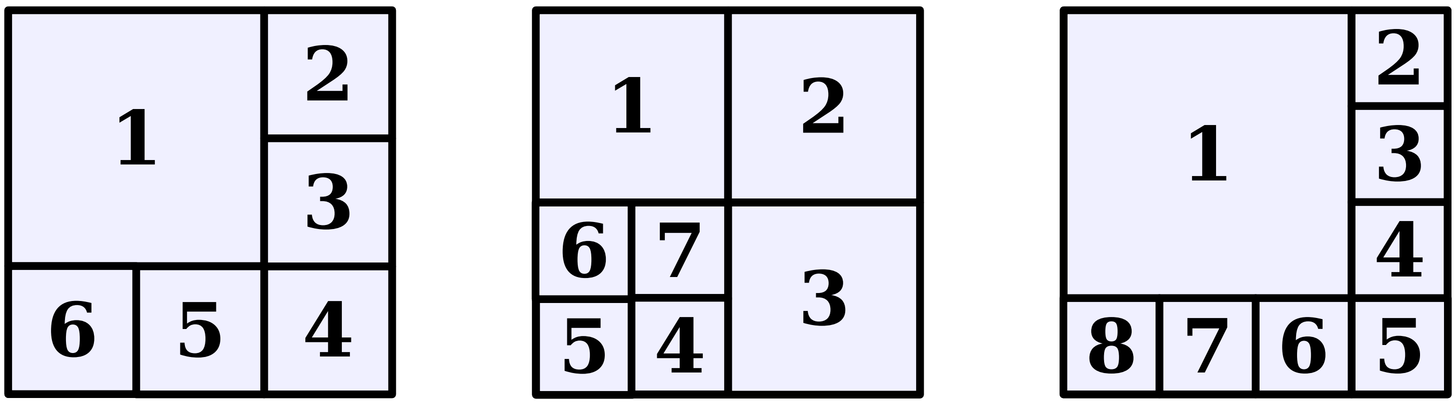

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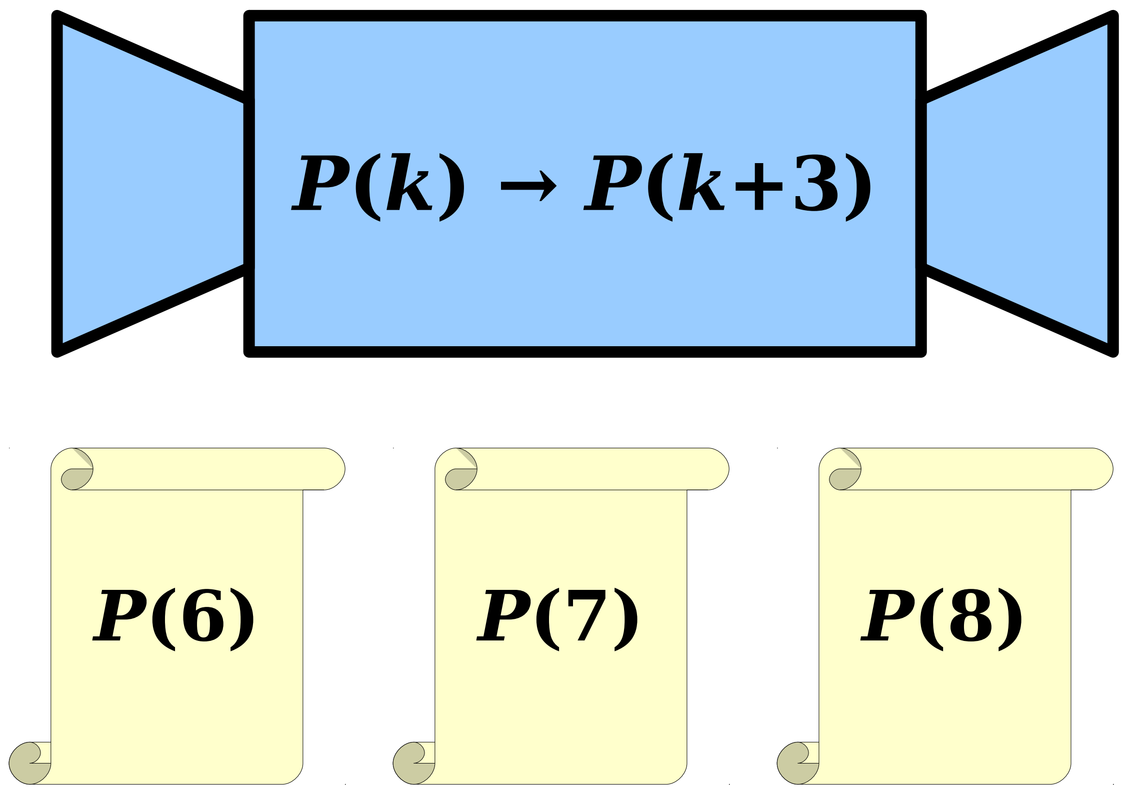
- This induction has three consecutive base cases and takes steps of size three.
- Thinking back to our “induction machine” analogy:



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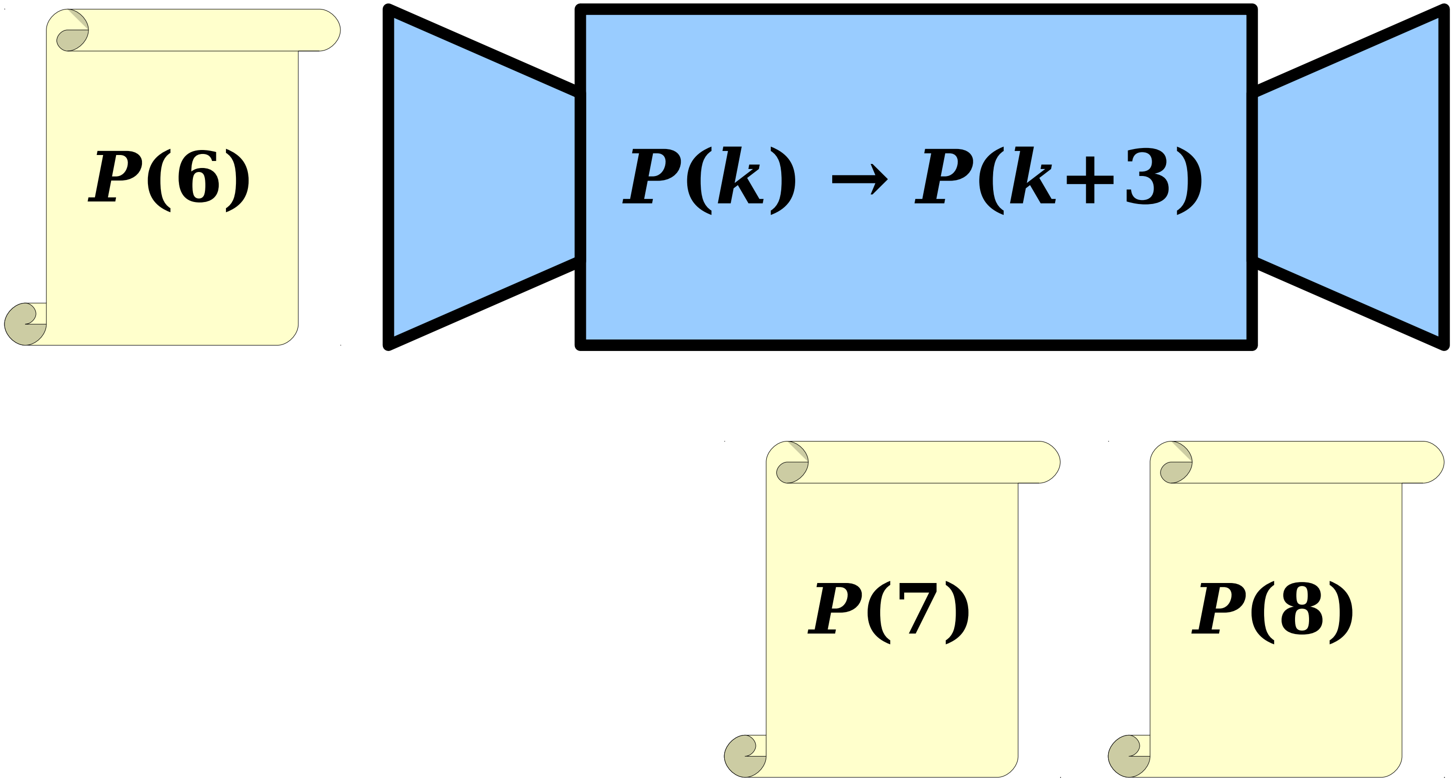

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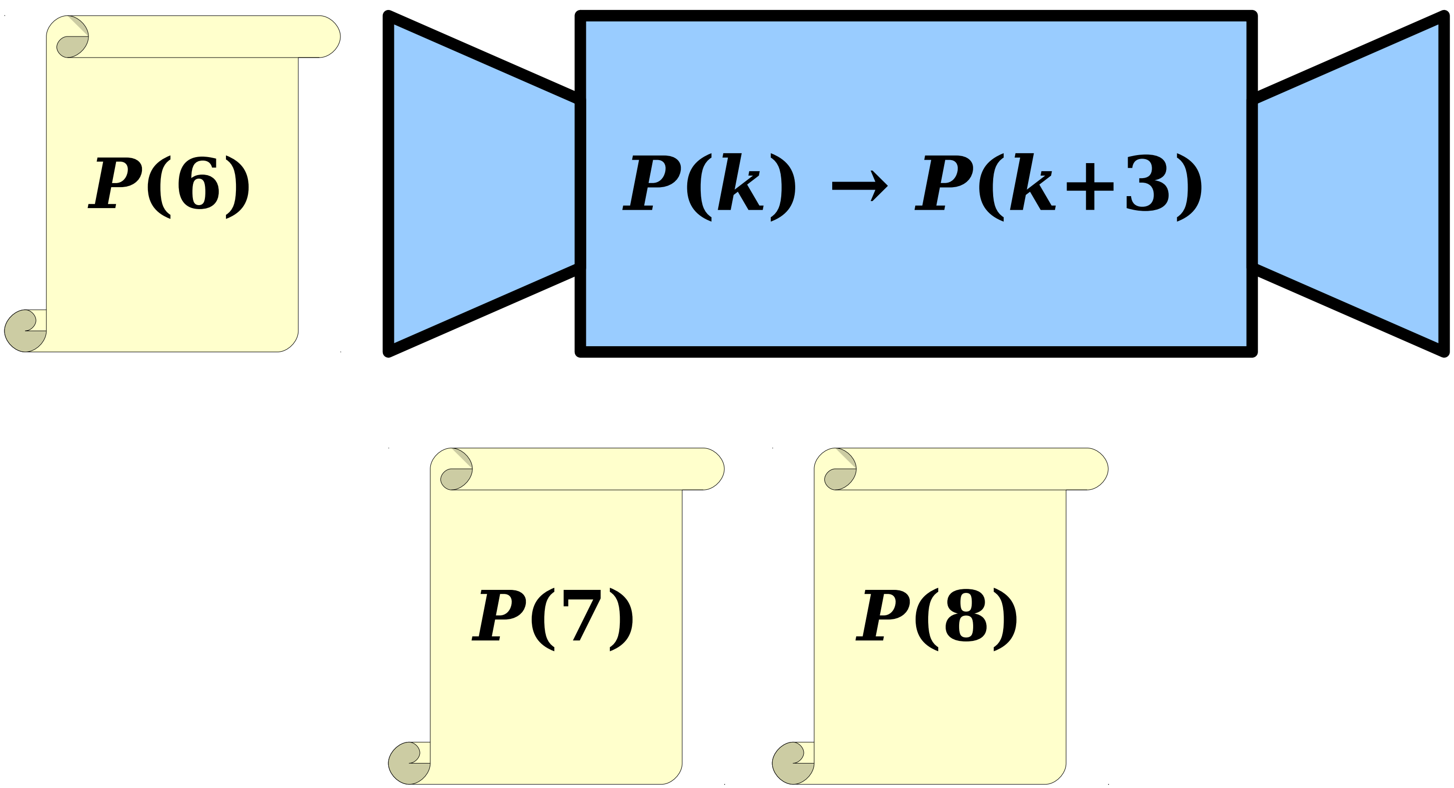

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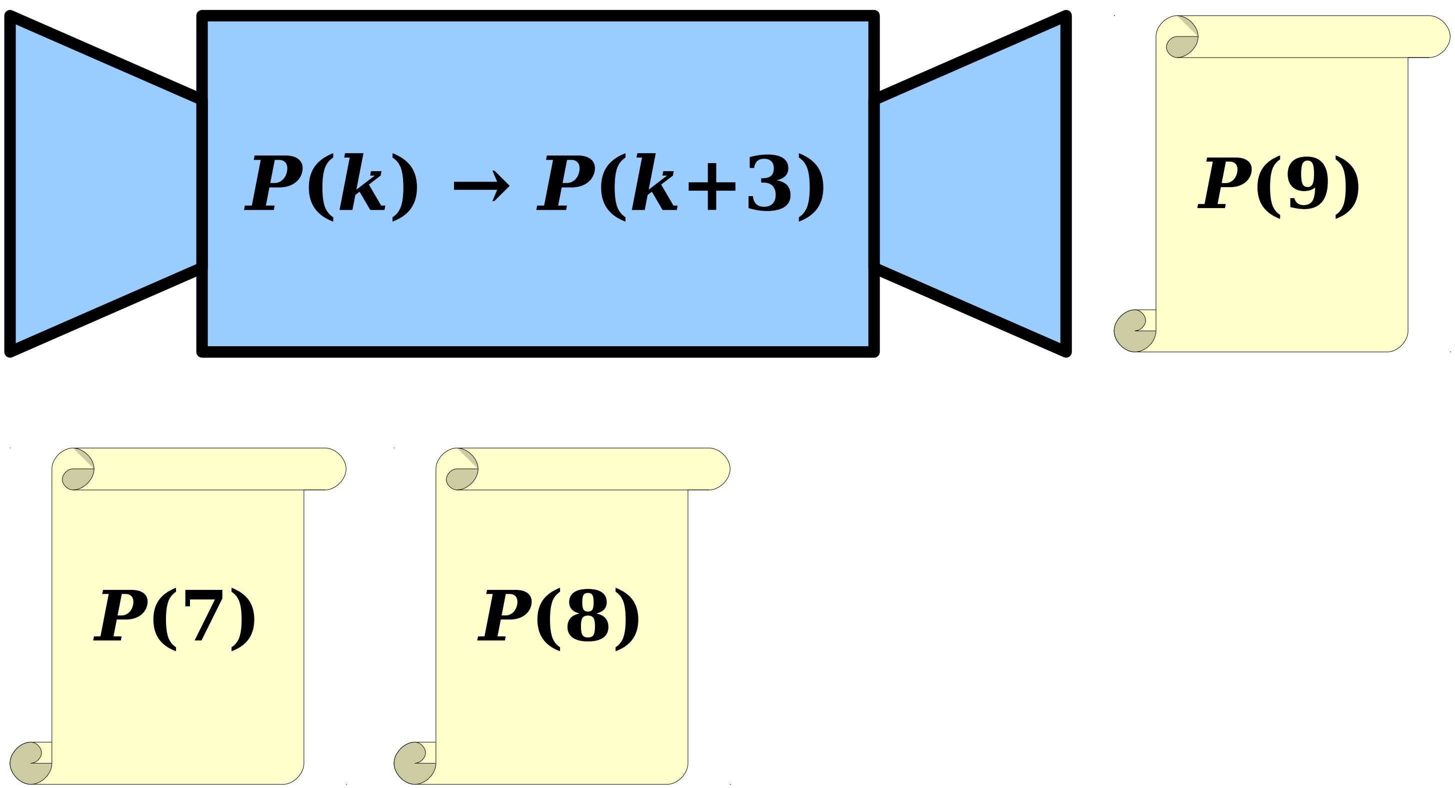

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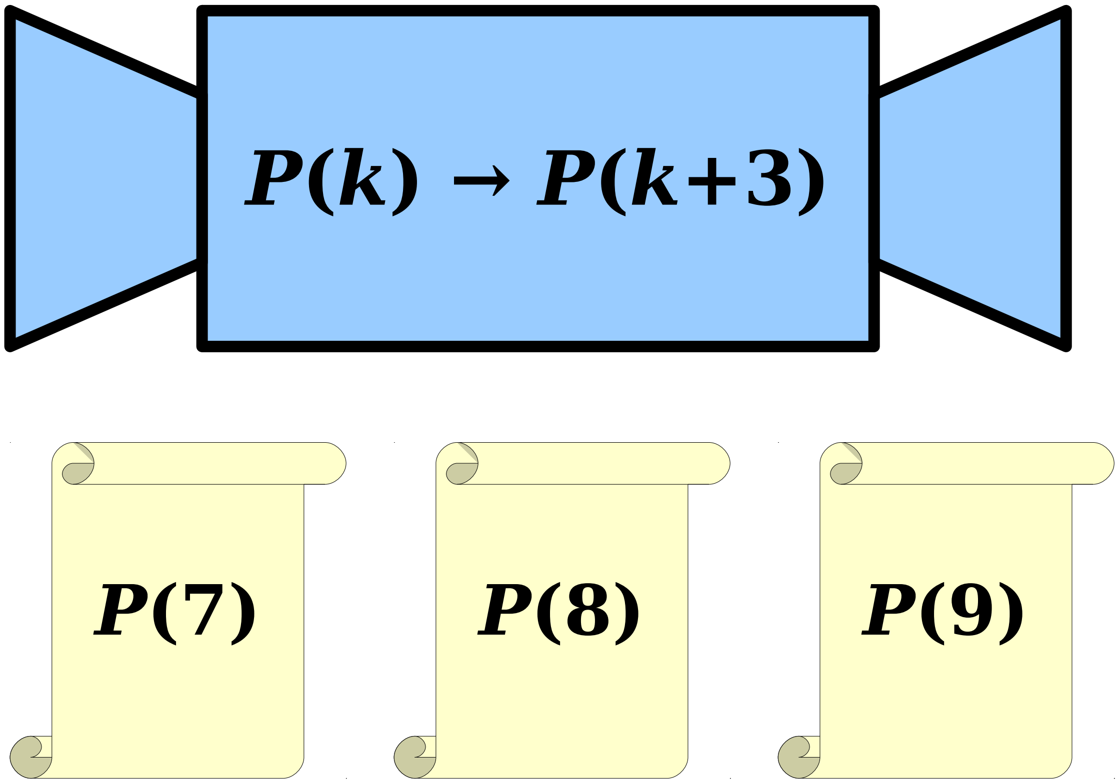

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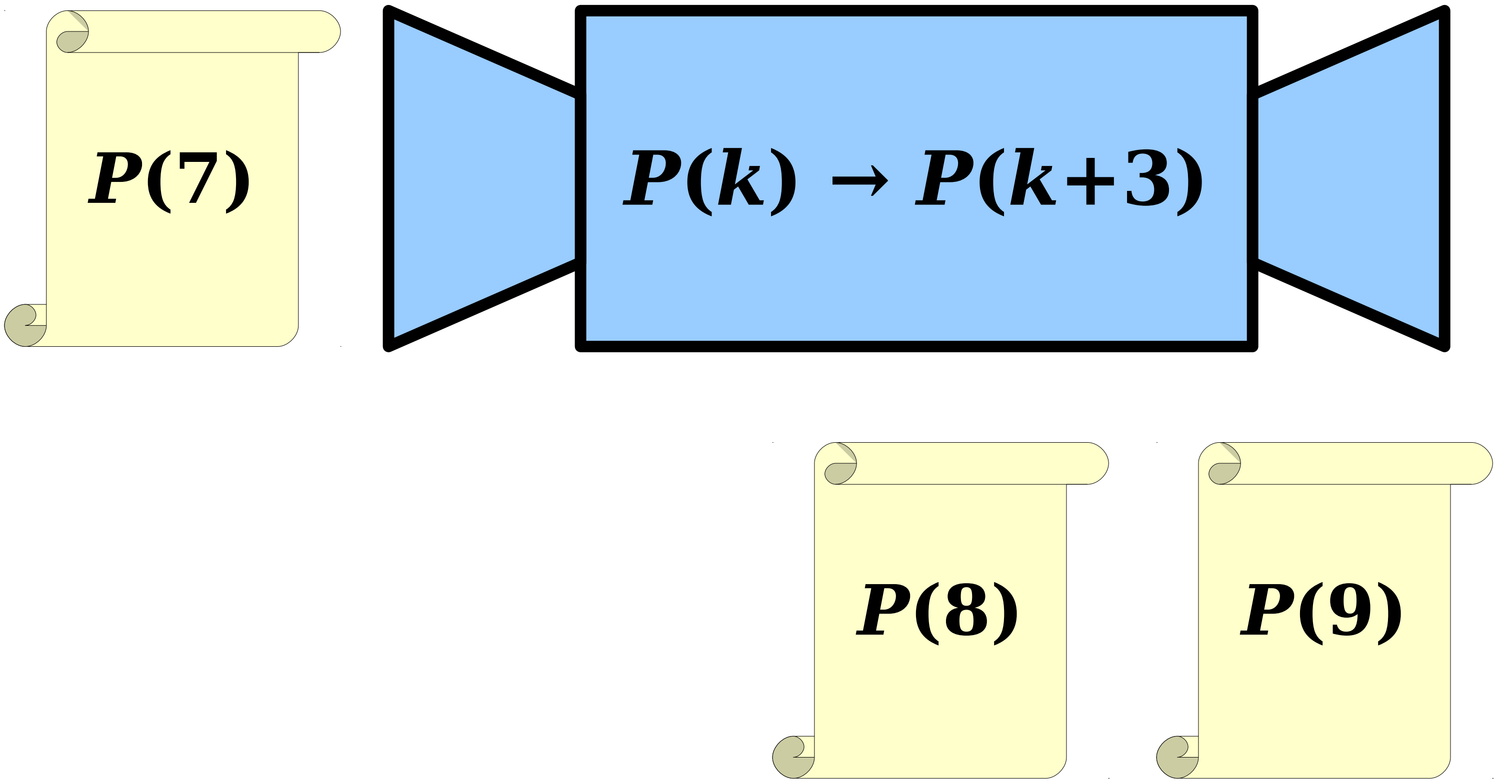

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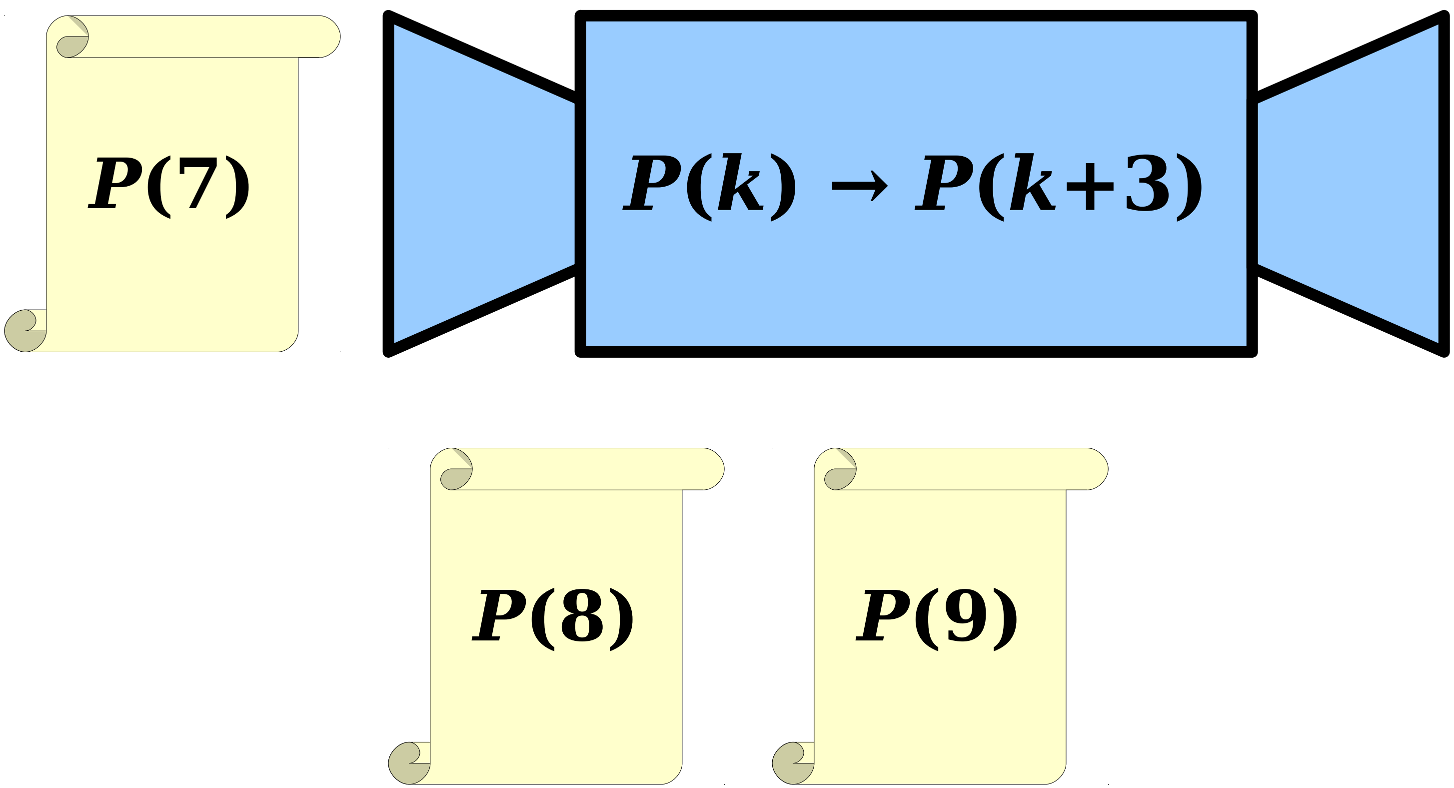

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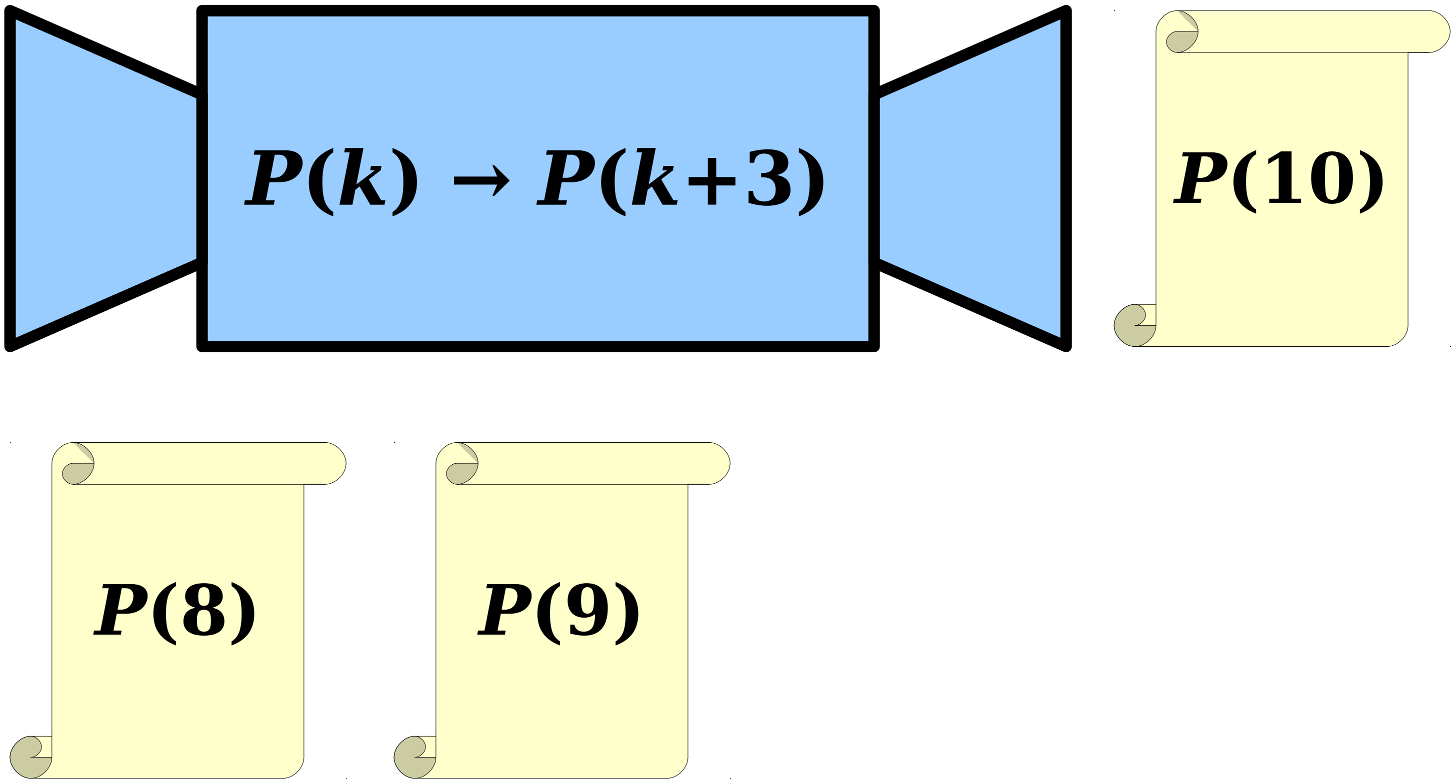

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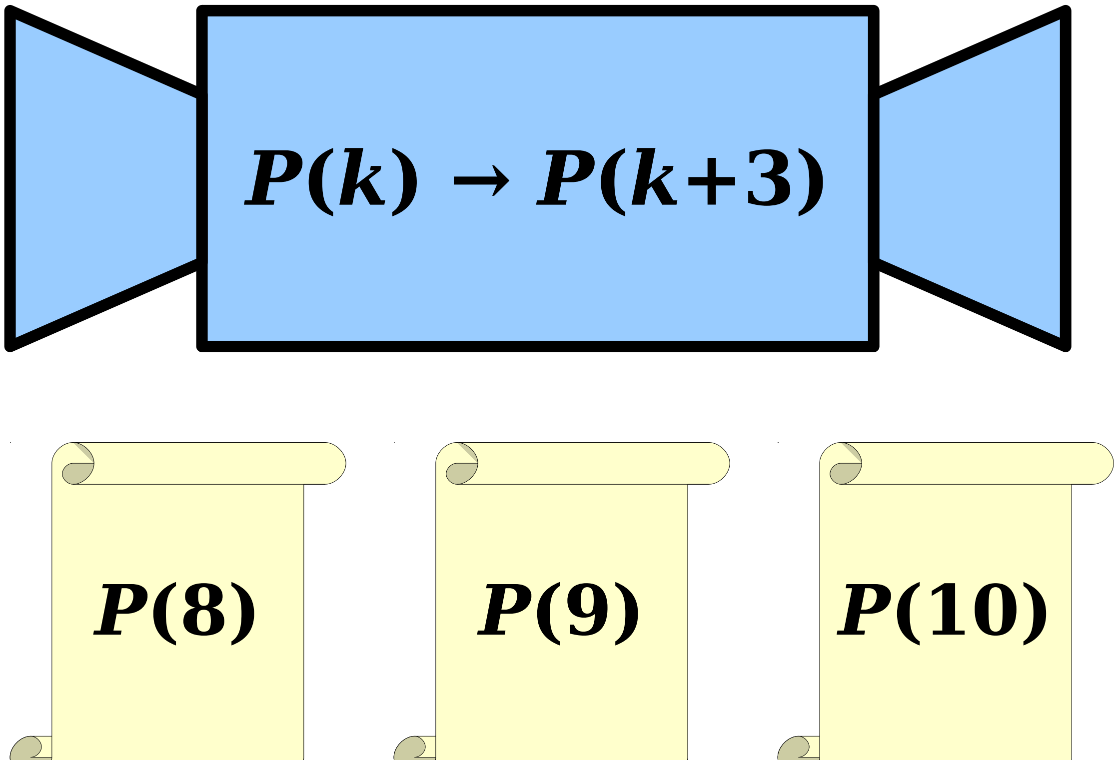

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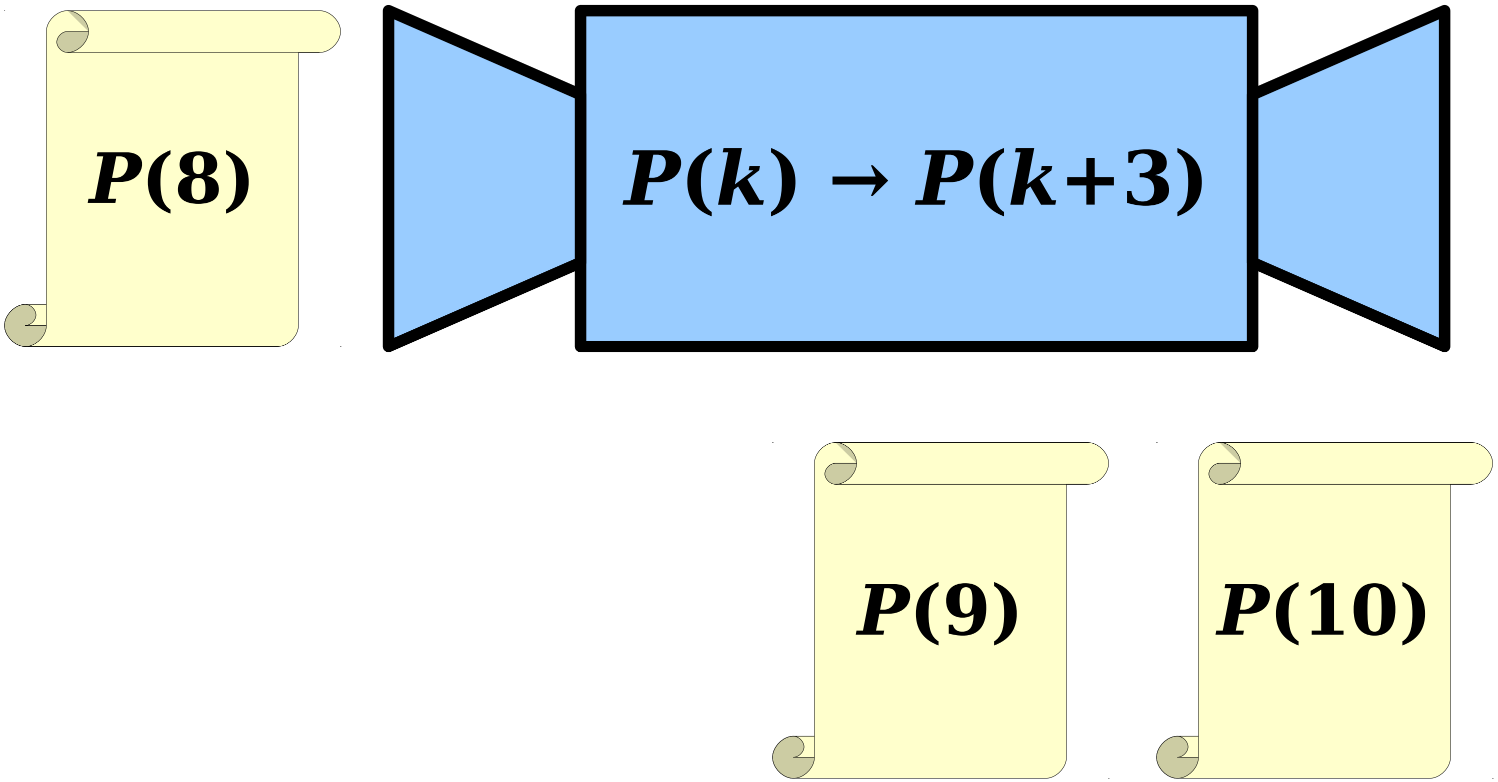

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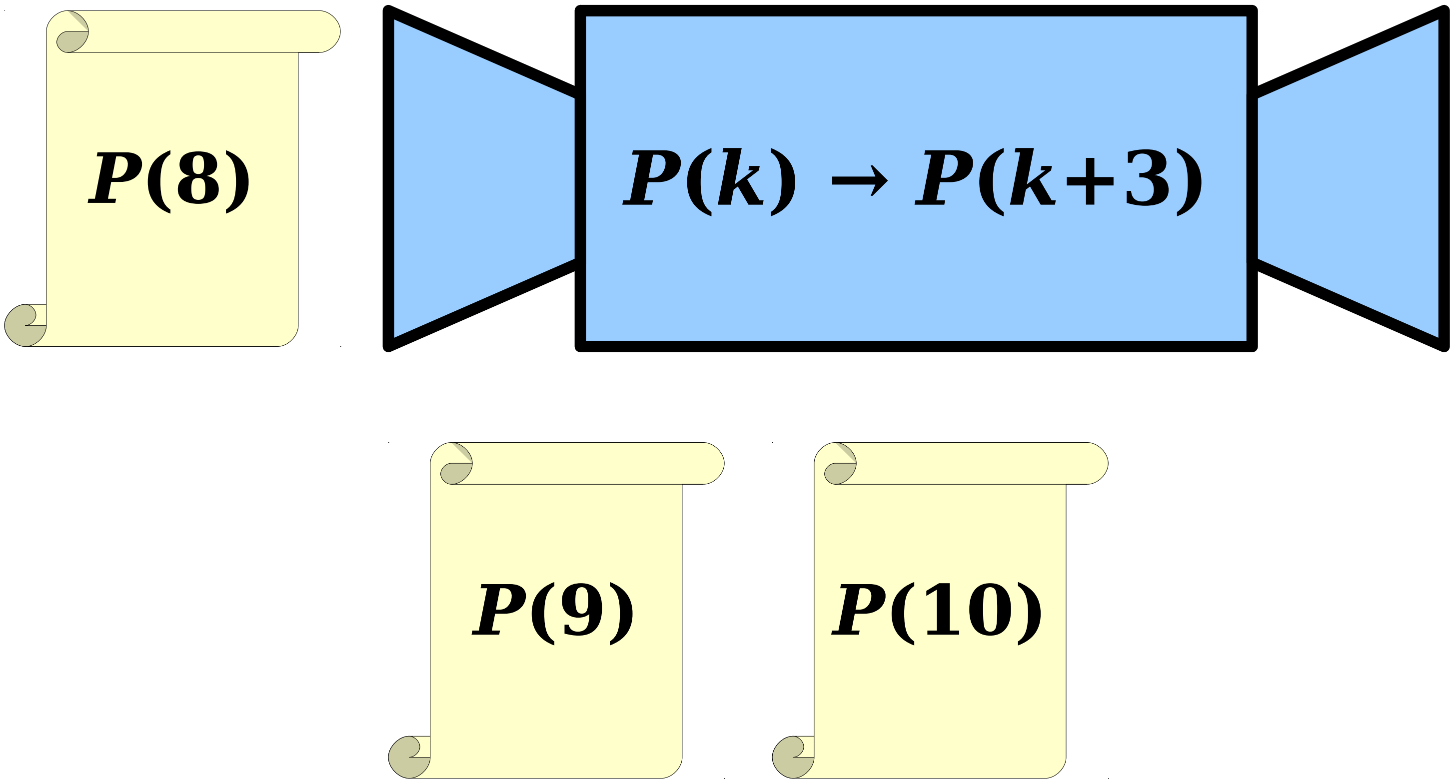

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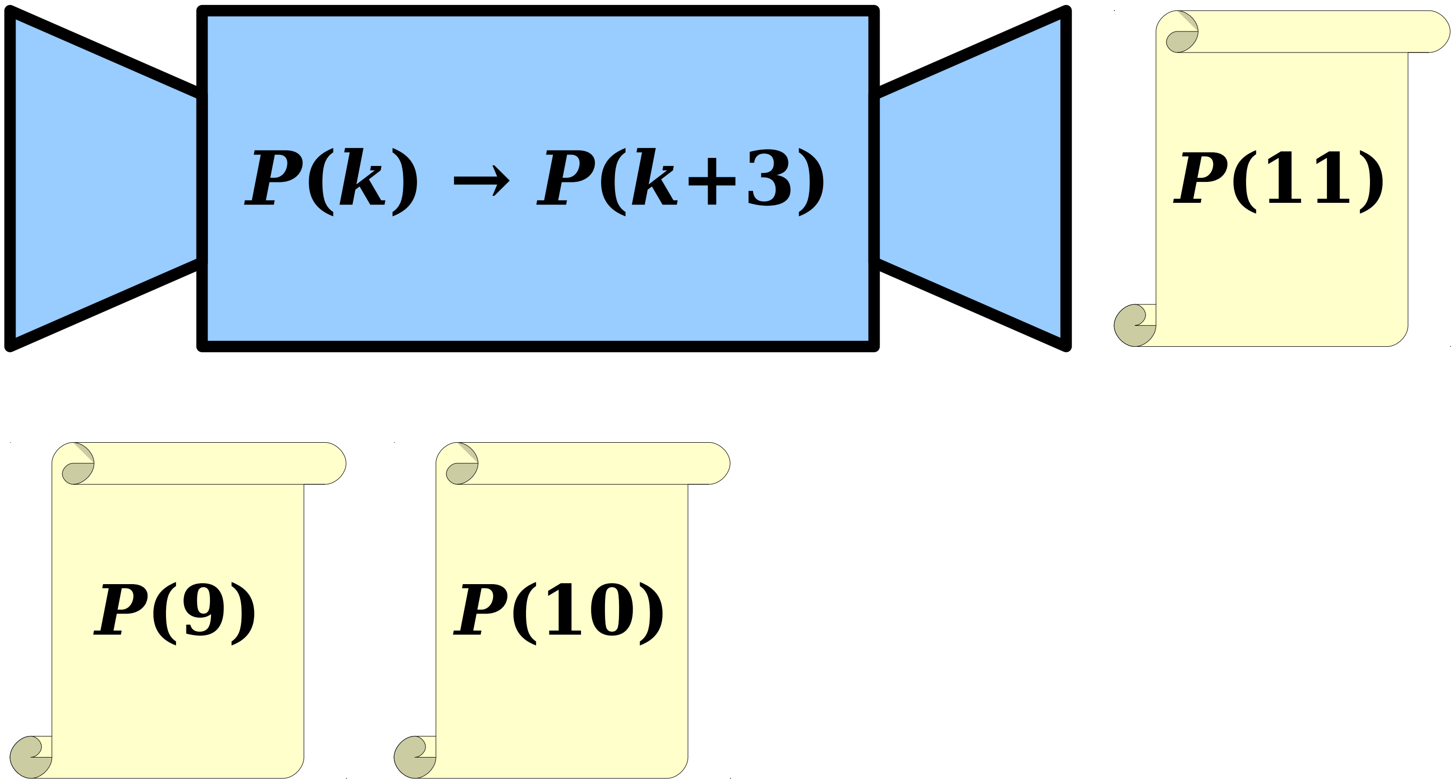

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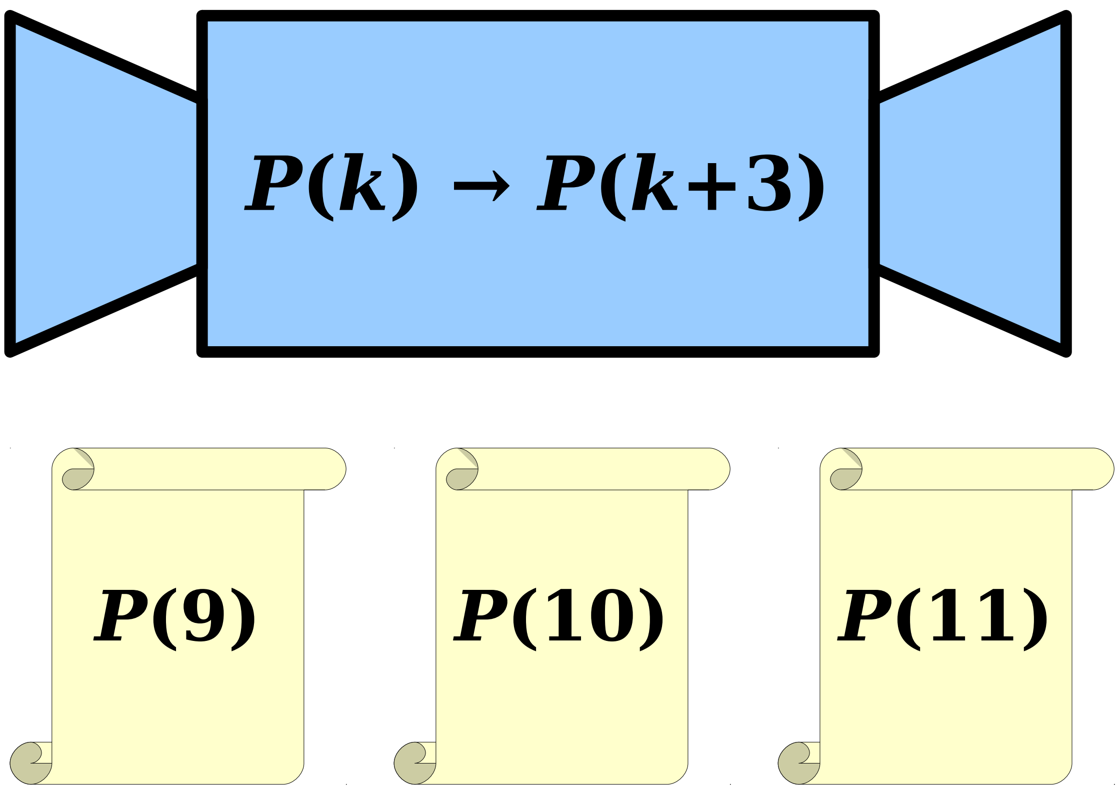


Generalizing Induction

- When doing a proof by induction:
  - Feel free to use multiple base cases.
  - Feel free to take steps of sizes other than one.
- Just be careful to make sure you cover all the numbers you think that you're covering!

Time-Out for Announcements!

Midterm Reminder

- Our first midterm is next Monday from 7PM – 10PM, location TBA.
- Closed-book, closed-computer, limited-notes.
  - You can have a double-sided 8.5” × 11” sheet of notes when you take the exam.
- Covers material from PS1 – PS3 and Lecture 00 – Lecture 09.

Practice Opportunities

- There's a practice midterm ***tonight*** from 7PM – 10PM in Lathrop 282 / 292. Purely optional, but highly recommended!
- Solutions to Extra Practice Problems 1 have been released.
- Extra Practice Problems 2 has just been posted.
- Want more review on certain topics? Let us know what you want to see more of!

CODE2040 Info Session

- CODE2040 is a group that focuses on help black and Latin@ students succeed in the tech industry.
- They have an amazing program.
- Interesting? Check out their info session tonight from **6PM – 7PM** in **Nitery 219**.
  - (Hey! That's right before the practice exam!)

Your Questions

“What would you tell yourself as an undergrad if you had the chance?”

Relax. Everything is going to be okay. Take some humanities and social science classes and don't just tunnel vision down CS. Try to get a better sense of people's life experiences and how that influences them. Strike up more conversations with people - they don't bite!

“There are so many private tech companies and startups that are being valued at extremely high multiples. Do you think we might be in a tech bubble? (many big tech firms are cutting jobs so might this be the end of it?)”

I'm pretty sure we're in a bubble now. I seriously doubt that the craziness in the Bay Area can keep up at the rate that it is. I think we'll have a huge correction soon.

That said, I think it's also likely that someone is going to point out that I said this ten years down the line and laugh at how wrong I was. ☺

“What's one law you'd like to break if you can walk free afterwards?”

“If you had only one more day to live in this world, what would you do?”

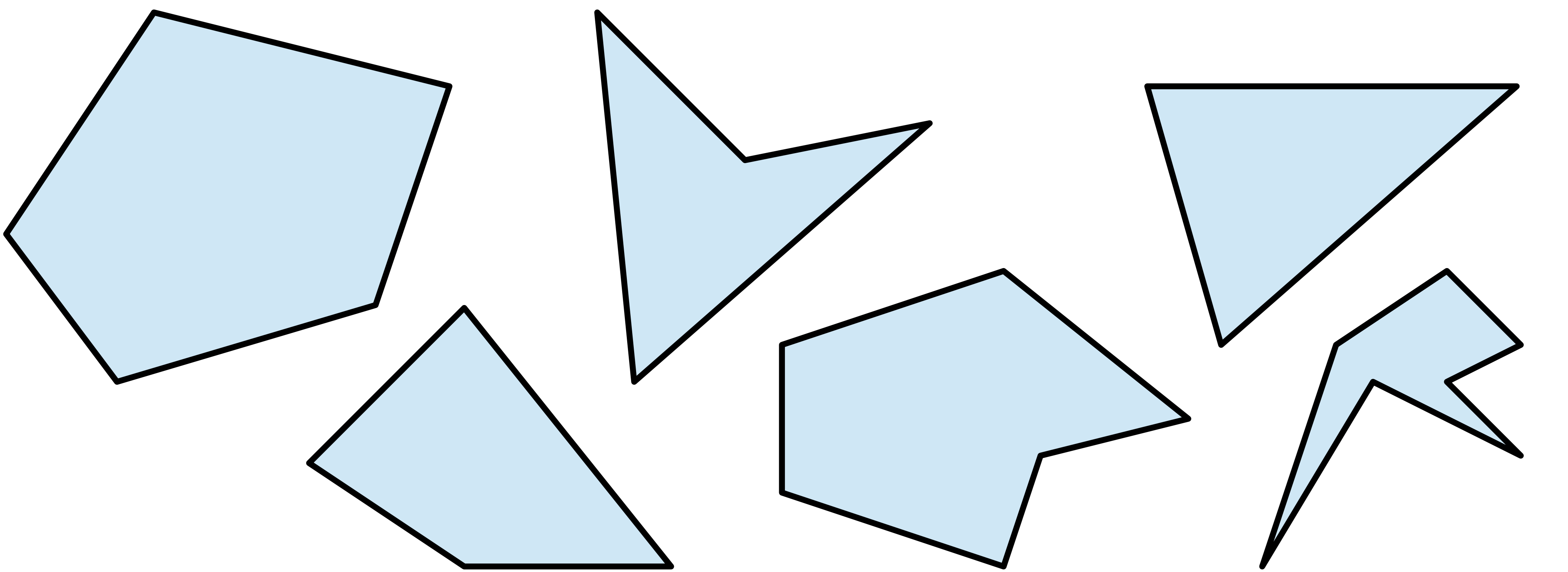
Oh man, that's hard. I thought about these for a long time and couldn't come up with anything good, but that might be because I'm not having the best week. Ask me again later and I'll try to come up with something good.

Back to CS103!

Application: **Polygons!**

Convex Polygons

- A ***convex polygon*** is a polygon where, for any two points in or on the polygon, the line between those points is contained within the polygon.



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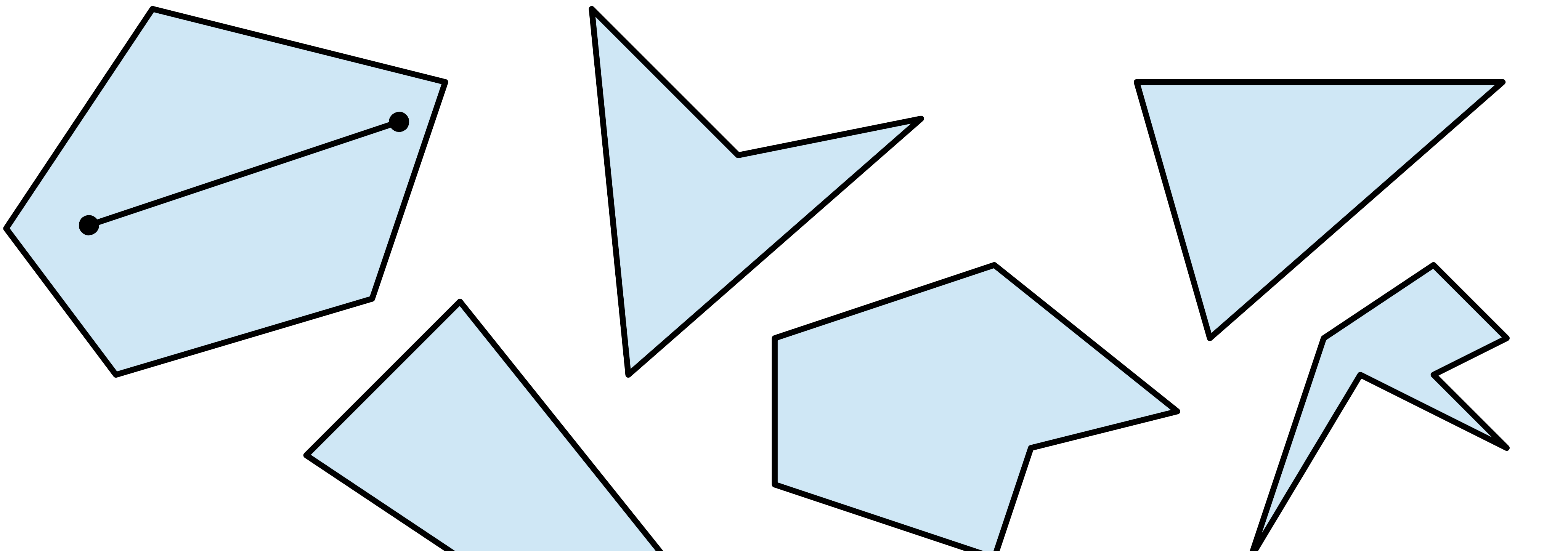

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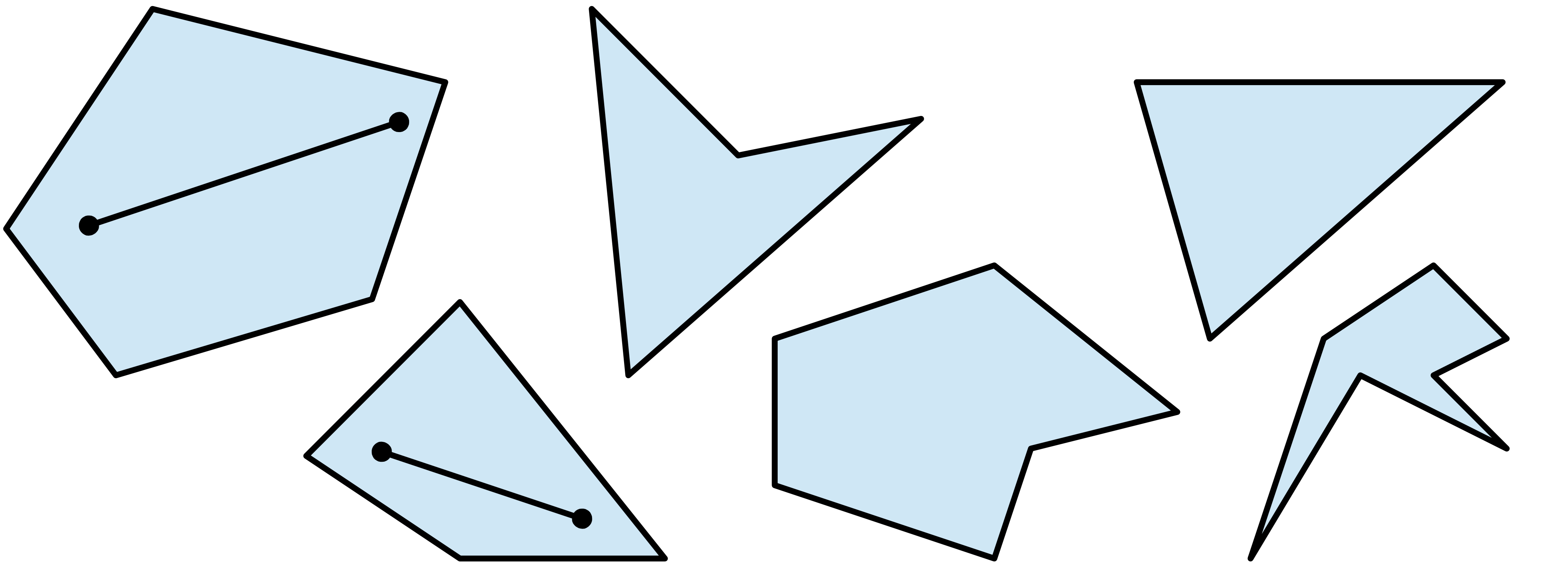

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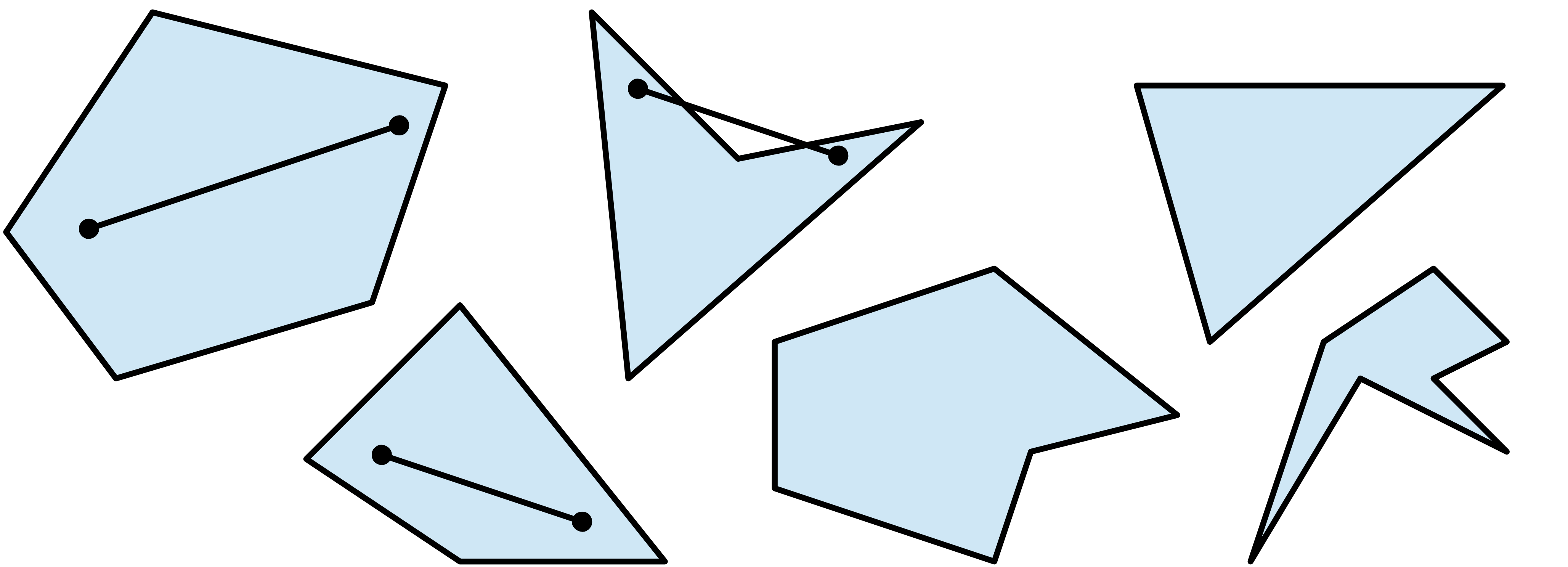

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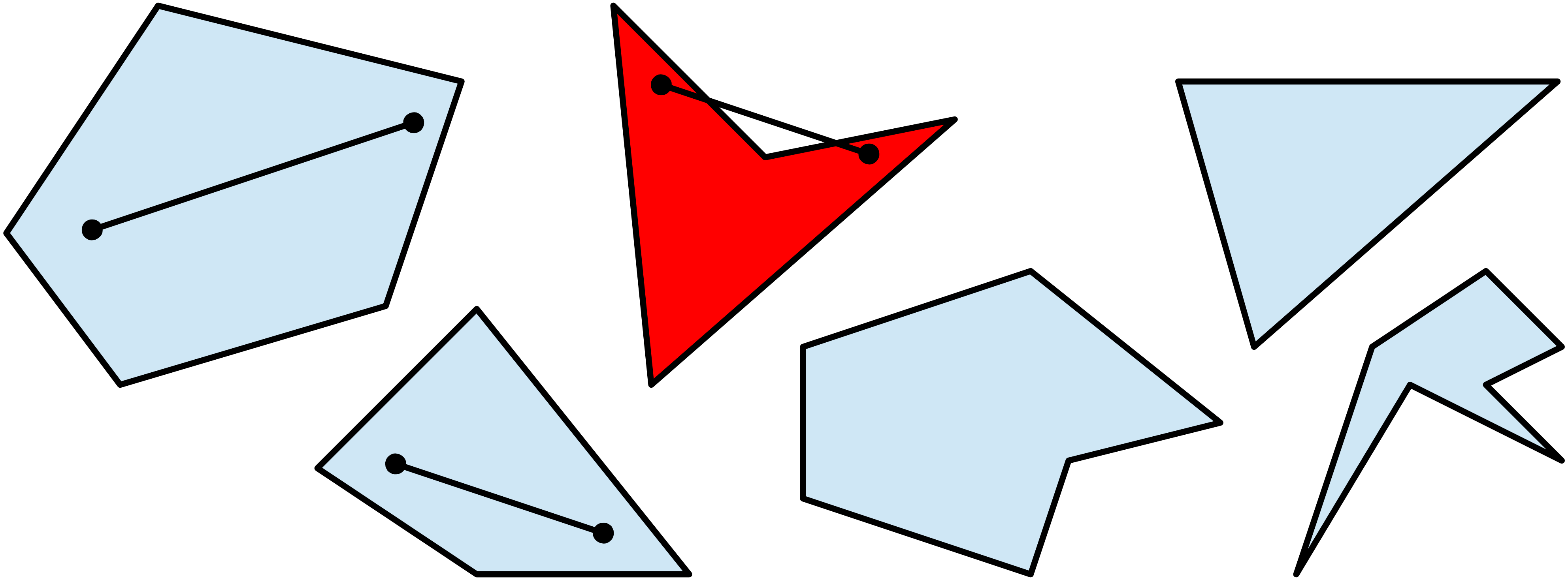

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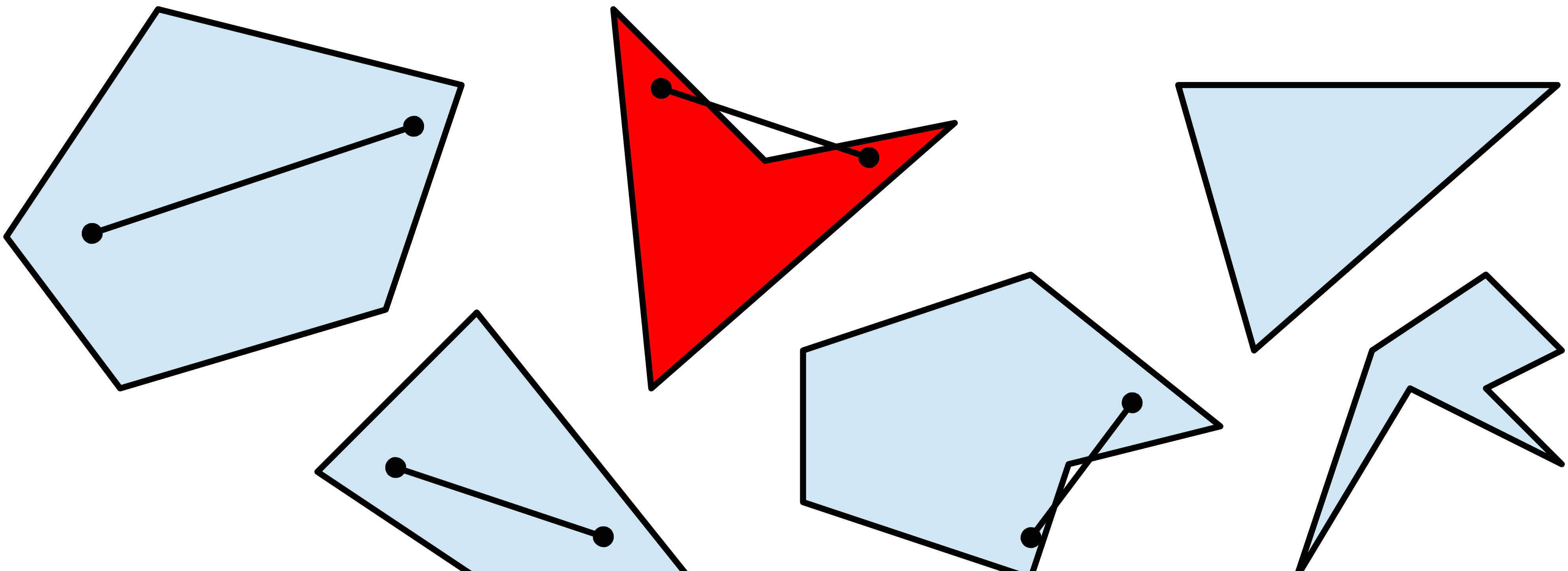

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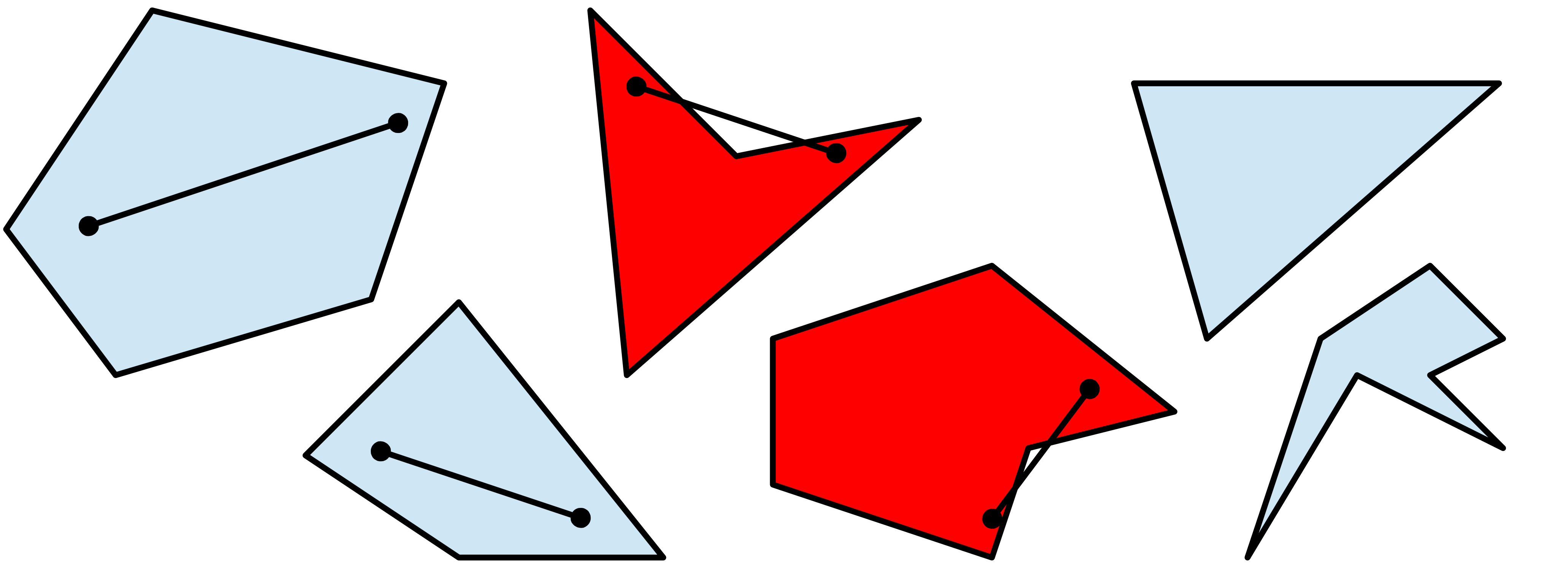

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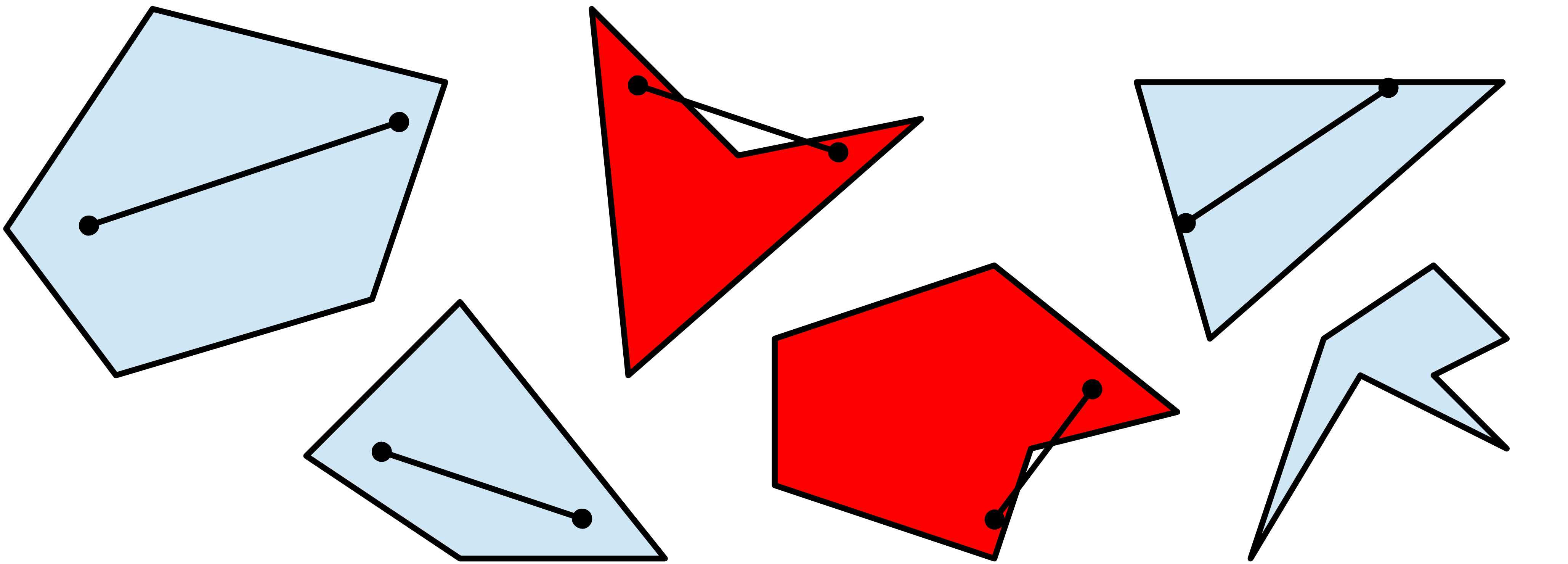

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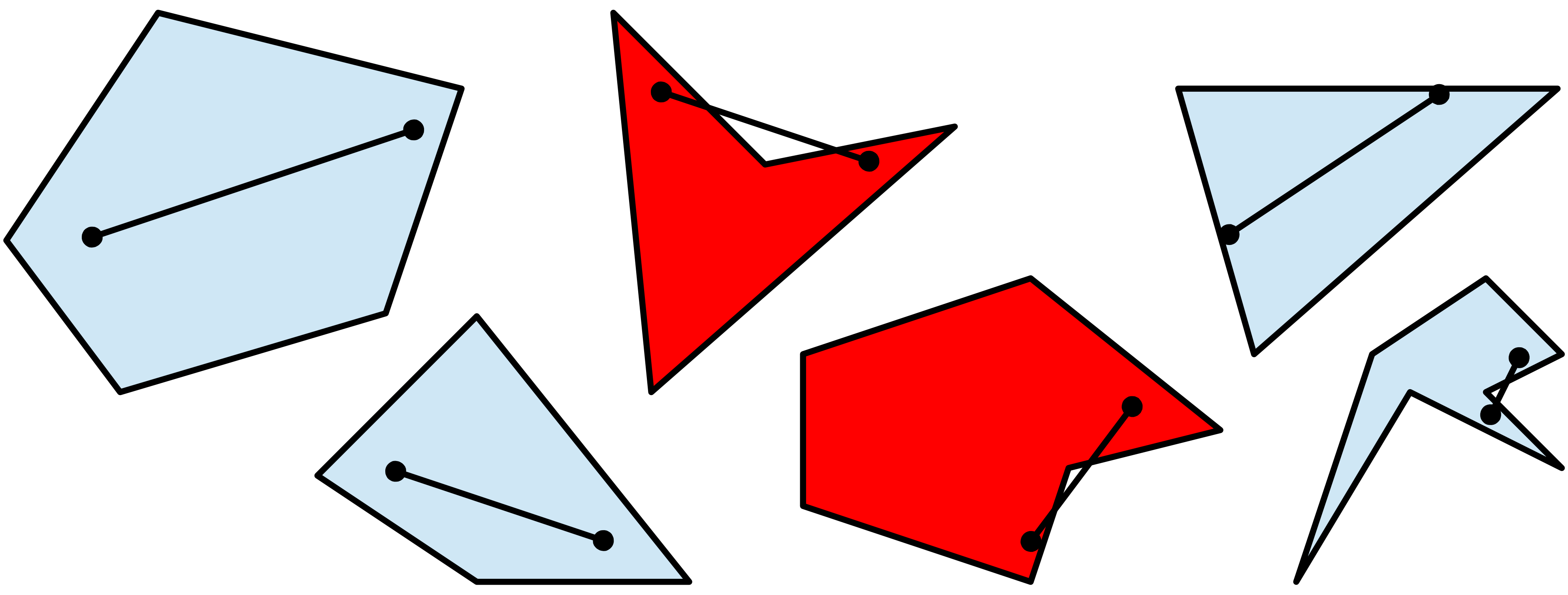

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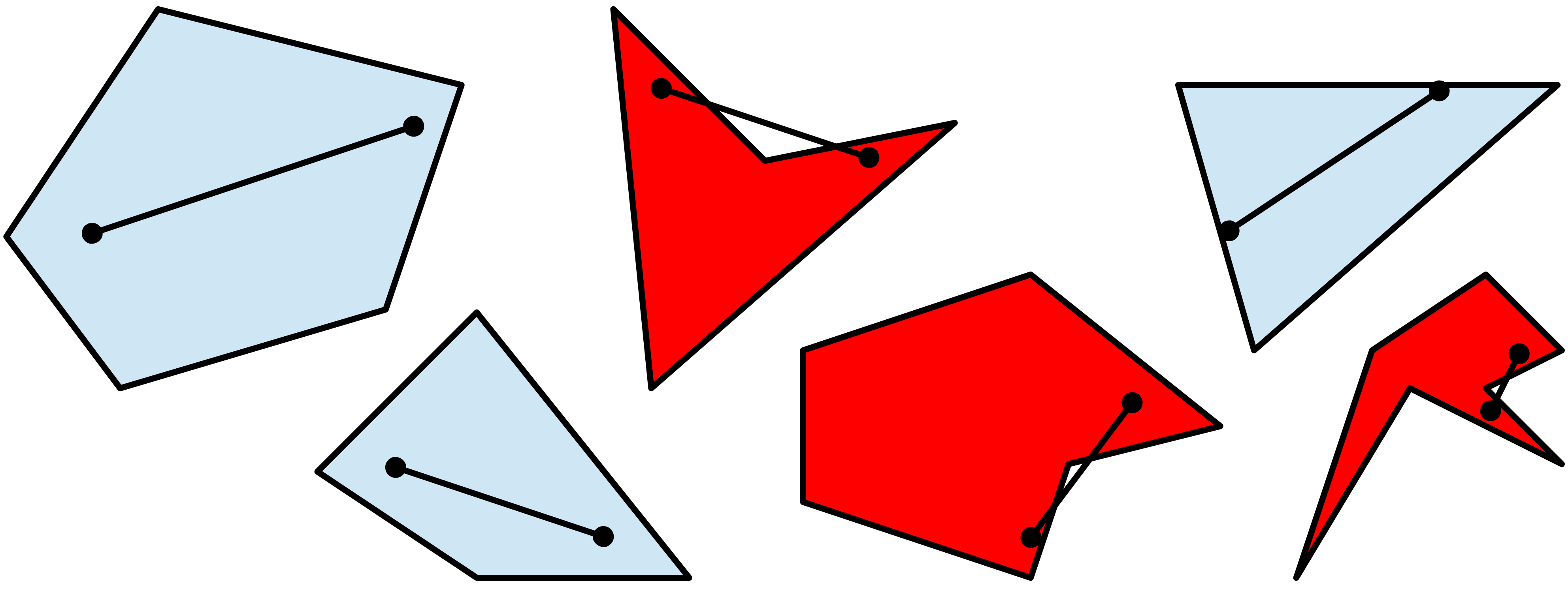
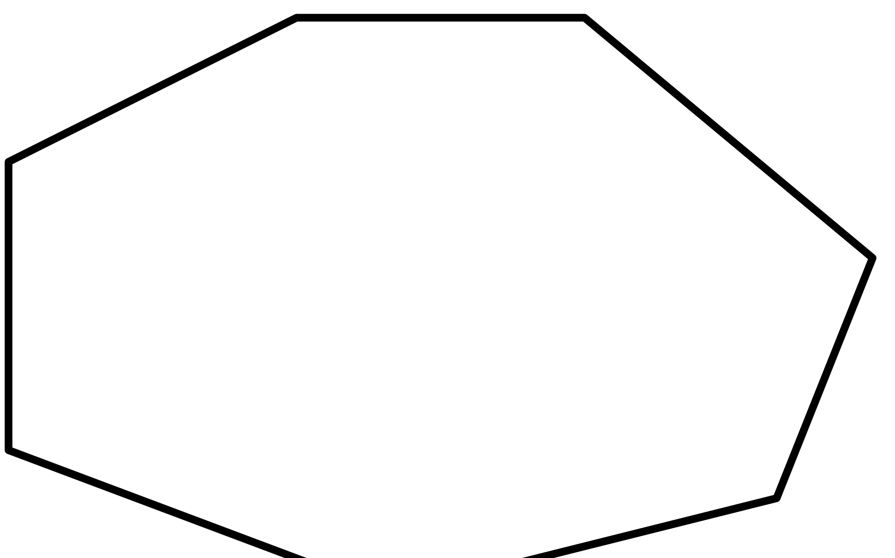


Splitting Polygons

- ***Theorem:*** If P is a convex polygon with $n \geq 4$ vertices, then any line connecting two non-adjacent vertices splits P into two smaller convex polygons, each of which have between 3 and n-1 vertices.



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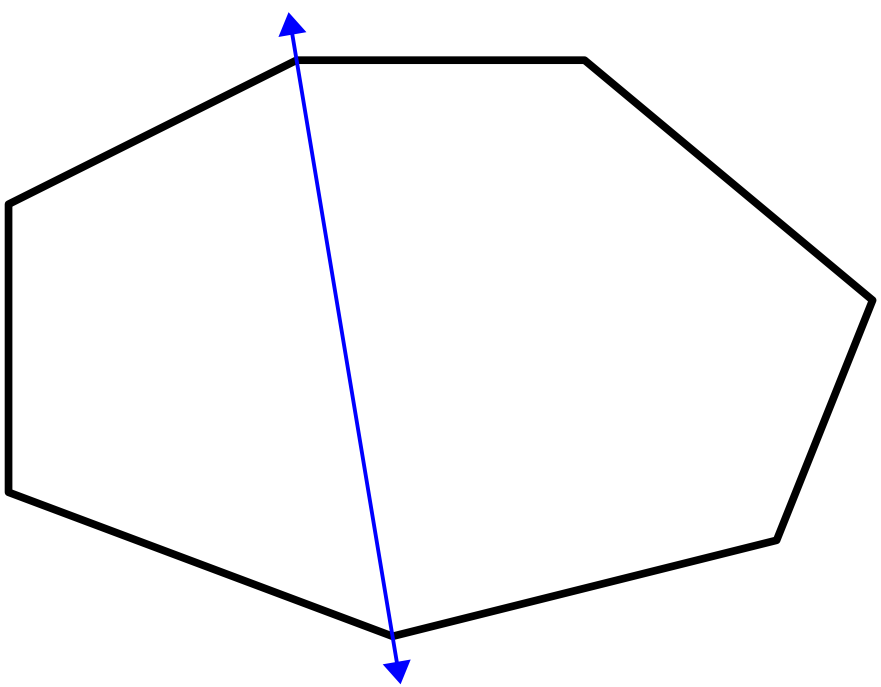

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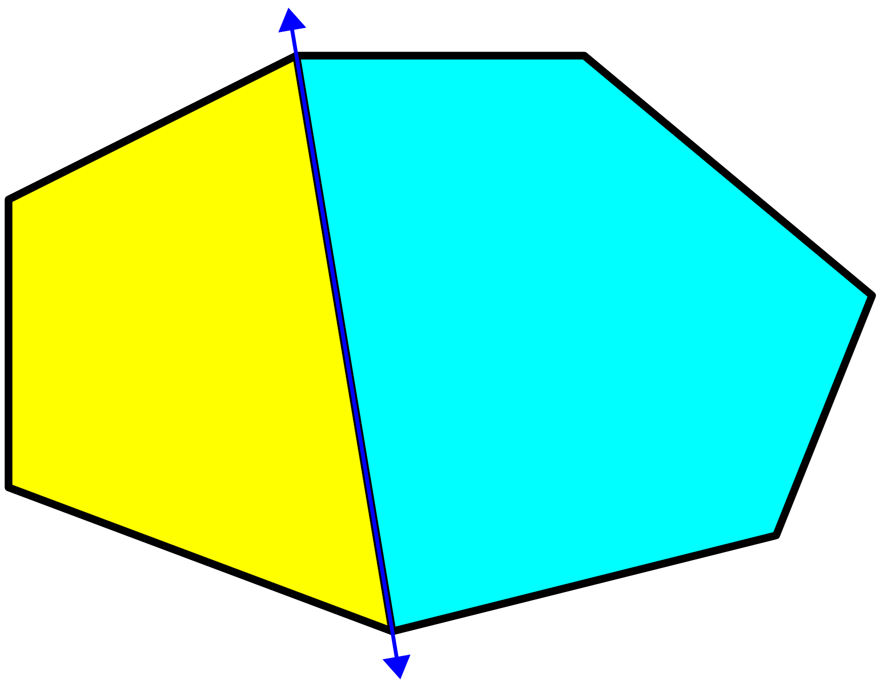
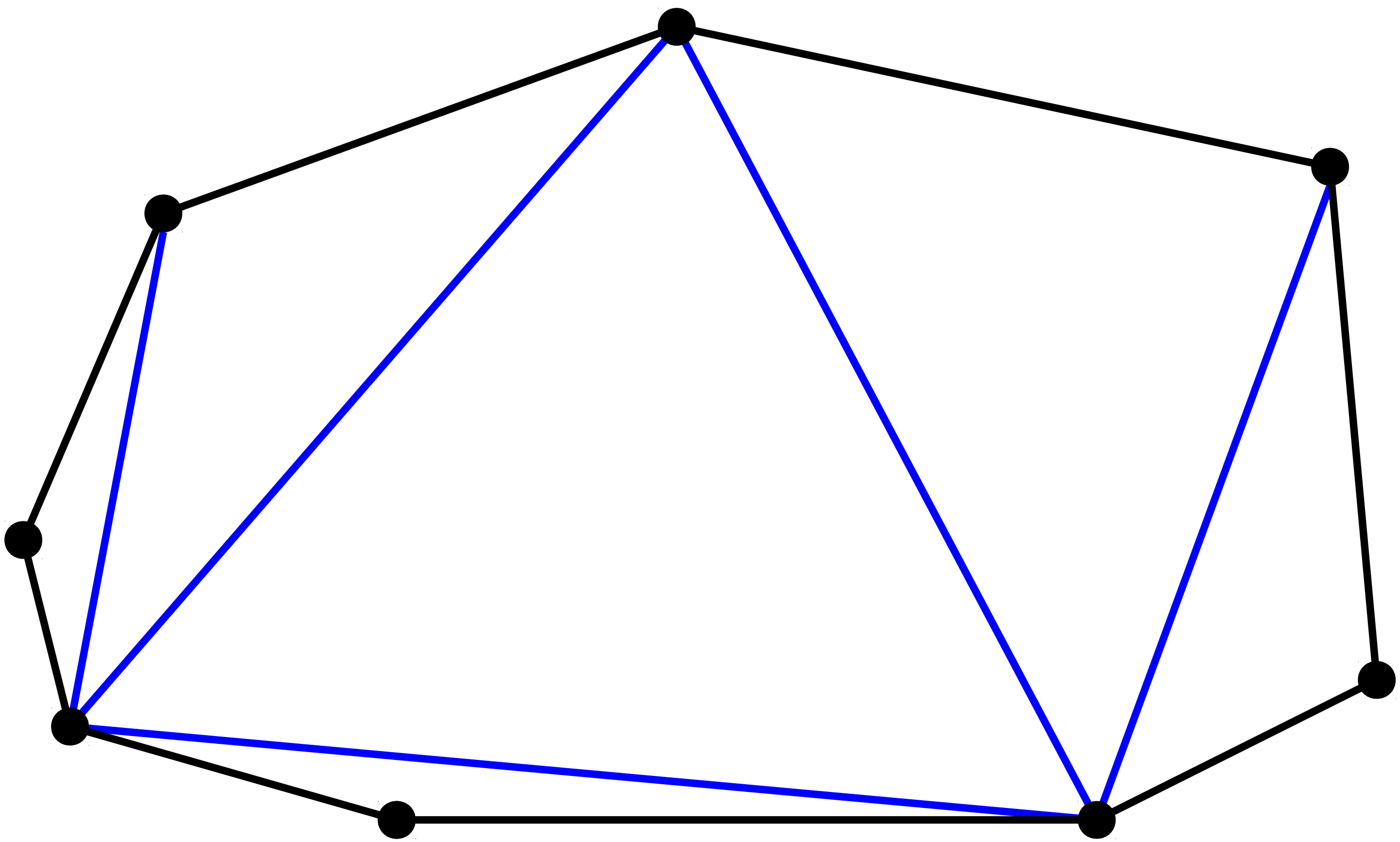


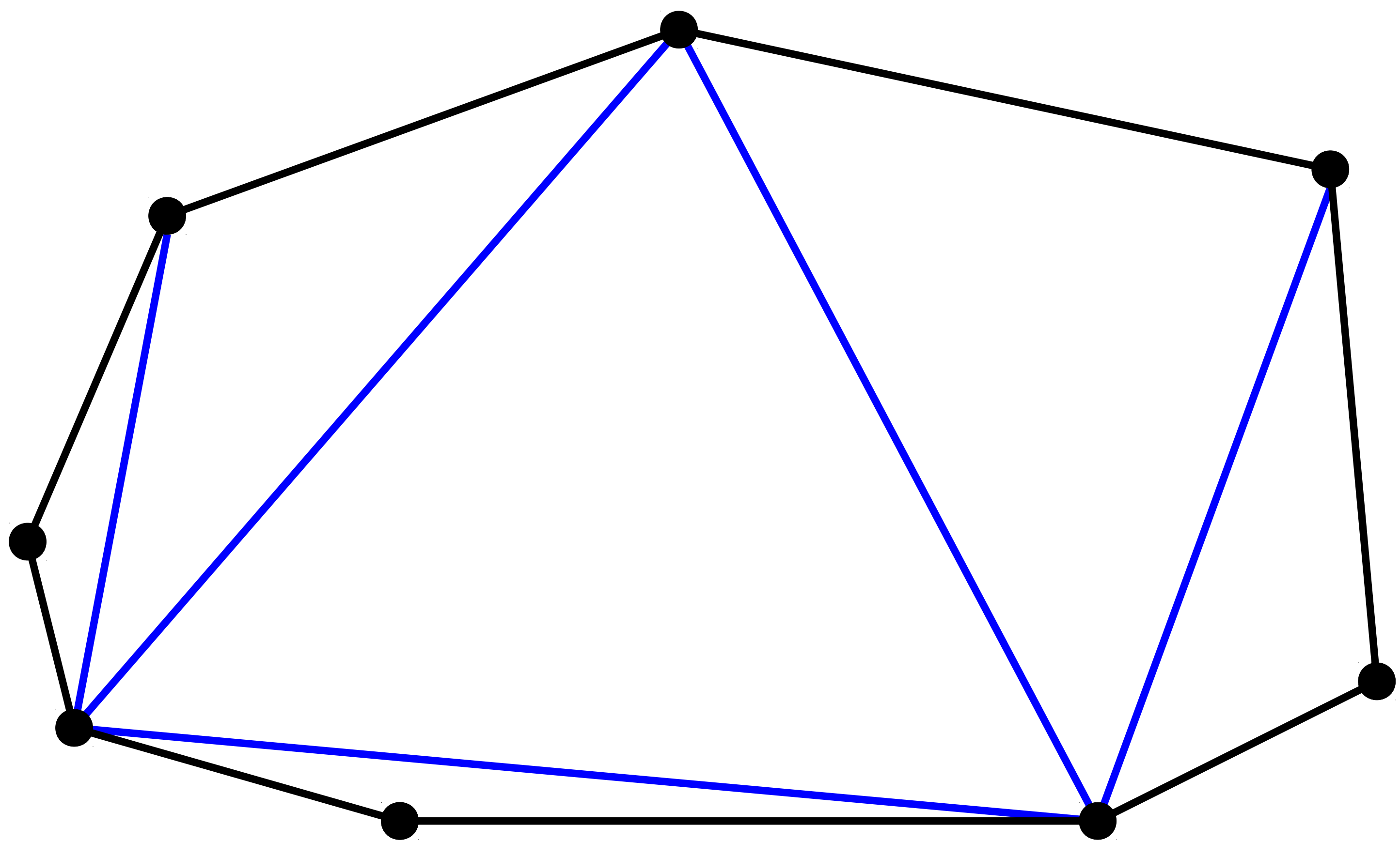
Polygon Triangulation



Polygon Triangulation

- Given a convex polygon, a ***triangulation*** of that polygon is a way of connecting the vertices with lines such that
  - no two lines intersect, and
  - the polygon is converted into a set of triangles.
- Question: How many lines do you have to draw to elementarily triangulate a convex polygon?

Triangulations



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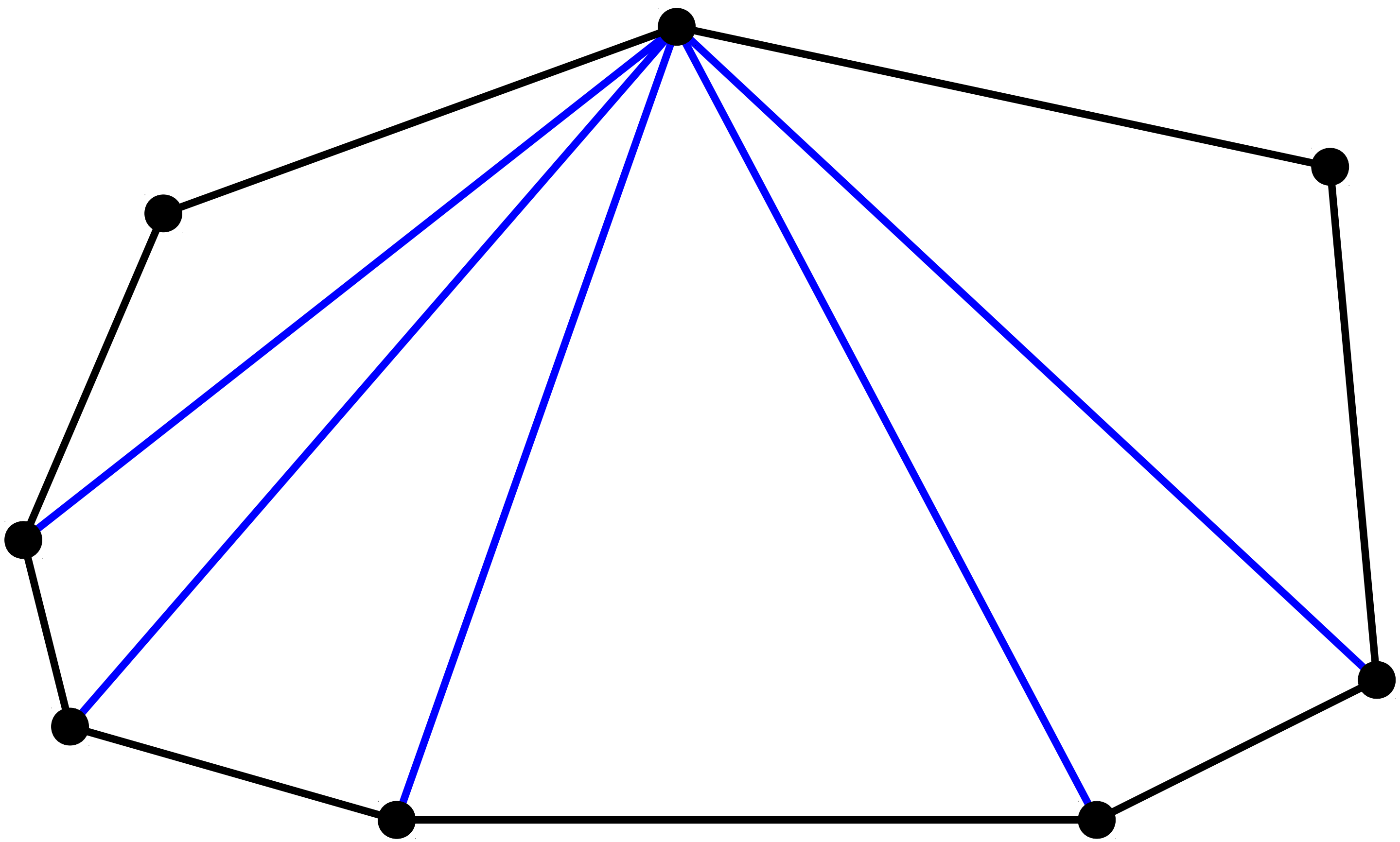

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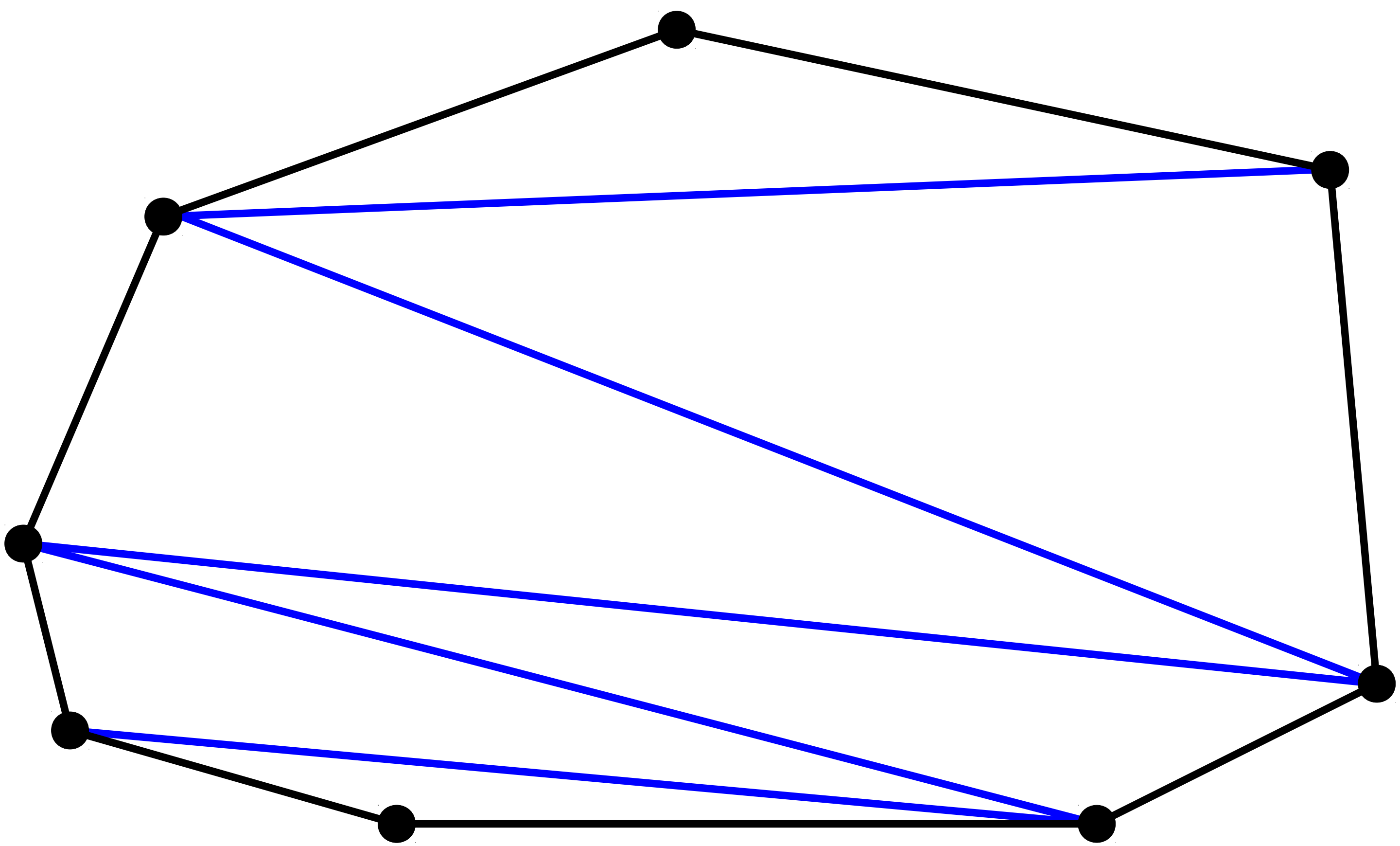

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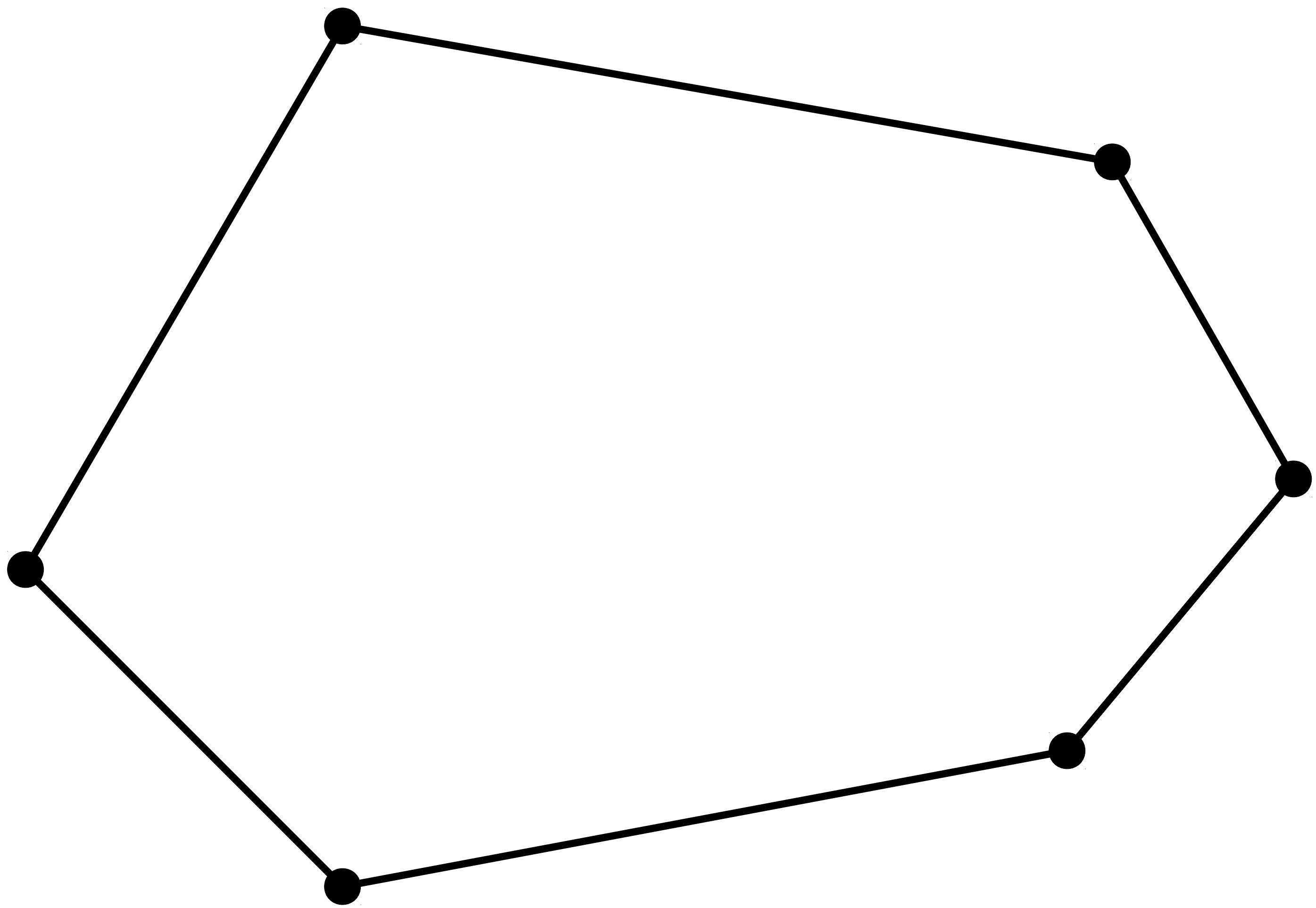

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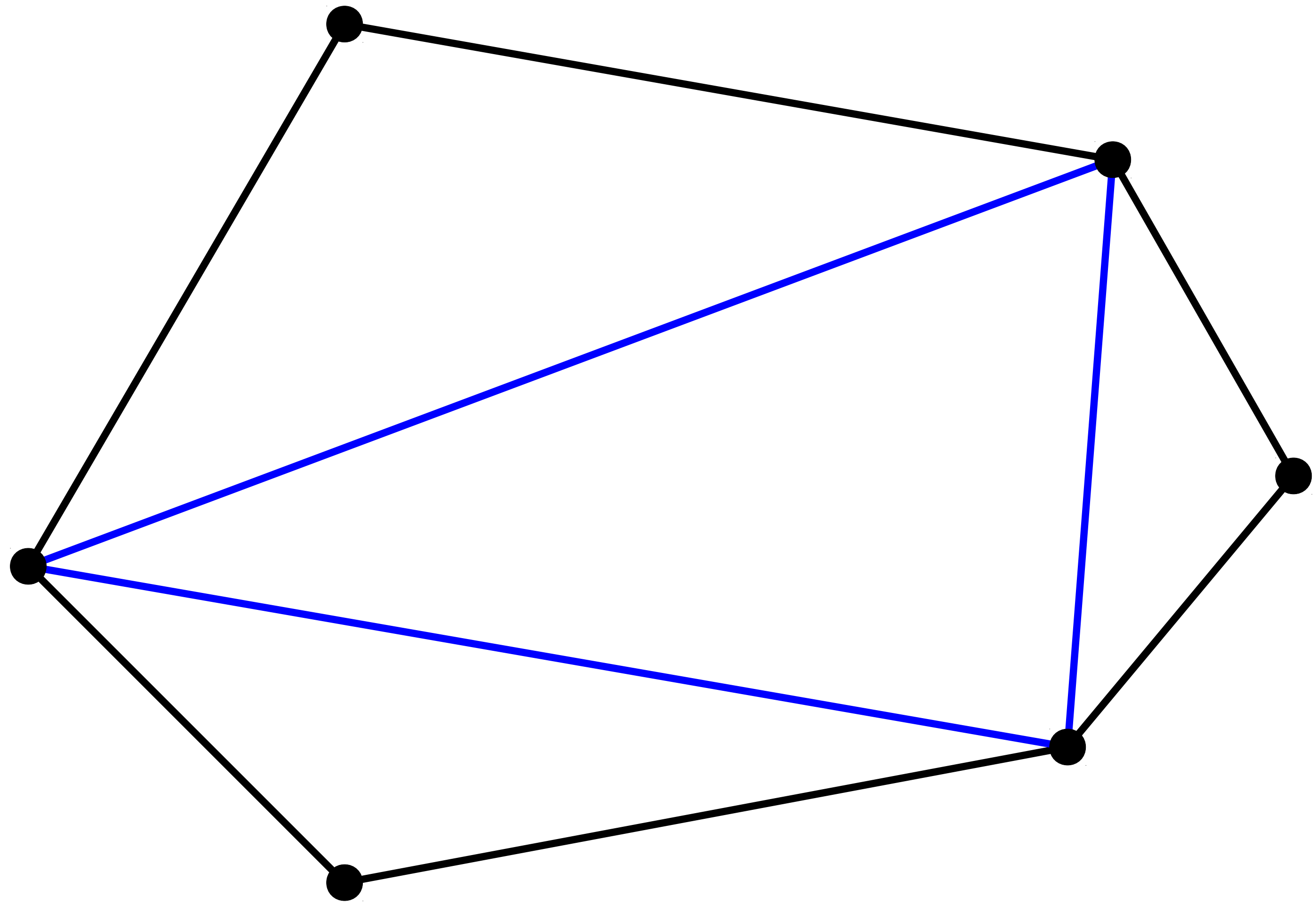

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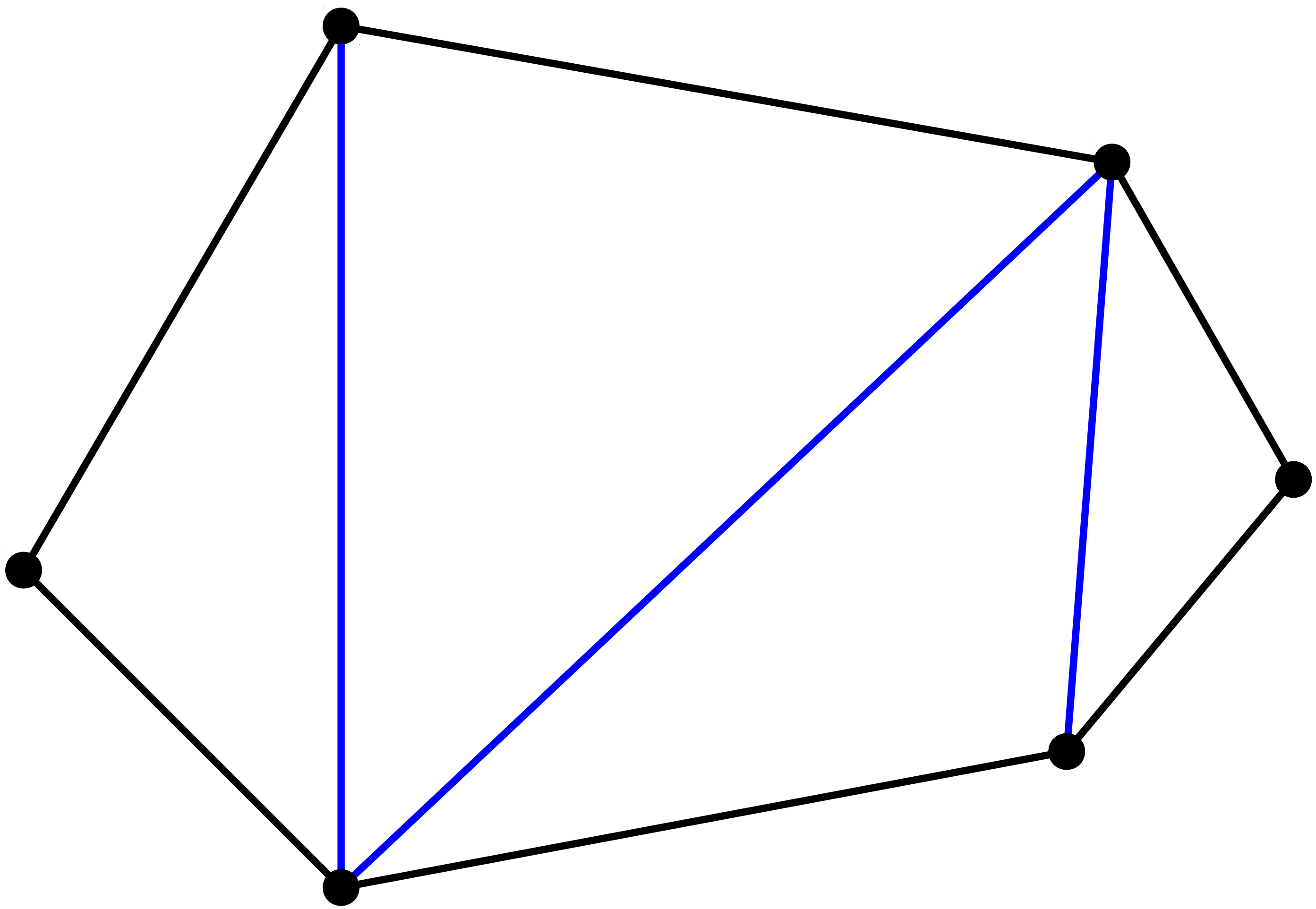

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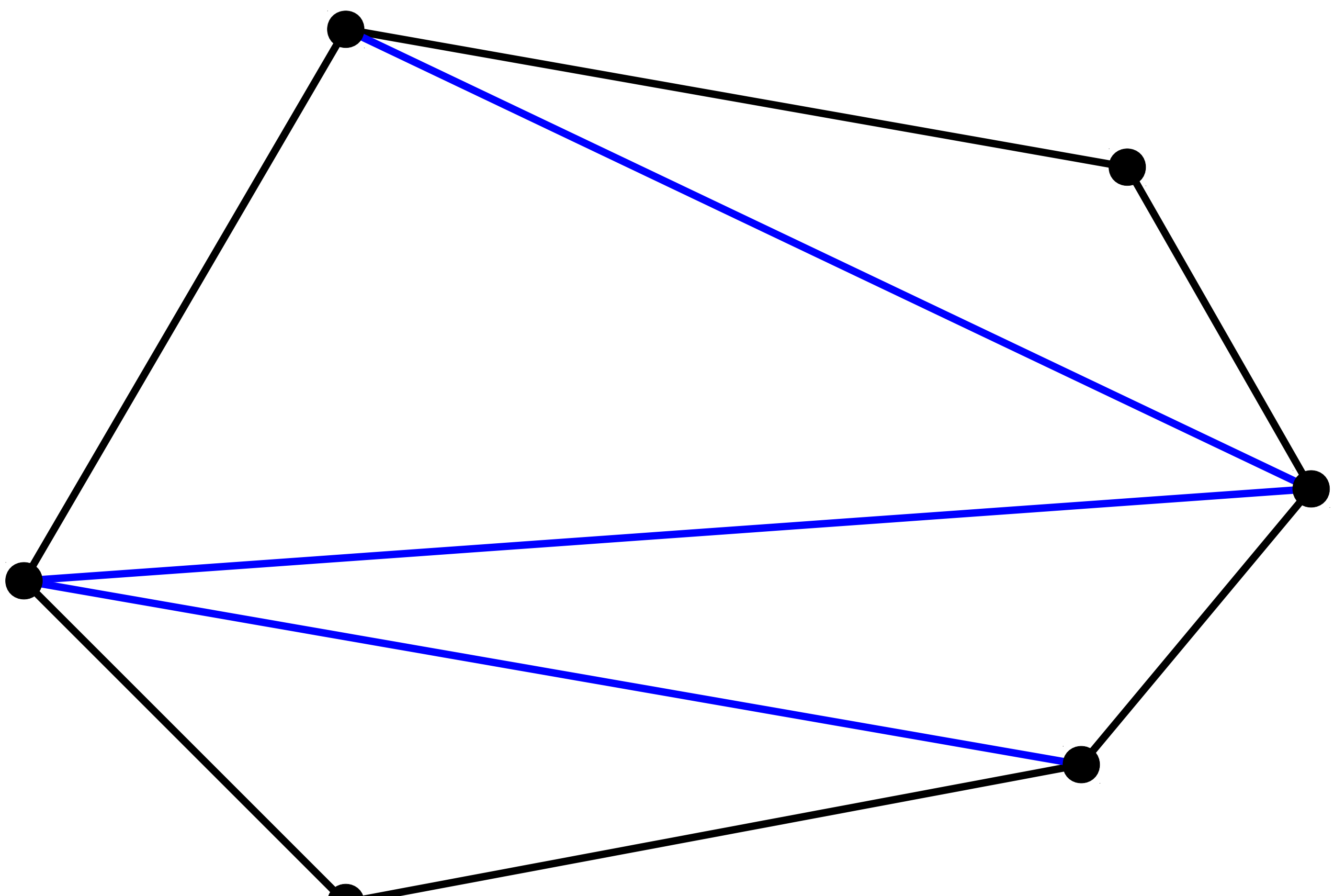

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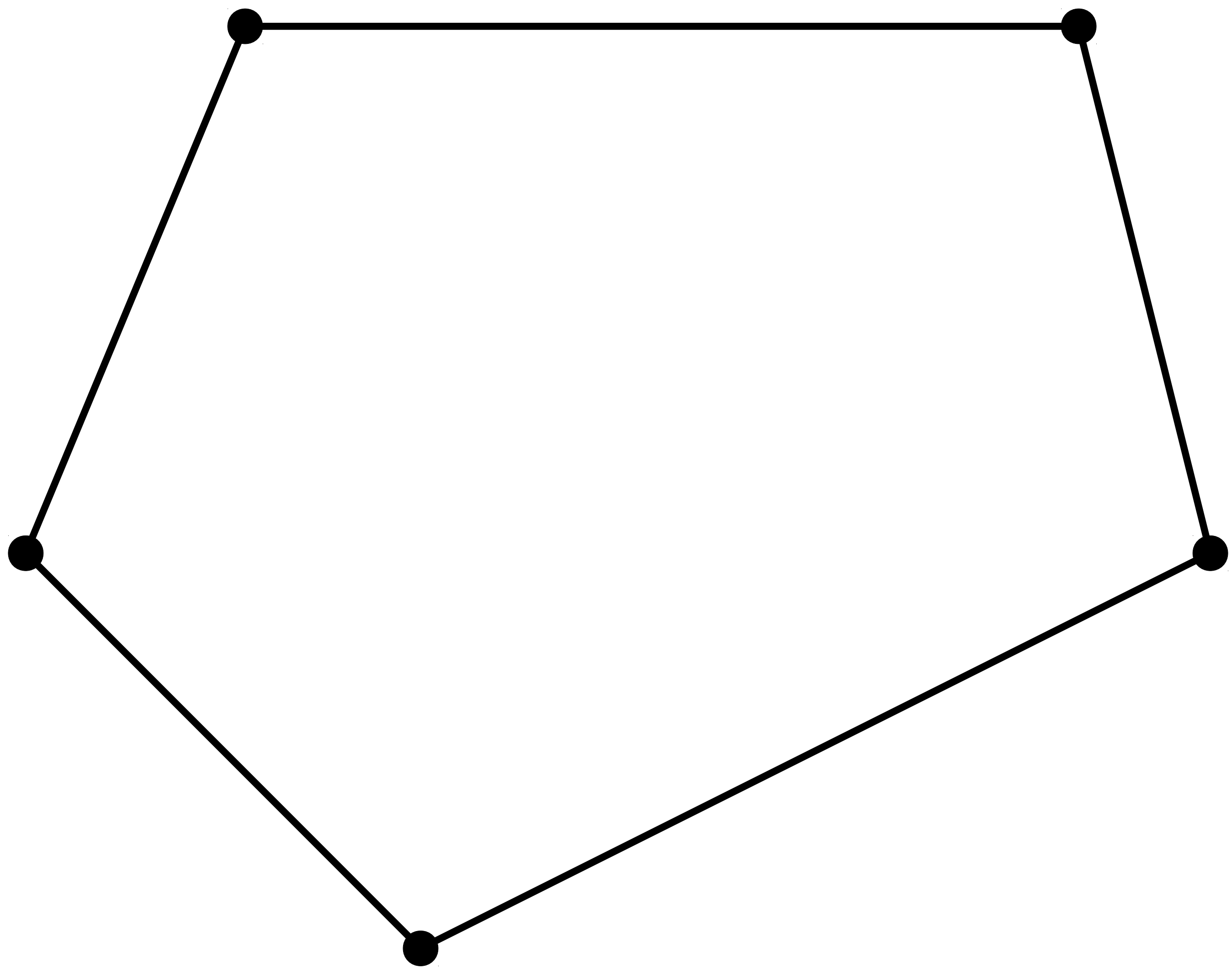

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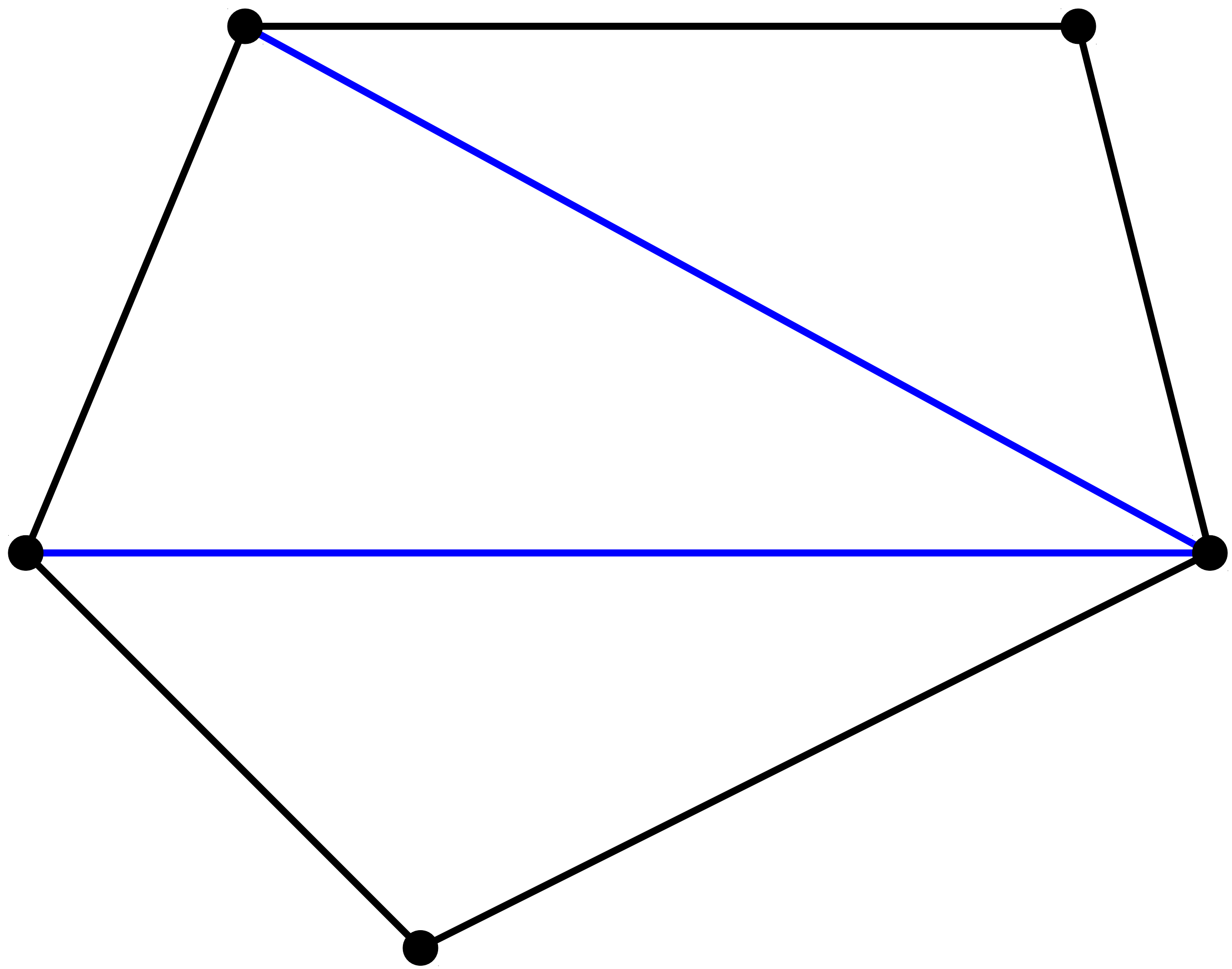

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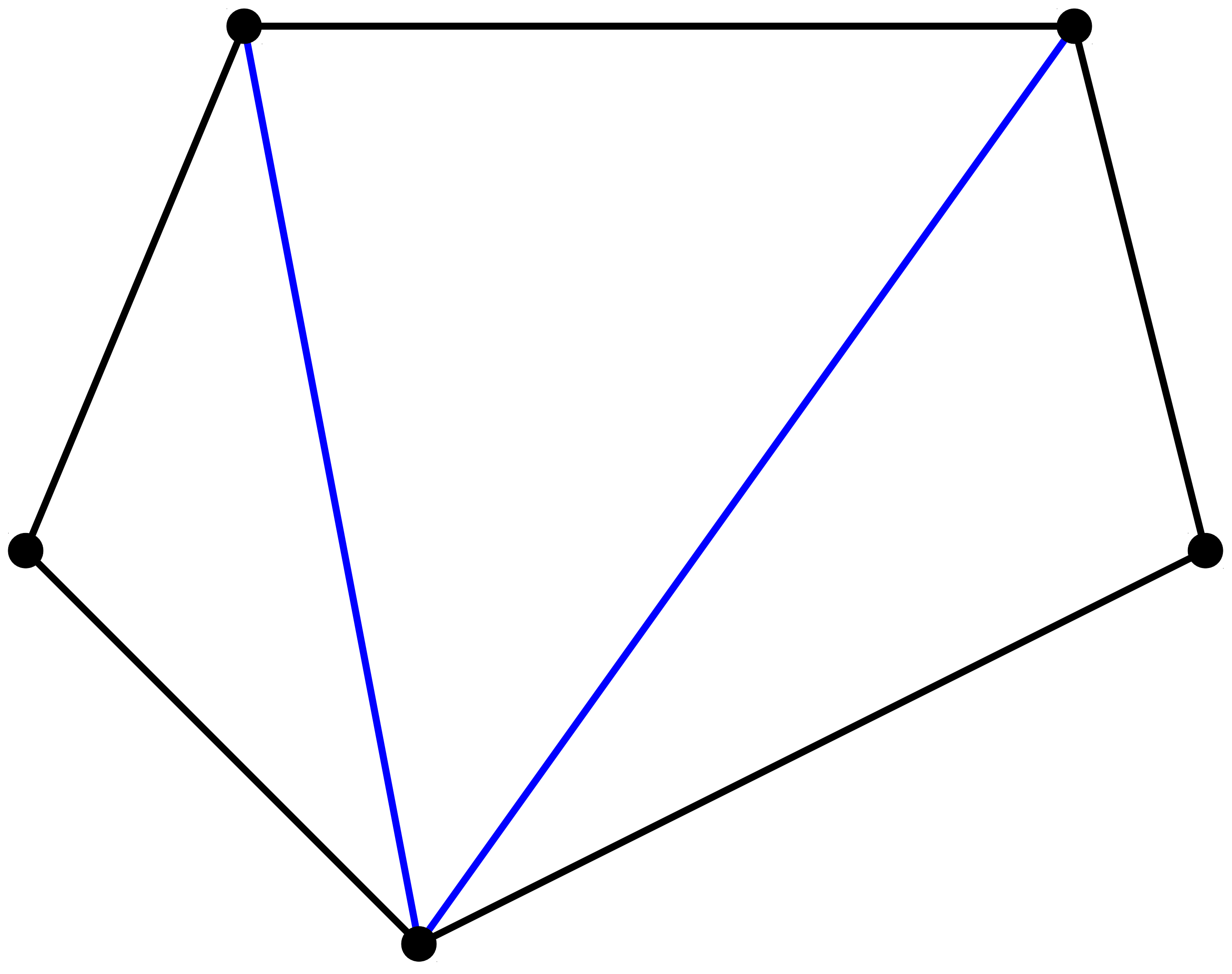
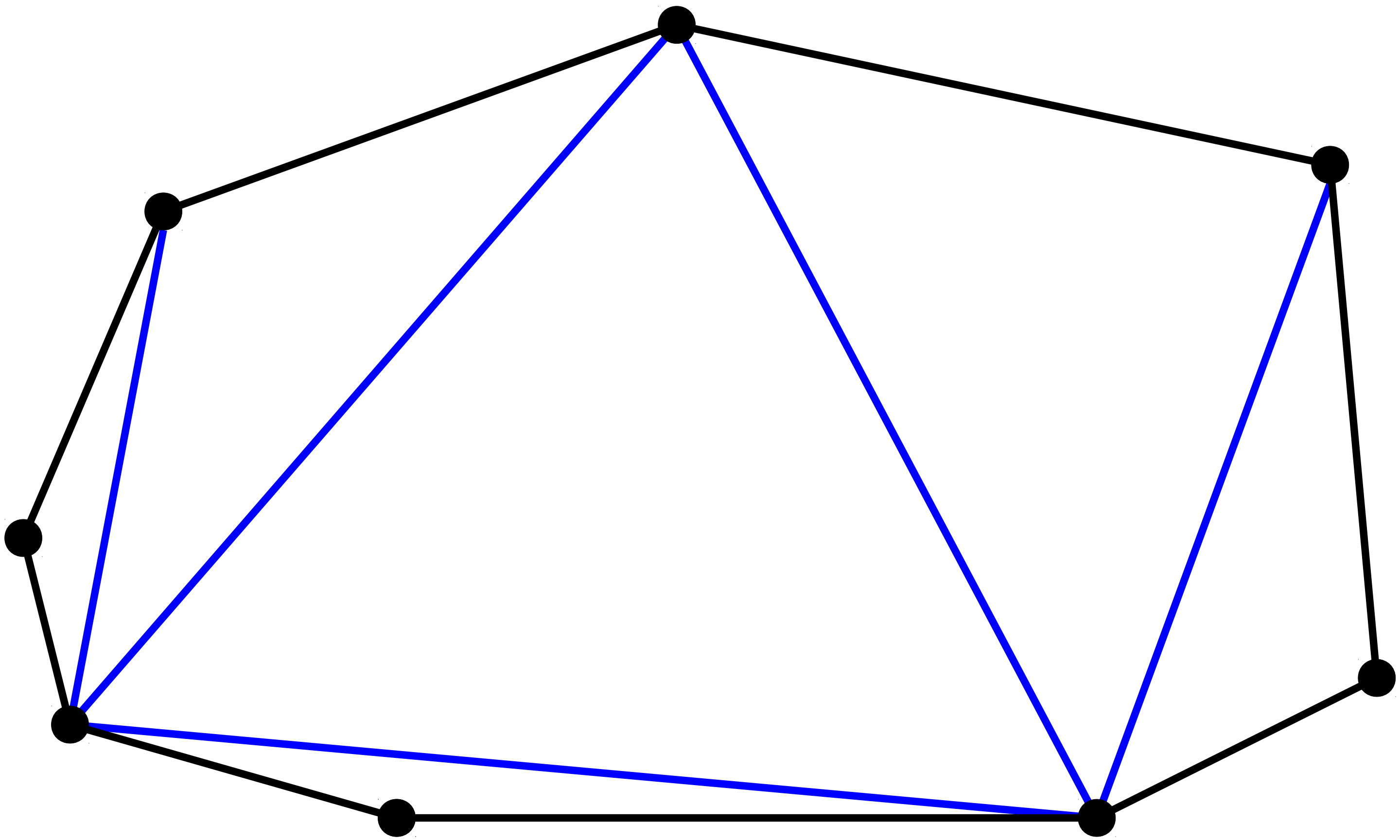


Some Observations

- Every triangulation of the same convex polygon seems to require the same number of lines.
- The number of lines depends on the number of vertices:
  - 5 vertices: 2 lines
  - 6 vertices: 3 lines
  - 8 vertices: 5 lines
- ***Conjecture*:** Every triangulation of an n-vertex convex polygon requires n – 3 lines.

Triangulations



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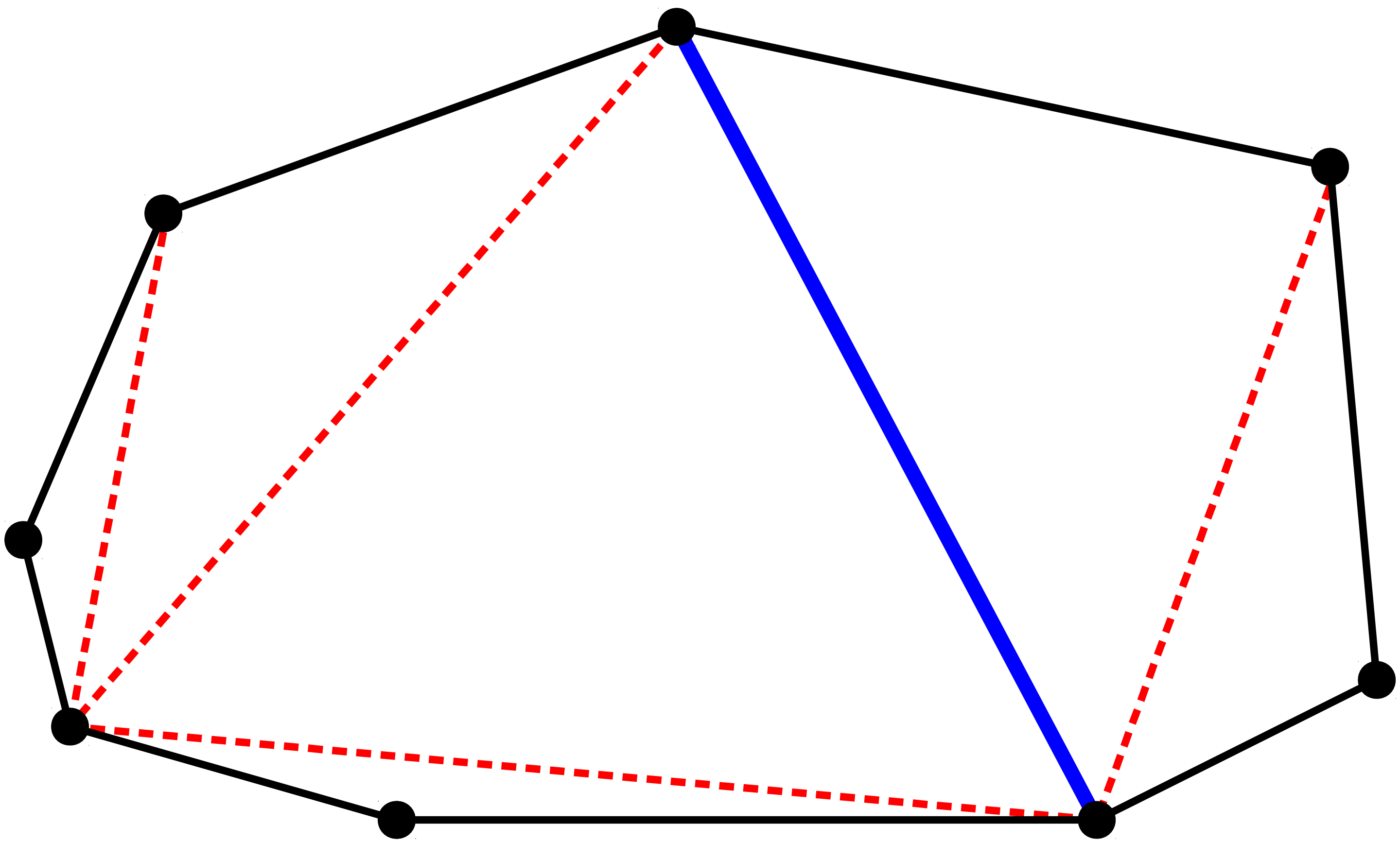

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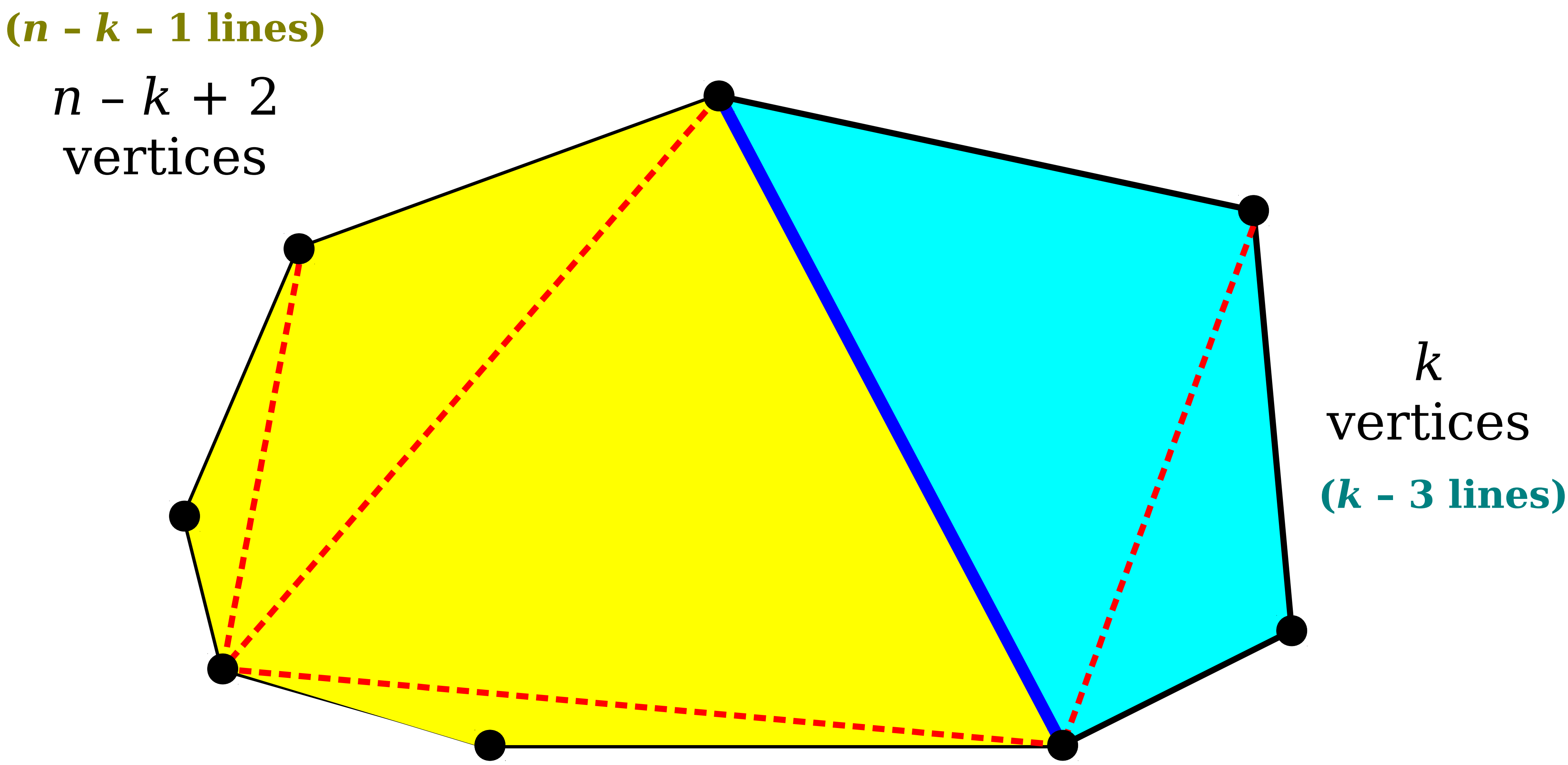

Total lines: $(n - k - 1) + (k - 3) + 1 = \boldsymbol{n - 3}$

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Let k be the number of vertices in B. Since A has $n+1$ vertices and k of them are in B, there are $(n+1)$-k vertices of A that are purely in C. Polygons B and C also share the two vertices connected by the line splitting them, so C has $(n+1)–k + 2 = (n–k) + 3$ vertices. Because B and C have between 3 and n vertices, by our inductive hypothesis any triangulations of B and C must use $k–3$ and $n–k$ lines, respectively. Thus our triangulation has $1 + (k\text{-}3) + (n\text{-}k) = n\text{-}2$ lines, as required.

Theorem: Every triangulation of a convex polygon with $n \geq 3$ vertices requires $n - 3$ lines.

Proof: We will prove this statement by induction on n. As a base case, we will prove that any triangulation of a 3-vertex polygon has 0 lines. Any 3-vertex polygon is a triangle, so no lines are required to triangulate it.

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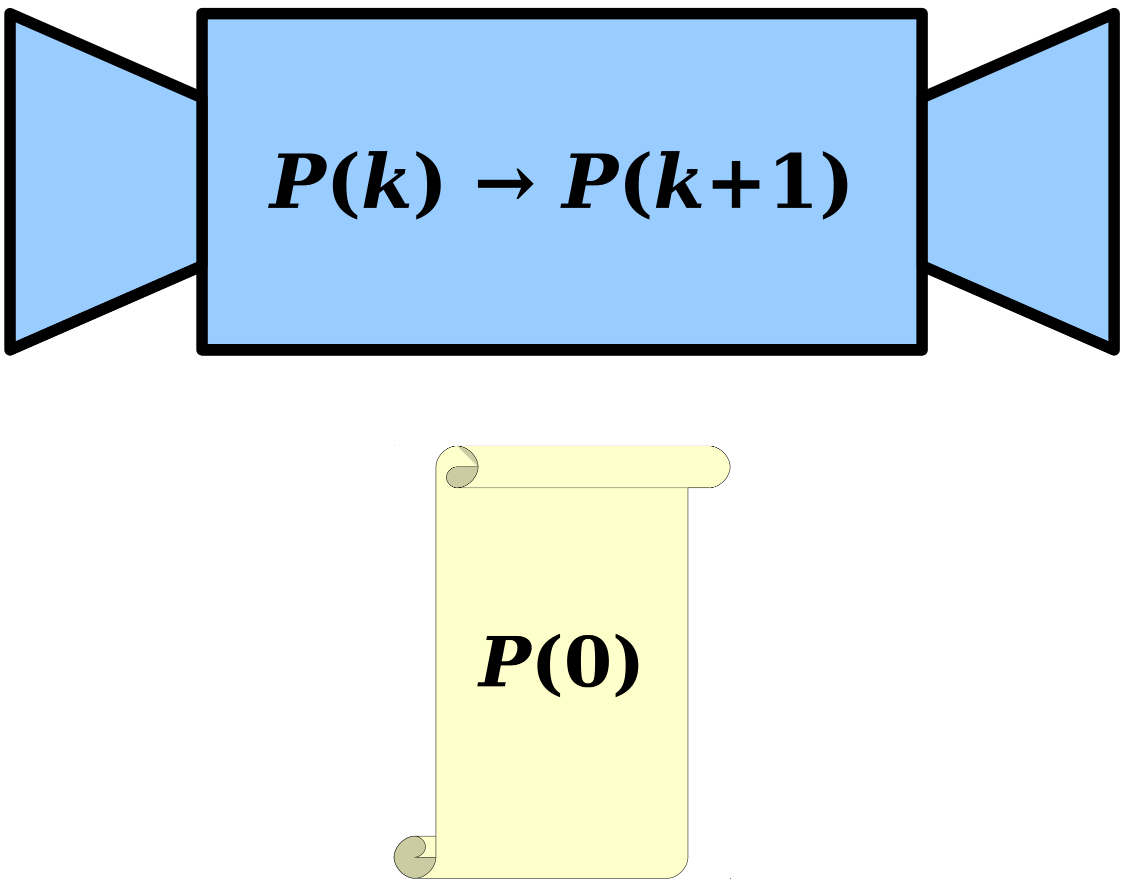
Complete Induction

- If the following are true:
  - $P(0)$ is true, and
  - If $P(0)$, $P(1)$, $P(2)$, …, $P(k)$ are true, then $P(k+1)$ is true as well.

  then $P(n)$ is true for all $n \in \mathbb{N}$.
- This is called the ***principle of complete induction*** or the ***principle of strong induction***.
  - (This also works starting from a number other than 0; just modify what you're assuming appropriately.)

Review: Induction as a Machine



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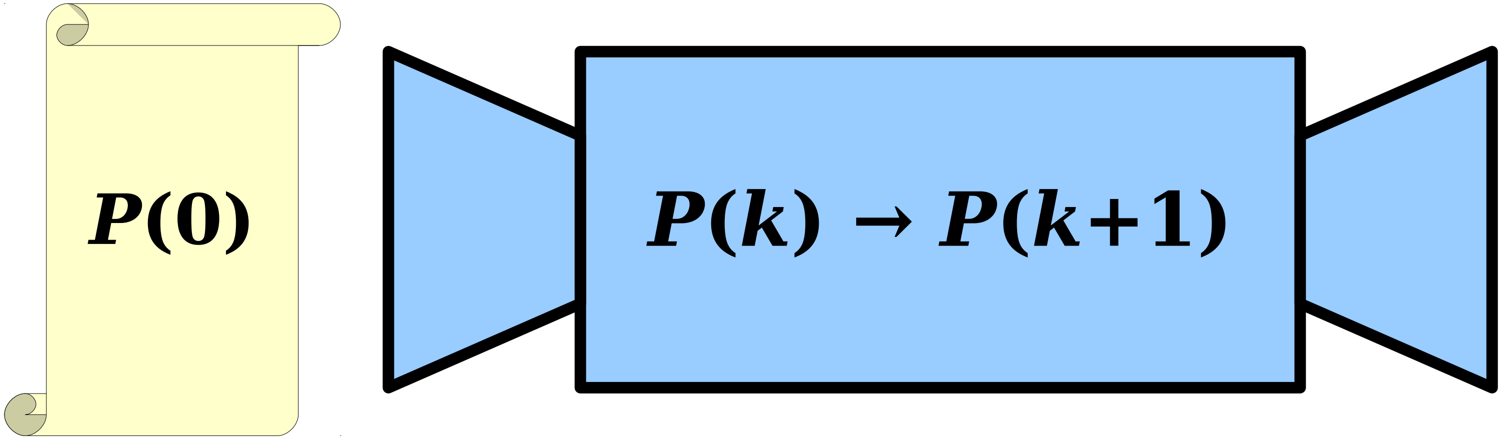

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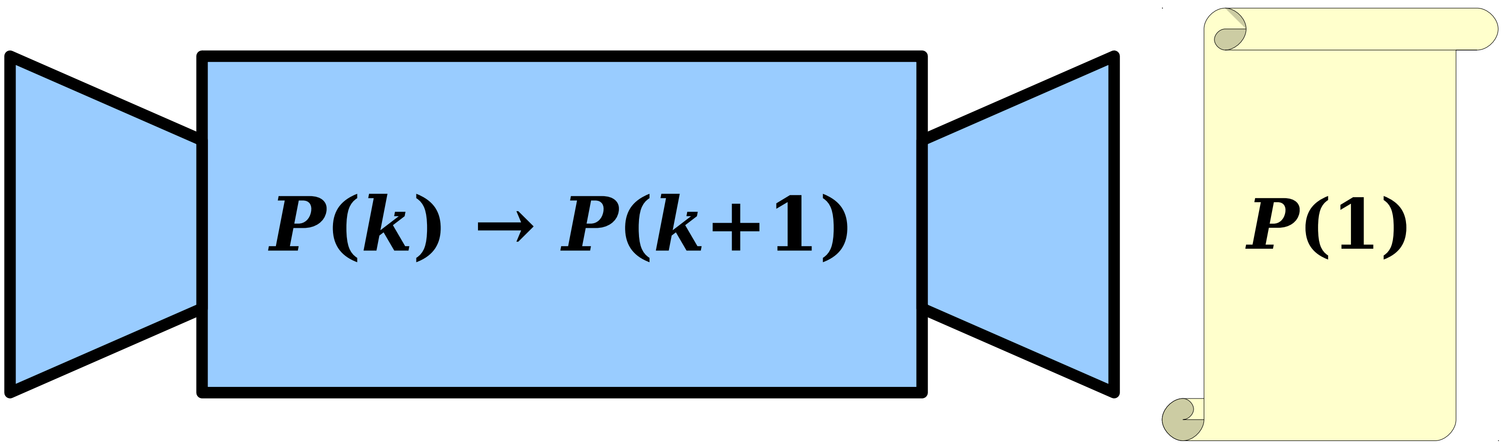

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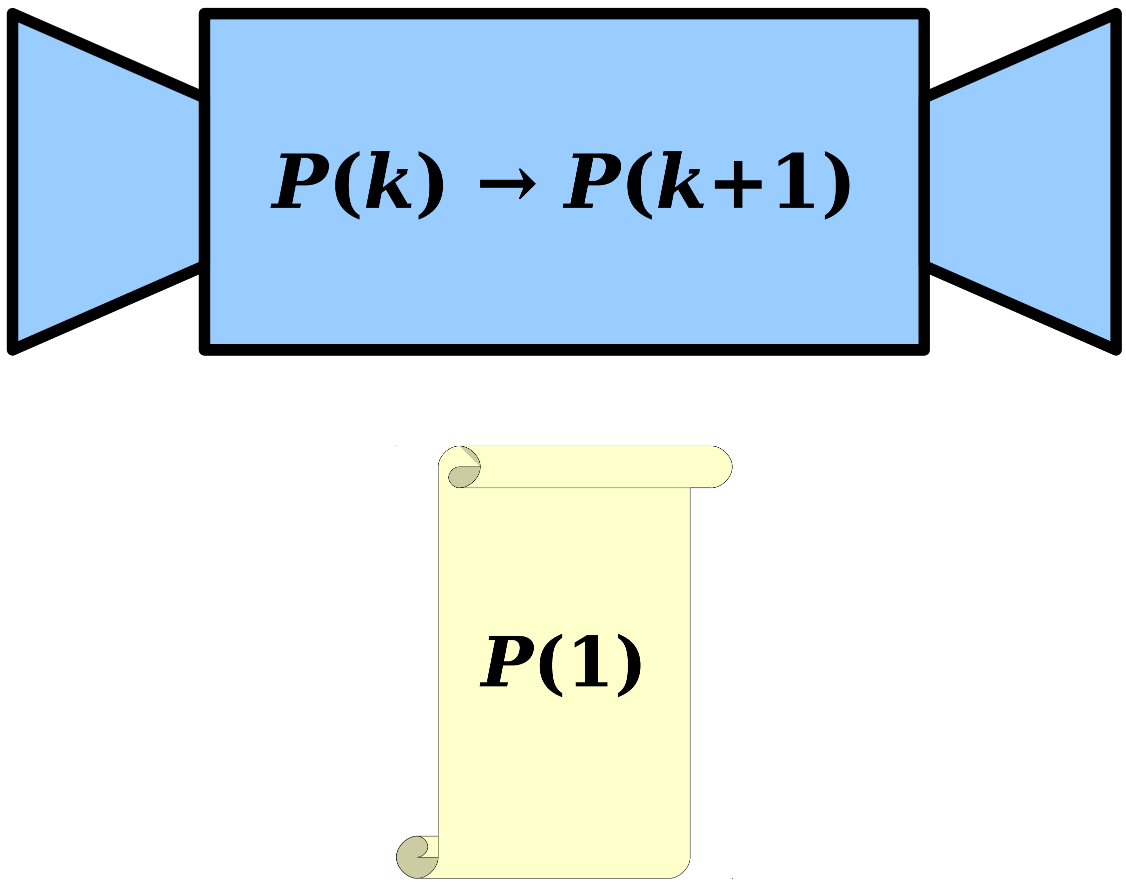
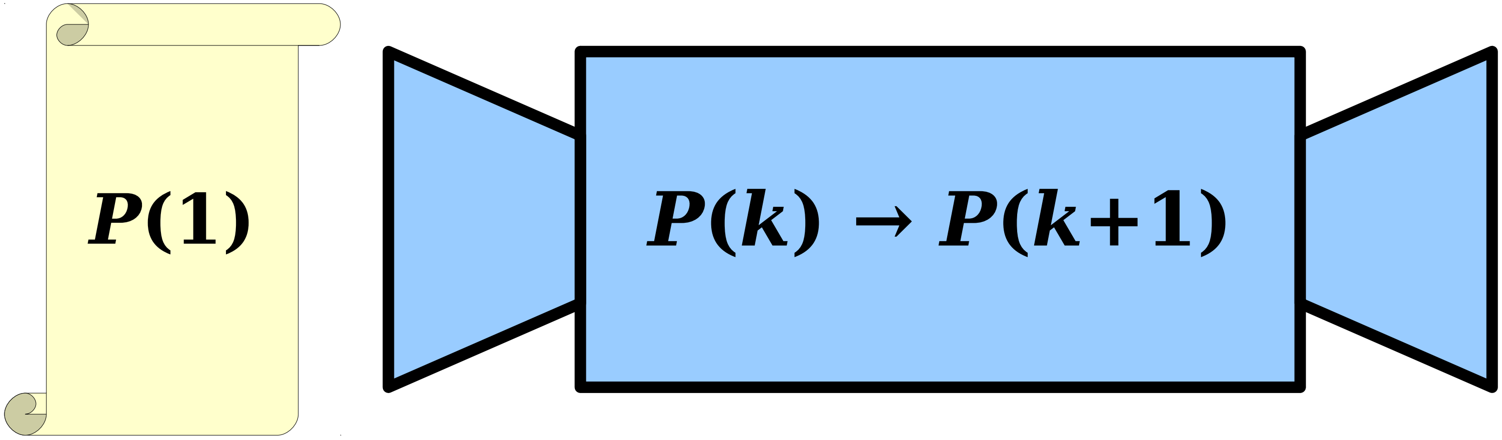


Review: Induction as a Machine



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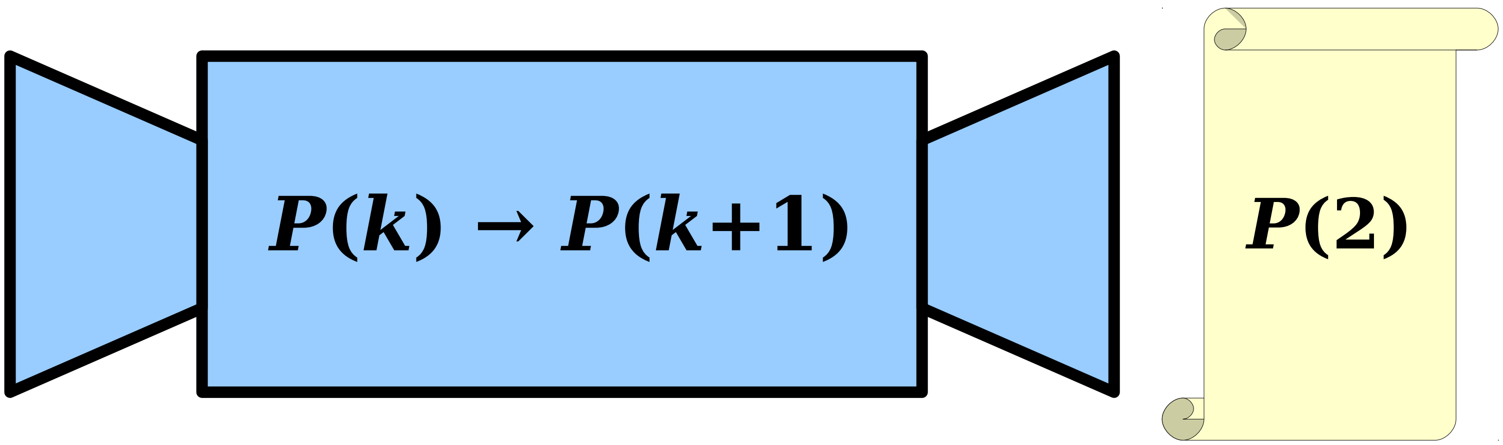

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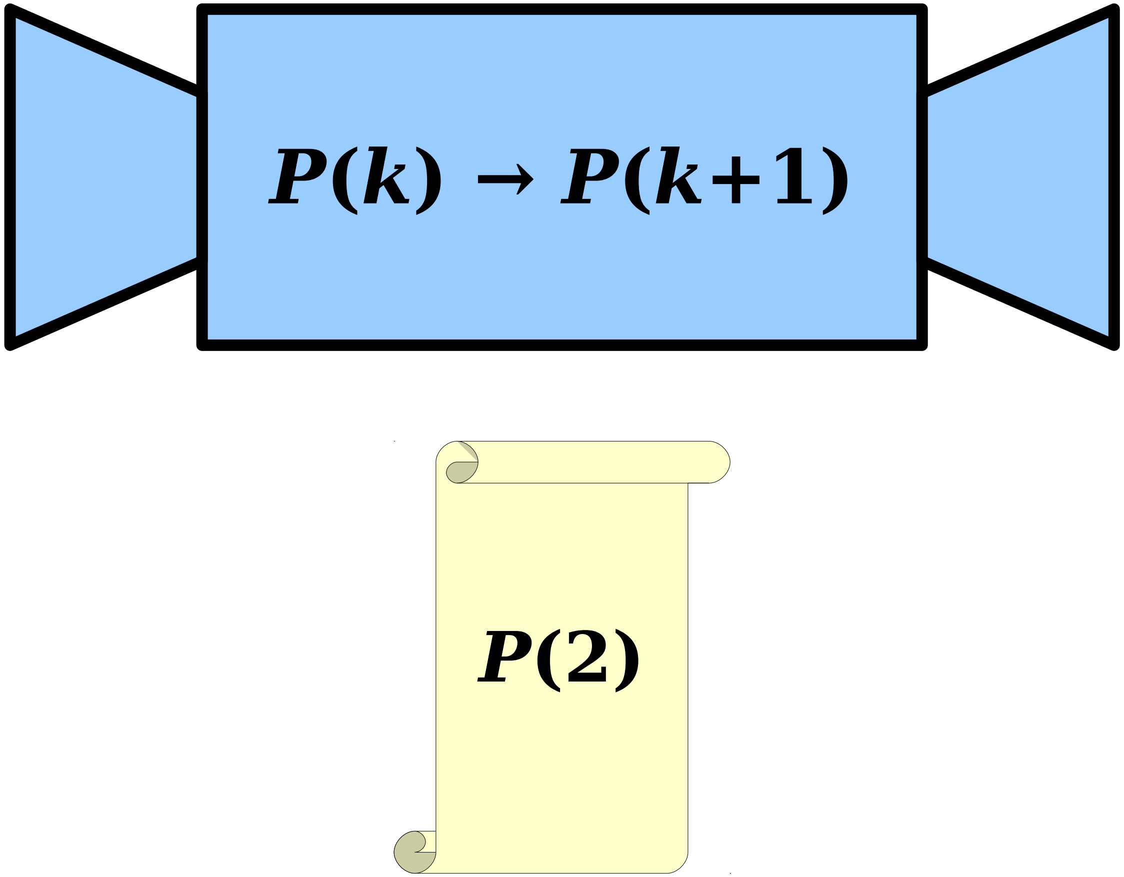

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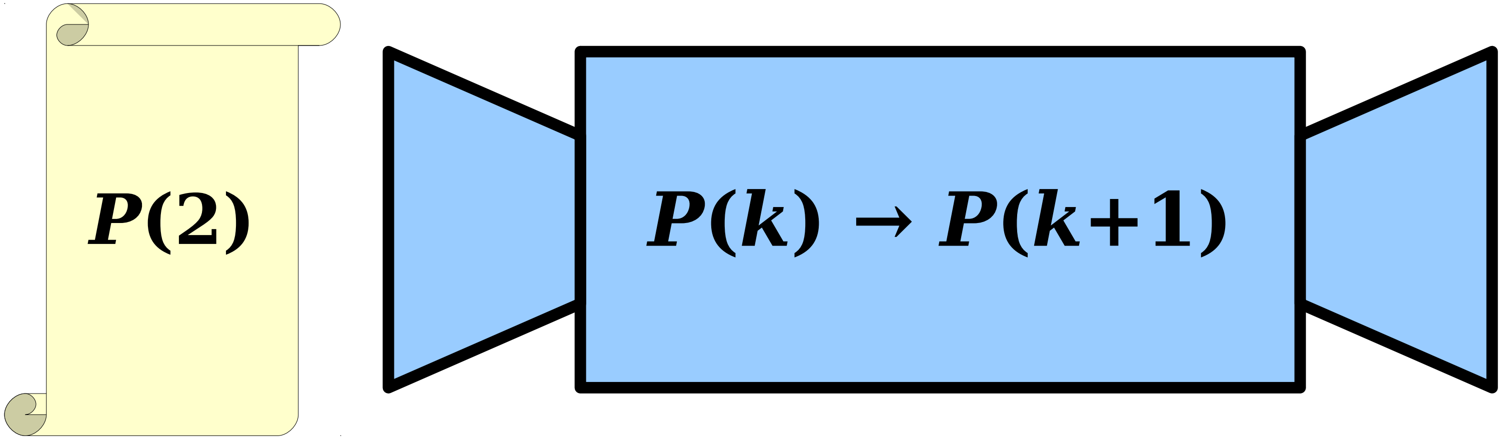

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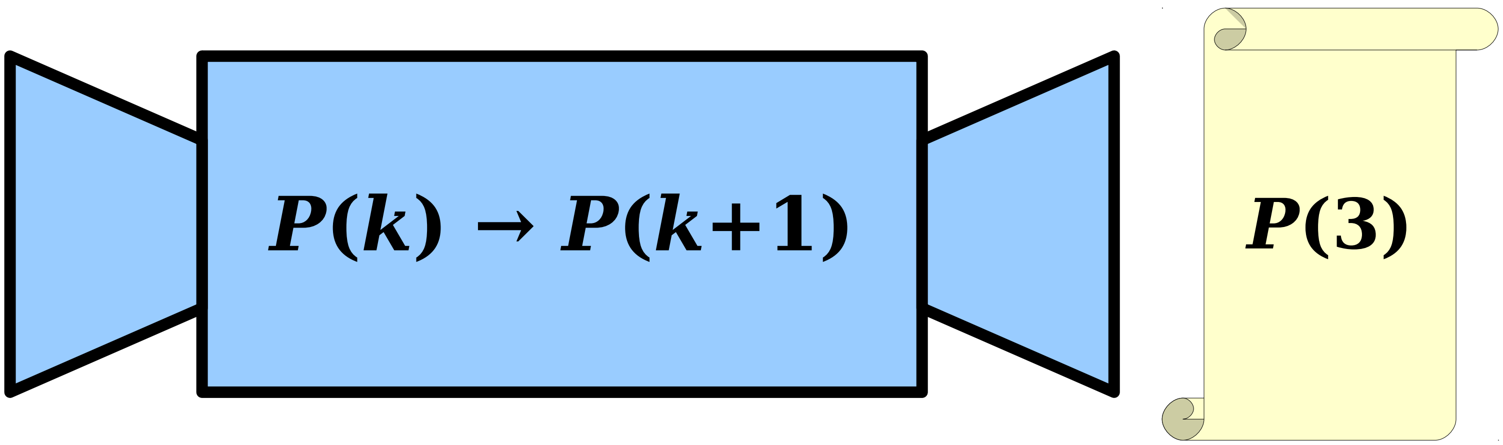

An Observation

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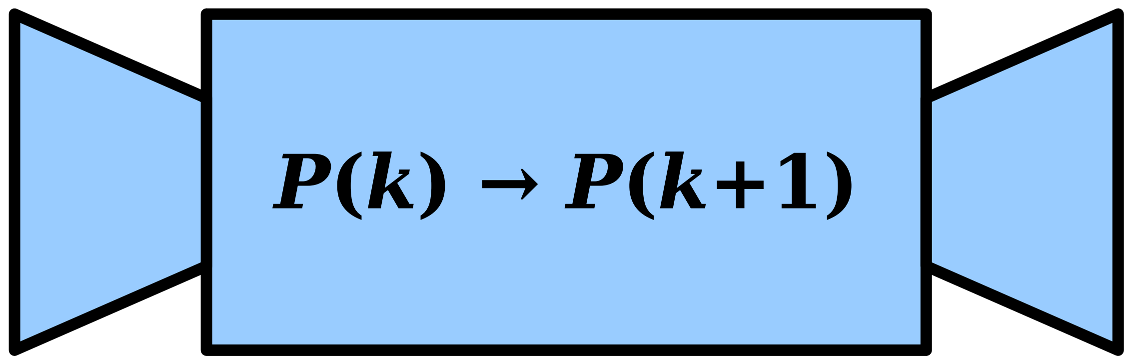

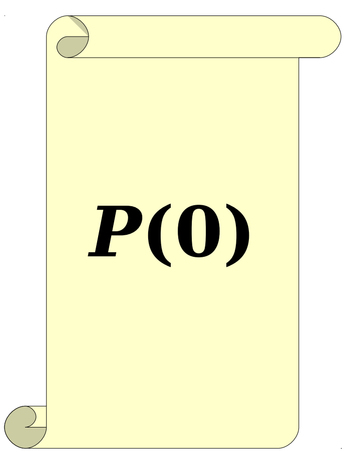

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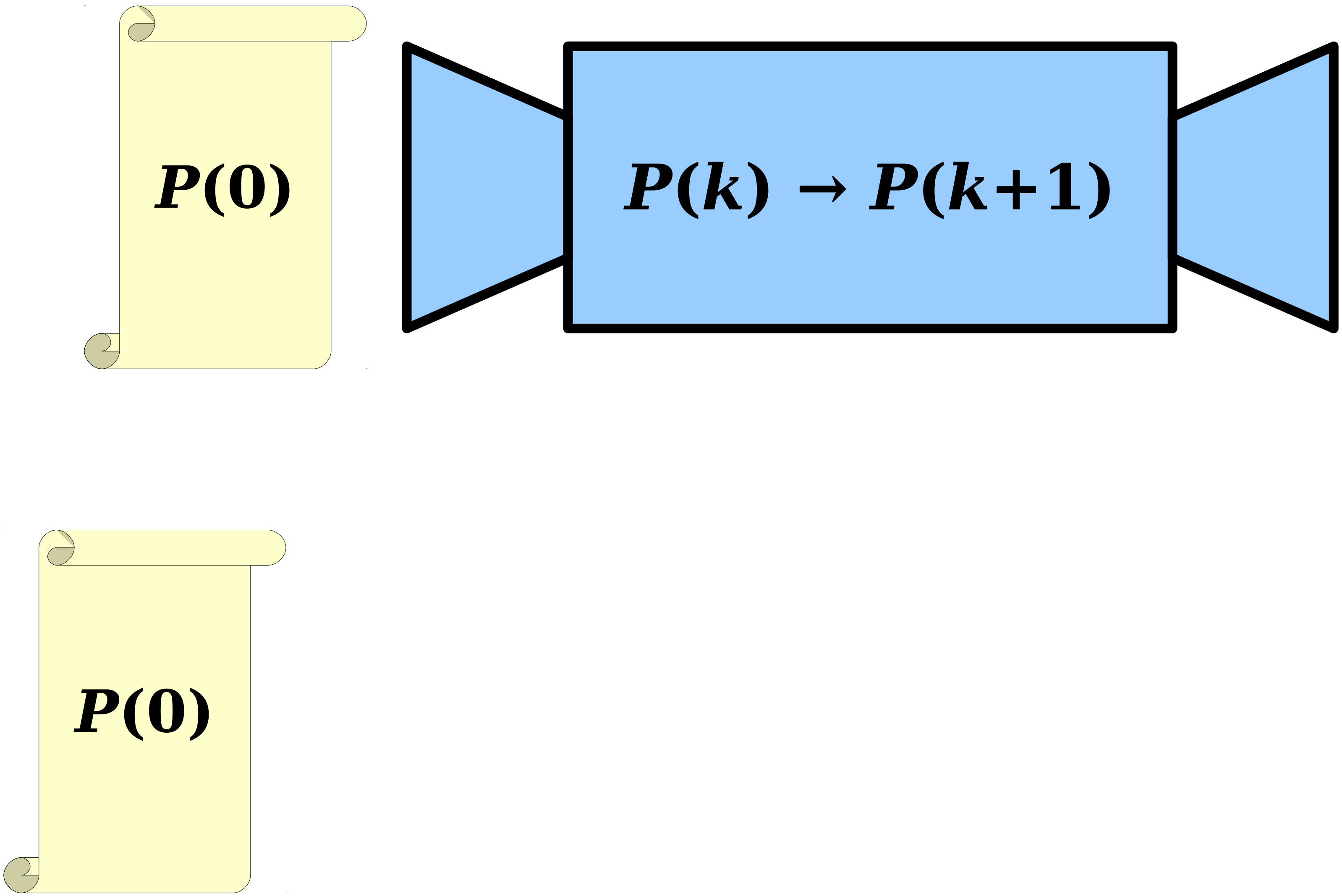

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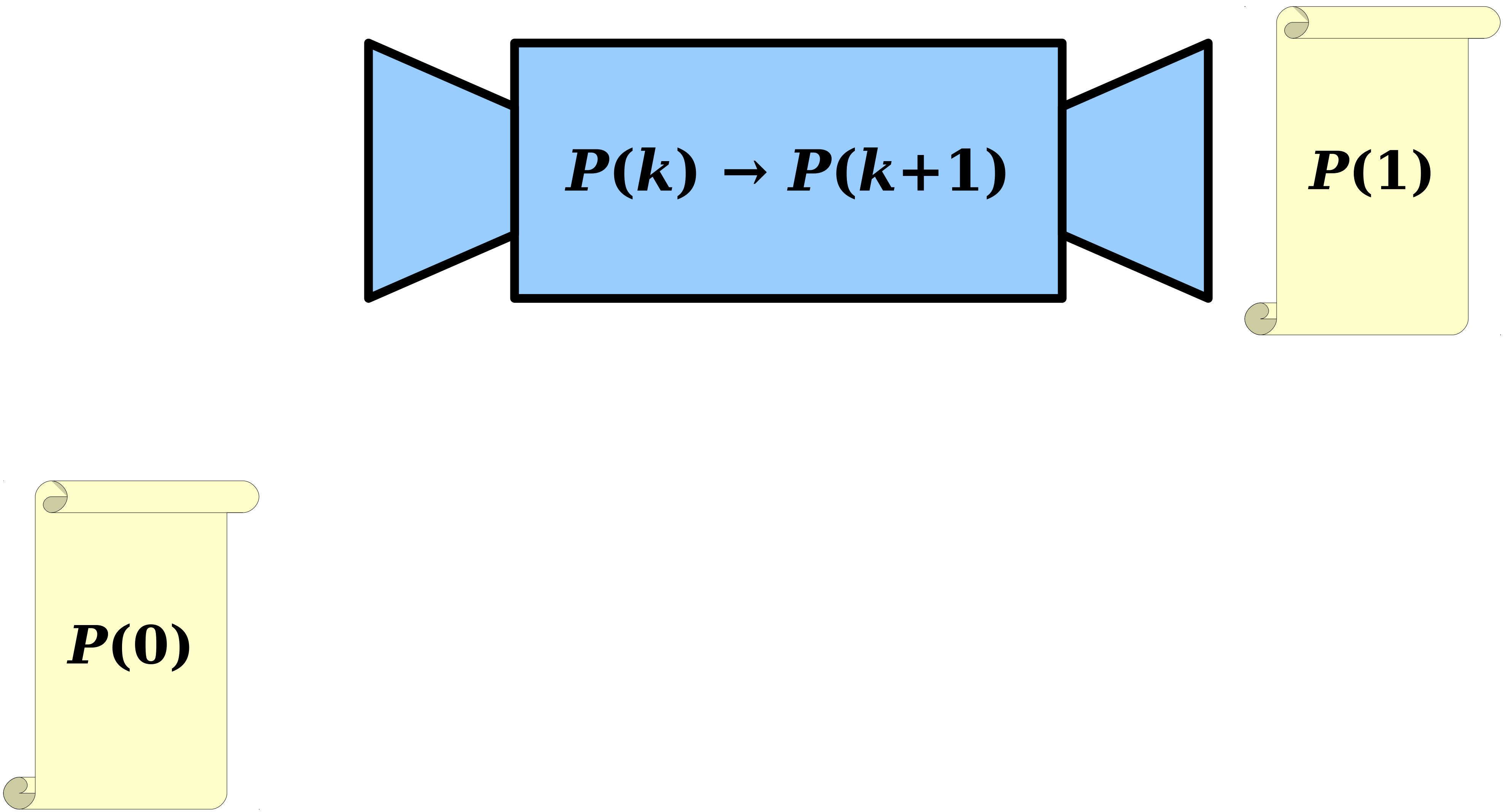

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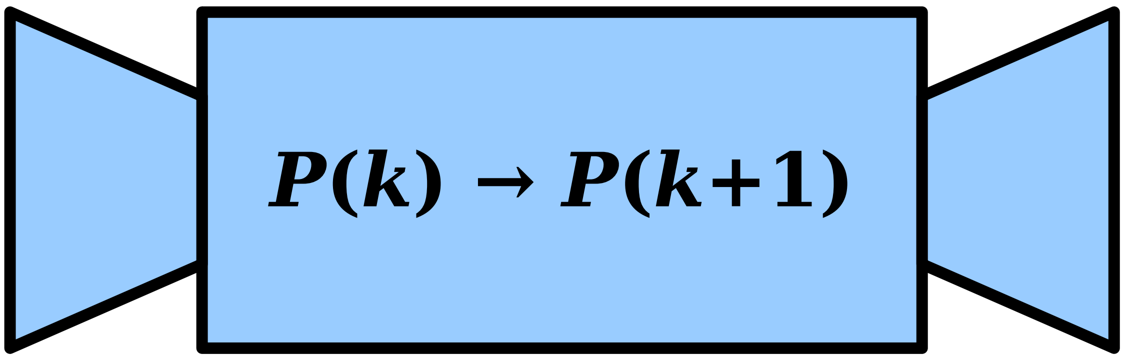

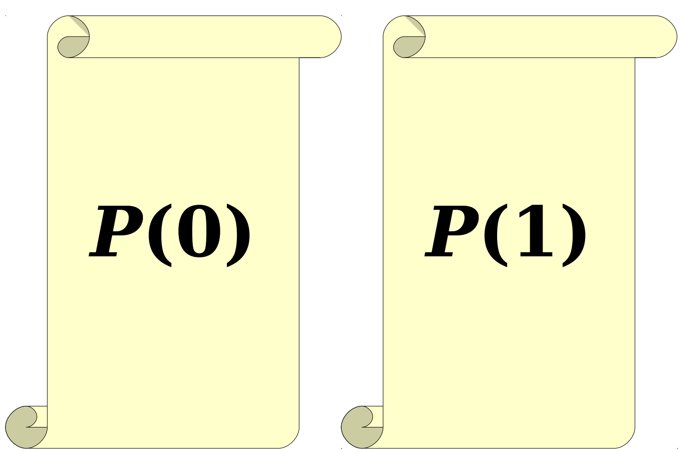

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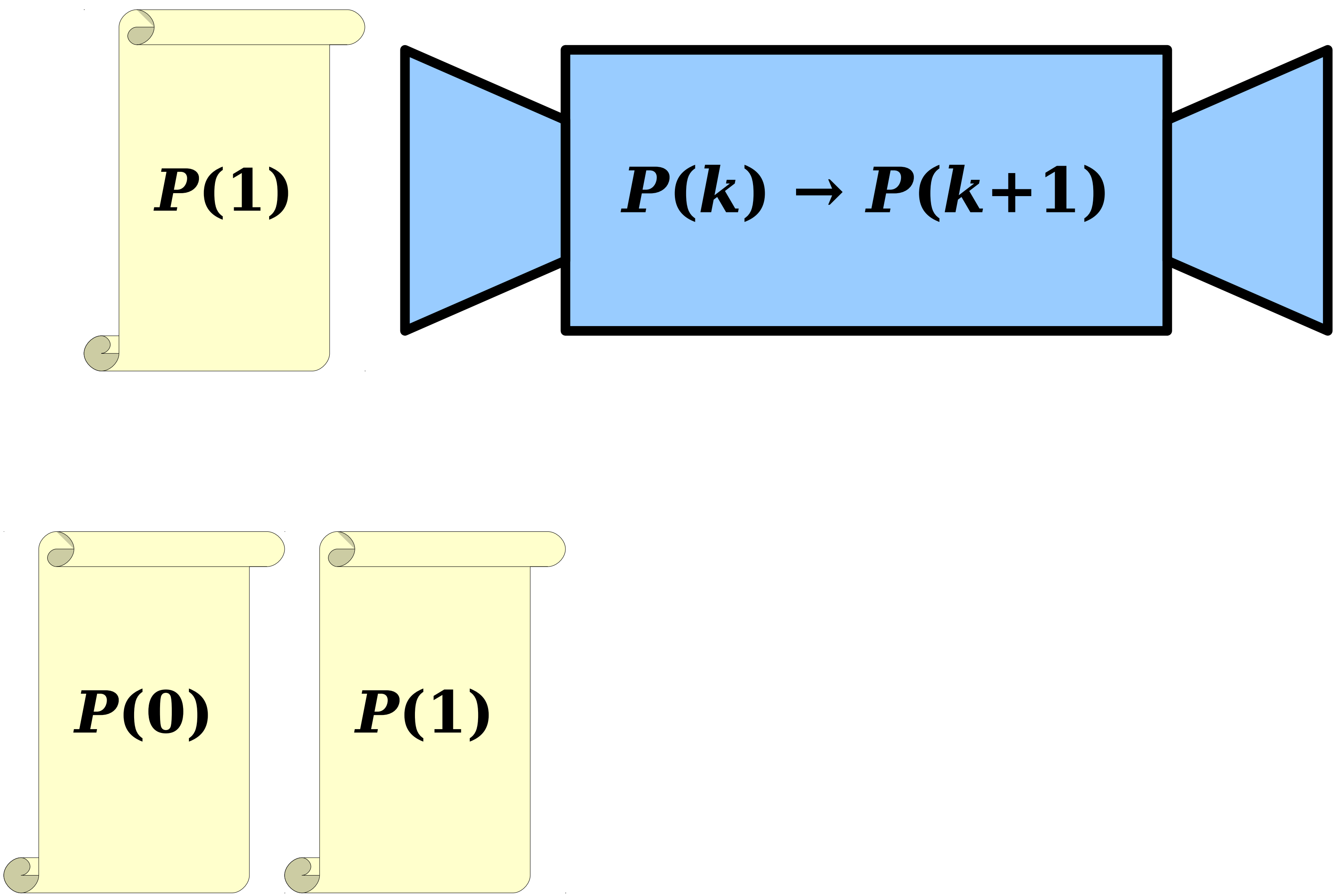

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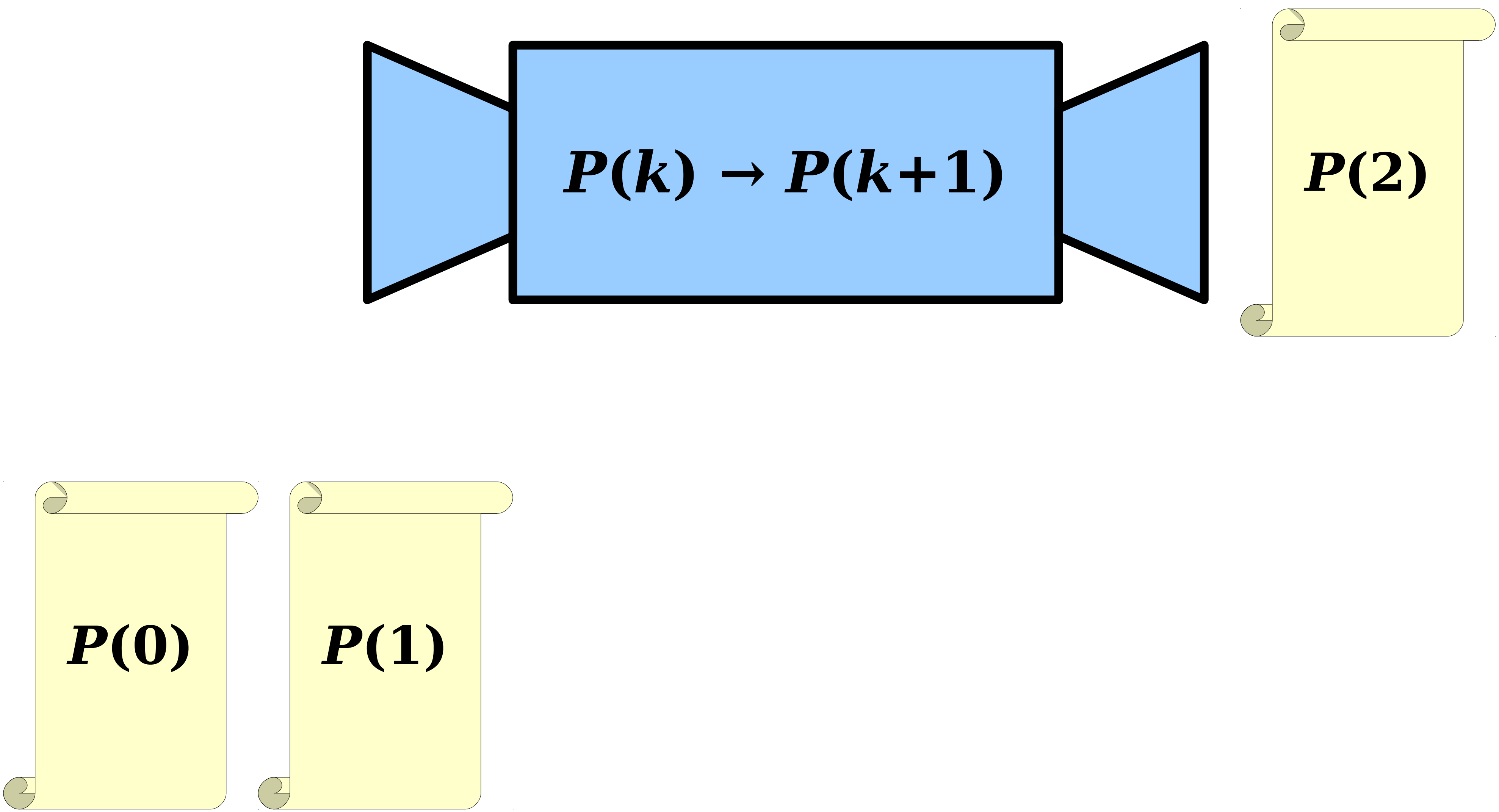

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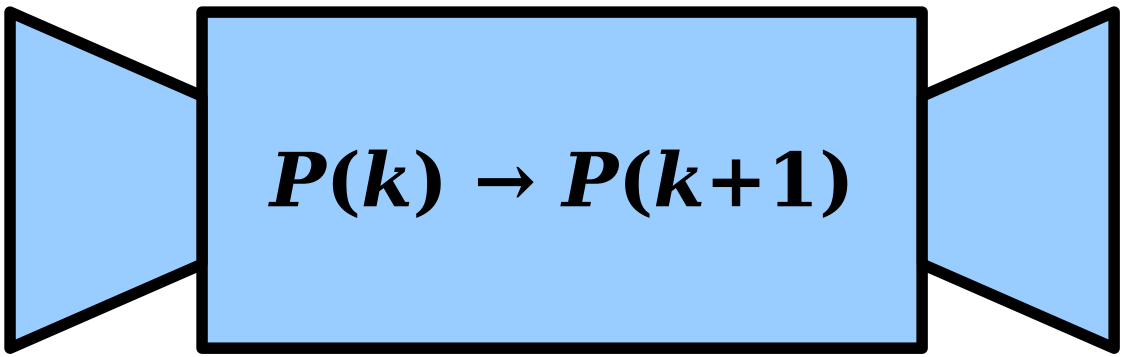

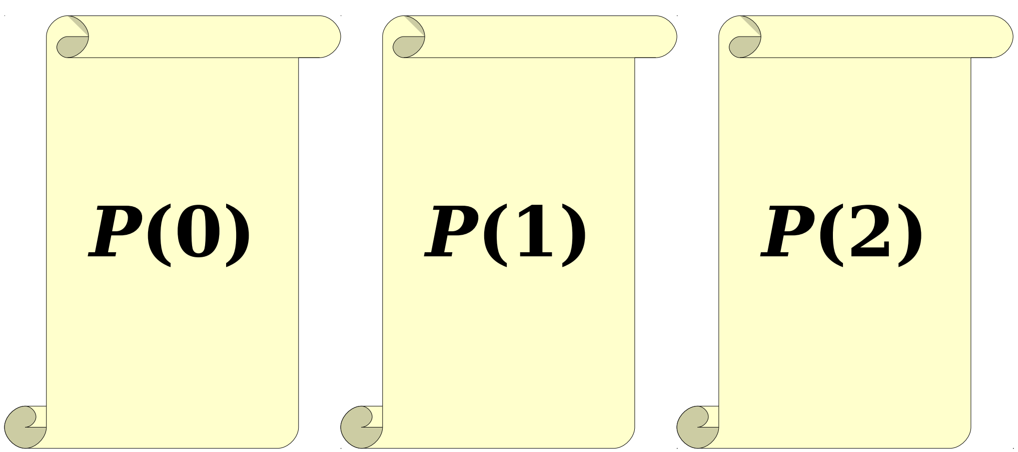

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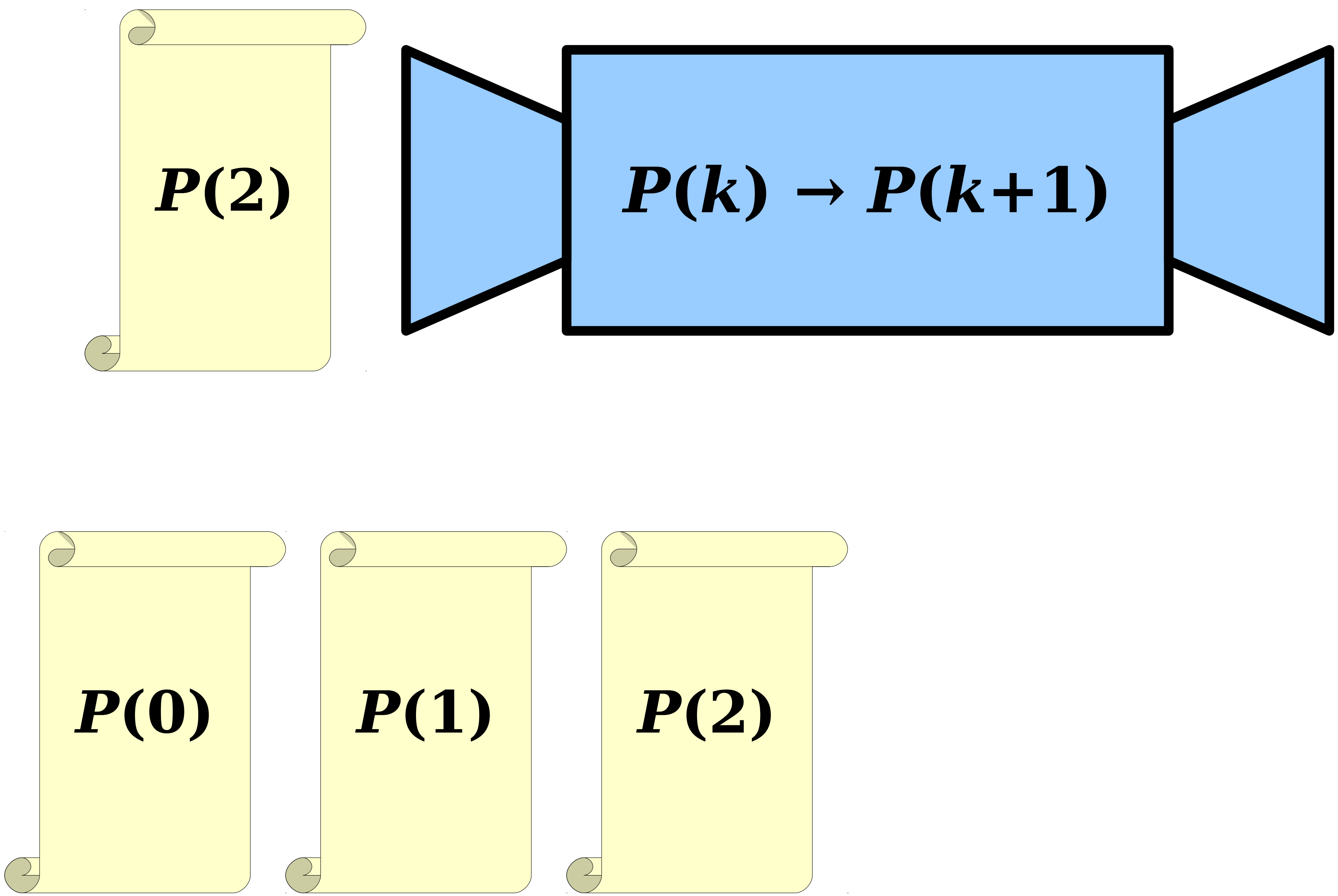

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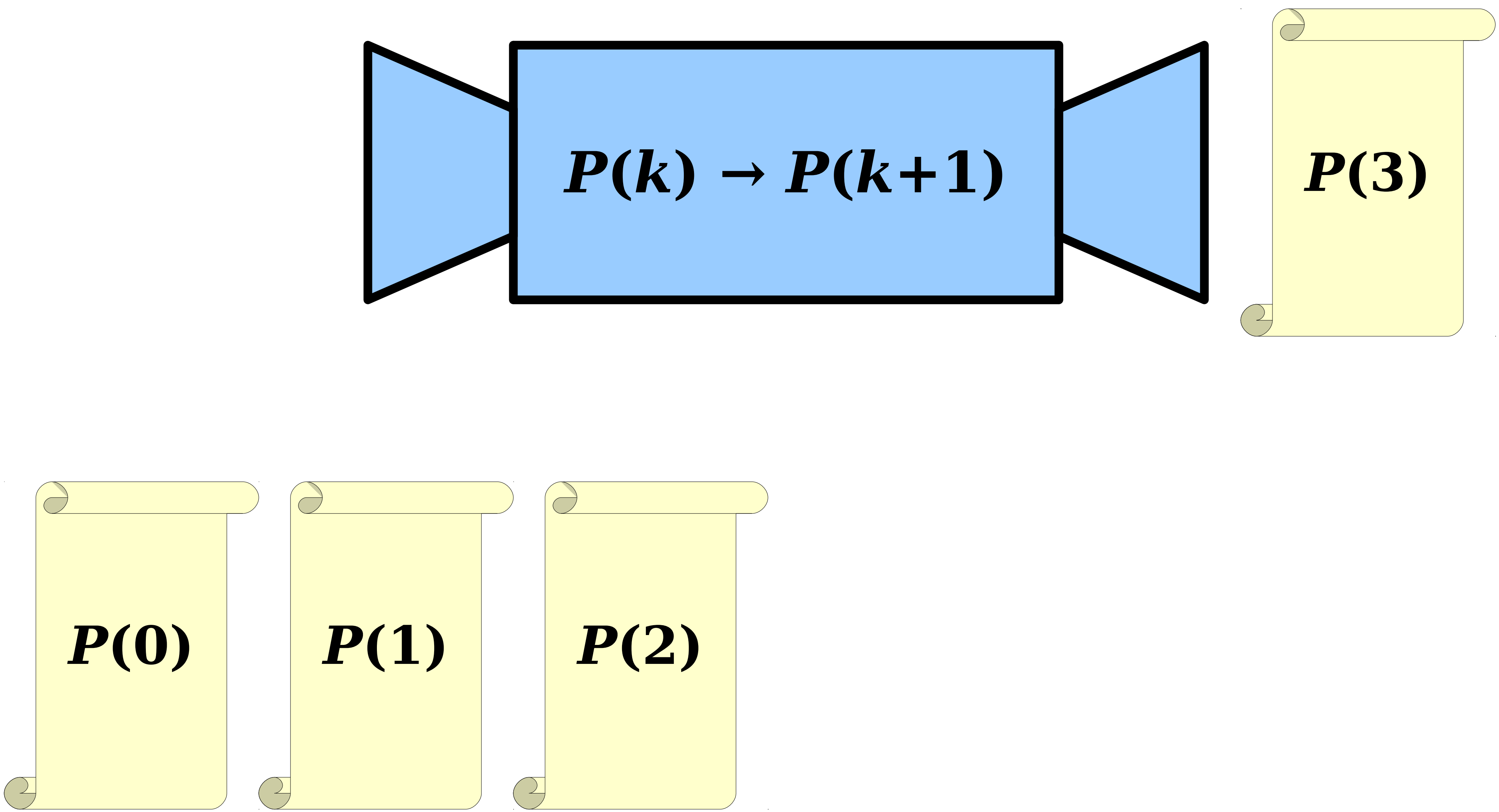

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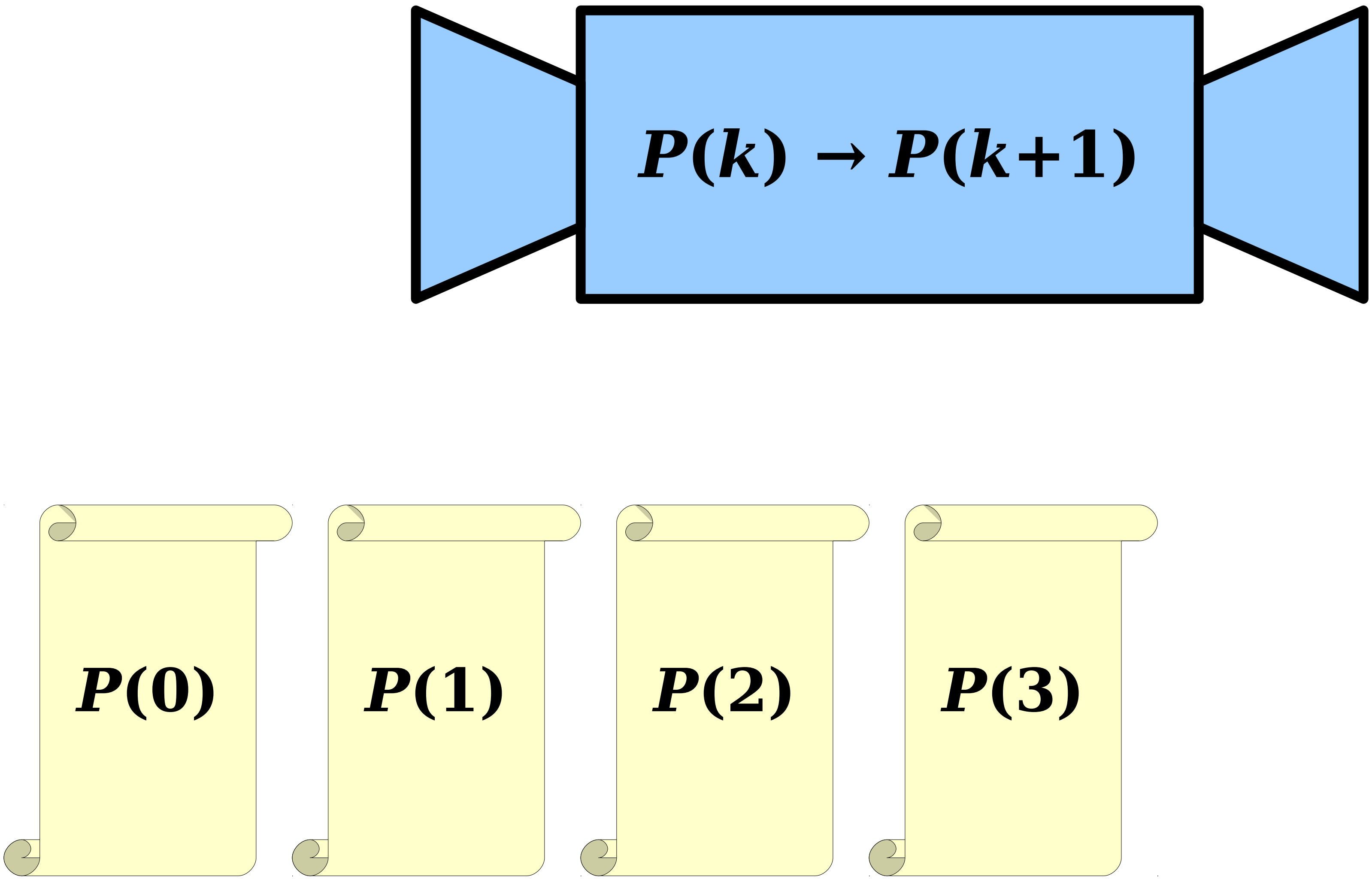

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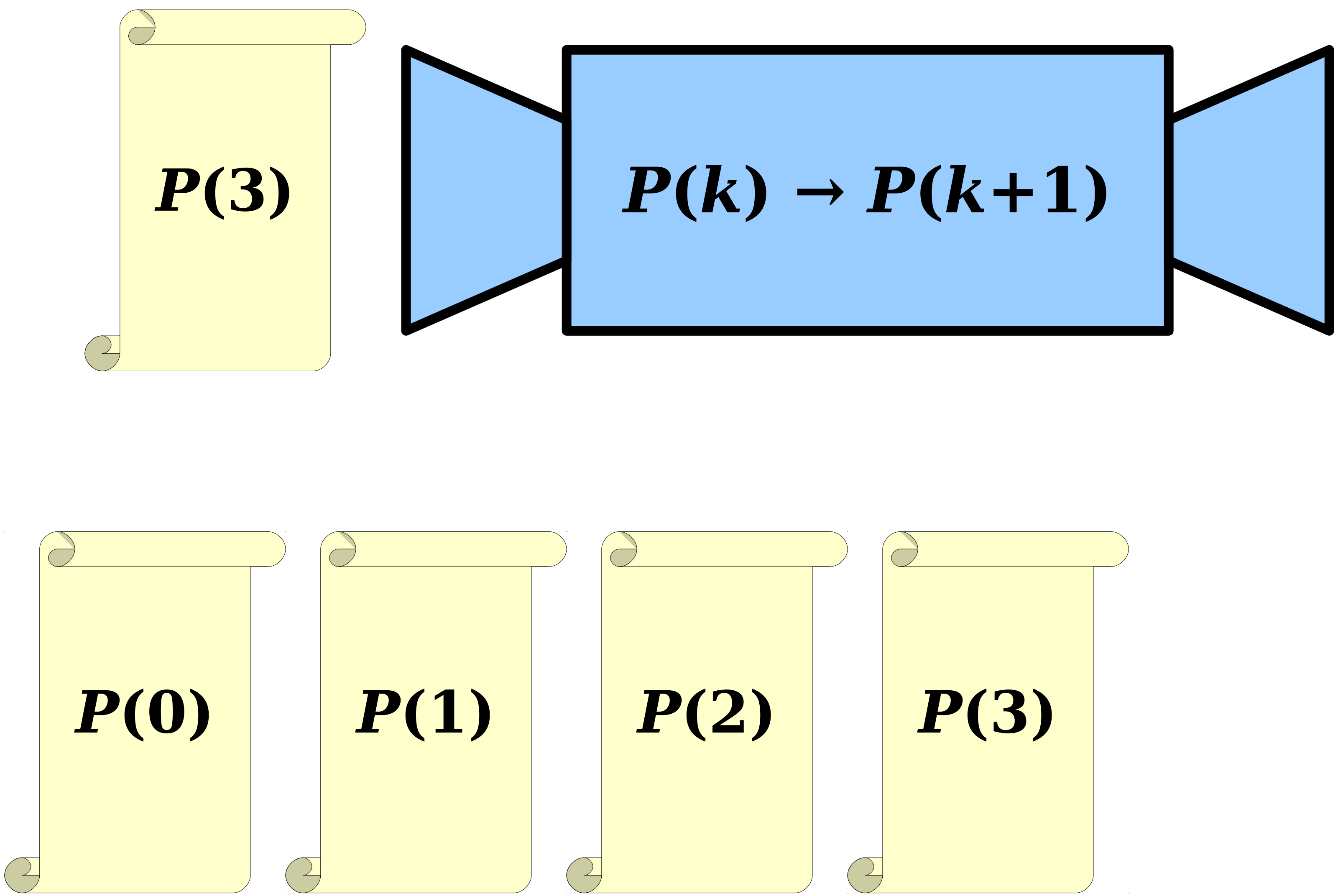

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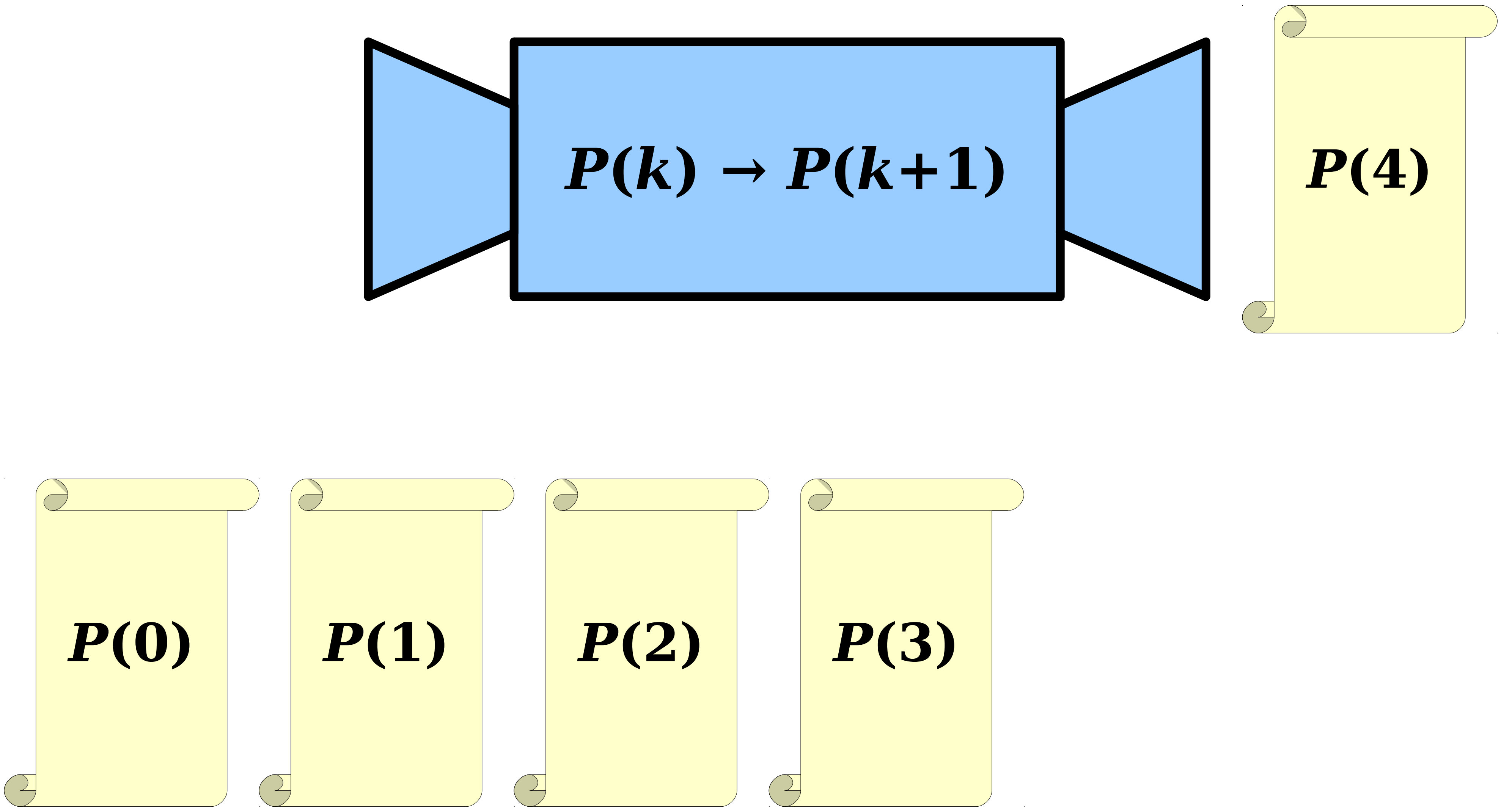

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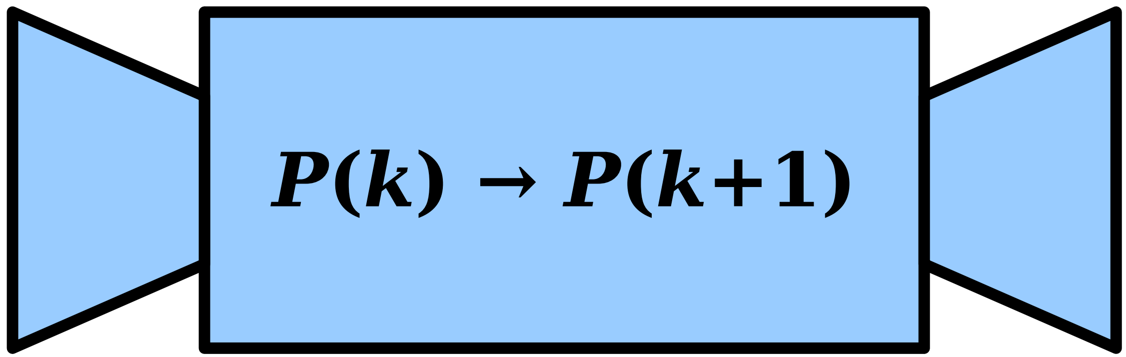

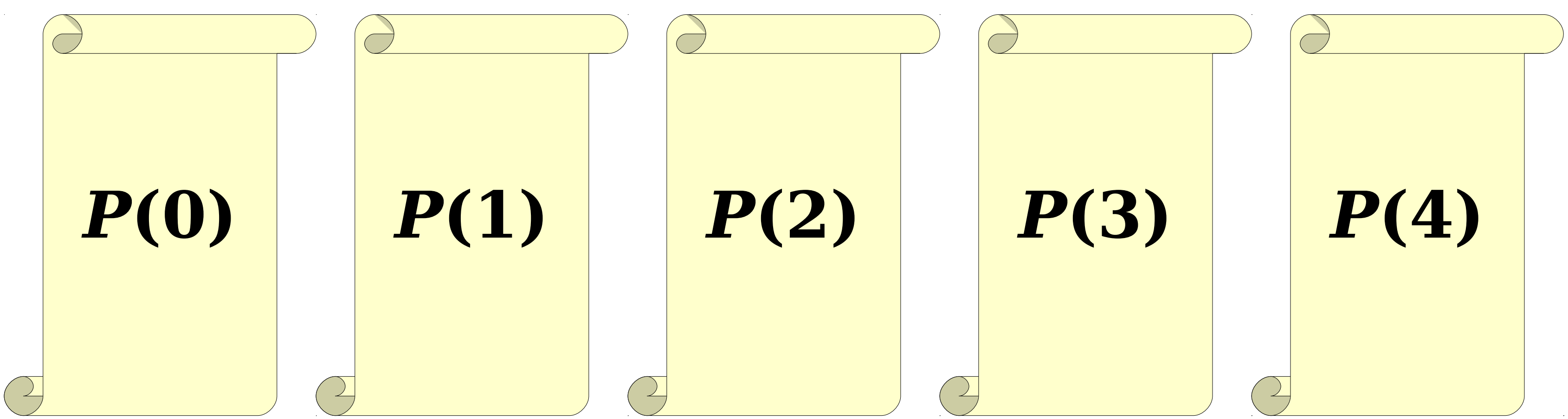

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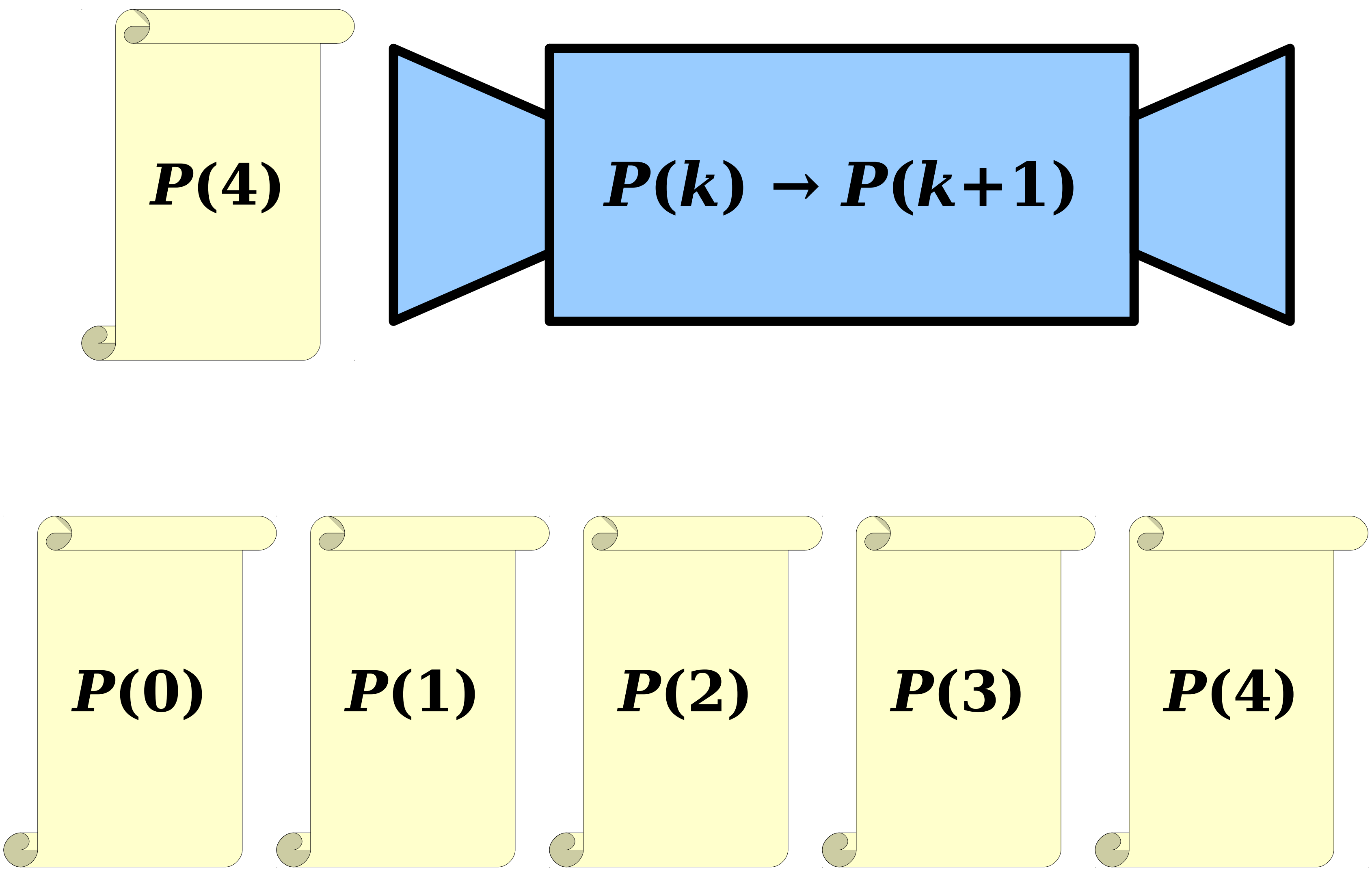

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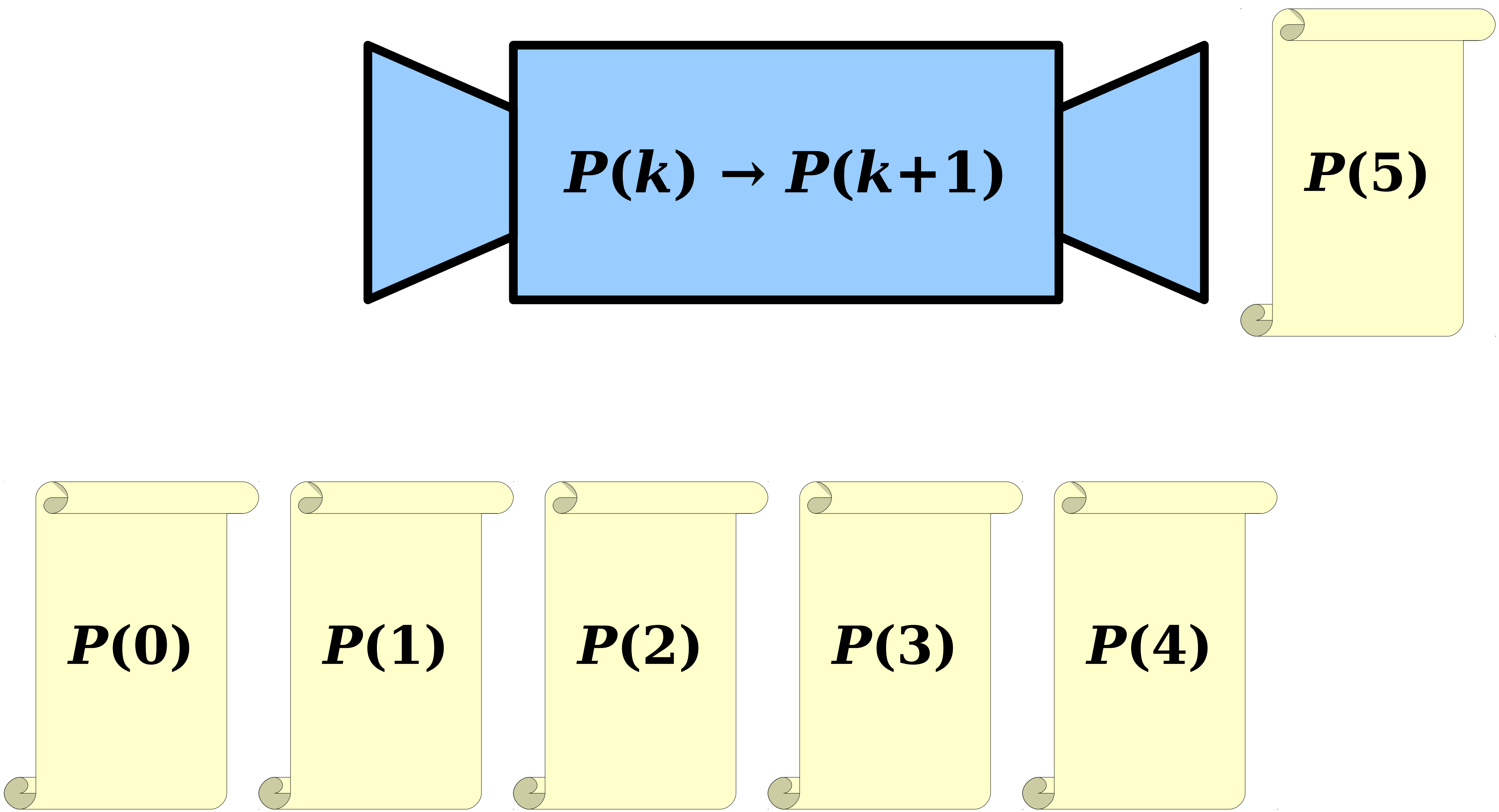

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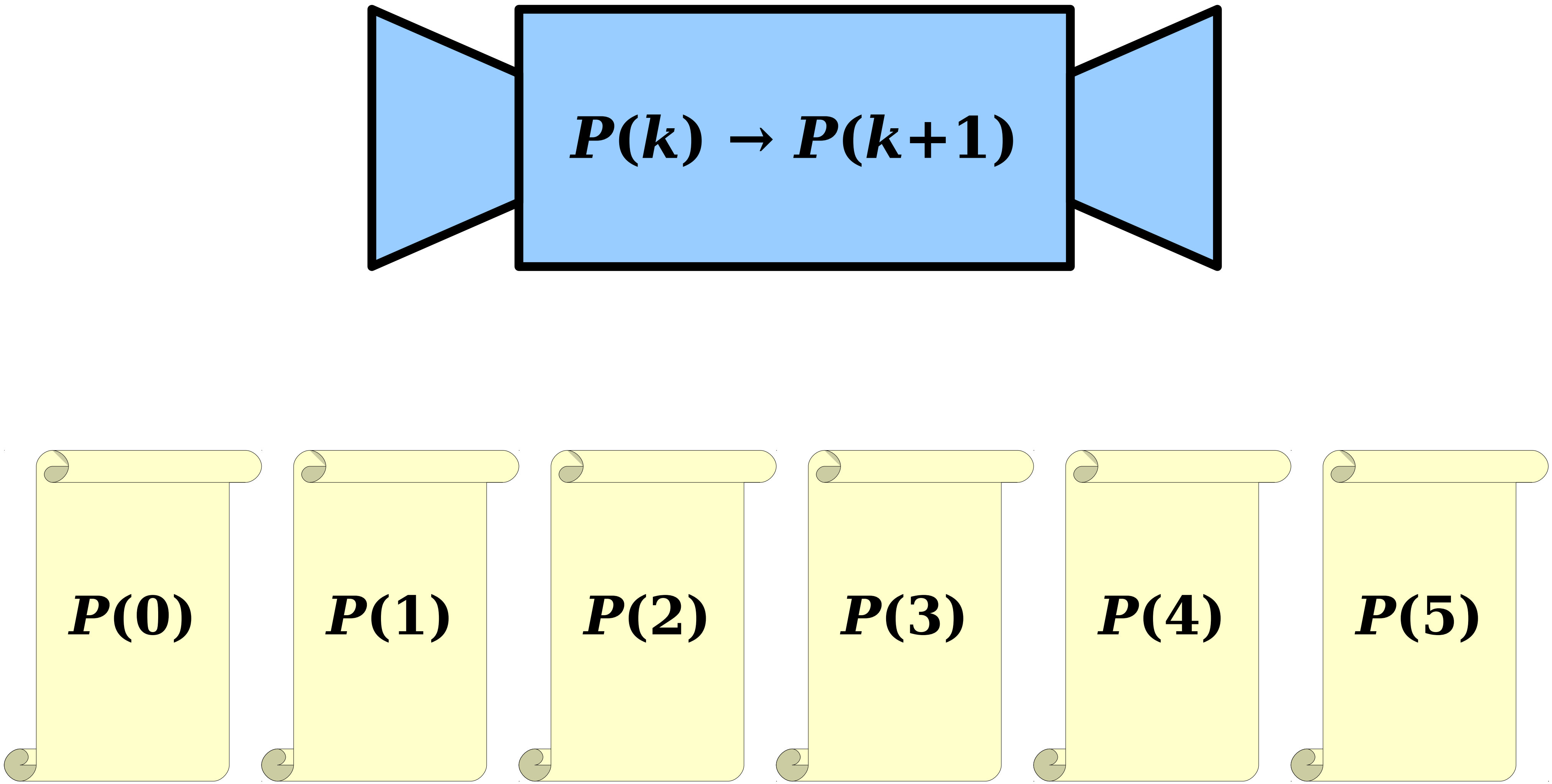
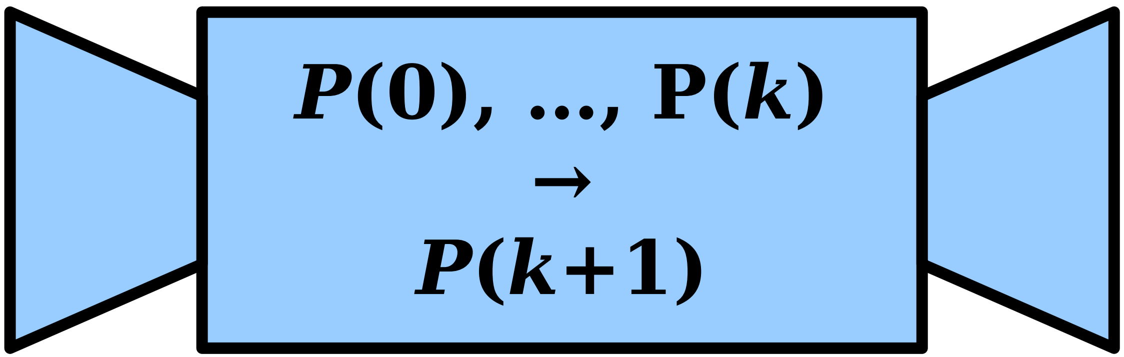
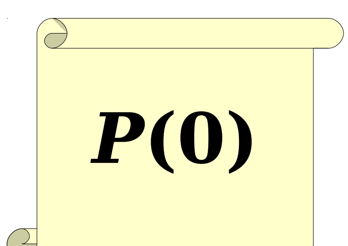


Intuiting Complete Induction





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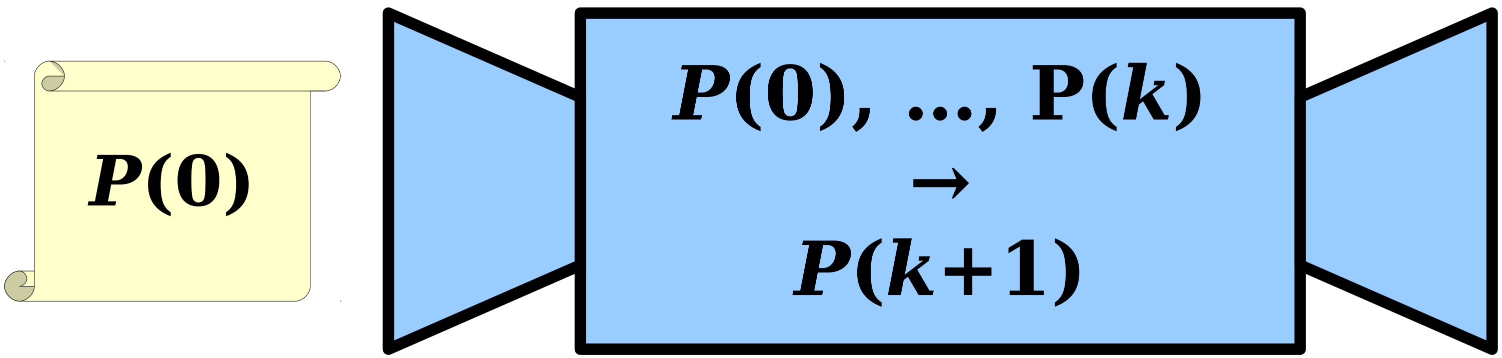

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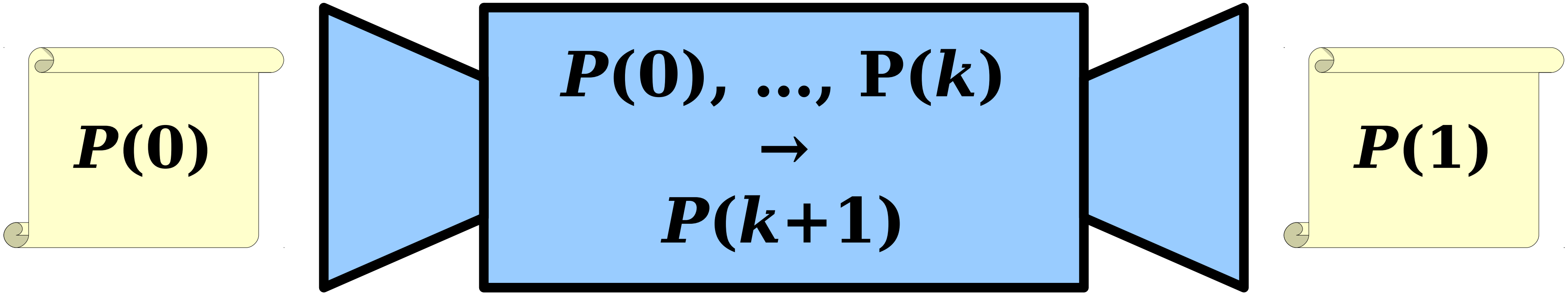

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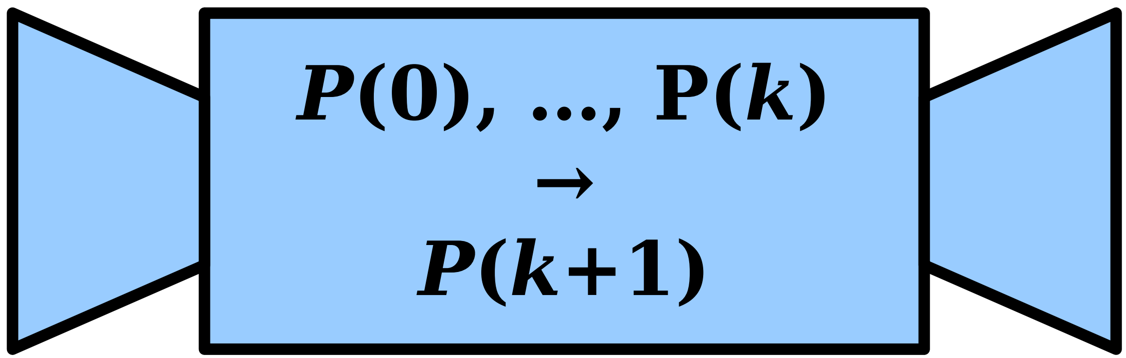

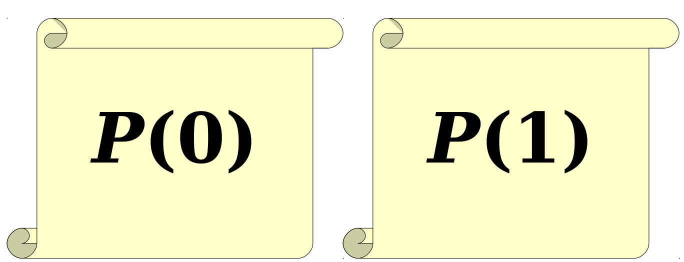

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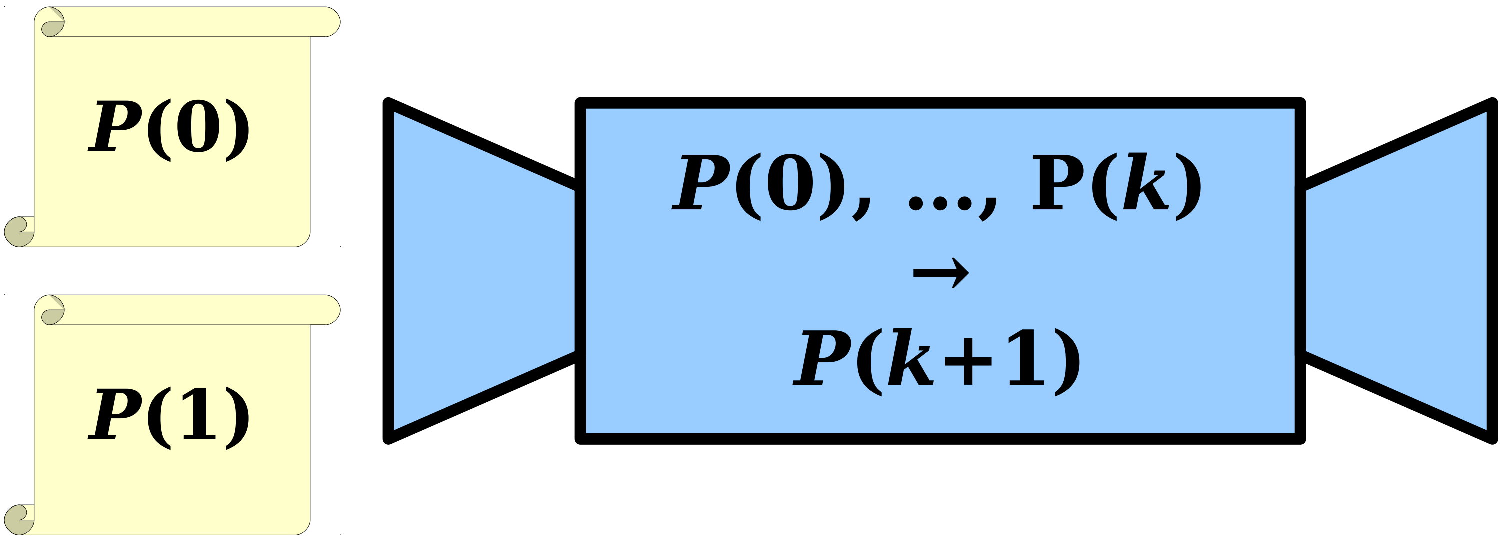

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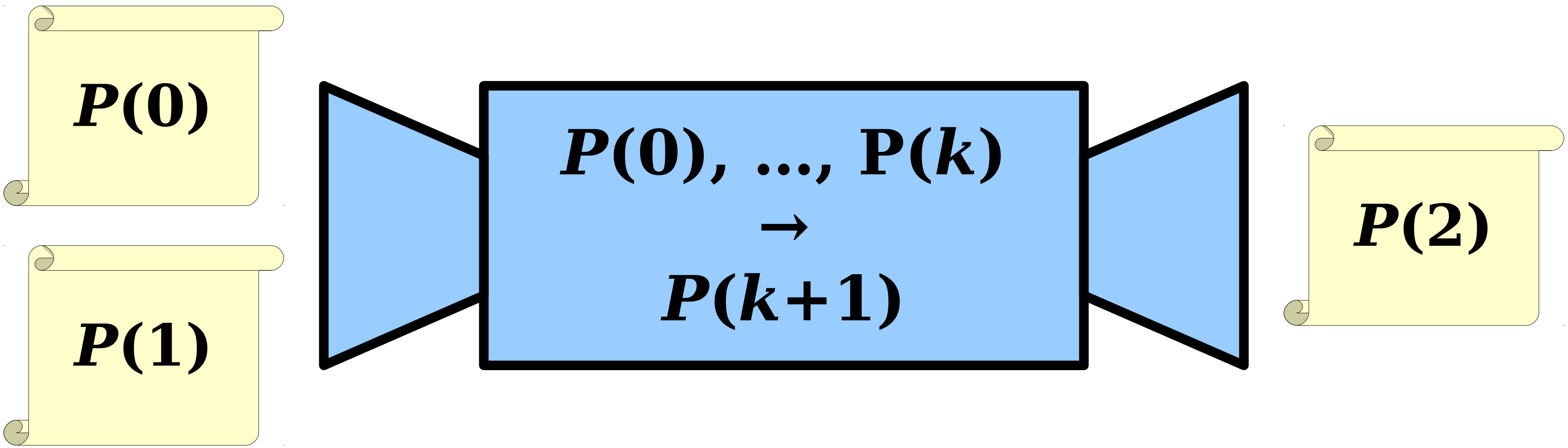

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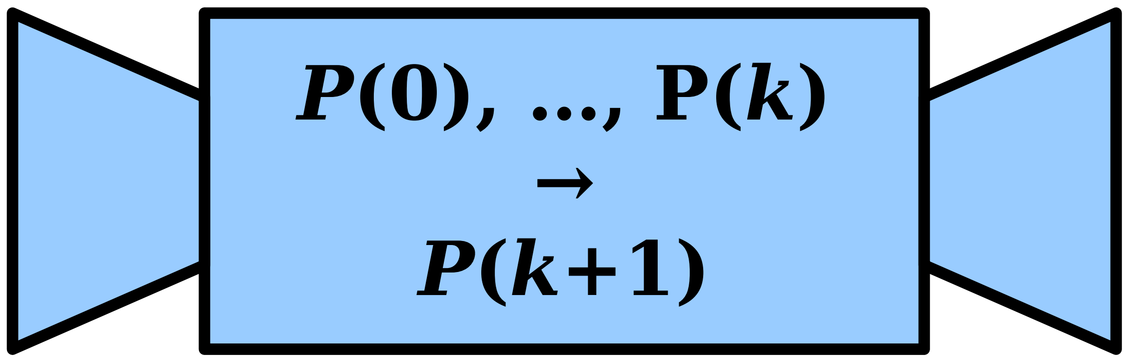

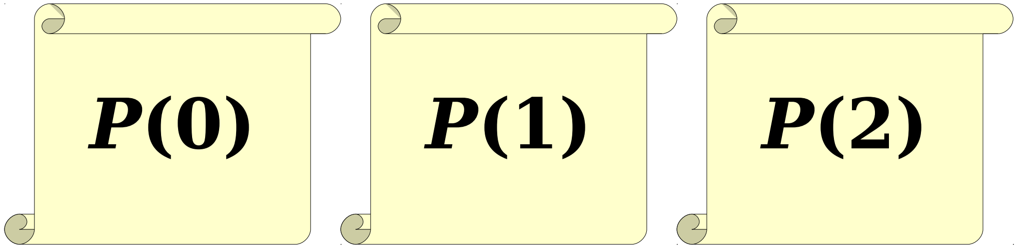

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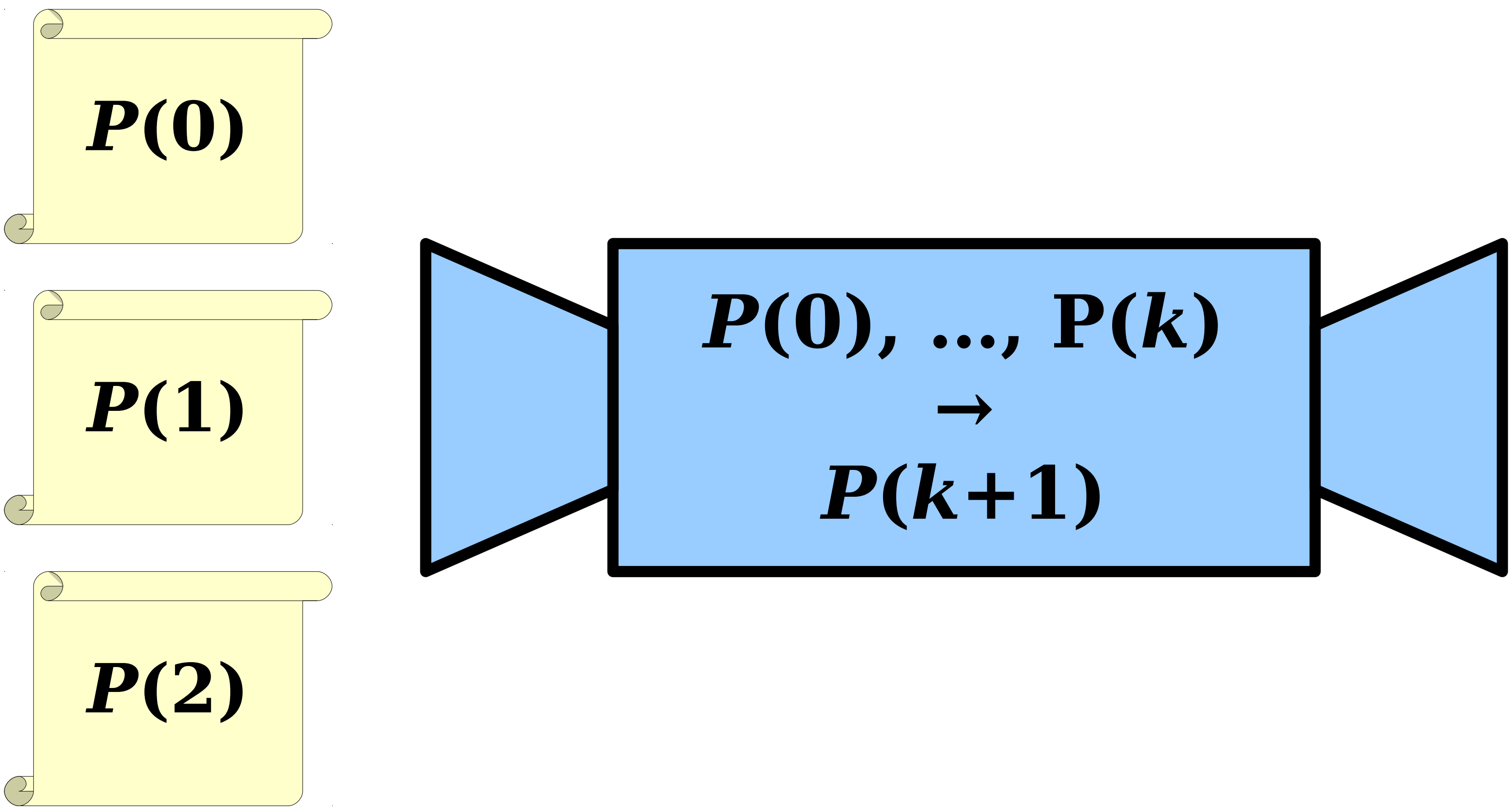

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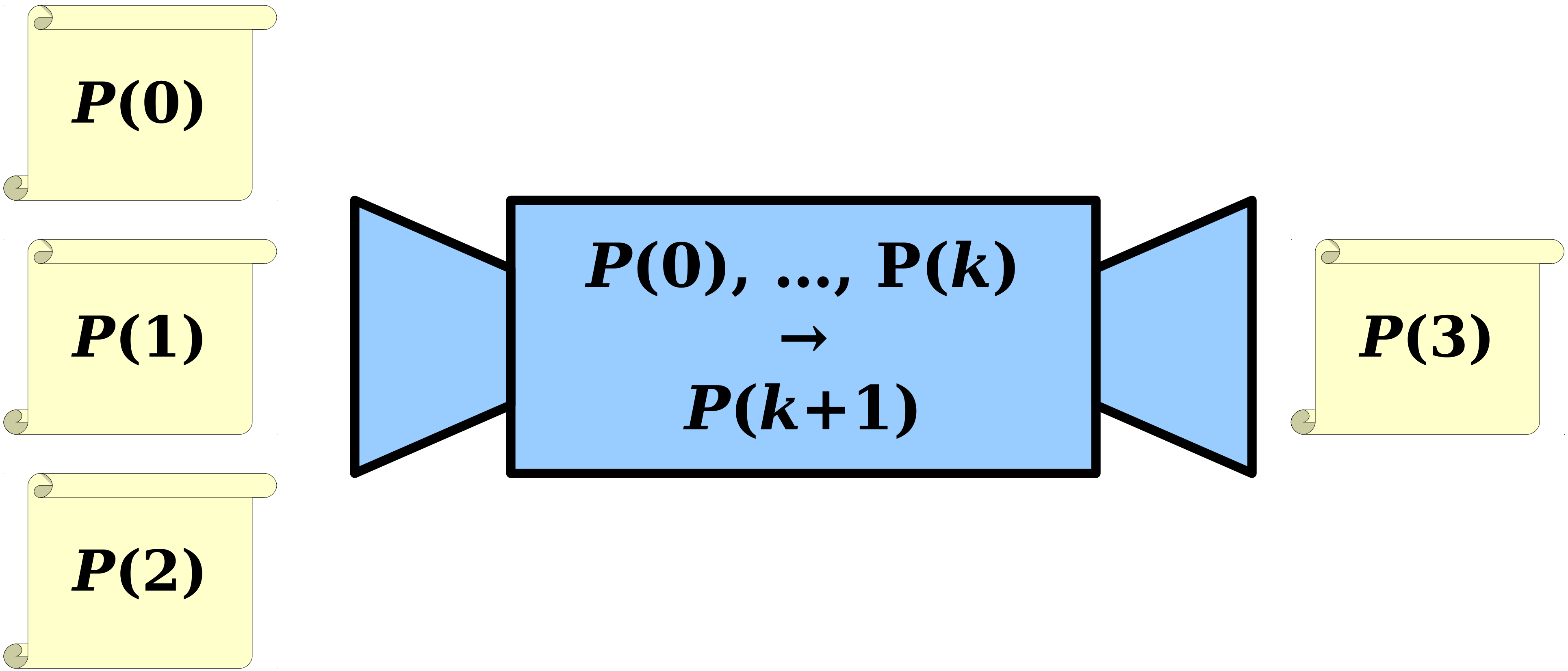

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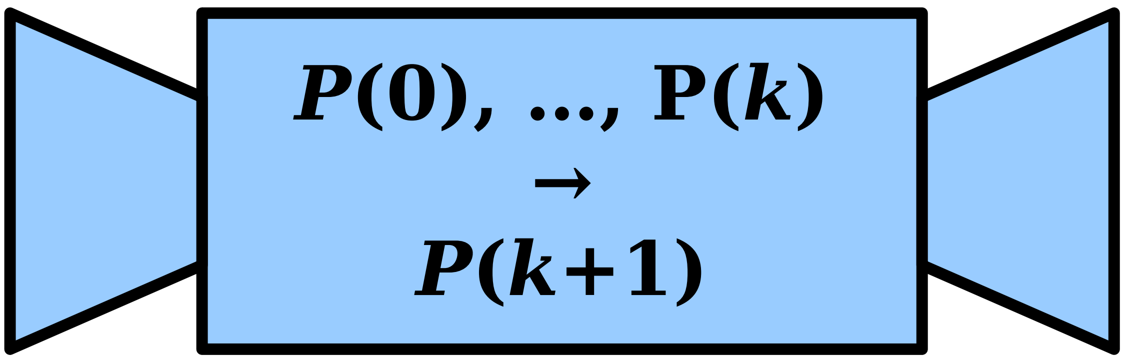
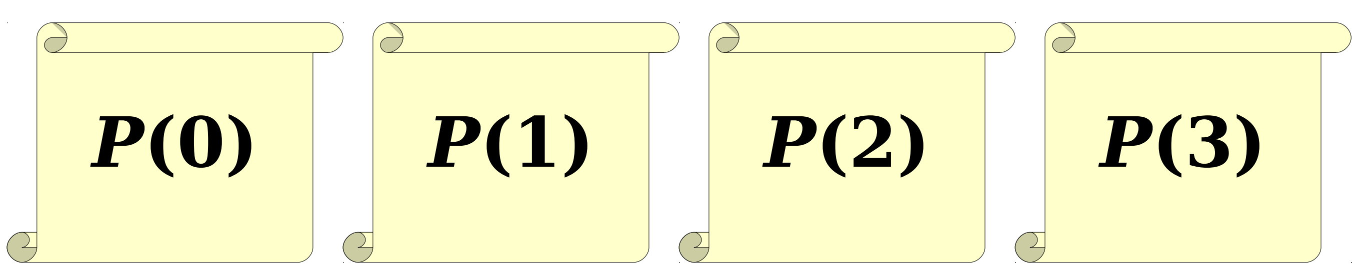
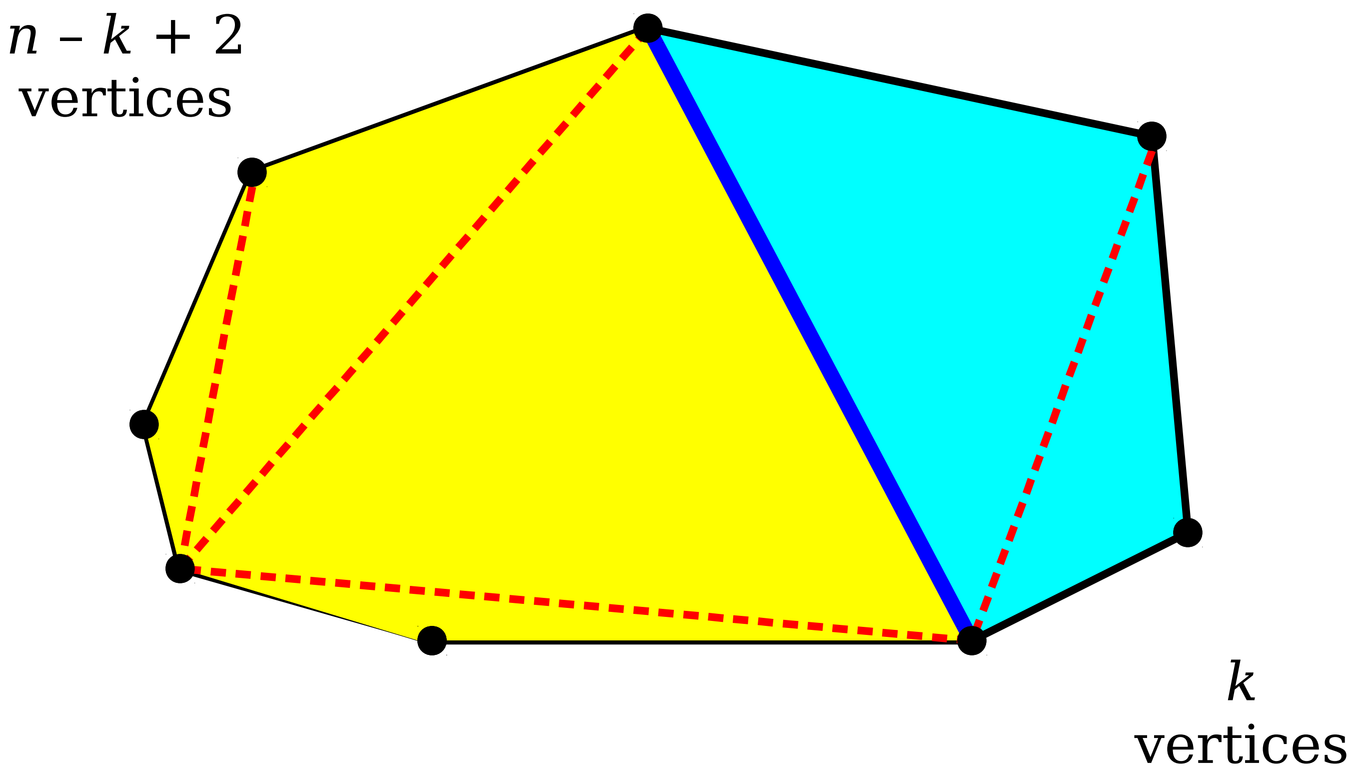




Complete Induction

- Complete induction is useful when you want to simplify something, but you don't know by how much.
- It ensures you can always apply the inductive hypothesis.



Application: **Continued Fractions**

Continued Fractions

$$\cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

Continued Fractions

$$\cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

Continued Fractions

$$\cfrac{1}{4 + \cfrac{1}{\cfrac{3}{2}}}$$

Continued Fractions

$$\cfrac{1}{4 + \cfrac{1}{\cfrac{3}{2}}}$$

Continued Fractions

$$\frac{1}{4 + \frac{2}{3}}$$

Continued Fractions

$$\frac{1}{4 + \frac{2}{3}}$$

Continued Fractions

$$\frac{1}{\frac{14}{3}}$$

Continued Fractions

$$\cfrac{1}{\cfrac{14}{3}}$$

Continued Fractions

$$\frac{3}{14}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{2}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{2}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{\cfrac{9}{2}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{\cfrac{9}{2}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{2}{9}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{2}{9}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{\cfrac{11}{9}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{\cfrac{11}{9}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{9}{11}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{9}{11}}$$

Continued Fractions

$$3 + \cfrac{1}{\cfrac{42}{11}}$$

Continued Fractions

$$3 + \cfrac{1}{\cfrac{42}{11}}$$

Continued Fractions

$$3 + \frac{11}{42}$$

Continued Fractions

$$3 + \frac{11}{42}$$

Continued Fractions

$$\frac{137}{42}$$

Continued Fractions

- A ***continued fraction*** is an expression of the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\dots + \cfrac{1}{a_n}}}}$$

- Formally, a continued fraction is either
  - An integer *x*, or
  - *x* + 1 / *F*, where *x* is an integer and *F* is a nonzero continued fraction.
- Continued fractions have numerous applications in number theory and computer science.
- (They're also really fun to write!)

Fun with Continued Fractions

- Every rational number has at least one continued fraction representation.
- Every *irrational* number has an infinite continued fraction representation.
- ***Interesting fact:*** If we truncate an infinite continued fraction for an irrational number, we can get progressively better approximations of that number.

π as a Continued Fraction

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\cdots}}}}}}}}}}$$

Approximating π

Approximating π

$\pi = 3$

$3 = \mathbf{3}.0000...$

Approximating π

$\pi = 3$

$3 = \textbf{3}.0000\ldots$

And he made the Sea of cast bronze, ten cubits from one brim to the other; it was completely round. [… A] line of thirty cubits measured its circumference.

1 Kings 7:23, New King James Translation

Approximating π

$$\pi = 3 + \cfrac{1}{7}$$

3 = **3**.0000...

22/7 = **3.14**2857...

Approximating π

$$\pi = 3 + \cfrac{1}{7}$$

3 = **3**.0000...

22/7 = **3.14**2857...

Greek mathematician ***Archimedes*** knew of this approximation, circa 250 BCE

Approximating π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15}}$$

3 = **3**.0000...

22/7 = **3.14**2857...

336/106 = **3.1415**094...

Approximating π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}}$$

3 = **3**.0000...

22/7 = **3.14**2857...

336/106 = **3.1415**094...

355/113 = **3.141592**92...

Approximating π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}}$$

3 = **3**.0000...

22/7 = **3.14**2857...

336/106 = **3.1415**094...

355/113 = **3.1415929**2...

Chinese mathematician 祖沖之 (Zu Chongzhi) discovered this approximation in the early fifth century; this was the best approximation of π for around one thousand years.

Approximating π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}}$$

3 = **3**.0000...

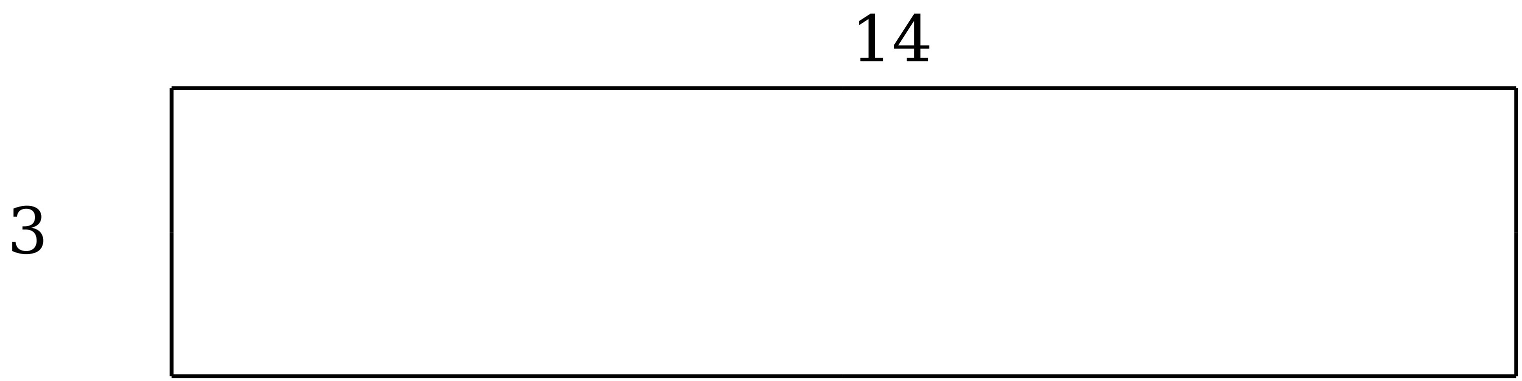
22/7 = **3.14**2857...

336/106 = **3.1415**094...

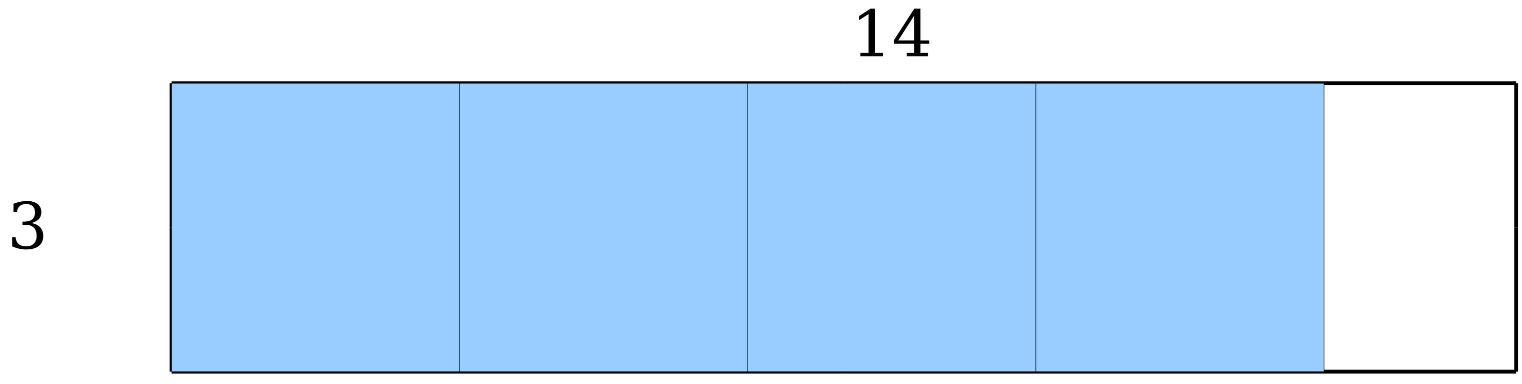
355/113 = **3.141592**92...

103993/33102 = **3.141592653**0...

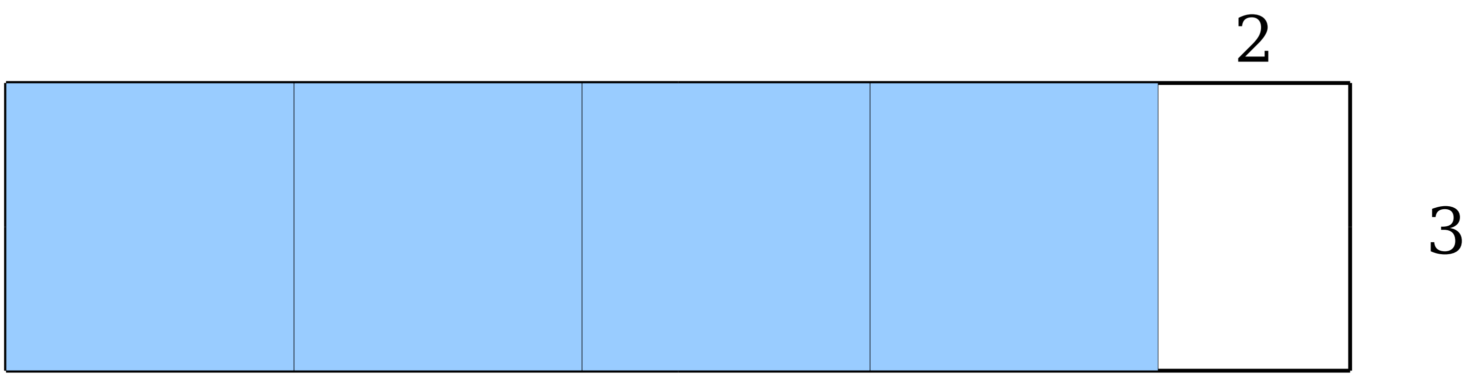
More Continued Fractions



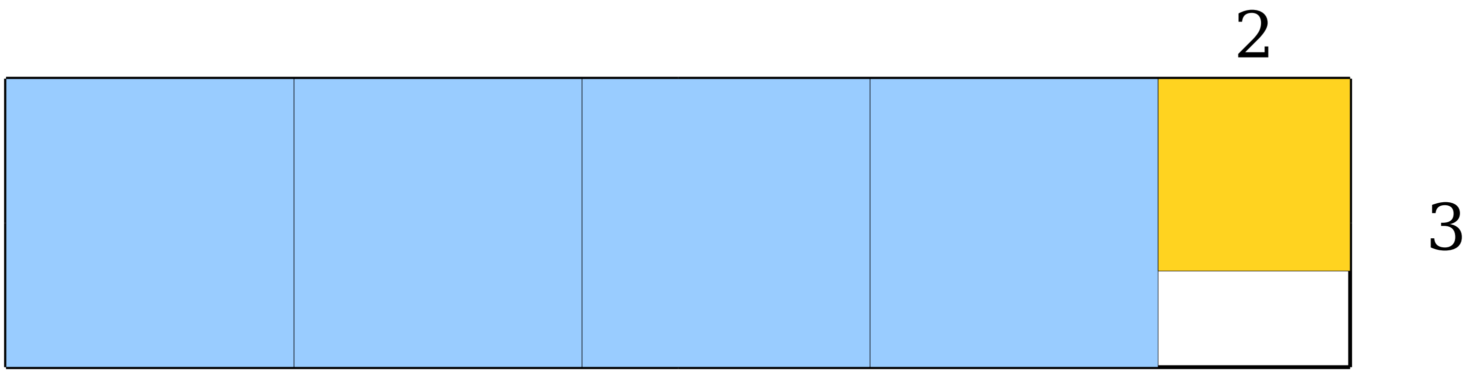
More Continued Fractions



More Continued Fractions



More Continued Fractions



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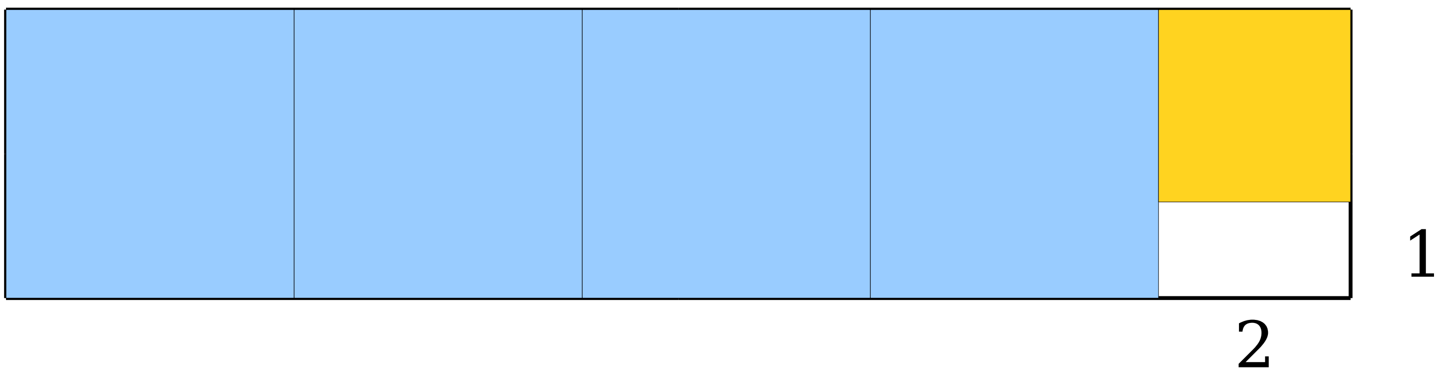

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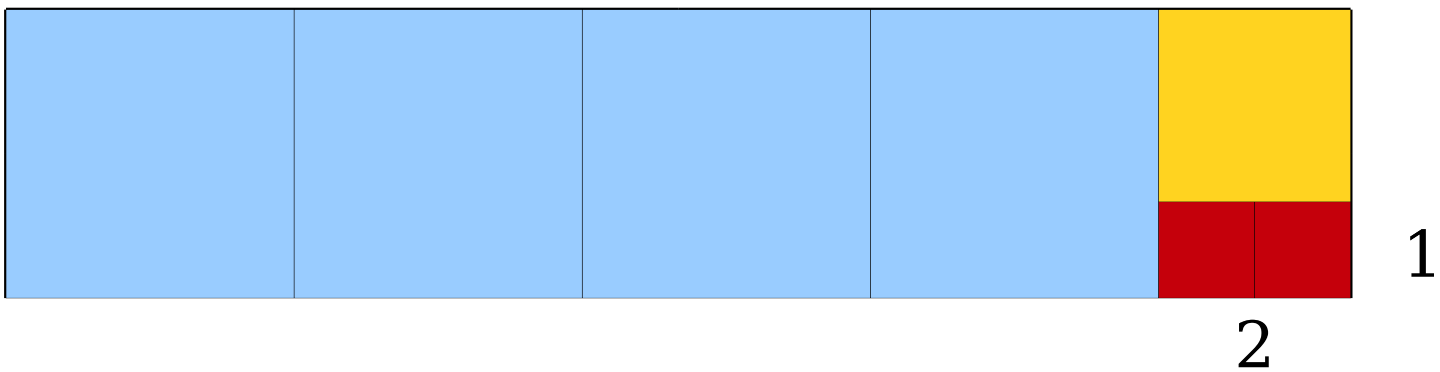

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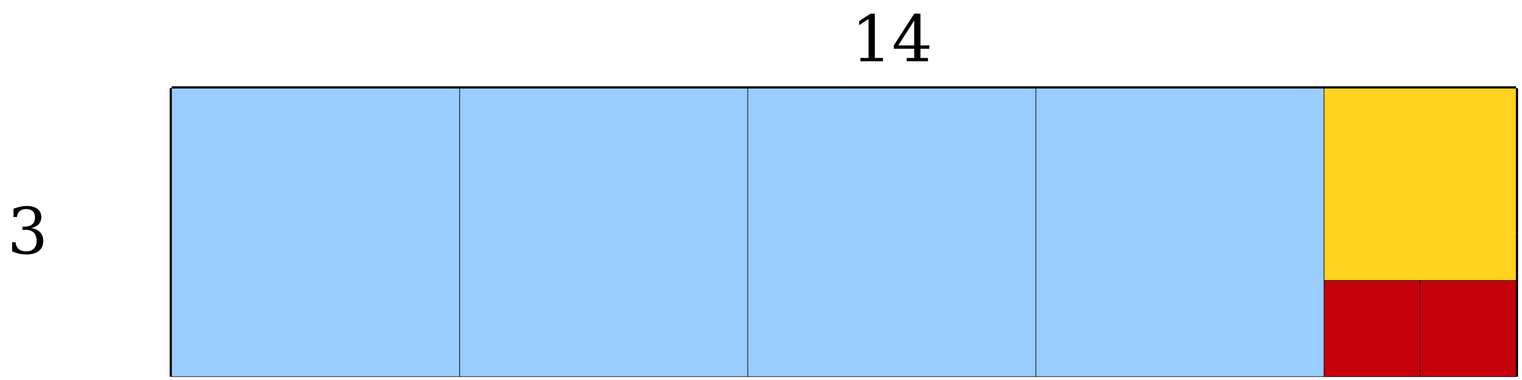

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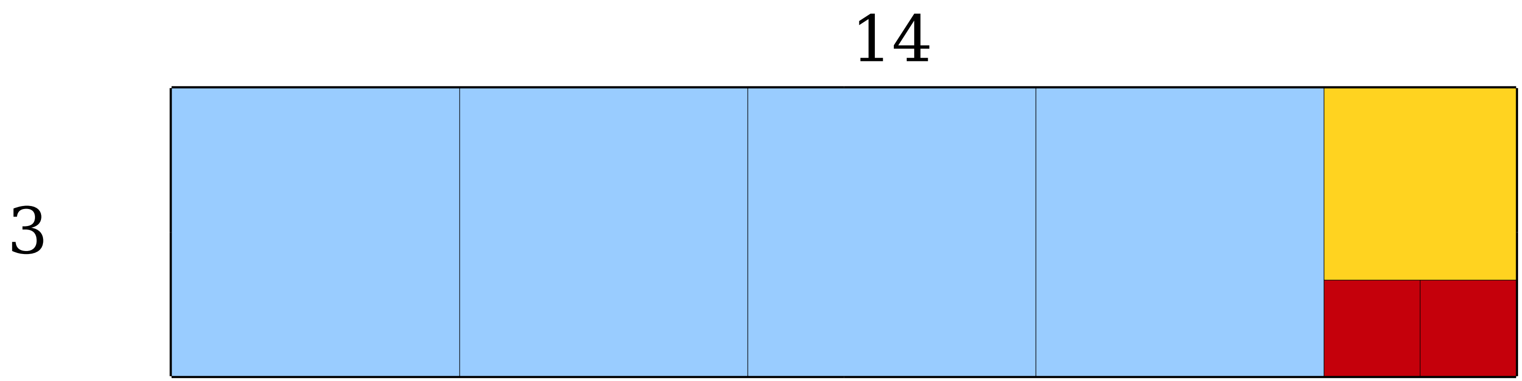

$$\frac{3}{14} = \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

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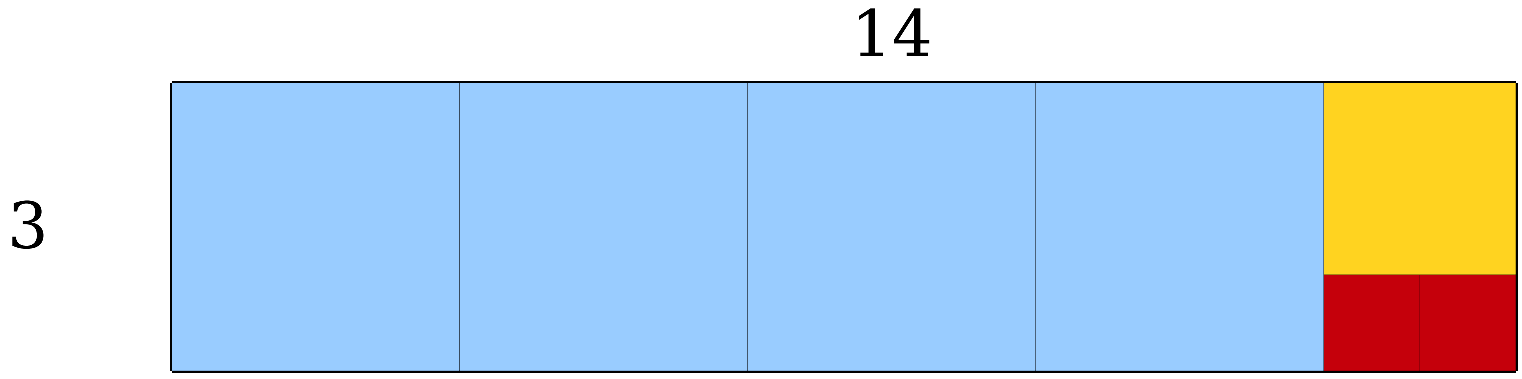

$$\frac{3}{14} = \cfrac{1}{\mathbf{4} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\mathbf{2}}}}$$

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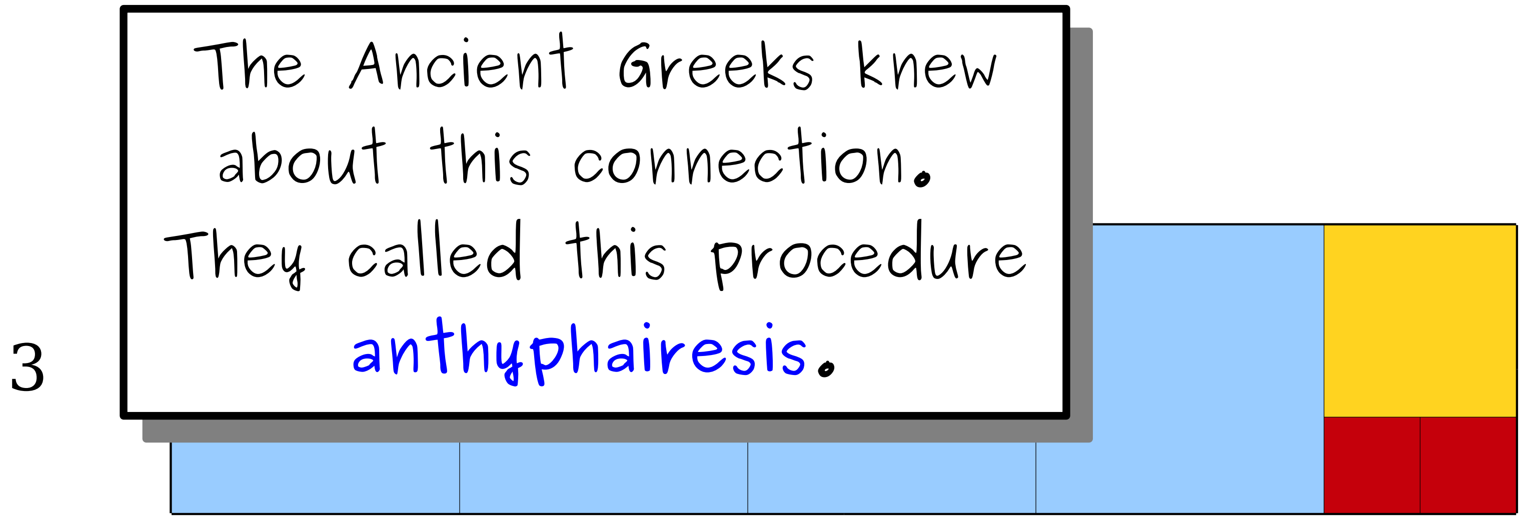

$$\frac{3}{14} = \cfrac{1}{\mathbf{4} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\mathbf{2}}}}$$

Theorem: Every rational number can be expressed as a continued fraction.

Where We're Going

- First, we're going to devise an *algorithm* for constructing a continued fraction from a rational number.
- Next, we're going to look at that algorithm to try to see why it works.
- Finally, we're going to prove that all rational numbers have continued fractions.
  - The proof will essentially describe the algorithm and use the justification we found in the second step.
- This approach is *very useful* for proving results inductively. We highly recommend it on the problem set!

Constructing a Continued Fraction

$$\frac{107}{103}$$

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$$\frac{107}{103} = 1 + \frac{4}{103}$$

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$$\frac{103}{4} = 25 + \frac{3}{4}$$

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Constructing a Continued Fraction

- Suppose we have rational number a / b.
- If a / b is an integer, it's its own continued fraction.
- Otherwise, compute the quotient q and remainder r of a / b and write

$$a / b = q + r / b$$

- Equivalently:

$$a / b = q + 1 / (b / r)$$

- Construct a continued fraction F for b / r.
- The overall continued fraction is then

$$\boldsymbol{a / b = q + 1 / F}$$

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Equivalently:

How do we know that this is possible?

- Construct a continued fraction F for b / r.

The overall continued fraction is then

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107 > 103 > 4 > 3

103 > 4 > 3 > 1

Observation: In this case, each rational number has a smaller numerator and denominator than the previous one.

This helps explain why we eventually “bottom out” - these numbers can't decrease forever.

Question: Is this just a coincidence?

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$$\frac{7}{9}$$

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$$\frac{2}{1}$$

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$$\frac{\mathbf{7}}{\mathbf{9}} = 0 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2}}}$$

$$\frac{\mathbf{9}}{\mathbf{7}} \rightarrow 1 + \cdots$$

$$\frac{\mathbf{7}}{\mathbf{2}} \rightarrow 3 + \cdots$$

$$\frac{\mathbf{2}}{\mathbf{1}} \rightarrow 2$$

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$$\frac{7}{9} = 0 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2}}}$$

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9 > 7 > 2 > 1

Observation: In this case, each rational number has a smaller ~~numerator and~~ denominator than the previous one.

This helps explain why we eventually "bottom out" - these numbers can't decrease forever.

Question: Is this just a coincidence?

$$\frac{a}{b}$$

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r is the remainder of *a* divided by *b*.
It's guaranteed to be smaller than *b*.

Fact: Using our algorithm, each continued fraction has a smaller denominator than the previous one.

This helps explain why we eventually “bottom out” - these numbers can't decrease forever.

Question: How do we turn this into a proof?

A Helpful Intuition

- If you see something of the form

  “keep repeating *X* until...”

  try proving it by induction.
- Use the inductive hypothesis to “assume away” future steps.
- Example: Counterfeit coins.
  - Process: “Keep splitting the coins into thirds and throwing away coins until only one's left.”
  - Proof: “Assume that it works for 3^k coins and prove that it works for 3^{k+1} coins.”

From Intuition to Proof

- In our case, the intuition is

"Keep constructing continued fractions until the denominator becomes 1."

- We'll prove this by using the following inductive hypothesis:

"We can construct a continued fraction for any rational number with denominator *k* or less."

- In our inductive step, we'll show that we can build continued fractions for rational numbers with denominator $k+1$ by using a continued fraction for a rational number with a smaller denominator.

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Take any rational number with denominator $k+1$; let it be $a / (k+1)$. Compute the quotient q and remainder r when a is divided by $k+1$.

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When Use Complete Induction?

- Normal induction is good for when you are shrinking the problem size by exactly one.
  - Peeling one final term off a sum.
  - Making one weighing on a scale.
  - Considering one more action on a string.
- Complete induction is good when you are shrinking the problem, but you can't be sure by how much.
  - Splitting a polygon into two smaller polygons.
  - Taking the remainder of one number divided by another.

For more on continued fractions:

http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/cfINTRO.html