Problem Set 3

This third problem set explores functions, cardinality, and equivalence relations. We've chosen these problems to help you get a sense for how to reason about these structures and how to write proofs using formal mathematical definitions. By the time you're done with these problems, you'll have a much more nuanced understanding of these structures and how to use them!

Good luck, and have fun!

Checkpoint due Monday, January 25 at the start of lecture.

Remaining problems due Friday, January 29 at the start of lecture.

This checkpoint problem is due on Monday at the start of lecture and should be submitted on GradeScope.

Checkpoint Problem: A Really Simple Bijection? (2 Points if Submitted)

Consider the function $f : \mathbb{N} \to \mathbb{Z}$ defined as $f(n) = n$.

i. Prove that f is not a bijection.

Below is a purported proof that f is a bijection:

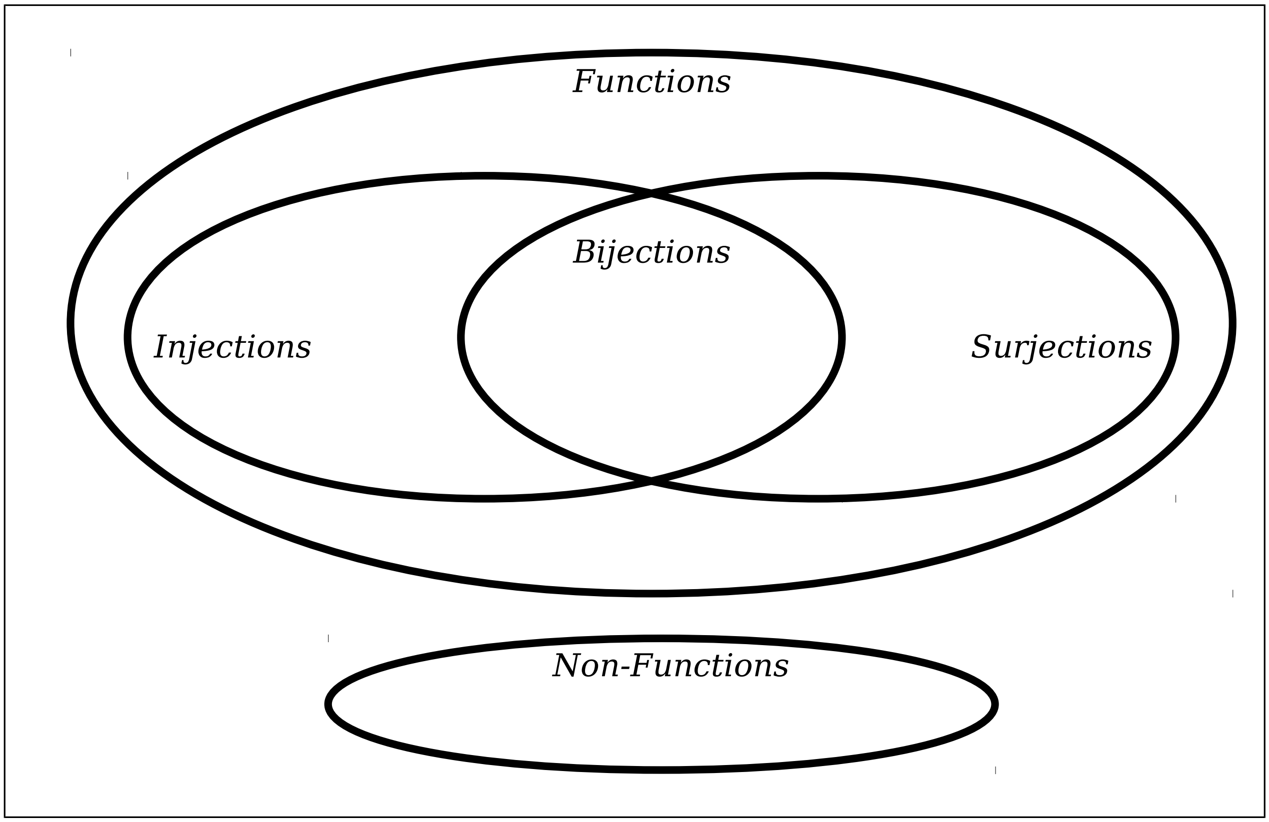
> ***Theorem:*** Let $f : \mathbb{N} \to \mathbb{Z}$ be defined as $f(n) = n$. Then f is a bijection.
>
> ***Proof:*** In lecture, we proved that $|\mathbb{N}| = |\mathbb{Z}|$. Since the sets $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality, we know that every function between them must be a bijection. In particular, this means that f must be a bijection, as required. ■

This proof has to be incorrect, since, as you proved in part (i), f isn't a bijection.

ii. What's wrong with this proof? Justify your answer.

Problem One: Properties of Functions (4 Points)

Consider the following Venn diagram:



Below is a list of purported functions. For each of those purported functions, determine where in this Venn diagram that object goes. No justification is necessary.

1. $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = n^2$
2. $f : \mathbb{Z} \to \mathbb{N}$ defined as $f(n) = n^2$
3. $f : \mathbb{N} \to \mathbb{Z}$ defined as $f(n) = n^2$
4. $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(n) = n^2$
5. $f : \mathbb{R} \to \mathbb{N}$ defined as $f(n) = n^2$
6. $f : \mathbb{N} \to \mathbb{R}$ defined as $f(n) = n^2$
7. $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = \sqrt{n}$.
8. $f : \mathbb{R} \to \mathbb{R}$ defined as $f(n) = \sqrt{n}$.
9. $f : \mathbb{R} \to \{ x \in \mathbb{R} \mid x \geq 0 \}$ defined as $f(n) = \sqrt{n}$.
10. $f : \{ x \in \mathbb{R} \mid x \geq 0 \} \to \{ x \in \mathbb{R} \mid x \geq 0 \}$ defined as $f(n) = \sqrt{n}$.
11. $f : \{ x \in \mathbb{R} \mid x \geq 0 \} \to \mathbb{R}$ defined as $f(n) = \sqrt{n}$.
12. $f : \mathbb{N} \to \wp(\mathbb{N})$, where f is some injective function.
13. $f : \{0, 1, 2\} \to \{3, 4\}$, where f is some surjective function.
14. $f : \{\text{breakfast, lunch, dinner}\} \to \{\text{shakshuka, soondubu, maafe}\}$, where f is an injection.

Problem Two: Left and Right Inverses (4 Points)

There's a weaker notion of an inverse function called a ***left inverse function***. Suppose that $f : A \to B$ is a function that isn't a bijection. It still might be possible to find a function $g : B \to A$ such that $g(f(a)) = a$ for all $a \in A$. A function meeting these requirements is called a ***left inverse*** of f.

i. Find examples of a function f and two *different* functions g and h such that both g and h are left inverses of f. This shows that left inverses don't have to be unique. (Two functions g and h are different if there is some x where $g(x) \neq h(x)$.) *(Hint: Define your functions through pictures.)*

ii. Prove that if f has a left inverse, then f is injective.

Another weaker notion of an inverse function is called a ***right inverse function***. Suppose that $f : A \to B$ is a function that isn't a bijection. It still might be possible to find a function $g : B \to A$ such that $f(g(b)) = b$ for all $b \in B$. A function meeting these requirements is called a ***right inverse*** of f.

iii. Find examples of a function f and two *different* functions g and h such that both g and h are right inverses of f. This shows that right inverses don't have to be unique.

iv. Prove that if f has a right inverse, then f is surjective.

Problem Three: Set Cardinalities (4 Points)

Let a and b be arbitrary objects such that $a \neq b$. Using the formal definition of equal cardinalities, prove that $|\mathbb{N} \times \{a, b\}| = |\mathbb{N}|$. Specifically, define a bijection $f : \mathbb{N} \times \{a, b\} \to \mathbb{N}$, then prove your function is a bijection. We are looking for a proof that calls back to the formal definition of bijections, so please be as rigorous and specific in your proof as possible.

We recommend that you draw out some pictures before trying to define a function so that you have an intuitive sense of how the bijection will pair the elements of $\mathbb{N} \times \{a, b\}$ and the elements of $\mathbb{N}$.

Problem Four: Understanding Diagonalization (4 Points)

Proofs by diagonalization are tricky and rely on nuanced arguments. In this problem, we'll ask you to review the diagonalization proof we covered in lecture to help you better understand how it works.

i. Consider the function $f : \mathbb{N} \to \wp(\mathbb{N})$ defined as $f(n) = \emptyset$. Trace through our proof of Cantor's theorem with this choice of f in mind. In the middle of the argument, the proof defines some set D in terms of f. Given that $f(n) = \emptyset$, what is that set D? Is it clear why $f(n) \neq D$ for any $n \in \mathbb{N}$?

ii. Find a set $S \subseteq \mathbb{N}$ such that $S \neq D$, but $f(n) \neq S$ for any $n \in \mathbb{N}$. Justify your answer. This shows that while the diagonalization proof will always find *some* set D that isn't covered by f, it won't find *every* set with this property.

iii. Repeat part (i) of this problem using the function $f : \mathbb{N} \to \wp(\mathbb{N})$ defined as

$$f(n) = \{ \ m \in \mathbb{N} \mid m \geq n \ \}$$

Now what do you get for the set D? Is it clear why $f(n) \neq D$ for any $n \in \mathbb{N}$?

iv. Repeat part (ii) using the function f from part (iii).

Problem Five: Simplifying Cantor's Theorem? (2 Points)

In lecture, we proved Cantor's theorem, that if S is a set, then $|S| < |\wp(S)|$. Our proof used a diagonal argument, which is clever but tricky. Below is a purported proof that $|S| \neq |\wp(S)|$ that doesn't use a diagonal argument:

> ***Theorem:*** If S is a set, then $|S| \neq |\wp(S)|$.
>
> ***Proof:*** Let S be any set and consider the function $f : S \to \wp(S)$ defined as $f(x) = \{x\}$. To see that this is a valid function from S to $\wp(S)$, note that for any $x \in S$, we have $\{x\} \subseteq S$. Therefore, $\{x\} \in \wp(S)$ for any $x \in S$, so f is a legal function from S to $\wp(S)$.
>
> Let's now prove that f is injective. Consider any $x_1, x_2 \in S$ where $f(x_1) = f(x_2)$. We'll prove that $x_1 = x_2$. Because $f(x_1) = f(x_2)$, we have $\{x_1\} = \{x_2\}$. Since two sets are equal if and only if their elements are the same, this means that $x_1 = x_2$, as required.
>
> However, f is not surjective. Notice that $\emptyset \in \wp(S)$, since $\emptyset \subseteq S$ for any set S, but that there is no x such that $f(x) = \emptyset$; this is because $\emptyset$ contains no elements and $f(x)$ always contains one element. Since f is not surjective, it is not a bijection. Thus $|S| \neq |\wp(S)|$. ■

Unfortunately, this proof is incorrect. What's wrong with this proof? Justify your answer.

Problem Six: Paradoxical Sets (5 Points)

What happens if we take *absolutely everything* and throw it into a set? If we do, we would get a set called the *universal set*, which we denote $\mathscr{U}$:

$$\mathscr{U} = \{ \ x \mid x \text{ exists } \}$$

Absolutely everything would belong to this set: $1 \in \mathscr{U}$, $\mathbb{N} \in \mathscr{U}$, philosophy $\in \mathscr{U}$, CS103 $\in \mathscr{U}$, etc. In fact, we'd even have $\mathscr{U} \in \mathscr{U}$, which is strange but not immediately a problem.

Unfortunately, the set $\mathscr{U}$ doesn't actually exist, as its existence would break mathematics.

i. Prove that if A and B are arbitrary sets where $A \subseteq B$, then $|A| \leq |B|$. Although this probably makes intuitive sense, to formally prove this result, you need to find an injection $f : A \to B$ and prove that your function is injective.

ii. Using your result from (i), prove that if $\mathscr{U}$ exists at all, then $|\wp(\mathscr{U})| \leq |\mathscr{U}|$.

iii. Using your result from (ii) and Cantor's Theorem, prove that $\mathscr{U}$ does not exist.

The result you've proven shows that there is a collection of objects (namely, the collection of everything that exists) that cannot be put into a set. When this was discovered at the start of the twentieth century, it caused quite a lot of chaos in the math world and led to a reexamination of logical reasoning itself and a more formal definition of what objects can and cannot be gathered into a set. If you're curious to learn more about what sets can and cannot be created, take Math 161 (Set Theory).

We will cover the material necessary to solve problems seven and eight in Monday's lecture.

Problem Seven: Odd Rational Numbers (5 Points)

Let's say that a rational number r is an ***odd rational number*** if it can be written as p / q where p and q are integers and q is an odd number. For example, the number 1.6 is an odd rational number because it can be written as $^8/_5$.

Consider the following binary relation ~ over the set $\mathbb{R}$:

$$x \sim y \quad \text{if} \quad y - x \text{ is an odd rational number.}$$

i. Prove that ~ is an equivalence relation.

ii. What is $[0]_\sim$? Prove it.

Problem Eight: Euclidean Relations (4 Points)

In Euclid's *The Elements*, considered a landmark work in ancient mathematics, the Greek mathematician Euclid states that "things which equal the same thing also equal one another." In honor of Euclid, we say that a binary relation R over a set A is ***Euclidean*** if

$$\forall x \in A.\ \forall y \in A.\ \forall z \in A.\ (xRy \land xRz \to yRz).$$

Let R be an arbitrary binary relation over some set A. Prove that R is an equivalence relation if and only if it is reflexive and Euclidean.

Extra Credit Problem: Finding a Bijection (1 Point Extra Credit)

In lecture, we used the Cantor-Bernstein-Schroder theorem to prove that the sets [0, 1] (the set of all real numbers between 0 and 1, inclusive) and (0, 1) (the set of all real numbers between 0 and 1, exclusive) have the same cardinality. This means that there must be a bijection between these two sets. However, we never actually found a specific example of a bijection between these two sets.

Give a specific example of a bijection $f : [0, 1] \to (0, 1)$, then prove that your function meets the necessary properties.