Extra Practice Problems 1

Over the course of the week, we will be releasing a series of practice problems to help you prepare for the first midterm exam. Specifically, we'll release three sets of practice problems, a practice midterm exam, and a set of extra challenge problems. We'll also hand out a set of practice problems in CS103A.

Here's an initial set of extra practice problems you can use to prep for the upcoming exam. We'll release solutions on Wednesday. In the meantime, feel free to ask questions on Piazza or stop by office hours if you have questions.

Problem One: Translating Out Of Logic

For each first-order statement below, write a short English sentence that describes what that sentence says. While you technically *can* literally translate these statements back into English, you'll probably have better luck translating them if you try to think about what they really mean. Then, determine whether the statement is true or false. No proofs are necessary.

- $\exists S.\ (Set(S) \land \forall x.\ x \notin S)$
- $\forall x.\ \exists S.\ (Set(S) \land x \notin S)$
- $\forall S.\ (Set(S) \to \exists x.\ x \notin S)$
- $\forall S.\ (Set(S) \land \exists x.\ x \notin S)$
- $\exists S.\ (Set(S) \land \exists x.\ x \notin S)$
- $\exists S.\ (Set(S) \to \forall x.\ x \in S)$
- $\exists S.\ (Set(S) \land \forall x.\ x \notin S \land \forall \mathrm{T}.\ (Set(T) \land S \neq T \to \exists x.\ x \in T))$
- $\exists S.\ (Set(S) \land \forall x.\ x \notin S \land \exists \mathrm{T}.\ (Set(T) \land \forall x.\ x \notin T \land S \neq T))$
- $\exists S.\ (Set(S) \land \forall x.\ x \notin S) \land \exists \mathrm{T}.\ (Set(T) \land \forall x.\ x \notin T))$

Problem Two: Binary Relations

On Problem Set Three, you explored the binary relation ~ over $\mathbb{R}$ defined as follows:

$$x \sim y \quad \text{if} \quad y - x \text{ is an odd rational number.}$$

Here, an odd rational number is a rational number that can be written with an odd denominator.

Prove that every element of $[\ \sqrt{2}\ ]_{\sim}$ is irrational.

Problem Three: Functions

Let $f : A \to B$ and $g : B \to C$ be functions. In class, we proved that if f and g are bijections, then $g \circ f$ is a bijection. This question asks whether the converse of this statement is true: if $g \circ f$ is a bijection, are g and f necessarily bijections? To help answer this question, we've broken this down into three steps.

i. Prove that if f *is* surjective and g is *not* a bijection, then $g \circ f$ is *not* a bijection.

ii. Prove that if f is *not* a bijection and g *is* injective, then $g \circ f$ is *not* a bijection.

iii. Find examples of functions f and g where neither f nor g is bijective, but $g \circ f$ is a bijection.

As hints for each of the above problems: we highly recommend starting off by drawing pictures to get a sense for how these properties interact, then turning some of your insights from those pictures into actual proofs.

Problem Four: General Discrete Math

If you'll recall from Problem Set Two, a ***tournament*** is a contest among $n \geq 0$ players in which each player plays one game against each other player. Every game ends with one player winning and one player losing, and no draws are allowed. A ***tournament winner*** is a player w in a tournament T where, for each other player p in T, either w beat p or w beat someone who beat p (or both). One of the major results about tournaments that you proved in Problem Set Two is that every tournament with at least one player has at least one tournament winner.

A ***pseudotournament*** is like a tournament, except that exactly one pair of people don't end up playing each other. It turns out that in this variation of a tournament, it's possible for no one to be a winner.

Prove that for all $n \geq 2$, there is a pseudotournament T with n players where no one is a winner.