Extra Practice Problems 2

Here's another set of practice problems you can work through to help prepare for the midterm. We'll release solutions to these problems on Friday. If you have any questions about them, please feel free to stop by office hours!

Problem One: First-Order Logic

Here's some more practice problems to help you get used to translating statements into first-order logic.

i. Given the predicates

   Person(p), which states that p is a person, and
   ParentOf(p_1, p_2), which states that p_1 is the parent of p_2,

   write a statement in first-order logic that says "someone is their own grandparent." (Paraphrased from an old novelty song.)

ii. Given the predicates

   Natural(n), which states that n is a natural number, and
   Integer(n), which states that n is an integer,

   along with the function symbol $f(n)$, which represents some particular function f, write a statement in first-order logic that says "$f : \mathbb{N} \to \mathbb{Z}$ is a bijection."

Problem Two: Functions

It's possible to find all sorts of weird functions from infinite sets into themselves. This question asks you to come up with functions with all sorts of properties from ℕ back to itself.

i. Find a function $f : \mathbb{N} \to \mathbb{N}$ that is both injective and surjective. You should briefly justify why your function has these properties, but no formal proof is necessary.

ii. Find a function $f : \mathbb{N} \to \mathbb{N}$ that is injective but not surjective. You should briefly justify why your function has these properties, but no formal proof is necessary.

iii. Find a function $f : \mathbb{N} \to \mathbb{N}$ that is surjective but not injective. You should briefly justify why your function has these properties, but no formal proof is necessary.

iv. Find a function $f : \mathbb{N} \to \mathbb{N}$ that is neither injective nor surjective. You should briefly justify why your function has these properties, but no formal proof is necessary.

Problem Three: Binary Relations

Let R be a binary relation over a set A. The ***square of R***, denoted R^2, is a binary relation over A defined as follows:

$$xR^2y \quad \text{if} \quad \text{there is some } z \in A \text{ such that } xRz \text{ and } zRy.$$

It turns out that if R is an equivalence relation, then R and R^2 end up being exactly the same relation.

Prove that if R is an equivalence relation over a set A and $a, b \in A$, then aRb if and only if aR^2b.

Problem Four: General Discrete Math

A ***Latin square*** is an $n \times n$ grid such that every natural number between 1 and n, inclusive, appears exactly once on each row and column. A ***symmetric Latin square*** is a Latin square that is symmetric across the main diagonal from the upper-left corner to the lower-right corner. Specifically, the elements at positions (i, j) and (j, i) are always the same.

Prove that in any $n \times n$ symmetric Latin square, where n is even, there is some number between 1 and n that appears nowhere on the diagonal.