Practice Midterm Exam

This exam is closed-book and closed-computer. You may have a double-sided, 8.5” × 11” sheet of notes with you when you take this exam. You may not have any other notes with you during the exam. You may not use any electronic devices during the course of this exam without prior authorization from the course staff. Please write all of your solutions on this physical copy of the exam.

You are welcome to cite results from the problem sets or lectures on this exam. Just tell us what you're citing and where you're citing it from. However, please do not cite results that are beyond the scope of what we've covered in CS103.

On the actual exam, there'd be space here for you to write your name and sign a statement saying you abide by the Honor Code. We're not collecting or grading this exam (though you're welcome to step outside and chat with us about it when you're done!) and this exam doesn't provide any extra credit, so we've opted to skip that boilerplate.

You have three hours to complete this practice midterm. There are 24 total points. This practice midterm is purely optional and will not directly impact your grade in CS103, but we hope that you find it to be a useful way to prepare for the exam. You may find it useful to read through all the questions to get a sense of what this practice midterm contains before you begin.

Question		**Points**	**Grader**
(1) Mathematical Logic	(6)	/ 6	
(2) Set Theory	(6)	/ 6	
(3) Binary Relations	(6)	/ 6	
(4) Functions	(6)	/ 6	
	(24)	**/ 24**	

Good luck!

Problem One: Mathematical Logic (6 Points)

(CS103 Midterm, Spring 2015)

Franz Kafka's *Before the Law* is an existentialist short story about a man who encounters an open gate guarded by a gatekeeper. The gatekeeper tells him that he should not enter, and over the years the man repeatedly tries and fails to persuade the gatekeeper to let him through. It is ultimately revealed that the man was the only person ever allowed to pass through this particular gate, though he never does so.

Before the Law only takes a few minutes to read, so if you haven't yet read it, I highly encourage you to do so after the exam. Right now, though, we'd like you to translate a summary of the story into first-order logic. It's existentialism meets existential quantifiers. ☺

i. **(3 Points)** Given the predicates

   $Person(p)$, which states that p is a person;
   $Gate(g)$, which states that g is a gate;
   $MayPass(p, g)$, which states that p is permitted to pass through g; and
   $WillPass(p, g)$, which states that p will eventually pass through g,

   write a statement in first-order logic that says "someone has a gate that they alone are permitted to pass through, but which they will never pass through."

(CS103 Midterm, Fall 2012)

Two candidates X and Y are running for office and are counting final votes. Candidate X argues that more people voted for him than for Candidate Y by making the following claim: "For every ballot cast for Candidate Y, there were two ballots cast for Candidate X." Candidate X states this in first-order logic as follows:

$$\begin{aligned}&\forall b.\ (BallotForY(b) \to \\ &\qquad \exists b_1.\ \exists b_2.\ (BallotForX(b_1) \land BallotForX(b_2) \land b_1 \neq b_2) \\ &)\end{aligned}$$

However, it is possible for the above first-order logic statement to be true even if Candidate X didn't get the majority of the votes.

ii. **(3 Points)** Give an example of a set of ballots such that

1. every ballot is cast for exactly one of Candidate X and Candidate Y,
2. the set of ballots obeys the rules described by the above statement in first-order logic, but
3. candidate Y gets strictly more votes than Candidate X.

You should justify why your set of ballots works, though you don't need to formally prove it. Make specific reference to the first-order logic statement in your justification.

Problem Two: Set Theory (6 Points)

(CS103 Midterm, Fall 2014)

In this question, we're going to introduce a special type of set called a ***hereditary set*** and then ask you to work with that definition.

Let's begin with the definition of hereditary sets:

A set *S* is a ***hereditary set*** if all its elements are hereditary sets.

This definition might seem strange because it's self-referential – it defines the hereditary sets in terms of other hereditary sets! However, it turns out that this is a perfectly reasonable definition to work with, and in this problem you'll explore properties of these types of sets.

i. **(1 Point)** Given the self-referential nature of the definition of hereditary sets, it's not even clear that hereditary sets even exist. As a starting point, prove that there is at least one hereditary set. *(Hint: Think about vacuous truths.)*

Here's the definition of hereditary sets from the previous page:

A set S is a ***hereditary set*** if all its elements are hereditary sets.

It's possible to use some of the standard set operations to transform hereditary sets into new hereditary sets. This question explores one example of this.

ii. **(5 Points)** Prove that if S is a hereditary set, then $\wp(S)$ is also a hereditary set.

Problem Three: Binary Relations (6 Points)

(CS103 Midterm, Fall 2015)

In this question, we're going to introduce a few new definitions pertaining to binary relations, then ask you to play around with these definitions and see how they relate to concepts you've seen in lecture and on Problem Set Three.

Let's begin with a new definition. We'll say that a binary relation R over a set A is called ***antitransitive*** if the following statement is true:

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (\neg aRb \land \neg bRc \to \neg aRc)$$

Next, we'll say that a binary relation R over a set A is called a ***nonequivalence relation*** if it is irreflexive, symmetric, and antitransitive.

Finally, if R is a binary relation over a set A, then we'll say that the ***complement of R***, denoted $\overline{R}$, is a binary relation over A defined as follows:

$$a\overline{R}b \quad \text{if} \quad a \not R b$$

Prove that if R is an equivalence relation over A, then $\overline{R}$ is a nonequivalence relation over A.

Problem Four: Functions (6 Points)

(CS103 Midterm, Fall 2015)

On Problem Set Three, we introduced ***right inverses*** and asked you to prove some properties about them. In this question, we're going to ask you to revisit right inverses and prove another property about them. In the course of doing so, you'll get a chance to demonstrate what you've learned about injections, surjections, and bijections.

Let's begin by refreshing some of the terminology from Problem Set Three. If $f : A \to B$ is a function, then we say that a function $g : B \to A$ is a ***right inverse of f*** if $f(g(b)) = b$ for every $b \in B$. If f is a function that has a right inverse, then we say that f is ***right-invertible***. On Problem Set Three, you proved that all right-invertible functions are surjective.

Let $f : A \to B$ be an arbitrary right-invertible function and let $g : B \to A$ be one of its right inverses. Prove that if g is surjective, then f is a bijection.