Challenge Problems 1

Inspired by a series of lecture notes from Tommy Jensen.

This packet of problems consists of a series of fourteen questions that build up to a surprising proof related to the Hadwiger-Nelson problem. We've tried to break the problem down into a series of manageable but nontrivial steps – they're definitely doable with the material we've covered so far, but are on average a little bit trickier than the problems we've encountered so far. If you'd like to push yourself to try a slightly harder set of problems, feel free to work through this handout.

I haven't put together a set of solutions to these problems; to be perfectly honest, I was too busy this week to write them up. However, if you have any questions about these problems, please feel free to ask on Piazza or to stop by office hours.

Have fun!

A ***Pythagorean triple*** is a triple of natural numbers (a, b, c) where $a^2 + b^2 = c^2$. You might be familiar with the Pythagorean triple (3, 4, 5), since $3^2 + 4^2 = 5^2$, or the triple (5, 12, 13). We're going to begin by exploring various properties of Pythagorean triples.

i. Suppose that (a, b, c) is a Pythagorean triple. Prove that at least one of a, b, and c is even.

ii. Suppose that (a, b, c) is a Pythagorean triple. Prove that it is not possible for exactly two of a, b, and c to be even.

A ***fundamental Pythagorean triple*** is a triple of natural numbers (a, b, c) where a, b, and c have no common factors other than ±1.

iii. Prove that if (a, b, c) is a fundamental Pythagorean triple, exactly one of a, b, and c is even.

iv. You can actually prove an even stronger result than what you showed in part (iii). Prove that if (a, b, c) is a fundamental Pythagorean triple, then c must be odd.

v. Prove that if (a, b, c) is a fundamental Pythagorean triple other than (0, 0, 0), then a/b and b/c are written in simplest form.

We'll say that a rational number r is an ***odd rational number*** if it can be written as $r = p/q$, where q is an odd number. A number that doesn't have this property is called an ***even rational number***.

vi. Suppose that (a_0, b_0) and (a_1, b_1) are points in the Cartesian plane where a_0, a_1, b_0, and b_1 are all rational numbers. Prove that if (a_0, b_0) is exactly one unit away from (a_1, b_1), then $a_1 - a_0$ is an odd rational number and $b_1 - b_0$ is an odd rational number. Additionally, show that if you write $a_1 - a_0$ and $b_1 - b_0$ with a common odd denominator, then one of the numerators will be even and the other will be odd.

You might be wondering why all this matters. It turns out that this whole exploration of Pythagorean triples has an interesting connection to the Hadwiger-Nelson problem. If you'll recall, the Hadwiger-Nelson problem is the following: how many colors do you need so that you can color every point in the Cartesian plane so that no two points at distance 1 are the same color?

Let's generalize this problem a bit.

vii. What is the set $\mathbb{R}^2$?

Let's suppose that $S \subseteq \mathbb{R}^2$. We'll say that the ***chromatic number of S***, denoted $\chi(S)$, is the minimum number of colors necessary to color all the points in S so that no two points in S that are exactly one unit apart have the same color. The notation $\chi(S)$ is designed to mirror the notation for the chromatic number of a graph. In mathematics, it's common to see the same symbols used to represent related concepts even when the core concepts are quite different.

The Hadwiger-Nelson problem then asks you to find $\chi(\mathbb{R}^2)$. From what you saw in class, we know that $4 \leq \chi(\mathbb{R}^2) \leq 7$, but we don't know exactly what the true value is.

viii. What is $\chi(\mathbb{Z}^2)$? Prove it.

The reason we had you do all that rigmarole with Pythagorean triples was to build up to part (vi), which helps us answer this question:

What is $\chi(\mathbb{Q}^2)$?

In other words, if you take all the points in the plane *whose coordinates are rational numbers*, how many colors do you need so that you can color those points so that no two of them are at distance 1? Well, you're about to find out.

Let's begin by defining a new binary relation ~ over $\mathbb{Q}^2$ as follows:

$$(a_0, b_0) \sim (a_1, b_1) \quad \text{if} \quad a_1 - a_0 \text{ is odd and } b_1 - b_0 \text{ is odd.}$$

Turns out this is an equivalence relation!

ix. Prove that ~ is an equivalence relation over $\mathbb{Q}^2$.

This equivalence relation partitions $\mathbb{Q}^2$ into a bunch of equivalence classes.

x. What is $[(0, 0)]_\sim$? Prove it.

You might be wondering why we care about this equivalence relation. Turns out it has a nifty connection to pairs of rational numbers that are one unit away from one another, which as you saw earlier has connections to Pythagorean triples.

xi. Let (a_0, b_0) and (a_1, b_1) be pairs of rational numbers. Prove that if (a_0, b_0) and (a_1, b_1) are exactly one unit away from one another, then $(a_0, b_0) \sim (a_1, b_1)$.

The contrapositive of your result from (xi) shows that if $(a_0, b_0) \sim (a_1, b_1)$ does not hold, then (a_0, b_0) and (a_1, b_1) aren't one unit apart. This is hugely important for determining $\chi(\mathbb{Q}^2)$, since it means that points in different equivalence classes of ~ aren't one unit apart. Therefore, we can focus our attempt to color all the points in $\mathbb{Q}^2$ on an equivalence-class-by-equivalence-class basis.

It turns out, oddly enough, that every equivalence class of ~ has the "same shape" as every other class in the following way.

xii. Let $[(a_0, b_0)]_\sim$ and $[(a_1, b_1)]_\sim$ be equivalence classes of ~. We'd like you to prove that $[(a_0, b_0)]_\sim$ and $[(a_1, b_1)]_\sim$ are ***translations*** of one another. Specifically, prove the following: there are rational numbers r and s such that the function $f : [(a_0, b_0)]_\sim \to [(a_1, b_1)]_\sim$ defined as $f(a, b) = (a + r, b + s)$ is a bijection. In other words, every equivalence class is basically the same shape as every other class, just shifted over by some amount.

We're almost there – home stretch! Because every equivalence class is a just a translated version of every other equivalence class, we can define a coloring of the entirety of $\mathbb{Q}^2$ by just finding a coloring of one equivalence class of ~ and then translating that coloring to every other equivalence class. For simplicity, we'll consider the class $[(0, 0)]_\sim$.

Consider the following way of coloring all the rational numbers in $[(0, 0)]_\sim$. Given a rational point (a, b) in $[(0, 0)]_\sim$, color it red if it can be written as a pair of odd rational numbers $(p/q, r/s)$ where $p + q$ is even. Color it blue otherwise.

xiii. Prove that in this coloring, no two points in $[(0, 0)]_\sim$ that are at distance 1 from one another are given the same color.

xiv. Putting everything together, prove that $\chi(\mathbb{Q}^2) = 2$

So there you have it. One gigantic example talking about even and odd numbers, chromatic numbers, bijections, equivalence classes, and rational numbers. And hey! You ended up proving something really surprising: even though we don't know what $\chi(\mathbb{R}^2)$ is, we do know that $\chi(\mathbb{Q}^2) = 2$. Crazy, isn't it?