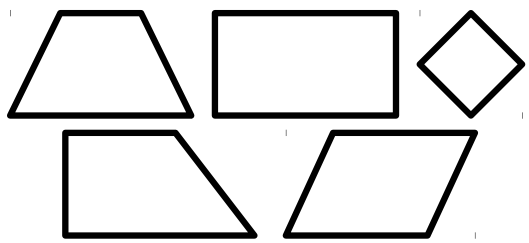
Extra Practice Problems 4

Here's a set of practice problems you can work through to help prepare for the upcoming midterm exam. We'll release solutions on Wednesday.

The Pigeonhole Principle

Suppose that you color every point in the real plane one of four colors (say, red, green, blue, and yellow). Prove that no matter how you color the plane, there will always be a trapezoid whose corners are all the same color. (Recall that a trapezoid is a quadrilateral with at least two parallel sides.) For example, all of the following figures are trapezoids:



(Hint: Try placing a specially-constructed object – say, a grid of dots – into the plane such that no matter how that object is colored, the object always contains a trapezoid whose corners are the same color.)

Induction and Set Theory

A set S is called an ***inductive set*** if the follow two properties are true about S:

- $0 \in S$.
- For any number $x \in S$, the number $x + 1$ is also an element of S.

This question asks you to explore various properties of inductive sets.

i. Find two different examples of inductive sets.

ii. Prove that the intersection of any two inductive sets is also an inductive set.

iii. Prove that if S is an inductive set, then $\mathbb{N} \subseteq S$.

iv. Prove that $\mathbb{N}$ is the *only* inductive set that's a subset of all inductive sets. This proves that $\mathbb{N}$ is, in a sense, the most "fundamental" inductive set. In fact, in foundational mathematics, the set $\mathbb{N}$ is sometimes defined as the one inductive set that's a subset of all inductive sets.

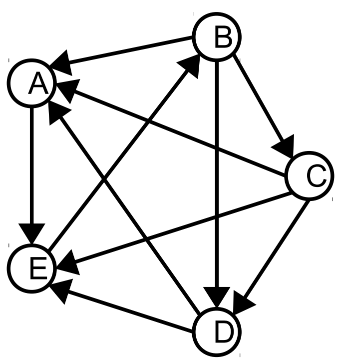
Induction and Graph Theory

A ***tournament graph*** is a graph of n nodes where every pair of distinct nodes has exactly one edge between them. A ***Hamiltonian path*** is a path in a graph that passes through every node in a graph exactly once. Prove that every tournament graph has a Hamiltonian path.

Binary Relations and Graph Theory

This question explores the interaction between binary relations and tournaments.

Let's quickly refresh a definition. A ***tournament*** is a contest between some number of players in which each player plays each other player exactly once. We assume that no games end in a tie, so each game ends in a win for one of the players.



Here's a new definition to work with. If p is a player in tournament T, then we can define the set $W(p) = \{ x \mid x \text{ is a player in } T \text{ and } p \text{ beat } x \}$. Intuitively, $W(p)$ is the set of all the players that player p beat. For example, in the tournament on the left, $W(B) = \{A, C, D\}$.

Now, let's define a new binary relation. Let T be a tournament. We'll say that $p_1 \sqsubset_T p_2$ if $W(p_1) \subset W(p_2)$. Intuitively, $p_1 \sqsubset_T p_2$ means that p_2 beat every player that p_1 beat, plus some additional players.

For example, in the tournament to the left, we have that $D \sqsubset_T C$ because $W(D) = \{A, E\}$ and $W(C) = \{A, D, E\}$. Similarly, we know $A \sqsubset_T D$ since $W(A) = \{E\}$ and $W(D) = \{A, E\}$.

Prove that if T is any tournament, then $\sqsubset_T$ is a strict order over the players in T.

Regular Languages

For each of the following, show that the given language is regular by designing a DFA or NFA for it ***and*** by designing a regular expression for it.

i. Let $\Sigma = \{\texttt{a}, \texttt{b}\}$. Show that Σ^* is regular via a DFA/NFA and a regular expression.

ii. Let $\Sigma = \{\texttt{a}, \texttt{b}, \texttt{c}\}$. Let $L = \{ w \in \Sigma^* \mid \text{any } \texttt{b}\text{'s in } w \text{ appear after the first } \texttt{c} \text{ in } w \}$. Show that L is regular via a DFA/NFA and a regular expression.

iii. Let $\Sigma = \{\texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}\}$ and let $L = \{ w \in \Sigma^* \mid \text{w is the base-10 representation of an even number and } w \text{ has no extraneous leading zeros} \}$. Show that L is regular via a DFA/NFA and a regular expression.