Challenge Problems 2

This packet of problems consists of some problems that will require you to play around with some mathematical objects, spot a pattern of some sort, then go prove a result about that pattern. We think this might be a fun way to get yourself thinking about the different terms, definitions, and concepts that we've covered so far in CS103. I haven't put together a set of solutions to these problems. However, if you have any questions about these problems, please feel free to ask on Piazza or to stop by office hours.

Have fun!

The Barberville Sequence

The ***Barberville sequence*** is an inductively-defined sequence of values. The first Barberville number, denoted B_0, is zero. Then, for all $n \in \mathbb{N}$, we define B_{n+1} as follows: B_{n+1} is defined as the smallest natural number that can't be written as $B_i + B_j$, where $0 \leq i < j \leq n$. In other words, B_{n+1} is the smallest natural number that can't be written as the sum of two previous Barberville numbers.

i. What are the first eight Barberville numbers?
ii. You may have noticed that the Barberville numbers eventually start following a pattern. Figure out what that pattern is, then use induction to prove that the pattern you found continues to all the remaining terms in the Barberville sequence.

Hasse Diagrams

Given a strict order relation $<_A$ over a set A, we can draw a Hasse diagram for $<_A$. This Hasse diagram can be thought of as an ***undirected*** graph with one node per element of A and where the edges are given by the lines drawn in the Hasse diagrams.

Let A be a finite set. Prove that there is a simple path through the Hasse diagram of the $\subsetneq$ relation (that is, the strict subset relation) over $\wp(A)$ that passes through every node once and exactly once.

Regular Languages

For any set $S \subseteq \mathbb{N}$, we can define a language L_S over the alphabet $\Sigma = \{\texttt{a}\}$:

$$L_S = \{ \texttt{a}^n \mid n \in \mathbb{N} \text{ and for every } m \in \mathbb{N}, \text{ if } m \geq n, \text{ then } m \in S \}$$

Prove that L_S is regular for all $S \subseteq \mathbb{N}$.