Another Practice Midterm Exam II

This exam is closed-book and closed-computer. You may have a double-sided, 8.5” × 11” sheet of notes with you when you take this exam. You may not have any other notes with you during the exam.

You are welcome to cite lectures from the problem sets or lecture on this exam. Just tell us what you're citing and where you're citing it from. However, please do not cite results that are beyond the scope of what we've covered in CS103.

On the actual exam, there'd be space here for you to write your name and sign a statement saying you abide by the Honor Code. We're not collecting or grading this exam (though you're welcome to step outside and chat with us about it when you're done!) and this exam doesn't provide any extra credit, so we've opted to skip that boilerplate.

You have three hours to complete this practice midterm. There are 24 total points. This practice midterm is purely optional and will not directly impact your grade in CS103, but we hope that you find it to be a useful way to prepare for the exam. You may find it useful to read through all the questions to get a sense of what this practice midterm contains before you begin.

Question		**Points**	**Grader**
(1) Strict Orders and the Pigeonhole Principle	(6)	/ 6	
(2) Induction	(6)	/ 6	
(3) Finite Automata and Regular Expressions	(6)	/ 6	
(4) Regular Languages	(6)	/ 6	
	(24)	**/ 24**	

Best of luck on the exam!

Problem One: Strict Orders and the Pigeonhole Principle (6 Points)

(Midterm Exam, Winter 2013)

Suppose that you have a sequence of distinct real numbers $S = \langle x_1, x_2, \ldots, x_n \rangle$. A ***subsequence*** of S is a sequence of elements from S where the elements appear in the same relative order as they do in S. For example, consider the sequence S defined as follows:

$$S = \langle 1, 2, 3, 4, 5, 6 \rangle$$

Then $\langle 1, 3, 6 \rangle$ is a subsequence of S, as is $\langle 2, 3, 4, 5 \rangle$. However, $\langle 3, 2, 1 \rangle$ is not a subsequence of S (its elements are not in the same relative order), nor is $\langle 1, 4, 7 \rangle$ (since 7 is not an element of S).

Given a sequence S, an ***ascending subsequence*** of S is a subsequence T where each element of T is strictly greater than all previous elements of T. A ***descending subsequence*** of S is a subsequence T where each element of T is strictly smaller than all previous elements of T. For example, given the sequence

$$S = \langle 106, 103, 107, 109, 110, 161 \rangle$$

The sequence $\langle 106, 103 \rangle$ is a descending subsequence of S, and the sequence $\langle 103, 109, 161 \rangle$ is an ascending subsequence of S. The sequence $\langle 107 \rangle$ is both an ascending and descending subsequence of S.

In this problem, you will prove a result called the ***Erdős–Szekeres Theorem***:

Any sequence of $rs + 1$ distinct real numbers contains an ascending subsequence of length $r + 1$ or a descending subsequence of length $s + 1$ (or both).

Suppose that our sequence of distinct real numbers is $S = \langle x_1, x_2, \ldots, x_{rs+1} \rangle$. Let's associate with each element x_k of this sequence a pair of natural numbers (I_k, D_k) with the following meaning:

I_k is the length of the longest ***increasing*** subsequence of S whose last element is x_k.

D_k is the length of the longest ***decreasing*** subsequence of S whose last element is x_k.

For example, consider the sequence $\langle 40, 20, 10, 30, 50 \rangle$. Then

$$(I_1, D_1) = (1, 1)$$

$$(I_2, D_2) = (1, 2)$$

$$(I_3, D_3) = (1, 3)$$

$$(I_4, D_4) = (2, 2)$$

$$(I_5, D_5) = (3, 1)$$

You might want to take a minute to check why these values are correct.

Feel free to tear this page out as a reference.

i. **(1 Point)** Let k be an arbitrary natural number where $1 \leq k \leq rs + 1$. Prove that $I_k \geq 1$ and $D_k \geq 1$.

ii. **(3 Points)** Let j and k be arbitrary natural numbers where $1 \leq j \leq rs + 1$ and $1 \leq k \leq rs + 1$. Prove that if $j \neq k$, then $(I_j, D_j) \neq (I_k, D_k)$. To keep your proof short, we recommend assuming without loss of generality that $j < k$.

(Additional space for Problem One, part (ii), if you need it)

iii. **(2 Points)** Using your results from parts (i) and (ii), prove that any sequence of $rs + 1$ distinct real numbers contains an ascending subsequence of length $r+1$ or a descending subsequence of length $s+1$. *(Hint: Proceed by contradiction. If the sequence does not have an ascending subsequence of length $r+1$ or a decreasing subsequence of length $s+1$, what do you know about the values of all the (I, D) pairs?)*

Problem Two: Induction (6 Points)

(Midterm Exam, Spring 2015)

This question explores ***jumbled inequalities*** and asks you to prove a surprising result about them.

A jumbled inequality is a puzzle consisting of a series of less-than and greater-than signs with blanks interspersed between them. Here are two examples:

__ < __ > __ < __ > __ > __

__ < __ > __ > __ > __ < __

The goal of the puzzle is to find a way to fill in the blanks with numbers so that all the inequalities are satisfied. For example, here are some solutions to the two above problems:

1 < 5 > 0 < 4 > 3 > 2

0 < 4 > 3 > 2 > 1 < 5

You can interpret these inequalities by reading them from left to right and just focusing on one pair of numbers at a time. For example, the first of these would be read as "one is less than five, and five is greater than zero, and zero is less than four, and four is greater than three, and three is greater than two." The second of these would be read "zero is less than four, and four is greater than three, and three is greater than two, and two is greater than one, and one is less than five."

Your task in this problem to prove a remarkable fact about these puzzles: for any $n \geq 1$, every jumbled inequality puzzle with $n+1$ blanks can be solved using just the numbers 0, 1, 2, 3, …, n and using each number exactly once. It doesn't matter which inequality signs are used or how they're ordered – you'll always be able to find at least one solution using just the natural numbers 0 through n inclusive.

Prove this result using a proof by induction.

A few clarifying points:

- The only inequality signs you need to consider are < and >.
- There can be any number of each kind of inequality sign and the inequality signs can be in any order.
- You only need to place the numbers so that they satisfy the inequality constraints immediately before and after them (if they exist).

Feel free to tear out this page as a reference, and write your answer on the next page.

(Space for your answer to Problem Two.)

Problem Three: Finite Automata and Regular Languages (6 Points)

Let $\Sigma = \{\texttt{a}, \texttt{b}\}$ and let $L = \{ w \in \Sigma^* \mid w \text{ does not contain } \texttt{bbb} \text{ as a substring} \}$.

i. (**3 Points**) Design a DFA for L.

ii. **(3 Points)** Write a regular expression for L.

Problem Four: Regular Languages (6 Points)

(Final Exam, Winter 2013)

The number of characters in a regular expression is defined to be the total number of symbols used to write out the regular expression. For example, `(a ∪ b)*` is a six-character regular expression, and `ab` is a two-character regular expression.

Let Σ = {a, b}. Find examples of all of the following:

- A regular language over Σ with a one-state NFA but no one-state DFA.
- A regular language over Σ with a one-state DFA but no one-character regular expression.
- A regular language over Σ with a one-character regular expression but no one-state NFA.

Prove that all of your examples have the required properties.