# Indirect Proofs

# Outline for Today

- **What is an Implication?**
  - Understanding a key type of mathematical statement.
- **Proof by Contrapositive**
  - What's a contrapositive?
  - Applications to bird storage.
- **Proof by Contradiction**
  - The basic method.
  - Applications to geometry.

# Logical Implication

# Implications

- An ***implication*** is a statement of the form

**If *P* is true, then *Q* is true.**

- Some examples:
  - If $n$ is an even integer, then $n^2$ is an even integer.
  - If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
  - If you like the way you look that much, (ohhh baby) then you should go and love yourself.

# Implications

- An ***implication*** is a statement of the form

**If *P* is true, then *Q* is true.**

- In the above statement, the term "*P* is true" is called the ***antecedent*** and the term "*Q* is true" is called the ***consequent***.

# What Implications Mean

- Consider the simple statement

  ***If I put fire near cotton, it will burn.***

- Some questions to consider:
  - Does this apply to all fire and all cotton, or just some types of fire and some types of cotton? *(Scope)*
  - Does the fire cause the cotton to burn, or does the cotton burn for another reason? *(Causality*)
- These are deeper questions than they might seem.
- To mathematically study implications, we need to formalize what implications really mean.

# Understanding Implications

**“If there's a rainbow,
then it's raining somewhere.”**

- Implication is *directional*.
  - The above statement doesn't mean that if it's raining somewhere, there has to be a rainbow.
- Implication only cares about cases where the antecedent is true.
  - If there's no rainbow, it doesn't mean that there's no rain.
- Implication says nothing about *causality*.
  - Rainbows do not cause rain. ☺

# Scoping Implications

- Consider the following statements:

**If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.**

**If $n$ is even, then $n^2$ is even.**

**If $A \subseteq B$ and $B \subseteq A$, then $A = B$.**

- In the above statements, what are $A$, $B$, $C$, and $n$? Are they *specific* objects? Or do these claims hold for all objects?

# Implications and Universals

- In discrete math, most* implications involving unknown quantities are, implicitly, universal statements.
- For example, the statement

**If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$**

actually means

**For any sets $A$, $B$, and $C$,**
**if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.**

** this will become clearer on Wednesday.*

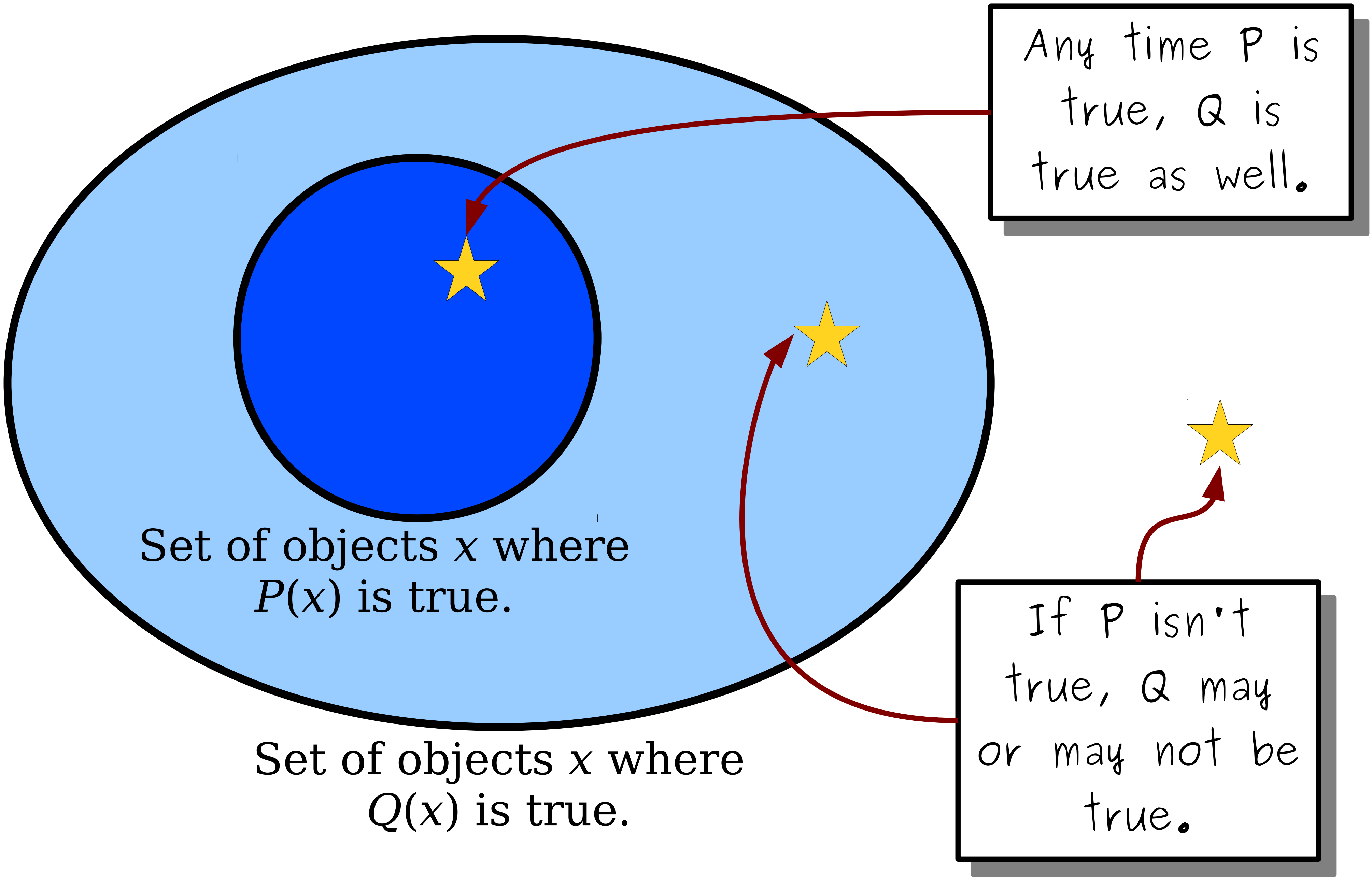
# What Implications Mean

- In mathematics, a statement of the form

  **For any $x$, if $P(x)$ is true, then $Q(x)$ is true**

  means that any time you find an object $x$ where $P(x)$ is true, you will find that $Q(x)$ is also true.
- There is no discussion of correlation or causation here. It simply means that if you find that $P(x)$ is true, you'll find that $Q(x)$ is true.

# Implication, Diagrammatically



# Negations

- The ***negation*** of a statement $X$ is a statement meaning the opposite of $X$.
- Examples:
  - The negation of “1 + 1 = 2” is “1 + 1 ≠ 2.”
  - The negation of “everything is awesome” is “something is not awesome.”
  - The negation of “there's something happenin' round here” is “nothing is happenin' round here.”
- Let's talk about the interplay between negations and implications.

# Puppies Are Adorable

- Consider the statement

**If *x* is a puppy, then I love *x*.**

- Can you explain why the following statement is *not* the negation of the original statement?

**If *x* is a puppy, then I don't love *x*.**

- This second statement is too strong.
  - The initial statement means “I love all puppies.”
  - The second statement says “I don't love *any* puppies.”
- Here's the correct negation:

**There is some puppy that I don't love.**

The negation of the statement

**“For any *x*, if *P*(*x*) is true, then *Q*(*x*) is true”**

is the statement

**“There is at least one *x* where *P*(*x*) is true and *Q*(*x*) is false.”**

***The negation of an implication is not an implication!***

# Proof by Contrapositive

# The Contrapositive

- The ***contrapositive*** of the implication "If $P$, then $Q$" is the implication "If ***not*** $Q$, then ***not*** $P$."
- For example:
  - "If Harry had opened the right book, then Harry would have learned about Gillyweed."
  - Contrapositive: "If Harry didn't learn about Gillyweed, then Harry didn't open the right book."
- Another example:
  - "If I store the cat food inside, then wild raccoons will not steal my cat food."
  - Contrapositive: "If wild raccoons stole my cat food, then I didn't store it inside."

To prove the statement

**“If *P* is true, then *Q* is true,”**

you may instead prove the statement

**“If *Q* is false, then *P* is false.”**

This is called a ***proof by contrapositive***.

***Theorem:*** For any $n \in \mathbb{Z}$, if $n^2$ is even, then $n$ is even.

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We're starting this proof by telling the reader that it's a proof by contrapositive. This helps cue the reader into what's about to come next.

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Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove. It also acts as a sanity check by forcing us to write out what we think the contrapositive is.

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We've said that we're going to prove this new implication, so let's go do it! The rest of this proof will look a lot like a standard direct proof.

***Theorem:*** For any $n \in \mathbb{Z}$, if $n^2$ is even, then $n$ is even.

***Proof:*** By contrapositive; we prove that if $n$ is odd, then $n^2$ is odd.

Since $n$ is odd, there is some integer $k$ such that $n = 2k + 1$.

***Theorem:*** For any $n \in \mathbb{Z}$, if $n^2$ is even, then $n$ is even.

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Since $n$ is odd, there is some integer $k$ such that $n = 2k + 1$. Squaring both sides of this equality and simplifying gives the following:

$$n^2 = (2k + 1)^2$$

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$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

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$n = 2$
and s

From
(nam
There

The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.

2. Explicitly state the contrapositive of what we want to prove.

3. Go prove the contrapositive.

# Biconditionals

- Combined with what we saw on Wednesday, we have proven that, if $n$ is an integer:

**If $n$ is even, then $n^2$ is even.**

**If $n^2$ is even, then $n$ is even.**

- Therefore, if $n$ is an integer:

**$n$ is even if and only if $n^2$ is even.**

- "If and only if" is often abbreviated ***iff***:

**$n$ is even iff $n^2$ is even.**

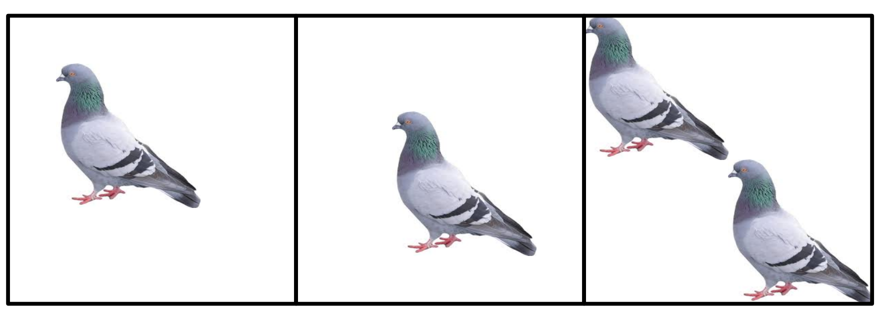
# Proving Biconditionals

- To prove ***P* iff *Q***, you need to prove that *P* implies *Q* and that *Q* implies *P*.
- You can use any proof techniques you'd like to show each of these statements.
  - In our case, we used a direct proof and a proof by contrapositive.
- Just make sure to cover both directions.

Application: ***The Pigeonhole Principle***

# The Pigeonhole Principle

- Suppose that you have *n* pigeonholes.
- Suppose that you have $m > n$ pigeons.
- If you put the *m* pigeons into the *n* pigeonholes, some pigeonhole must have more than one pigeon in it.



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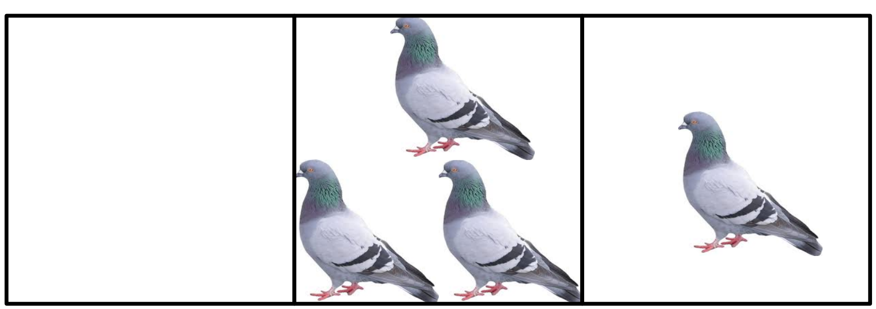

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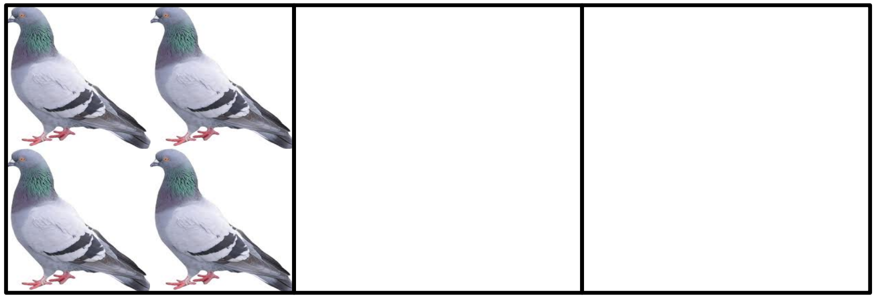

# The Pigeonhole Principle

- Suppose that $m$ objects are distributed into $n$ bins.
- We want to prove the statement

  **If $m > n$, then some bin contains at least two objects.**

- What is the contrapositive of this statement?

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**If "some bin contains at least two objects" is false, then "$m > n$" is false.**

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**If "some bin contains at least two objects" is false, then $m \leq n$.**

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Is this a universal statement or an existential statement?

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There is a bin that has two or more objects in it.

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**If "some bin contains at least two objects" is false,**
**then *m* ≤ *n*.**

There is a bin that doesn't have 0 or 1 objects in it

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Every bin

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There is a bin that doesn't have 0 or 1 objects in it

Every bin has at most one object in it.

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Suppose that *m* objects are distributed into *n* bins.

We want to prove the statement

**If *m* > *n*, then some bin contains at least two objects.**

What is the contrapositive of this statement?

**If every bin contains at most one object,**
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- Suppose that $m$ objects are distributed into $n$ bins.
- We want to prove the statement

**If $m > n$, then some bin contains at least two objects.**

- What is the contrapositive of this statement?

**If every bin contains at most one object, then $m \leq n$.**

- Look at the definitions of $m$ and $n$. Does this make sense?

**Theorem:** Let $m$ objects be distributed into $n$ bins. If $m > n$, then some bin contains at least two objects.

**Proof:** By contrapositive; we prove that if every bin contains at most one object, then $m \leq n$.

Let $x_i$ denote the number of objects in bin $i$. Since $m$ is the number of total objects, we see that

$$m = x_1 + x_2 + \ldots + x_n.$$

We're assuming every bin has at most one object. In our notation, this means that $x_i \leq 1$ for all $i$. Using this inequality, we get the following:

$$\begin{aligned} m &= x_1 + x_2 + \ldots + x_n \\ &\leq 1 + 1 + \ldots + 1 \qquad (n \textit{ times}) \\ &= n. \end{aligned}$$

So $m \leq n$, as required. ■

# Some Simple Applications

- Any group of 367 people must have a pair of people that share a birthday.
  - 366 possible birthdays (pigeonholes)
  - 367 people (pigeons)
- Two people in San Francisco have the exact same number of hairs on their head.
  - Maximum number of hairs ever found on a human head is no greater than 500,000.
  - There are over 800,000 people in San Francisco.
- Each day, two people in New York City drink the same amount of water, to the thousandth of a fluid ounce.
  - No one can drink more than 50 gallons of water each day.
  - That's 6,400 fluid ounces. This gives 6,400,001 possible numbers of thousands of fluid ounces.
  - There are about 8,000,000 people in New York City proper.

# Time-Out for Announcements!

# Handouts

- There are six total handouts for today, three of which are available outside in hard copy:
  - Handout 05: Problem Set Policies
  - Handout 06: Honor Code Policies
  - Handout 07: Guide to Proofs
  - Handout 08: Mathematical Vocabulary
  - Handout 09: Guide to Indirect Proofs
  - Handout 10: Problem Set One
- Be sure to read over Handouts 05 – 09; there's a lot of really important information in there!

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# Announcements

- Problem Set 1 goes out today!
- ***Checkpoint*** due Monday, January 11.
  - Grade determined by attempt rather than accuracy. It's okay to make mistakes – we want you to give it your best effort, even if you're not completely sure what you have is correct.
  - We will get feedback back to you with comments on your proof technique and style.
  - The more effort you put in, the more you'll get out.
- ***Remaining problems*** due Friday, January 15.
  - Feel free to email us with questions, stop by office hours, or ask questions on Piazza!

# Submitting Assignments

- This quarter, we will be using GradeScope to handle assignment submissions. Visit www.gradescope.com and enter code `942VB9`.
- Summary of the late policy:
  - Everyone has *three* 24-hour late days.
  - Late days can't be used on checkpoints.
  - Nothing may be submitted more than three days past the due date.
- Because submission times are recorded automatically, we're strict about the submission deadlines.
- ***Very good idea:*** Leave at least two hours buffer time for your first assignment submission, just in case something goes wrong.
- ***Very bad idea:*** Wait until the last minute to submit.

# Working in Groups

- You can work on the problem sets individually, in a pair, or in a group of three.
- Each group should only submit a single problem set, and should submit it only once.
- Full details about the problem sets, collaboration policy, and Honor Code can be found in Handouts 05 and 06.

# A Note on the Honor Code

Office hours start tonight!

Schedule is available
on the course website.

Back to CS103!

# Proof by Contradiction

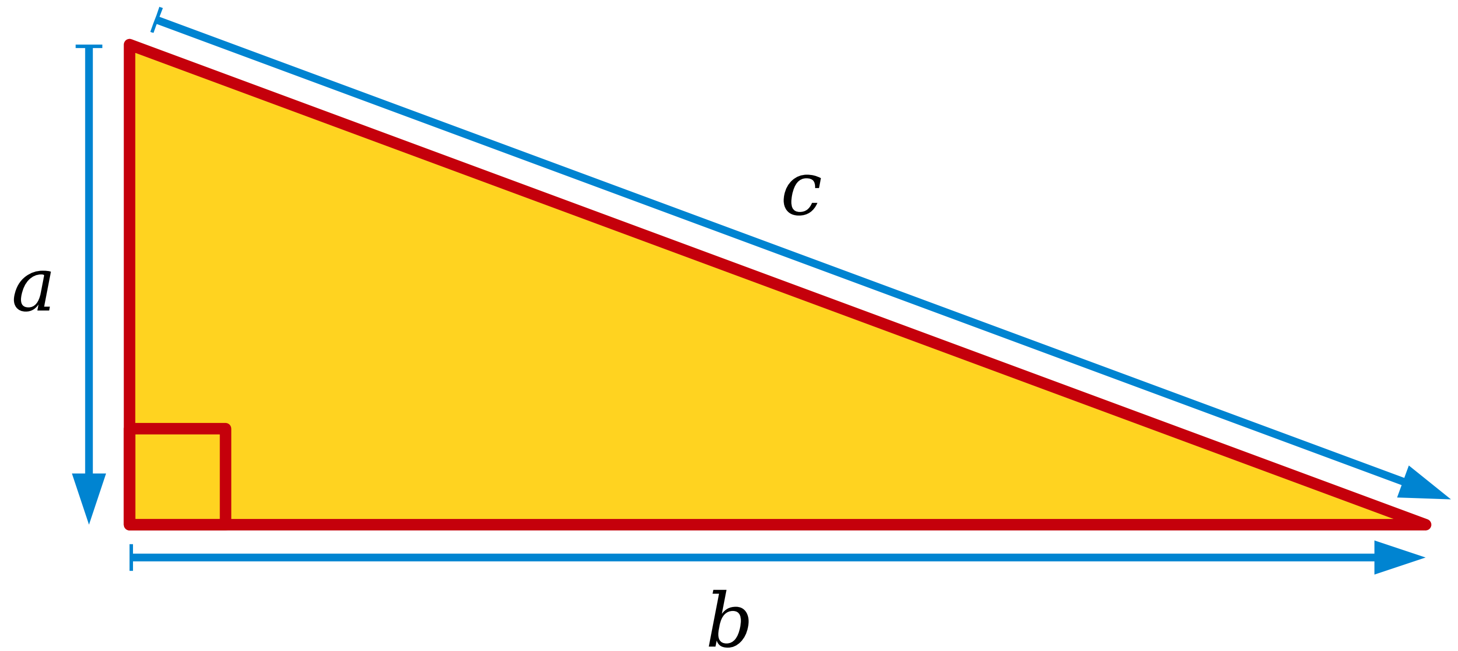
“When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth.”

- Sir Arthur Conan Doyle, *The Adventure of the Blanched Soldier*

# Proof by Contradiction

- A ***proof by contradiction*** is a proof that works as follows:
  - To prove that $P$ is true, assume that $P$ is *not* true.
  - Starting with this assumption, use logical reasoning to conclude something that is clearly impossible.
    - For example, that $1 = 0$, that $x \in S$ and $x \notin S$, etc.
  - This means that if $P$ is false, something impossible happens.
  - Therefore, $P$ can't be false, so it must be true.

# Right Triangles




$$a^2 + b^2 = c^2$$

***Claim:*** $a + b \geq c$

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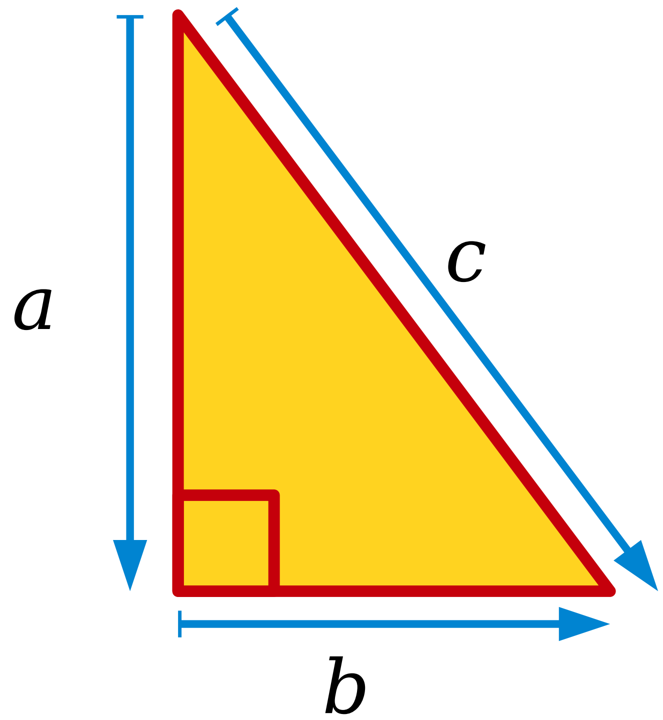


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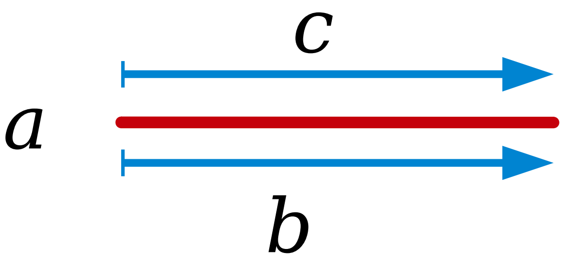


$$a^2 + b^2 = c^2$$

**Claim:** $a + b \geq c$

***Theorem:*** For all natural numbers $a$, $b$, and $c$ where $a^2+b^2 = c^2$, we have $a + b \geq c$.

***Theorem:*** For all natural numbers $a$, $b$, and $c$ where $a^2+b^2 = c^2$, we have $a + b \geq c$.

To prove this by contradiction, we're going to assume the negation of the above statement.

***Theorem:*** For all natural numbers $a$, $b$, and $c$ where $a^2 + b^2 = c^2$, we have $a + b \geq c$.

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What is its negation?

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"For all natural numbers $a$, $b$, and $c$ where $a^2 + b^2 = c^2$, we have $a + b \geq c$"

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Notice that we're announcing

1. that this is a proof by contradiction, and
2. what, specifically, we're assuming.

This helps the reader understand where we're going. Remember – proofs are meant to be read by other people!

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We're numbering our intermediate stages to make it easier to refer to them later. If you have a calculation-heavy proof, we recommend structuring it like this.

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The three key pieces:

1. State that the proof is by contradiction.
2. State what you are assuming is the negation of the statement to prove.
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In CS103, please include all these steps in your proofs!

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# Proving Implications

- To prove the implication

**"If *P* is true, then *Q* is true."**

- you can use these three techniques:
  - **Direct Proof.**
    - Assume *P* and prove *Q*.
  - **Proof by Contrapositive**
    - Assume not *Q* and prove not *P*.
  - **Proof by Contradiction**
    - … what does this look like?

# Negating Implications

- To prove the statement

  **"For any *x*, if *P*(*x*), then *Q*(*x*)"**

  by contradiction, we do the following:
  - Assume this statement is false.
  - Derive a contradiction.
  - Conclude that the statement is true.
- What is the negation of this statement?

  **"There is an *x* where *P*(*x*) is true and *Q*(*x*) is false"**

# Contradictions and Implications

- To prove the statement

  **"If *P* is true, then *Q* is true"**

  using a proof by contradiction, do the following:
  - Assume that *P* is true and that *Q* is false.
  - Derive a contradiction.
  - Conclude that if *P* is true, *Q* must be as well.

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***Proof:*** Assume for the sake of contradiction that $n$ is an integer and that $n^2$ is even, but that $n$ is odd.

Since $n$ is odd we know that there is an integer $k$ such that

$$n = 2k + 1. \tag{1}$$

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# Rational and Irrational Numbers

# Rational and Irrational Numbers

- A number $r$ is called a ***rational number*** if it can be written as

$$r = \frac{p}{q}$$

where $p$ and $q$ are integers and $q \neq 0$.

- A number that is not rational is called ***irrational***.

# Simplest Forms

- *By definition,* if $r$ is a rational number, then $r$ can be written as $p / q$ where $p$ and $q$ are integers and $q \neq 0$.
- ***Theorem:*** If $r$ is a rational number, then $r$ can be written as $p / q$ where $p$ and $q$ are integers, $q \neq 0$, and $p$ and $q$ have no common factors other than 1 and -1.
  - That is, $r$ can be written as a fraction in simplest form.
- We're just going to take this for granted for now, though with the techniques you'll see later in the quarter you'll be able to prove it!

***Question:*** Are all real numbers rational?

***Theorem:*** $\sqrt{2}$ is irrational.

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***Theorem:*** $\sqrt{2}$ is irrational.

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***Theorem:*** $\sqrt{2}$ is irrational.

***Proof:*** Assume for the sake of contradiction that $\sqrt{2}$ is rational. This means that there must be integers $p$ and $q$ where $q \neq 0$, where $p$ and $q$ have no common divisors other than 1 and -1, and where

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***Theorem:*** √2 is irrational.

***Proof:*** Assume for the sake of contradiction that √2 is rational. This means that there must be integers $p$ and $q$ where $q \neq 0$, where $p$ and $q$ have no common divisors other than 1 and -1, and where

$$p \,/\, q = \sqrt{2}. \qquad (1)$$

Multiplying both sides of equation (1) by $q$ and squaring both sides

The three key pieces:

1. State that the proof is by contradiction.
2. State what you are assuming is the negation of the statement to prove.
3. State you have reached a contradiction and what the contradiction entails.

In CS103, please include all these steps in your proofs!

Equation (3) shows that $q^2$ is even. Our earlier theorem tells us that, because $q^2$ is even, $q$ must also be even. But this is not possible – we know that $p$ and $q$ have no common factors other than 1 and -1, but we've shown that $p$ and $q$ must have two as a common factor.

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***Theorem:*** $\sqrt{2}$ is irrational.

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Vi Hart on Pythagoras and
the Square Root of Two:

**http://www.youtube.com/watch?v=X1E7I7_r3Cw**

# What We Learned

- ***What's an implication?***
  - It's statement of the form "if *P*, then *Q*," and states that if *P* is true, then *Q* is true.
- ***What is a proof by contrapositive?***
  - It's a proof of an implication that instead proves its contrapositive.
  - (The contrapositive of "if *P*, then *Q*" is "if not *Q*, then not *P*.")
- ***What's a proof by contradiction?***
  - It's a proof of a statement *P* that works by showing that *P* cannot be false.

# Next Time

- **Mathematical Logic**
  - How do we formalize the reasoning from our proofs?
- **Propositional Logic**
  - Reasoning about simple statements.
- **Propositional Equivalences**
  - Simplifying complex statements.

# Appendix: Negating Statements

## Negating Universal Statements

**"For all *x*, P(*x*) is true"**

becomes

**"There is an *x* where P(*x*) is false."**

## Negating Existential Statements

**"There exists an *x* where P(*x*) is true"**

becomes

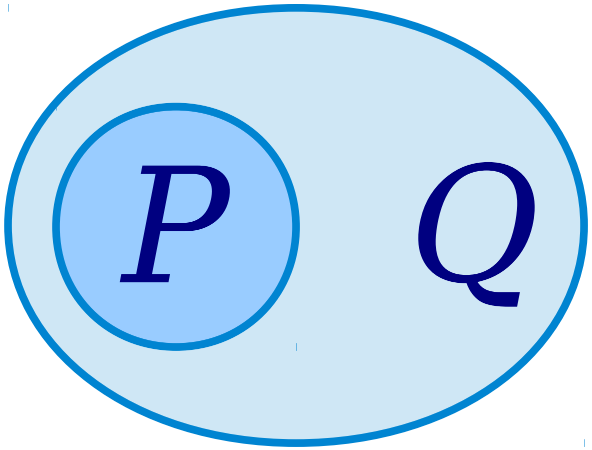
**"For all *x*, P(*x*) is false."**

## Negating Implications

**"For every *x*, if *P*(*x*) is true, then *Q*(*x*) is true"**

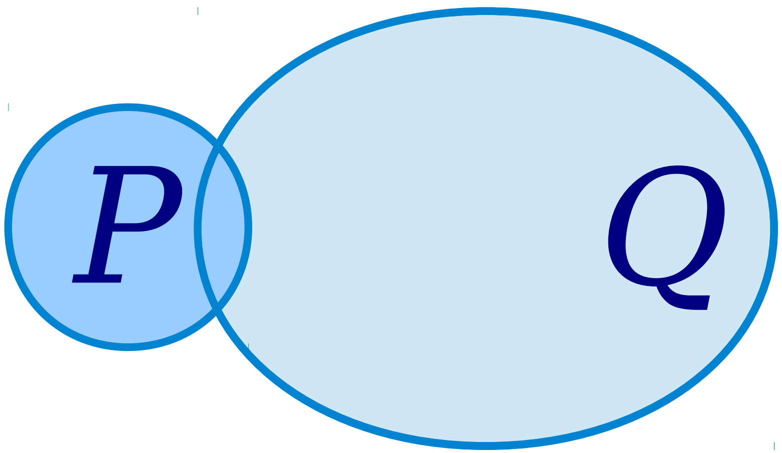
becomes

**"There is an *x* where *P*(*x*) is true and *Q*(*x*) is false"**




**$P(x)$ implies $Q(x)$**

"If $P(x)$ is true, then $Q(x)$ is true."

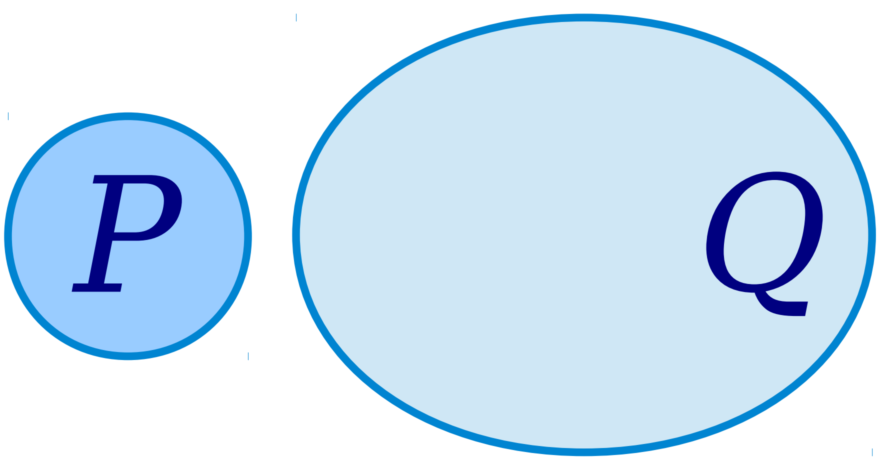



**$P(x)$ does not imply $Q(x)$**

**-and-**

**$P(x)$ does not imply not $Q(x)$**

"Sometimes $P(x)$ is true and $Q(x)$ is true,
-and-
sometimes $P(x)$ is true and $Q(x)$ is false."




**$P(x)$ implies not $Q(x)$**

If $P(x)$ is true, then $Q(x)$ is false