Mathematical Logic

Part Three

Recap from Last Time

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects, and
  - ***functions*** that map objects to one another,
  - ***quantifiers*** that allow us to reason about multiple objects simultaneously.

"For any natural number n,
n is even iff n^2 is even"

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow Even(n^2)))$$

∀ is the ***universal quantifier***
and says "for any choice of n,
the following is true."

Some muggles are intelligent.

$$\exists m.\ (Muggle(m) \land Intelligent(m))$$

∃ is the ***existential quantifier*** and says "for some choice of m, the following is true."

“All *P*'s are *Q*'s”

translates as

$$\forall x.\ (P(x) \to Q(x))$$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$$\forall x.\ (P(x) \to Q(x))$$

If x is a counterexample, it *must* have property P but not have property Q.

“Some *P* is a *Q*”

translates as

$$\exists x.\ (P(x) \land Q(x))$$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$$\exists x.\ (P(x) \land Q(x))$$

If x is an example, it *must* have property P on top of property Q.

Good Pairings

- The ∀ quantifier *usually* is paired with →.
- The ∃ quantifier *usually* is paired with ∧.
- In the case of ∀, the → connective prevents the statement from being *false* when speaking about some object you don't care about.
- In the case of ∃, the ∧ connective prevents the statement from being *true* when speaking about some object you don't care about.

The Art of Translation

Using the predicates

- *Person*(p), which states that p is a person, and
- *Loves*(x, y), which states that x loves y,

write a sentence in first-order logic that means “everybody loves someone else.”

Everybody loves someone else

Every person loves some other person

Every person p loves some other person

$\forall p. \ (Person(p) \rightarrow$

p loves some other person

$)$

$\forall p.\ (Person(p) \rightarrow$

there is some other person that p loves

$)$

$\forall p. \ (Person(p) \rightarrow$

there is a person other than p that p loves

$)$

$\forall p. \ (Person(p) \rightarrow$

there is a person q, other than p, where p loves q

$)$

$\forall p.\ (Person(p) \rightarrow$

there is a person q, other than p, where

p loves q

$)$

$\forall p.\ (Person(p) \rightarrow$

$\exists q.\ (Person(q) \land$*, other than p, where*

p loves q

$)$

$)$

$\forall p. \ (Person(p) \rightarrow$

$\quad \exists q. \ (Person(q) \land p \neq q \land$

$\quad\quad$ *p loves q*

$\quad)$

$)$

$$
\begin{aligned}
&\forall p.\ (Person(p) \to \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

Using the predicates

- *Person*(p), which states that p is a person, and
- *Loves*(x, y), which states that x loves y,

write a sentence in first-order logic that means “there is a person that everyone else loves.”

There is a person that everyone else loves

There is a person p where everyone else loves p

$\exists p. \ (Person(p) \ \wedge$

everyone else loves p

$)$

$\exists p. \; (Person(p) \; \wedge$

every other person q loves p

$)$

$\exists p.\ (Person(p)\ \wedge$

every person q, other than p, loves p

$)$

$\exists p.\ (Person(p)\ \land$

$\quad \forall q.\ (Person(q)\ \land\ p \neq q \rightarrow$

q loves p

$)$

$$
\begin{aligned}
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Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
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For Comparison

$$\forall p.\ (Person(p) \to \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

For every person,

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$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \to Loves(q, p)))$$

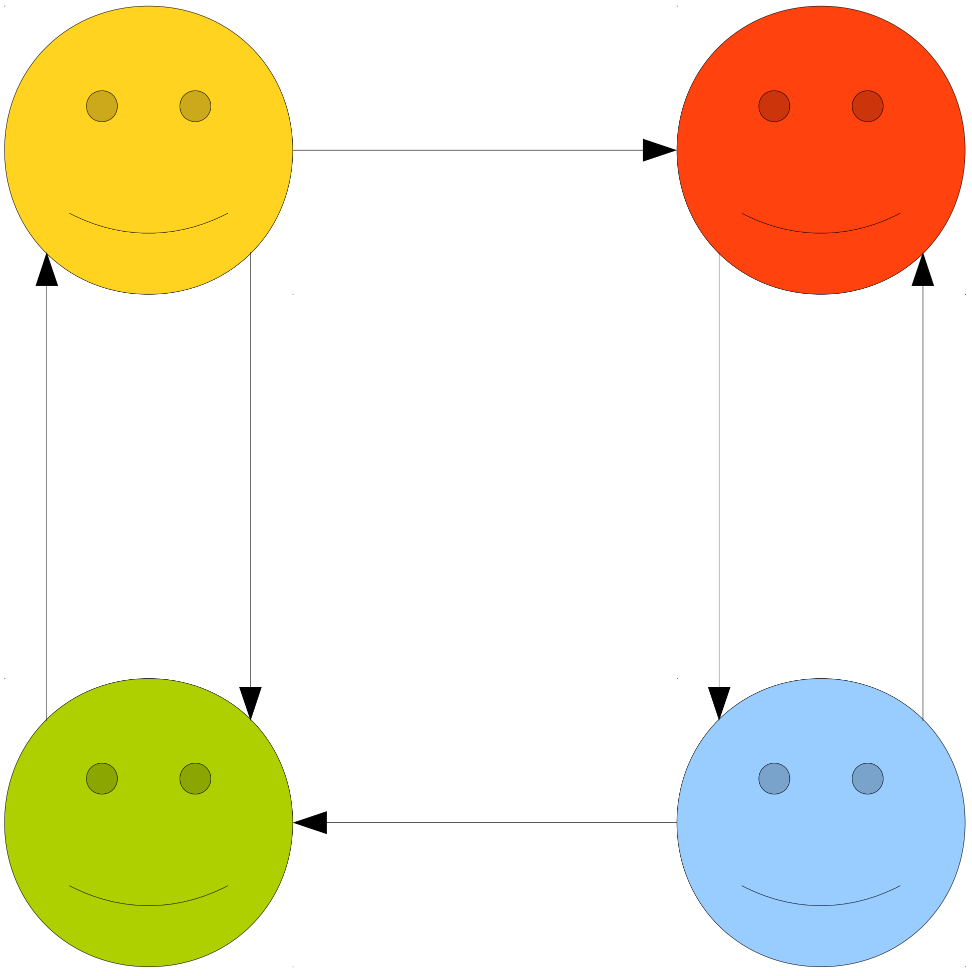
There is some person

who everyone

who isn't them

loves.

Everyone Loves Someone Else



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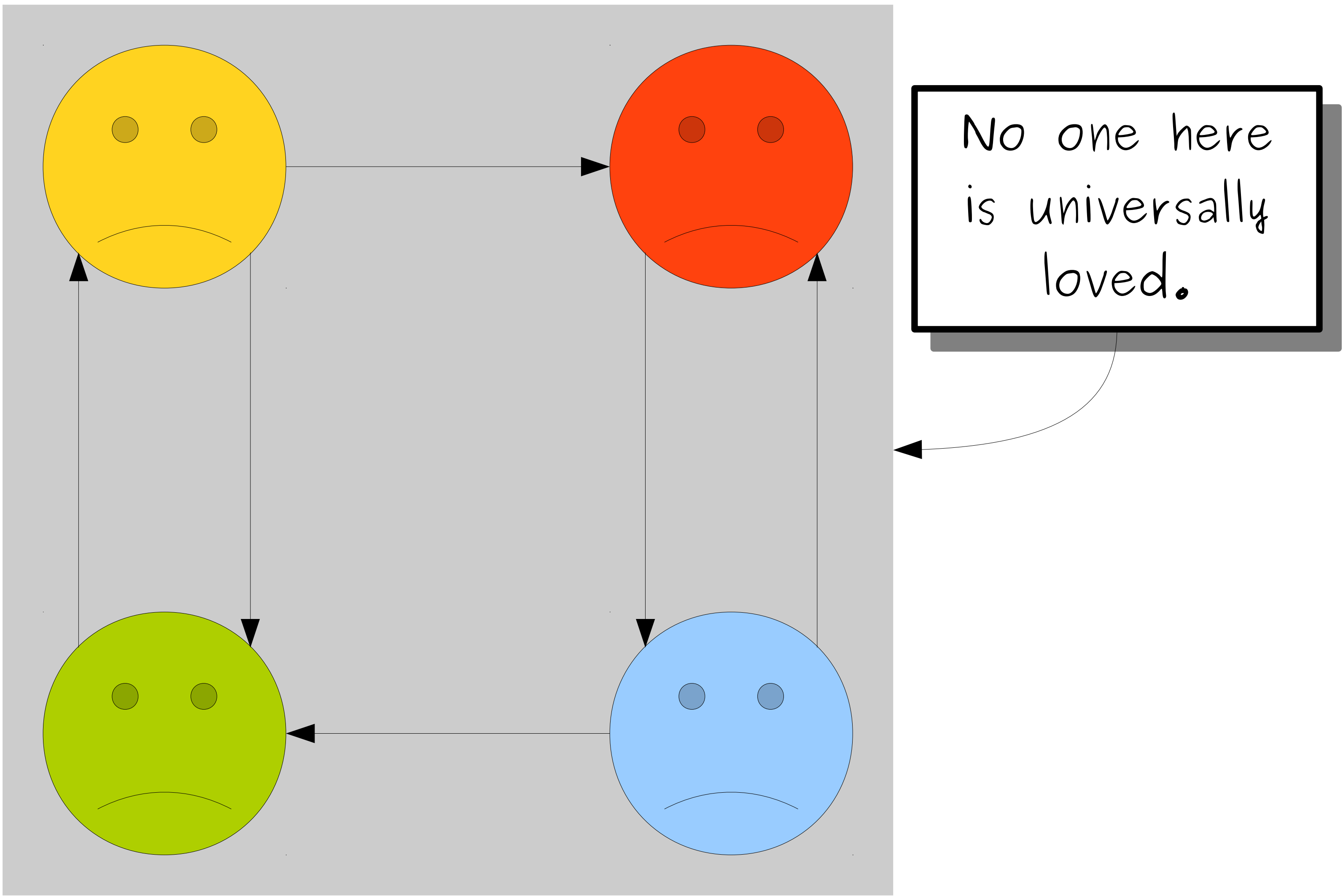
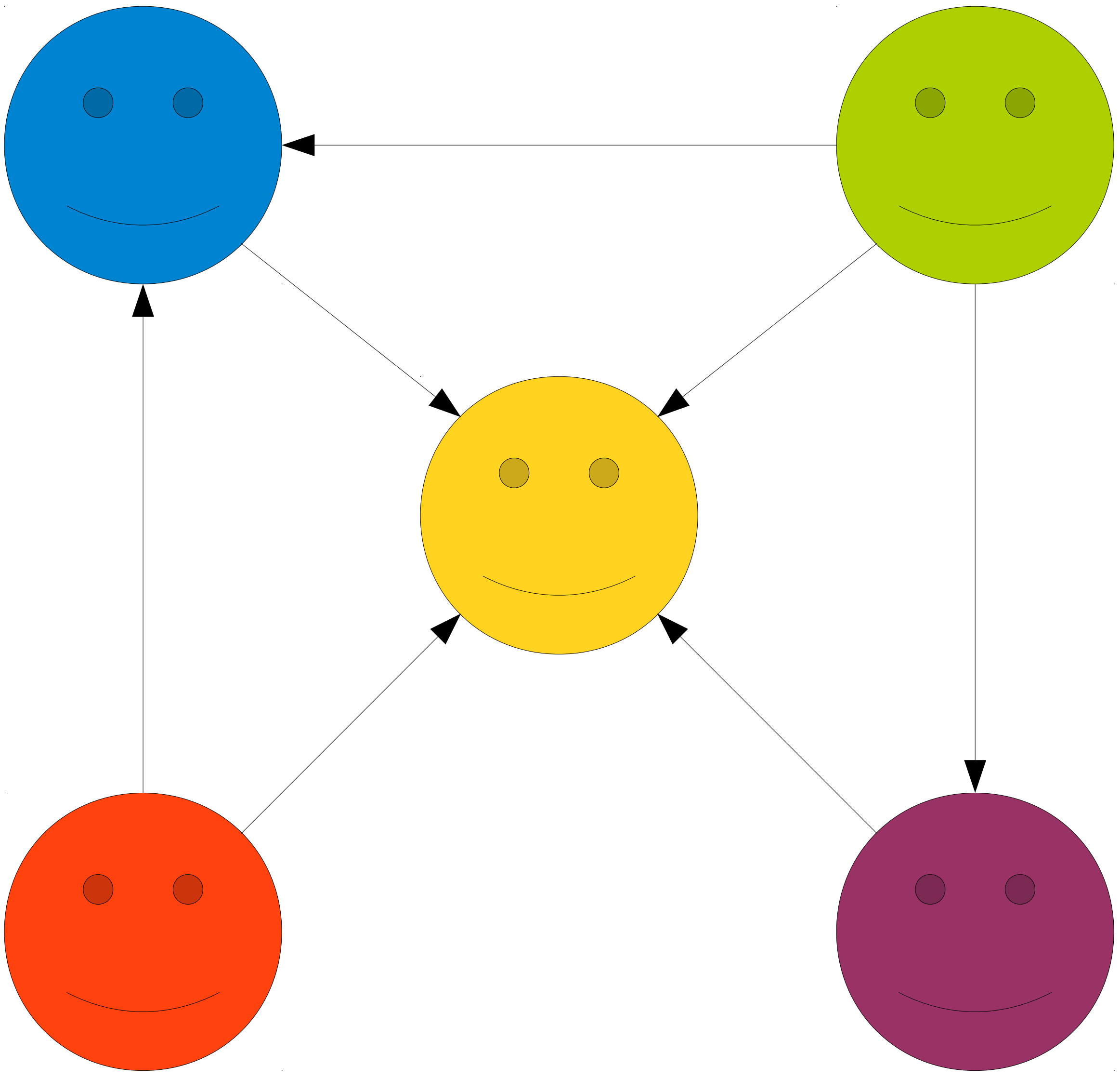


There is Someone Everyone Else Loves



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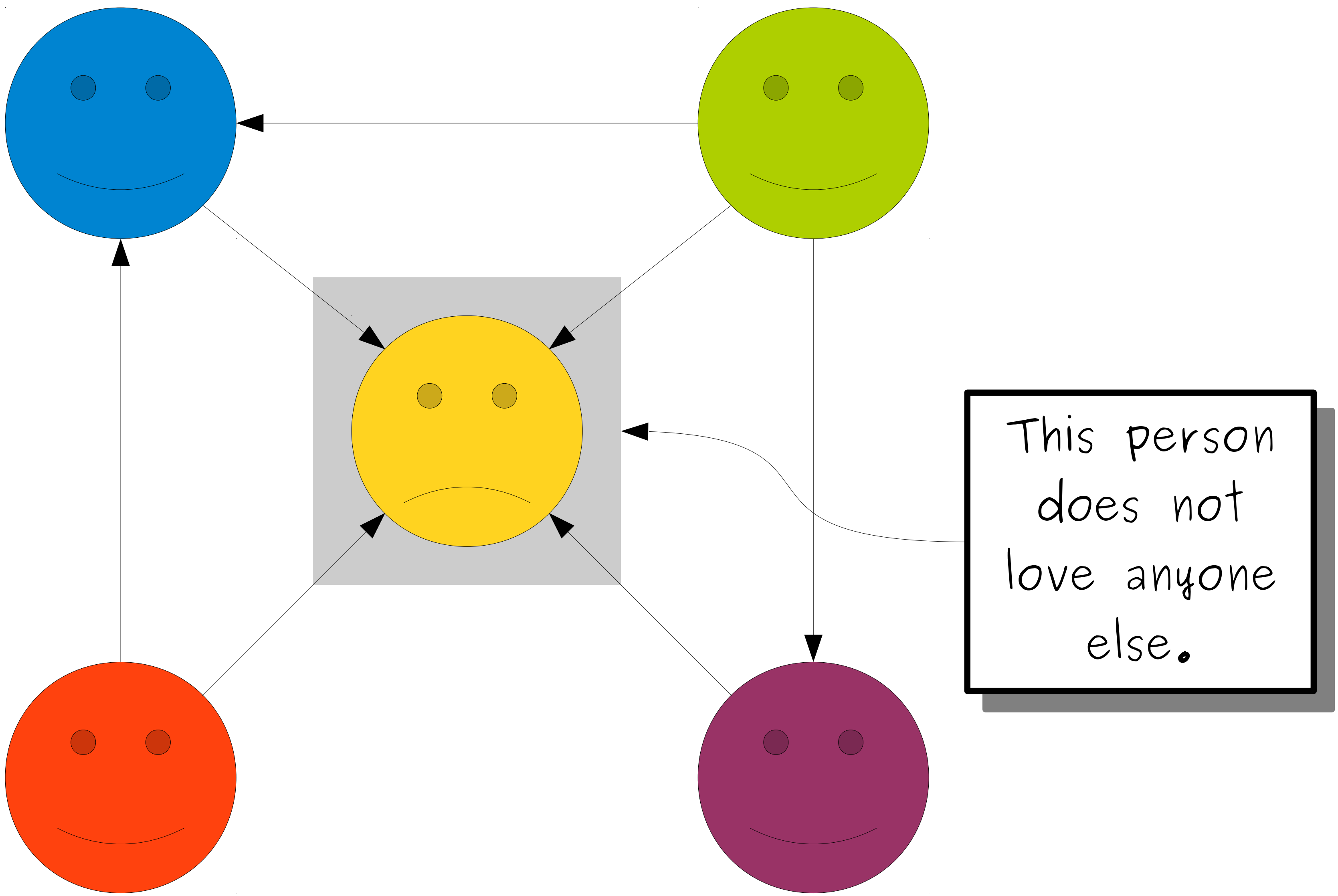
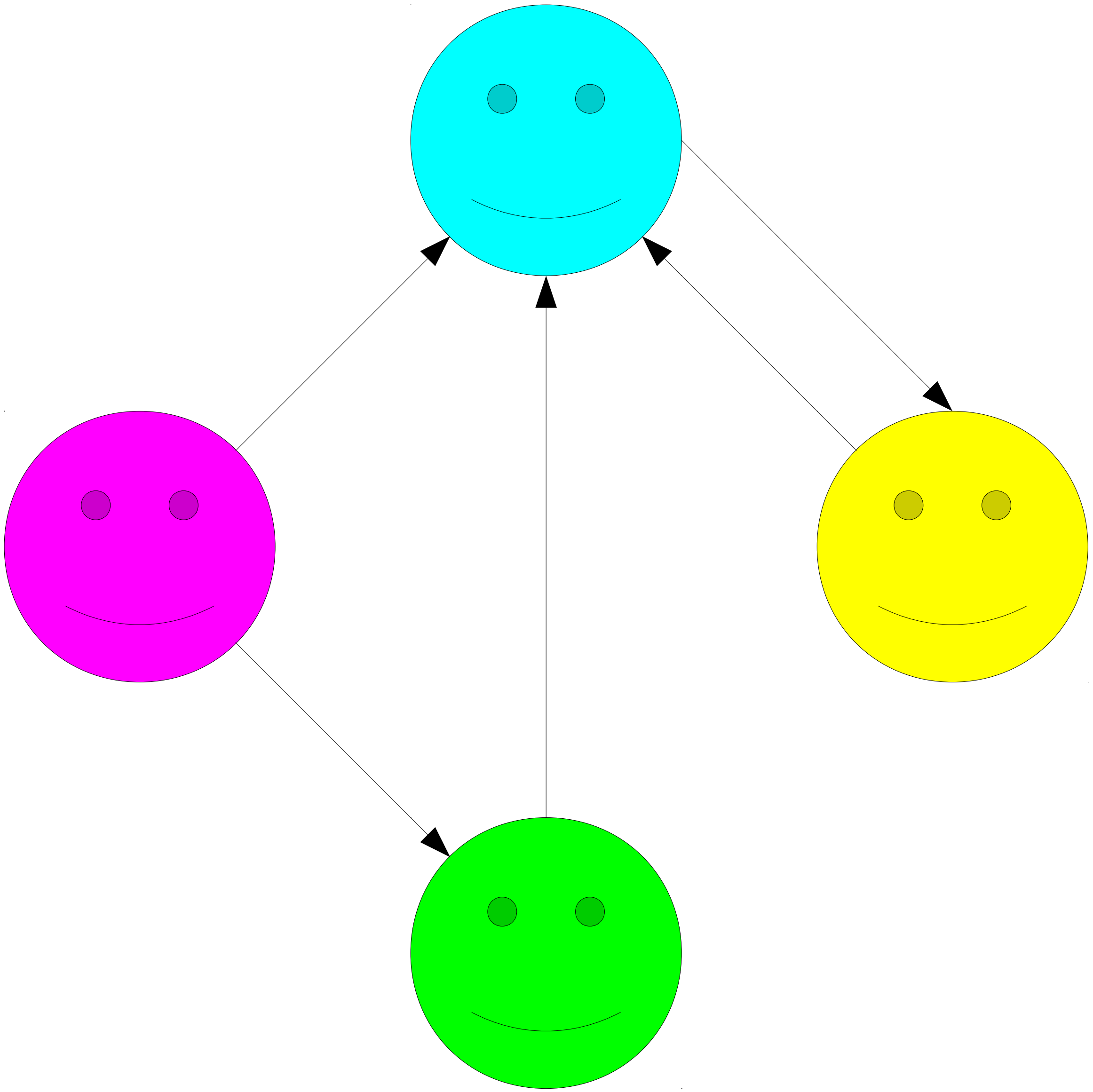


Everyone Loves Someone Else ***and*** There is Someone Everyone Else Loves



$$\forall p.\ (Person(p) \rightarrow \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

For every person,

there is some person

who isn't them

that they love.

$$\land$$

$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \rightarrow Loves(q, p)))$$

There is some person

who everyone

who isn't them

loves.

Quantifier Ordering

- The statement

$$\forall x.\ \exists y.\ P(x, y)$$

  means "for any choice x, there's some y where $P(x, y)$ is true."
- The choice of y can be different every time and can depend on x.

Quantifier Ordering

- The statement

$$\exists x.\ \forall y.\ P(x, y)$$

  means "there is some x where for any choice of y, we get that $P(x, y)$ is true."
- Since the inner part has to work for any choice of y, this places a lot of constraints on what x can be.

Order matters when mixing existential and universal quantifiers!

Time-Out for Announcements!

Problem Set Two

- Problem Set One was due at 3:00PM today. You can submit late up until Monday at 3:00PM.
- Problem Set Two goes out now.
  - Checkpoint due Tuesday at 3:00PM (there's no class on Monday).
  - Remaining problems due Friday.
- Play around with propositional logic, first-order logic, and their applications!
- As always, feel free to ask us questions!

Your Questions

“Hi Keith, how logical are you in real life, and has there ever been a time that it didn't serve you too well?”

I am not a very logical person. ☺

I see logic as a great tool for reasoning about mathematics and thinking about argumentative structure, but I don't see it as a guiding principle of how to live one's life. Since I think logic is something we invented, I wouldn't put too much faith in it as the absolute truth about the world.

"If you can prove something with one method of proof, is it always possible to prove it with the other methods of proof?"

Not necessarily! There are some proofs that cannot be done without using contradiction, and there are some proofs that only work with a specific set of starting assumptions. Look up "intuitionistic logic" or "reverse mathematics" for how we know this!

"What software do you use to create the slides and handouts? I'm working on my p-set in latex, but my equations look different from those in the CS 103 documents."

Um… LibreOffice. I should switch to LaTeX. ☺

"Did you say in lecture that an existential statement is considered not true until proof is found? Wouldn't there be an "unproven" state rather than a "not true/false" state?"

I may have misspoken on this one. In classical logic, there's an "objective truth" to every statement (it's either true or false). For an existential statement, if there truly are no positive examples where it holds, the statement is false.

Independently, there's the question of whether we know whether the statement is true. Just because we can't find a positive example doesn't mean none exists!

"I want to work at a company this summer, but I also want to do research. How should I decide between the two?"

This is a good question, but I don't have time for it today. Can someone ask this again for next time?

Back to CS103!

Set Translations

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means "the empty set exists."

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write a sentence in first-order logic that means “the empty set exists.”

First-order logic doesn't have set operators or symbols “built in.” If we only have the predicates given above, how might we describe this?

The empty set exists.

There is some set S that is empty.

$\exists S. \ (Set(S) \ \land$

S is empty.

$)$

$\exists S. \; (Set(S) \; \wedge$

there are no elements in S

$)$

$\exists S.\ (Set(S)\ \land$

$\neg$ *there is an element in S*

$)$

$\exists S.\ (Set(S)\ \wedge$

$\neg$*there is an element x in S*

$)$

$$\exists S.\ (Set(S) \land$$
$$\quad \neg \exists x.\ x \in S$$
$$)$$

$$\exists S. \ (Set(S) \land \neg \exists x. \ x \in S)$$

$$\exists S. \ (Set(S) \land \neg \exists x. \ x \in S)$$

$\exists S. \ (Set(S) \land$

there are no elements in S

$)$

$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$

$\exists S.\ (Set(S) \land$

every object does not belong to S

$)$

$\exists S. \, (Set(S) \land \neg \exists x. \, x \in S)$

$\exists S. \, (Set(S) \land$

every object x does not belong to S

)

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land$$
$$\quad \forall x.\ \neg(x \in S)$$
$$)$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ \neg (x \in S))$$

Both of these translations are correct. Just like in propositional logic, there are many different equivalent ways of expressing the same statement in first-order logic.

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means “two sets are equal if and only if they contain the same elements.”

Two sets are equal if and only if they have the same elements.

Any two sets are equal if and only if they have the same elements.

Any two sets S and T are equal if and only if they have the same elements.

$\forall S. \ (Set(S) \rightarrow$

$\quad \forall T. \ (Set(T) \rightarrow$

S and T are equal if and only if they have the same elements.

$\quad)$

$)$

$\forall S.\ (Set(S) \rightarrow$

$\quad \forall T.\ (Set(T) \rightarrow$

$\quad\quad (S = T$ *if and only if they have the same elements.*$)$

$\quad)$

$)$

$\forall S.\ (Set(S) \rightarrow$

$\quad \forall T.\ (Set(T) \rightarrow$

$\quad\quad (S = T \leftrightarrow$ *they have the same elements.*$)$

$\quad)$

$)$

$\forall S.\ (Set(S) \to$

$\quad \forall T.\ (Set(T) \to$

$\quad\quad (S = T \leftrightarrow$ *S and T have the same elements.*$)$

$\quad)$

$)$

$\forall S.\ (Set(S) \rightarrow$

$\quad \forall T.\ (Set(T) \rightarrow$

$\quad\quad (S = T \leftrightarrow$ *every element of S is an element of T and vice-versa*)

$\quad)$

$)$

$\forall S. \ (Set(S) \rightarrow$

$\quad \forall T. \ (Set(T) \rightarrow$

$\quad\quad (S = T \leftrightarrow$ *x is an element of S if and only if x is an element of T*)

$\quad)$

$)$

$$
\begin{aligned}
&\forall S.\ (Set(S) \rightarrow \\
&\quad \forall T.\ (Set(T) \rightarrow \\
&\qquad (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T)) \\
&\quad) \\
&)
\end{aligned}
$$

$$
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&)
\end{aligned}
$$

You sometimes see the universal quantifier pair with the ↔ connective. This is especially common when talking about sets because two sets are equal when they have precisely the same elements.

Mechanics: Negating Statements

Negating Quantifiers

- We spent much of Monday's lecture discussing how to negate propositional constructs.
- How do we negate statements with quantifiers in them?

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	For some choice of x, $\neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For any choice of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For any choice of x, $P(x)$

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$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	$\forall x.\ P(x)$

Negating First-Order Statements

- Use the equivalences

$$\neg \forall x.\ \varphi \quad \equiv \quad \exists x.\ \neg \varphi$$

$$\neg \exists x.\ \varphi \quad \equiv \quad \forall x.\ \neg \varphi$$

  to negate quantifiers.
- Mechanically:
  - Push the negation across the quantifier.
  - Change the quantifier from ∀ to ∃ or vice-versa.
- Use techniques from propositional logic to negate connectives.

Taking a Negation

$$\forall x.\ \exists y.\ \mathit{Loves}(x, y)$$

("Everyone loves someone.")

$$\neg\forall x.\ \exists y.\ \mathit{Loves}(x, y)$$
$$\exists x.\ \neg\exists y.\ \mathit{Loves}(x, y)$$
$$\exists x.\ \forall y.\ \neg\mathit{Loves}(x, y)$$

("There's someone who doesn't love anyone.")

Two Useful Equivalences

- The following equivalences are useful when negating statements in first-order logic:

$$\neg(p \land q) \equiv p \to \neg q$$

$$\neg(p \to q) \equiv p \land \neg q$$

- These identities are useful when negating statements involving quantifiers.
  - $\land$ is used in existentially-quantified statements.
  - $\to$ is used in universally-quantified statements.
- When pushing negations across quantifiers, we *strongly recommend* using the above equivalences to keep $\to$ with $\forall$ and $\land$ with $\exists$.

Negating Quantifiers

- What is the negation of the following statement, which says “there is a cute puppy”?

$$\boldsymbol{\exists x. (Puppy(x) \land Cute(x))}$$

- We can obtain it as follows:

$$\neg \exists x. (Puppy(x) \land Cute(x))$$

$$\forall x. \neg (Puppy(x) \land Cute(x))$$

$$\forall x. (Puppy(x) \to \neg Cute(x))$$

- This says “every puppy is not cute.”
- Do you see why this is the negation of the original statement from both an intuitive and formal perspective?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

("There is a set that doesn't contain anything")

$$\neg\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

$$\forall S.\ \neg(Set(S) \land \forall x.\ \neg(x \in S))$$

$$\forall S.\ (Set(S) \to \neg\forall x.\ \neg(x \in S))$$

$$\forall S.\ (Set(S) \to \exists x.\ \neg\neg(x \in S))$$

$$\forall S.\ (Set(S) \to \exists x.\ x \in S)$$

("Every set contains at least one element")

These two statements are *not* negations of one another. Can you explain why?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

("There is a set that doesn't contain anything")

$$\forall S.\ (Set(S) \land \exists x.\ (x \in S))$$

("Everything is a set that contains something")

Remember: ∀ usually goes with →, not ∧

Restricted Quantifiers

Quantifying Over Sets

- The notation

$$\forall x \in S.\ P(x)$$

  means "for any element x of set S, $P(x)$ holds."
- This is not technically a part of first-order logic; it is a shorthand for

$$\forall x.\ (x \in S \to P(x))$$

- How might we encode this concept?

$$\exists x \in S.\ P(x)$$

  Answer: $\exists x.\ (x \in S \land P(x))$.

Note the use of ∧ instead of → here.

Quantifying Over Sets

- The syntax

$$\forall x \in S.\ \varphi$$

$$\exists x \in S.\ \varphi$$

  is allowed for quantifying over sets.

- In CS103, feel free to use these restricted quantifiers, but please do not use variants of this syntax.
- For example, don't do things like this:

⚠ $\forall x$ with $P(x)$. $Q(x)$ ⚠

⚠ $\forall y$ such that $P(y) \land Q(y)$. $R(y)$. ⚠

⚠ $\exists P(x)$. $Q(x)$ ⚠

Expressing Uniqueness

Using the predicate

- $Level(l)$, which states that l is a level,

write a sentence in first-order logic that means “there is only one level.”

A fun diversion:

http://www.onemorelevel.com/game/there_is_only_one_level

There is only one level.

Something is a level, and nothing else is.

Some thing l is a level, and nothing else is.

Some thing l is a level, and nothing besides l is a level

$\exists l.\ (Level(l)\ \wedge$

nothing besides l is a level.

$)$

$\exists l.\ (Level(l)\ \land$

anything that isn't l isn't a level

$)$

$\exists l.\ (Level(l)\ \wedge$

any thing x that isn't l isn't a level

$)$

$$
\begin{aligned}
&\exists l.\ (Level(l)\ \land \\
&\quad \forall x.\ (x \neq l \rightarrow \textit{x isn't a level}) \\
&)
\end{aligned}
$$

$$
\exists l.\ (Level(l)\ \wedge \\
\quad \forall x.\ (x \neq l \rightarrow \neg Level(x)) \\
)
$$

$$
\begin{aligned}
&\exists l.\ (Level(l)\ \land \\
&\quad \forall x.\ (x \neq l \rightarrow \neg Level(x)) \\
&)
\end{aligned}
$$

$$\exists l.\ (Level(l)\ \land$$
$$\quad \forall x.\ (Level(x) \rightarrow x = l)$$
$$)$$

Expressing Uniqueness

- To express the idea that there is exactly one object with some property, we write that
  - there exists at least one object with that property, and that
  - there are no other objects with that property.
- You sometimes see a special "uniqueness quantifier" used to express this:

$$\exists! x.\ \varphi$$

- For the purposes of CS103, please do not use this quantifier. We want to give you more practice using the regular $\forall$ and $\exists$ quantifiers.

Next Time

- **A Long Weekend!**
- **Functions**
- **Classes of Functions**
- **First-Order Definitions**
- **Formalizing Cardinality**