Binary Relations

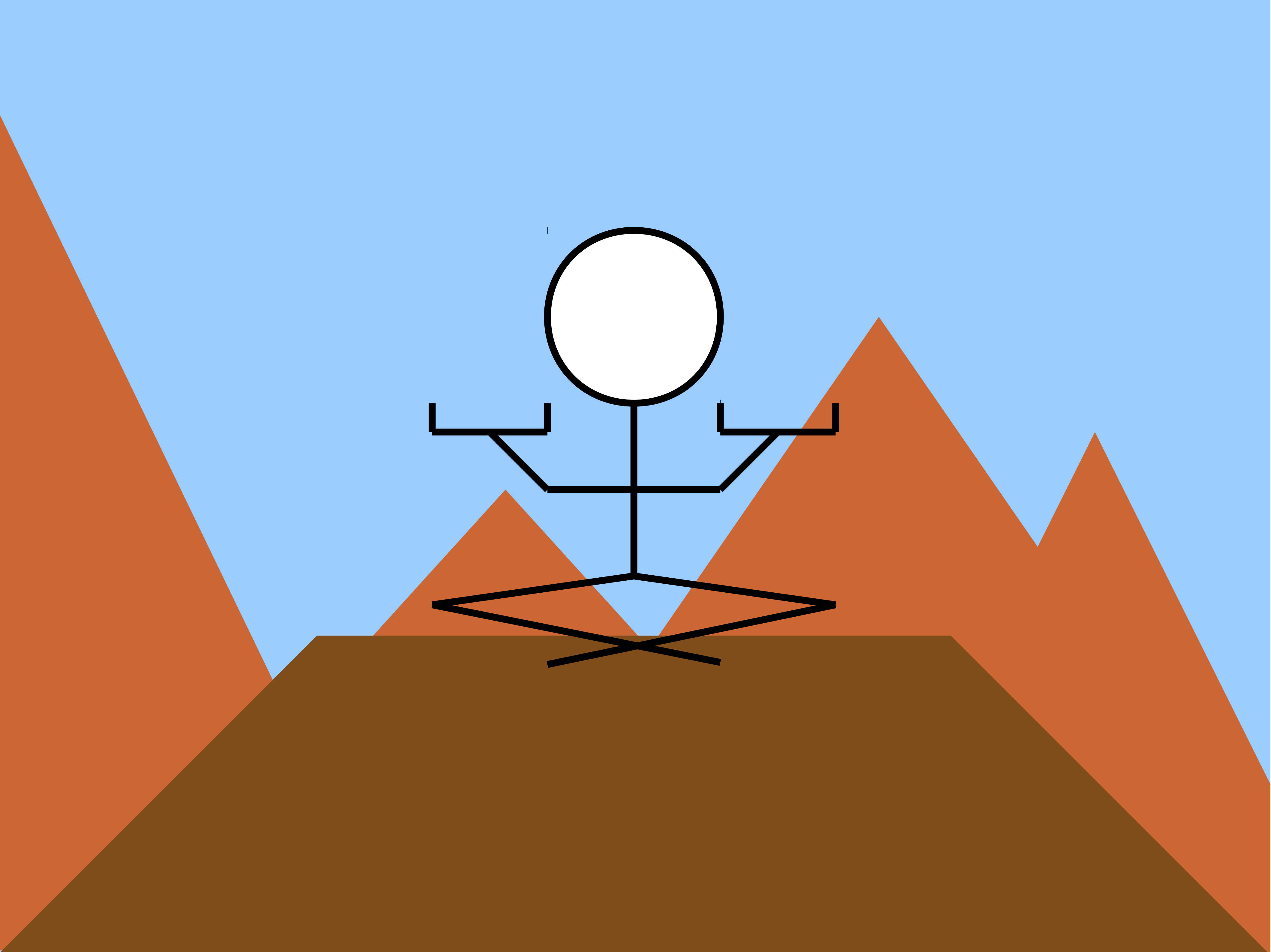
Outline for Today

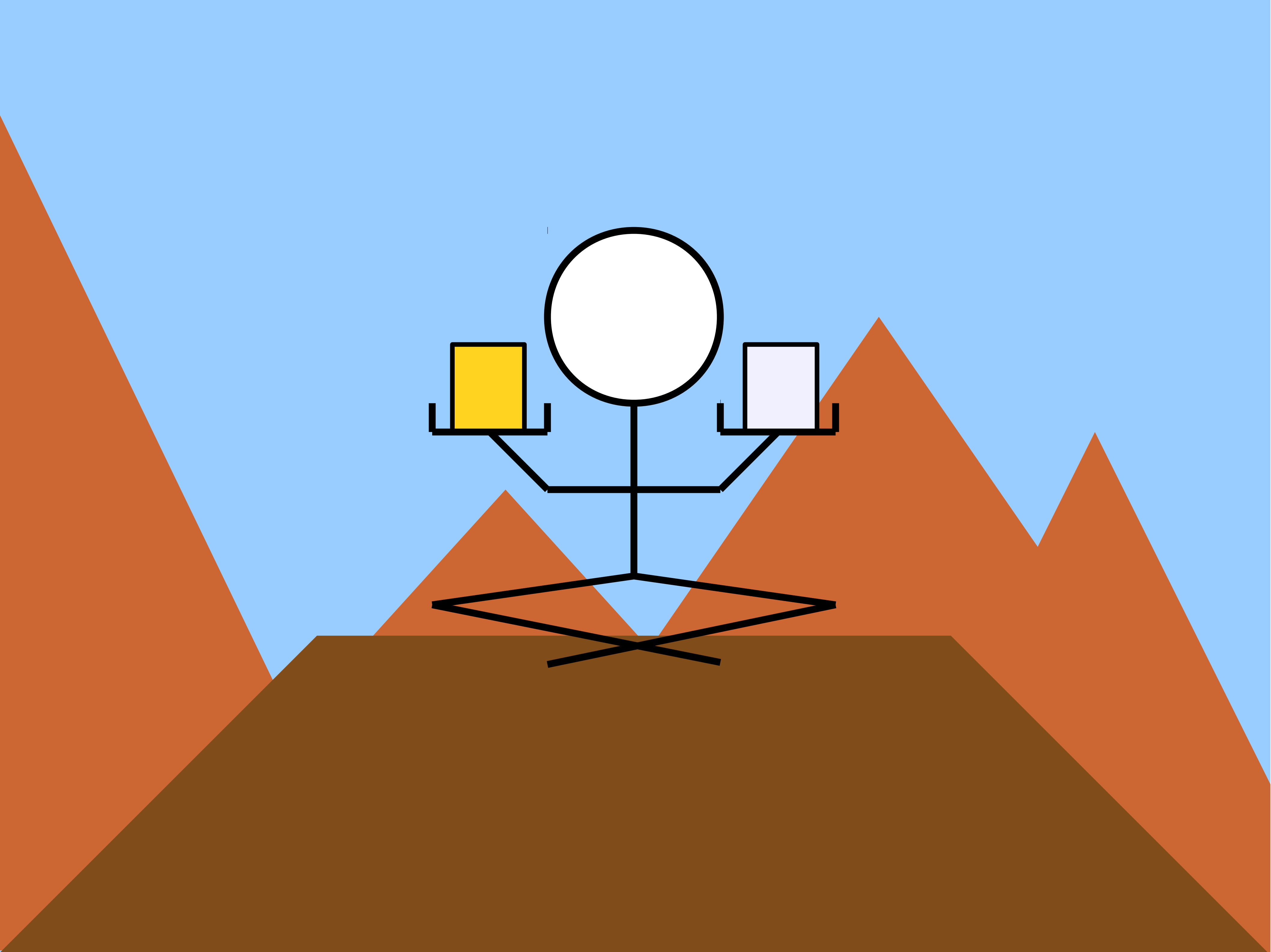
- **Binary Relations**
  - Reasoning about connections between objects.
- **Equivalence Relations**
  - Reasoning about clusters.
- **Strict Orders**
  - Reasoning about prerequisites.

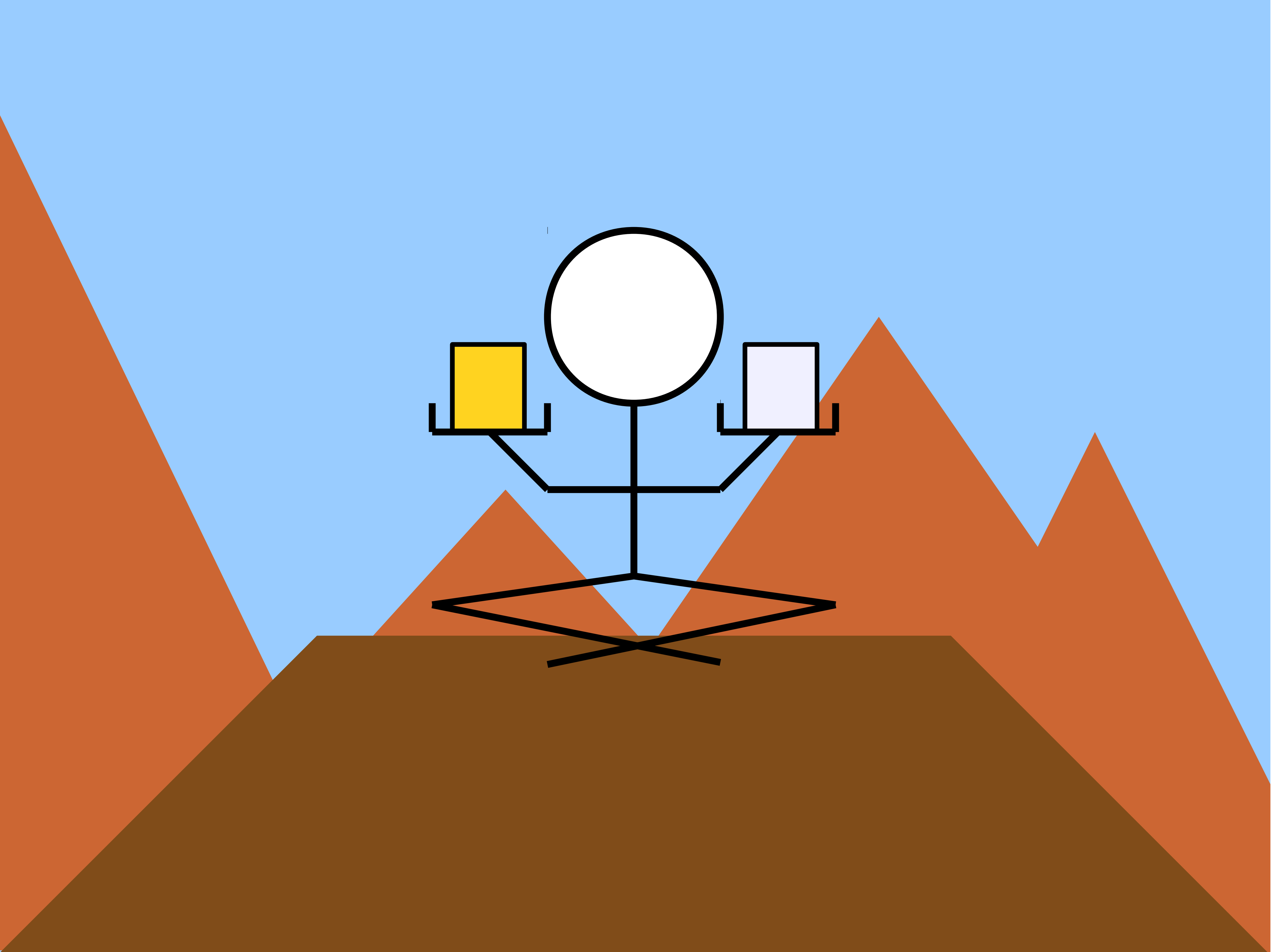
Relationships

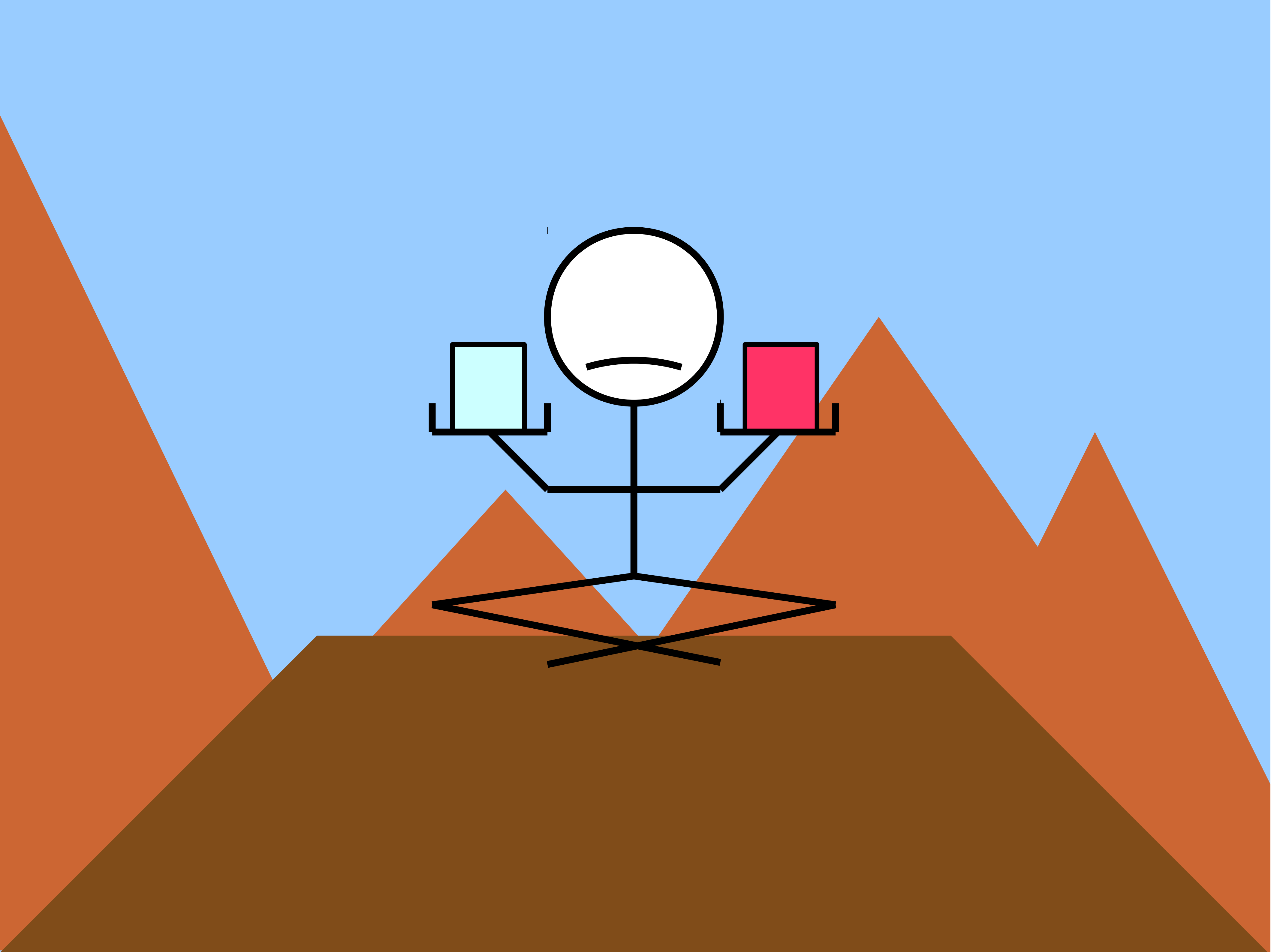
- In CS103, you've seen examples of relationships
  - between sets:
    - $A \subseteq B$
  - between numbers:
    - $x < y \quad x \equiv_k y \quad x \leq y$
  - between people:
    - p loves q
- Since these relations focus on connections between two objects, they are called ***binary relations***.

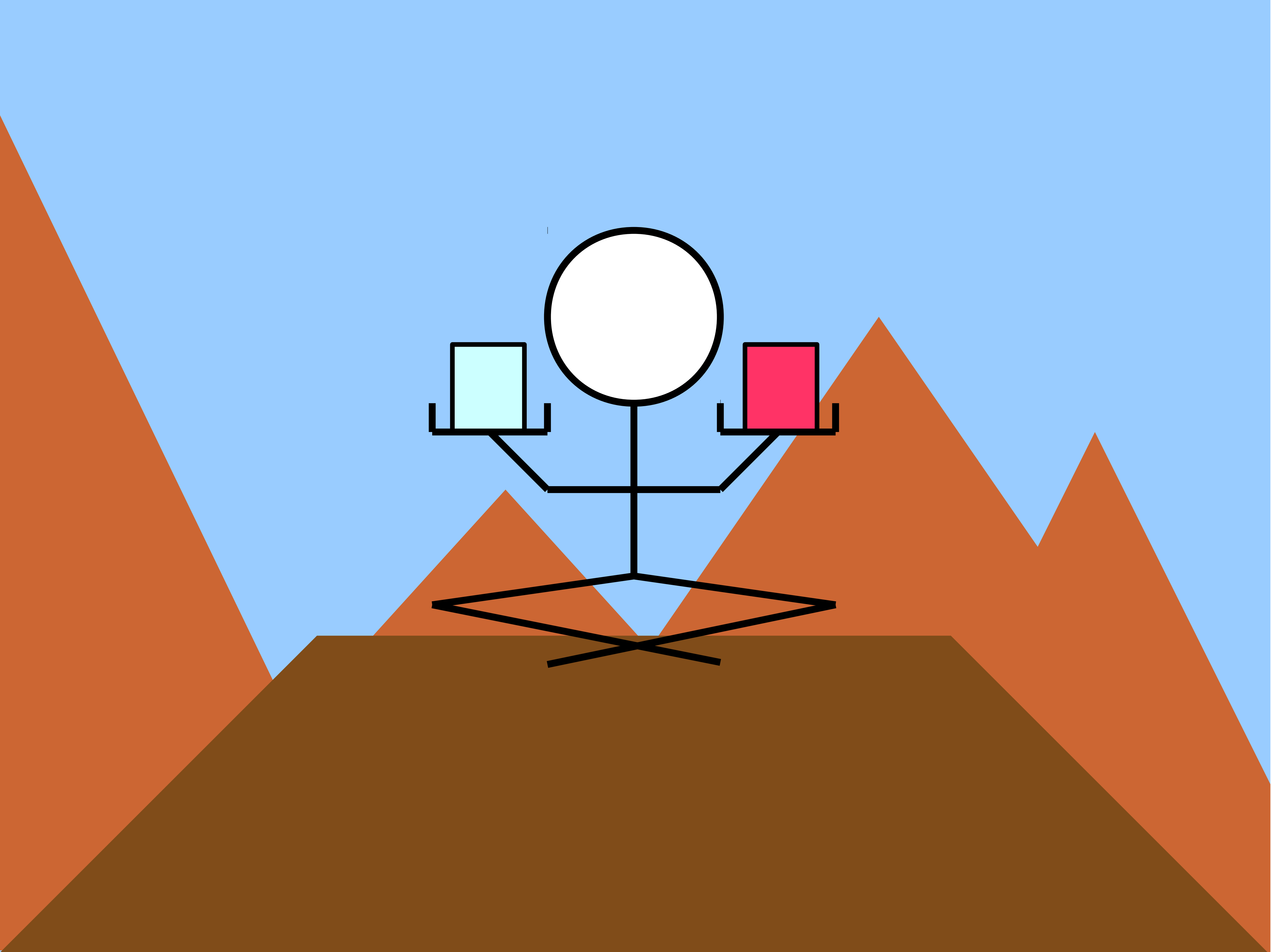
What exactly is a binary relation?

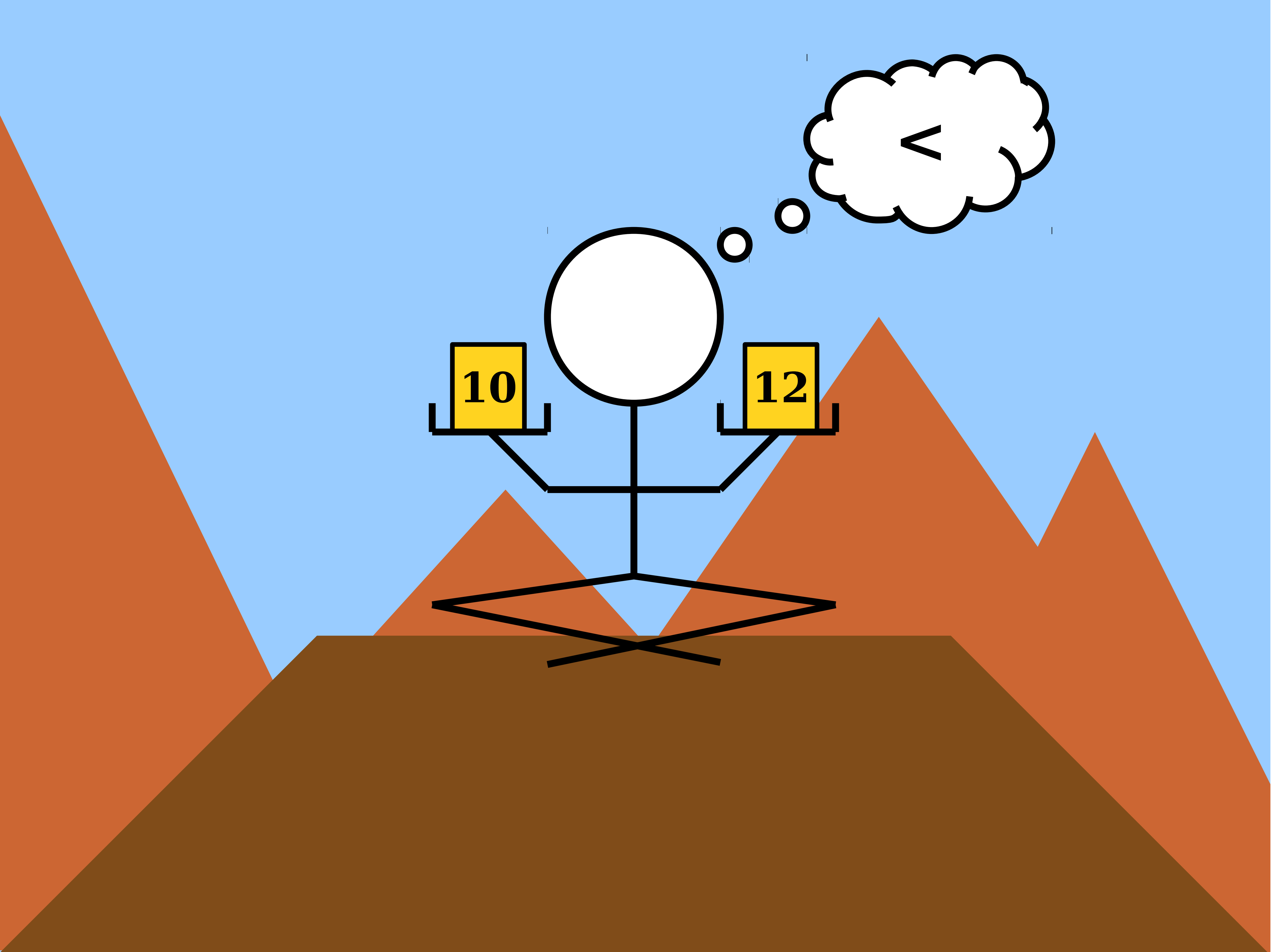








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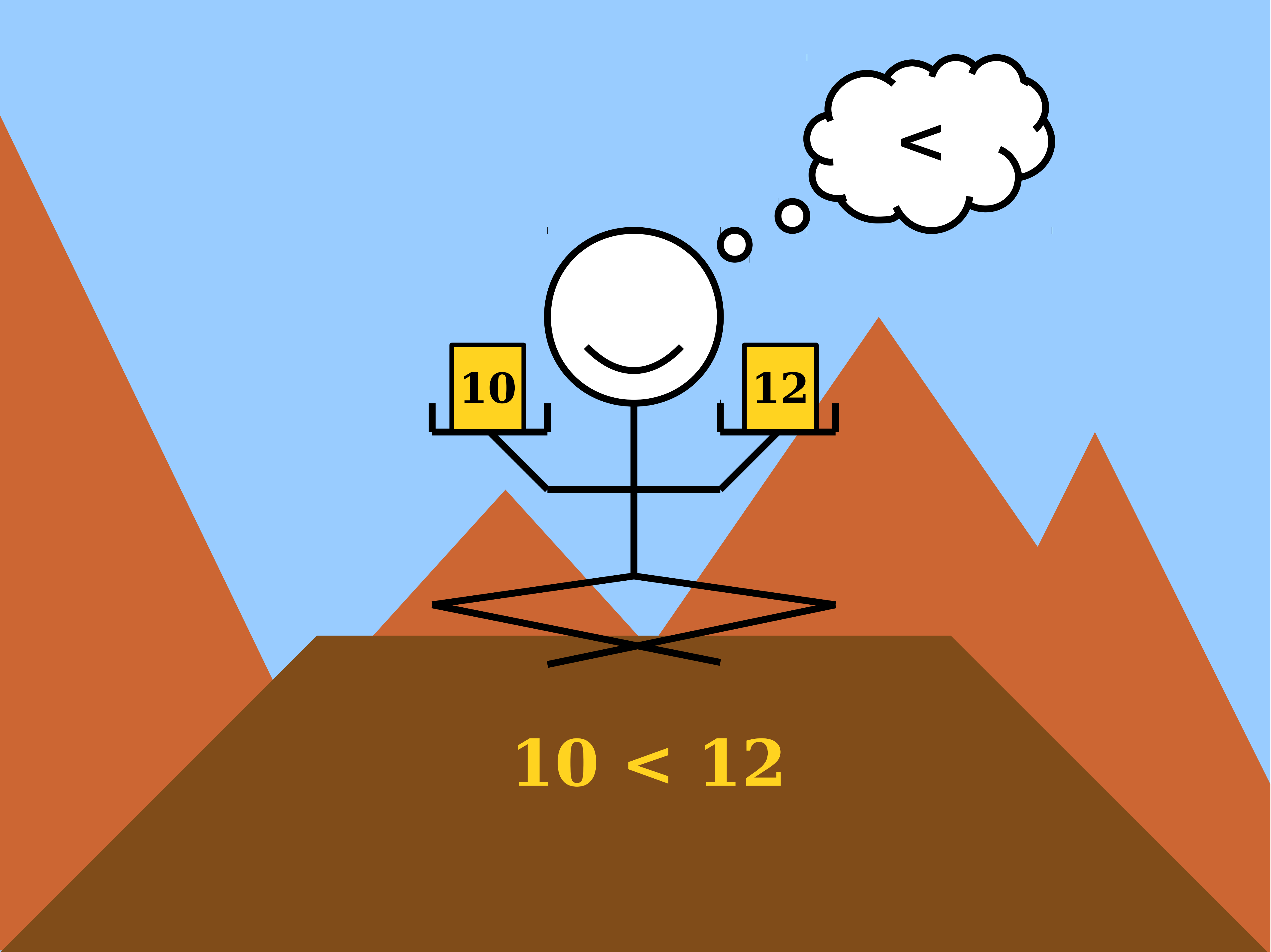
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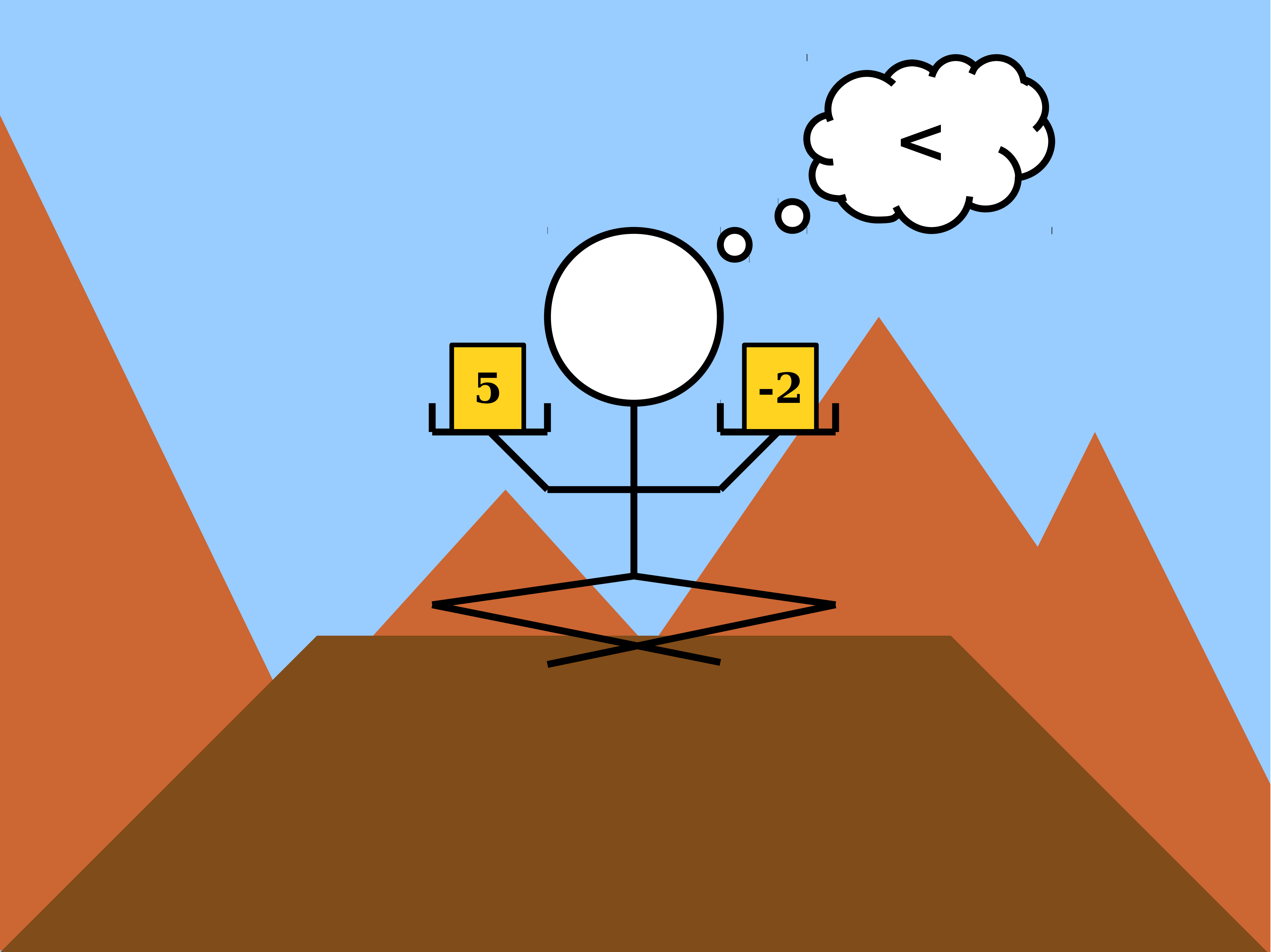
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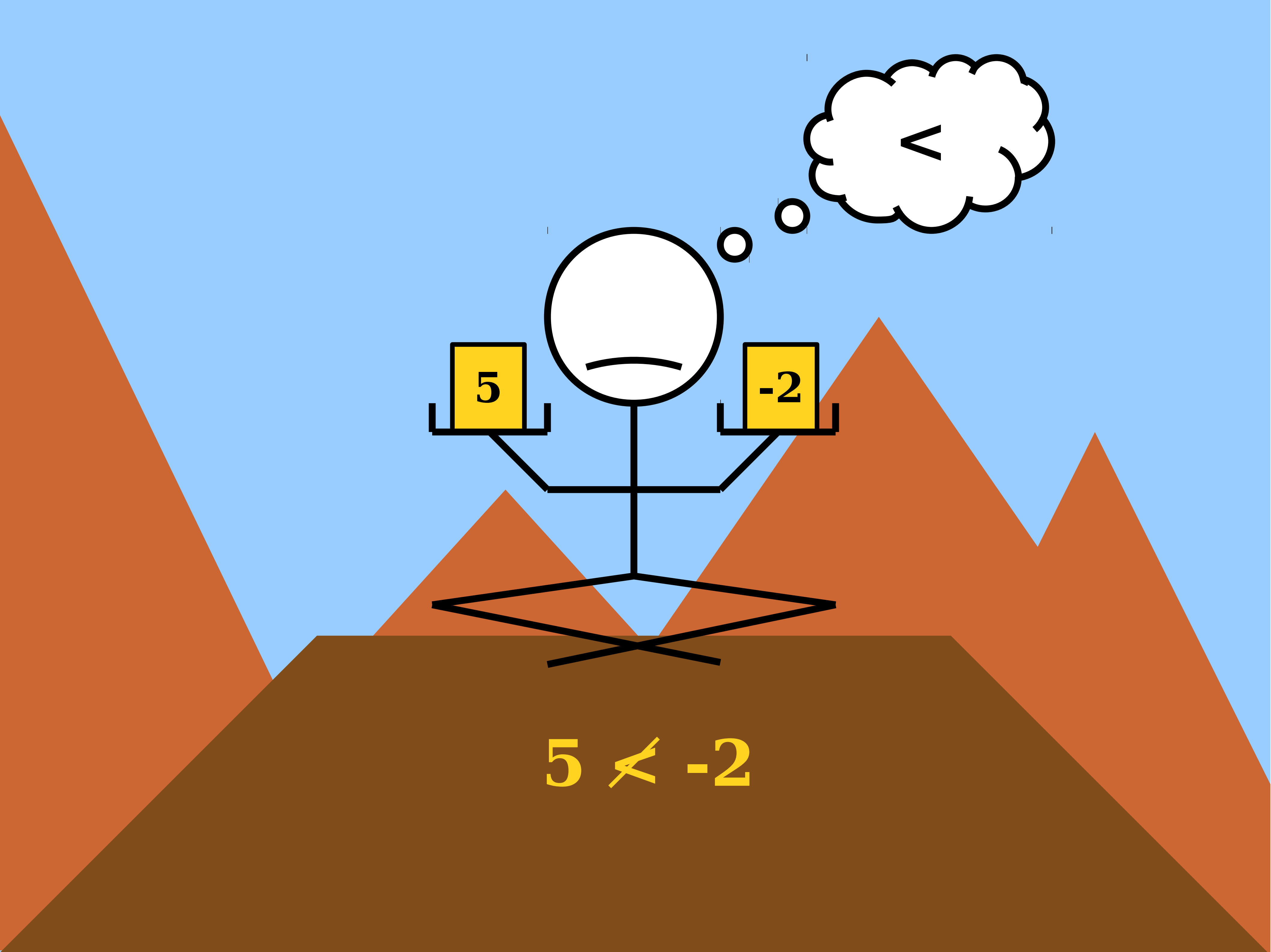
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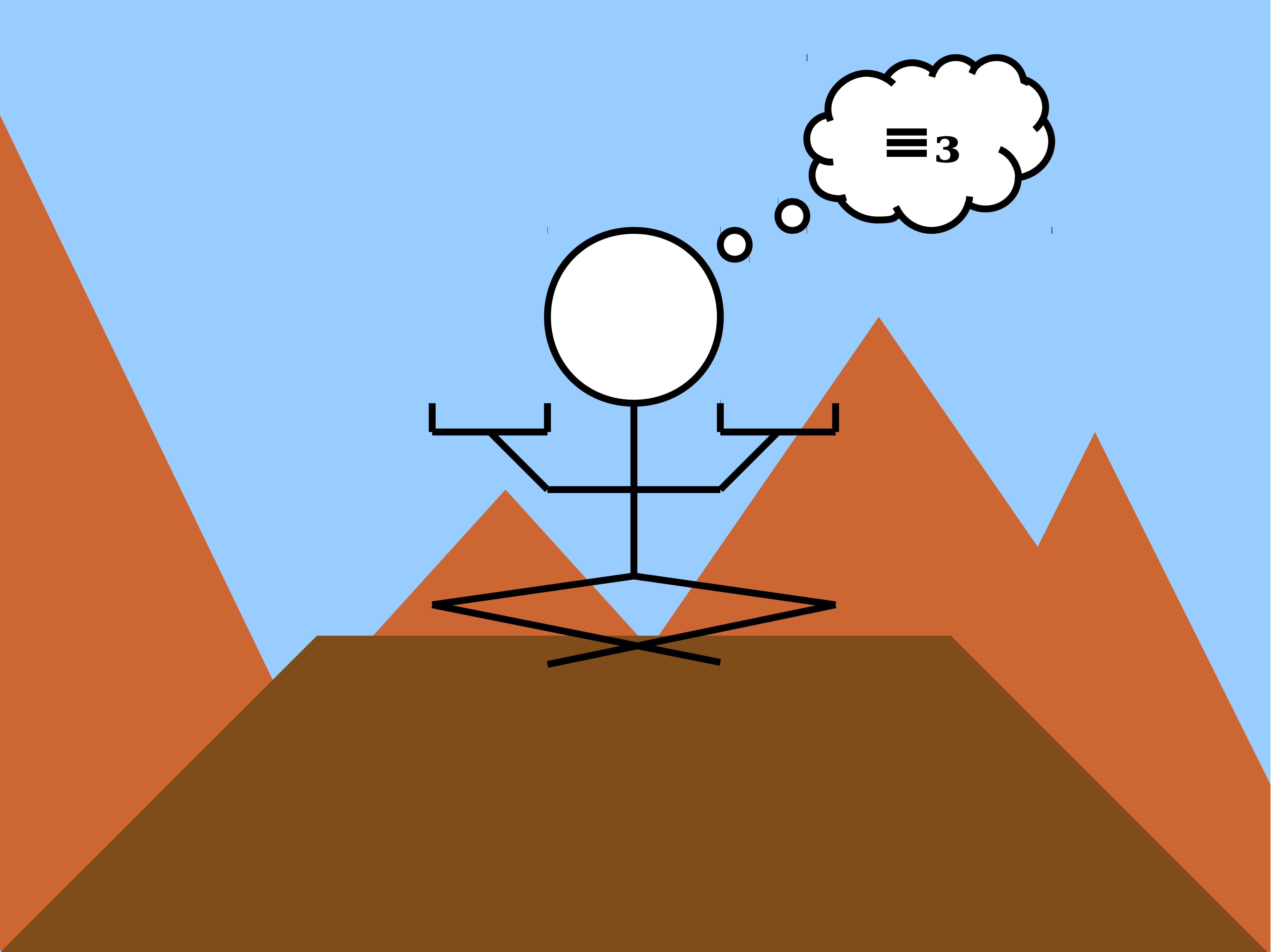
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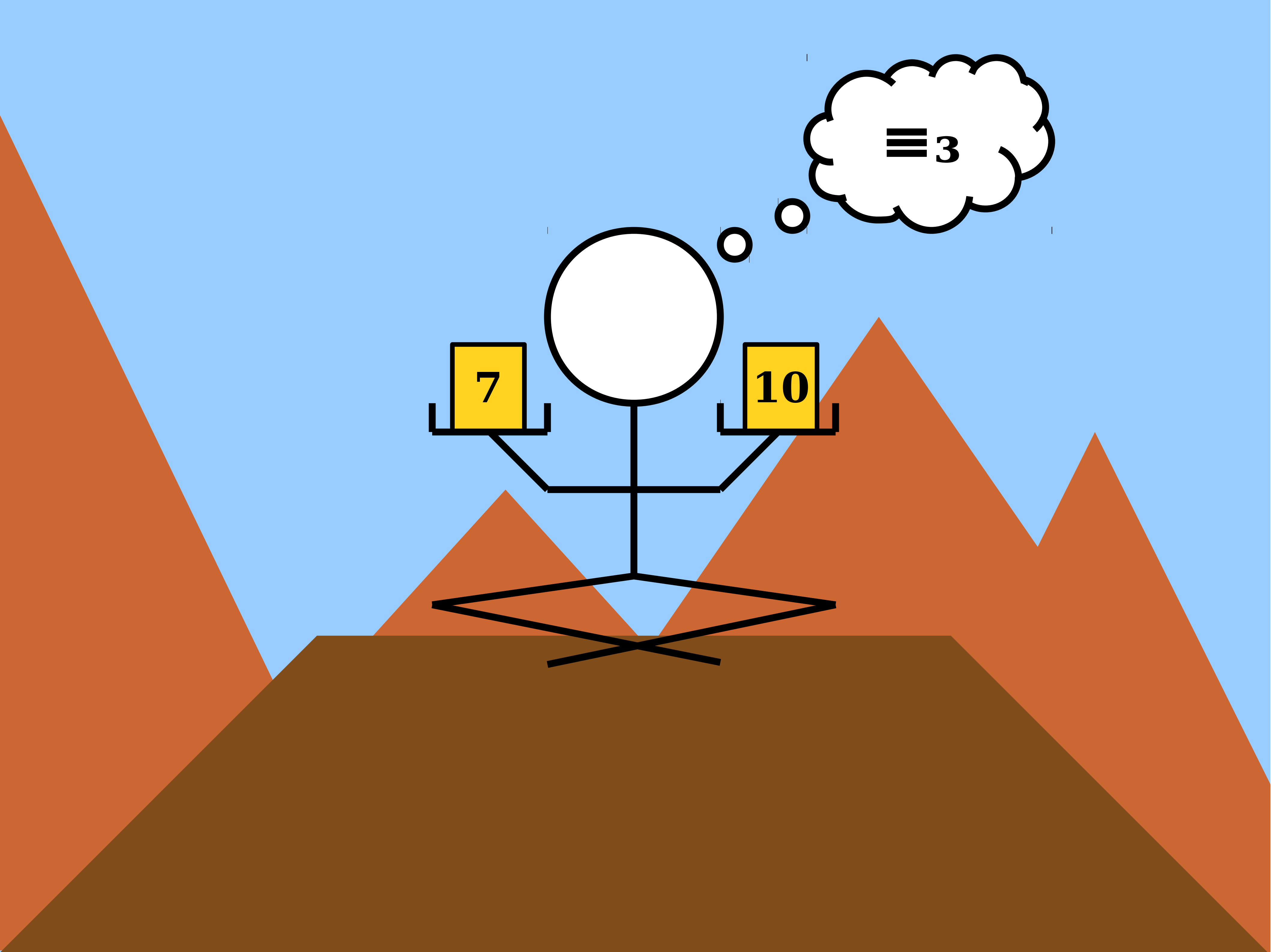
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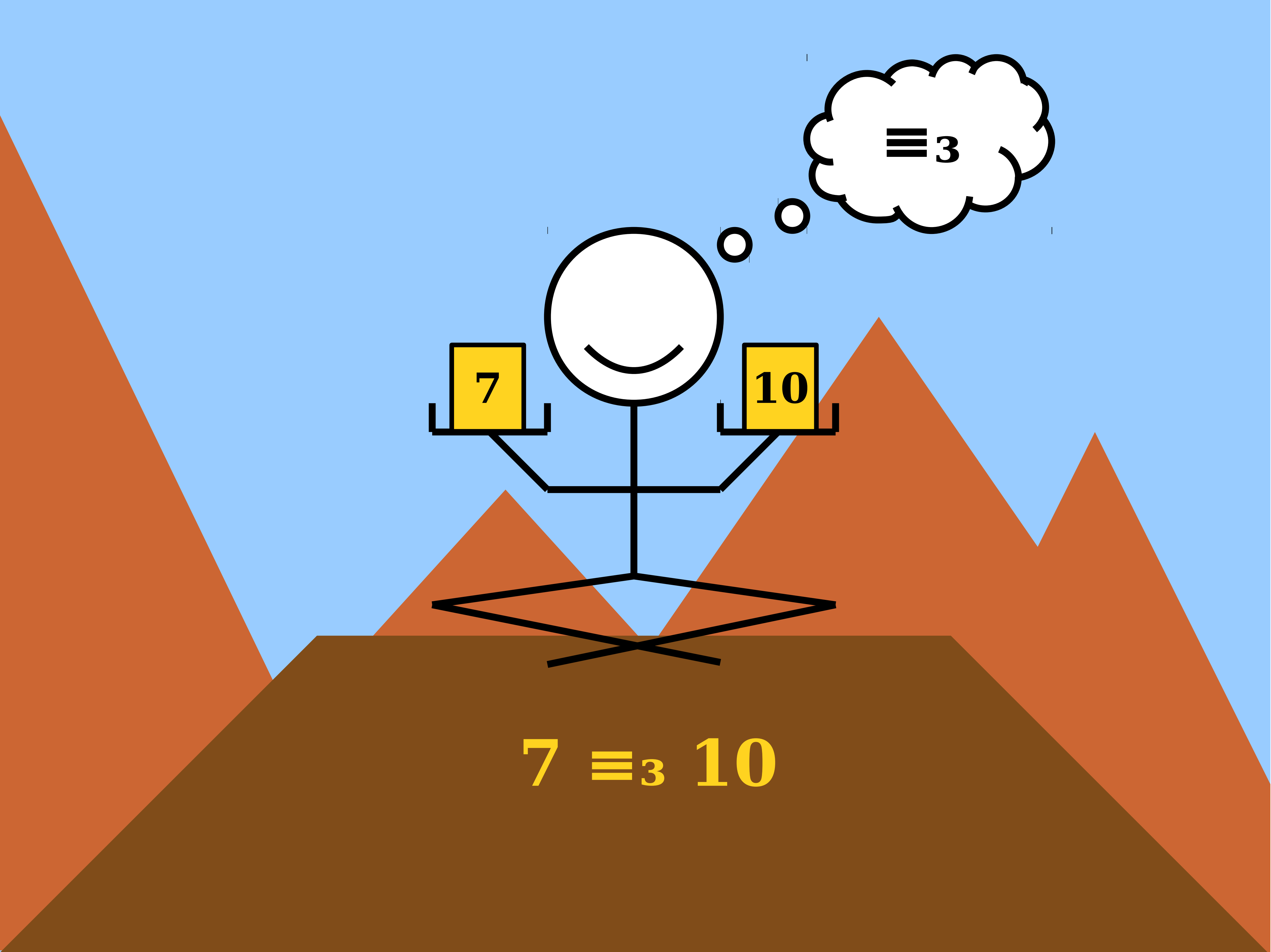

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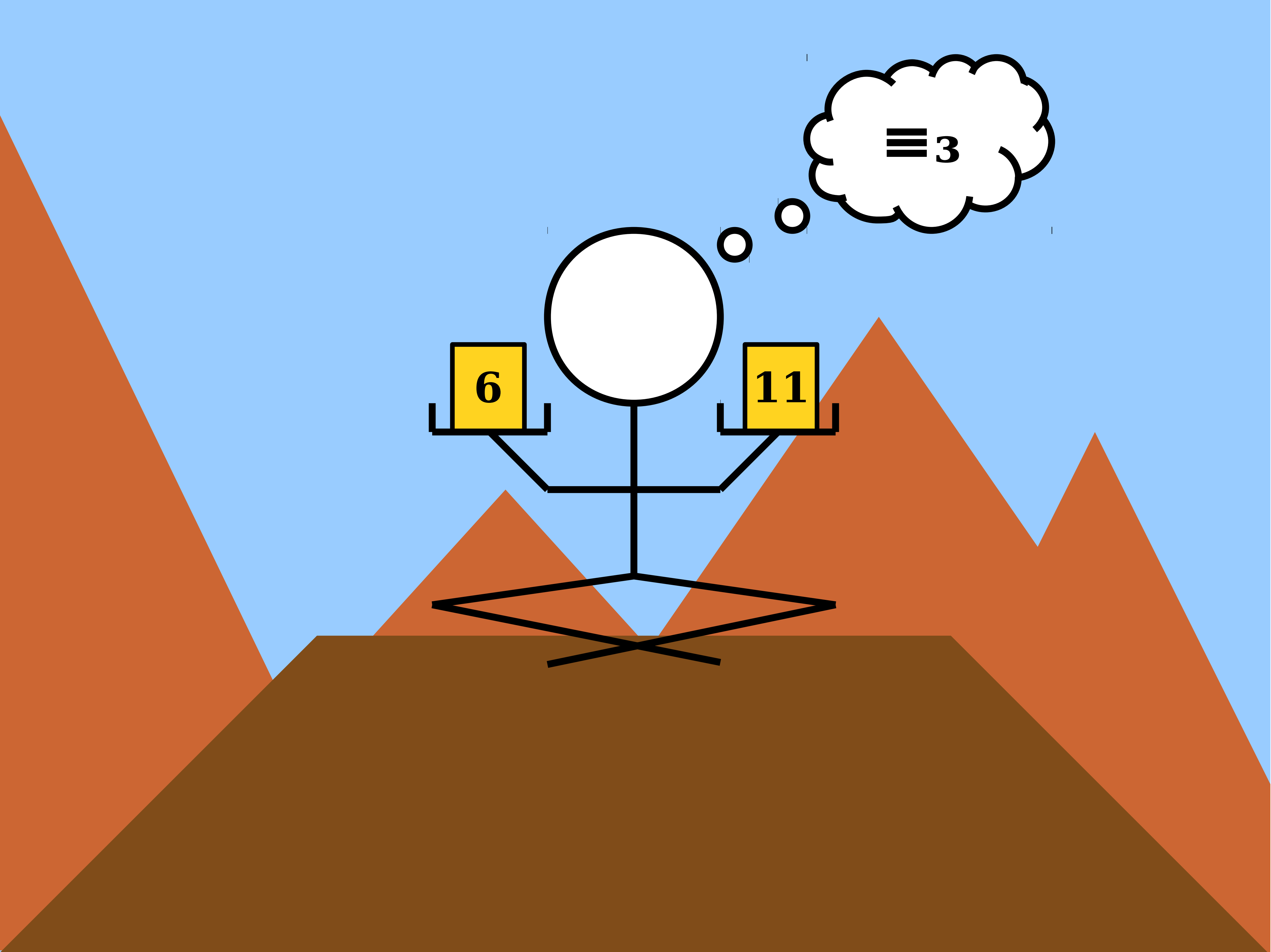

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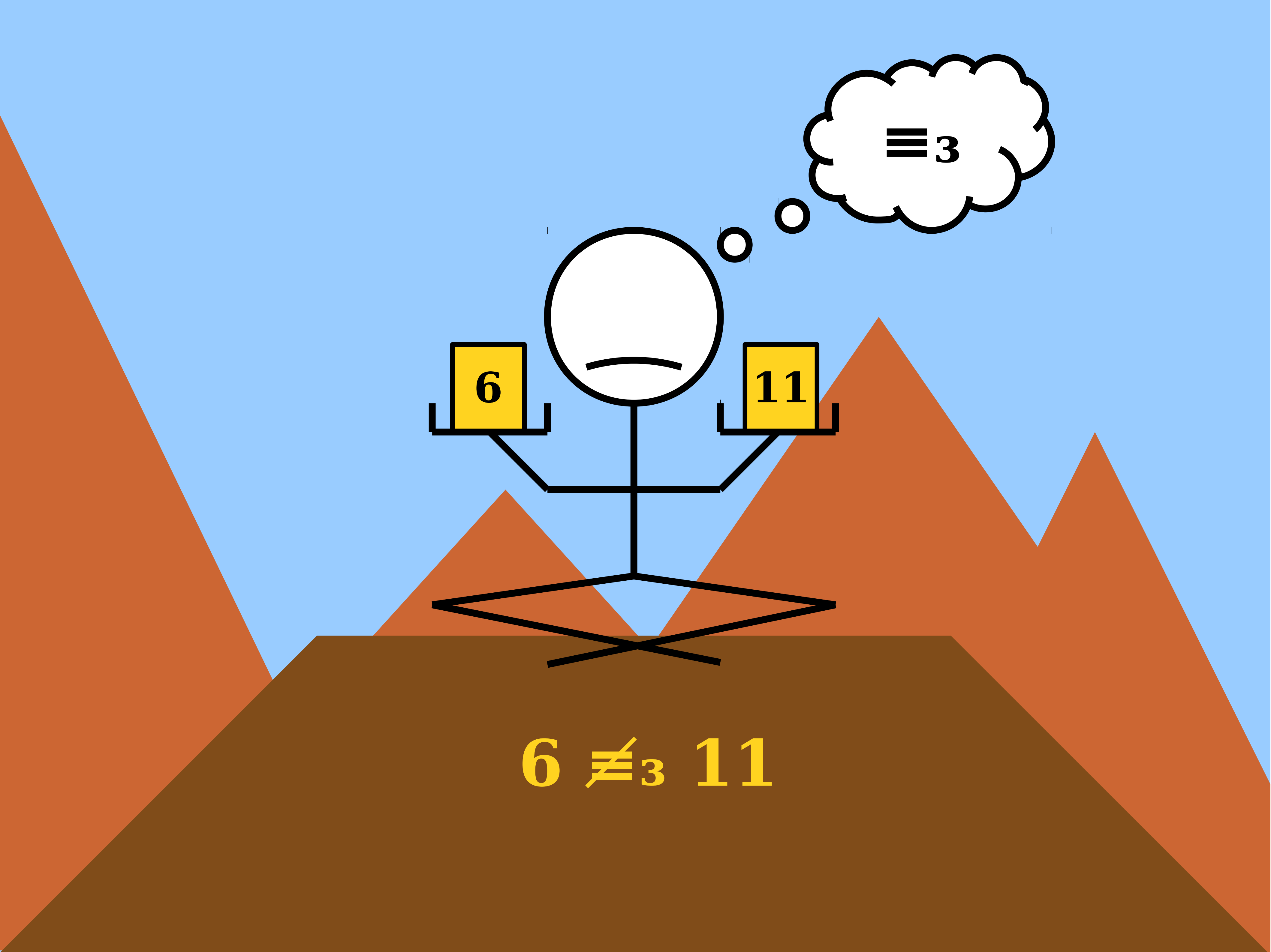
$\equiv_3$
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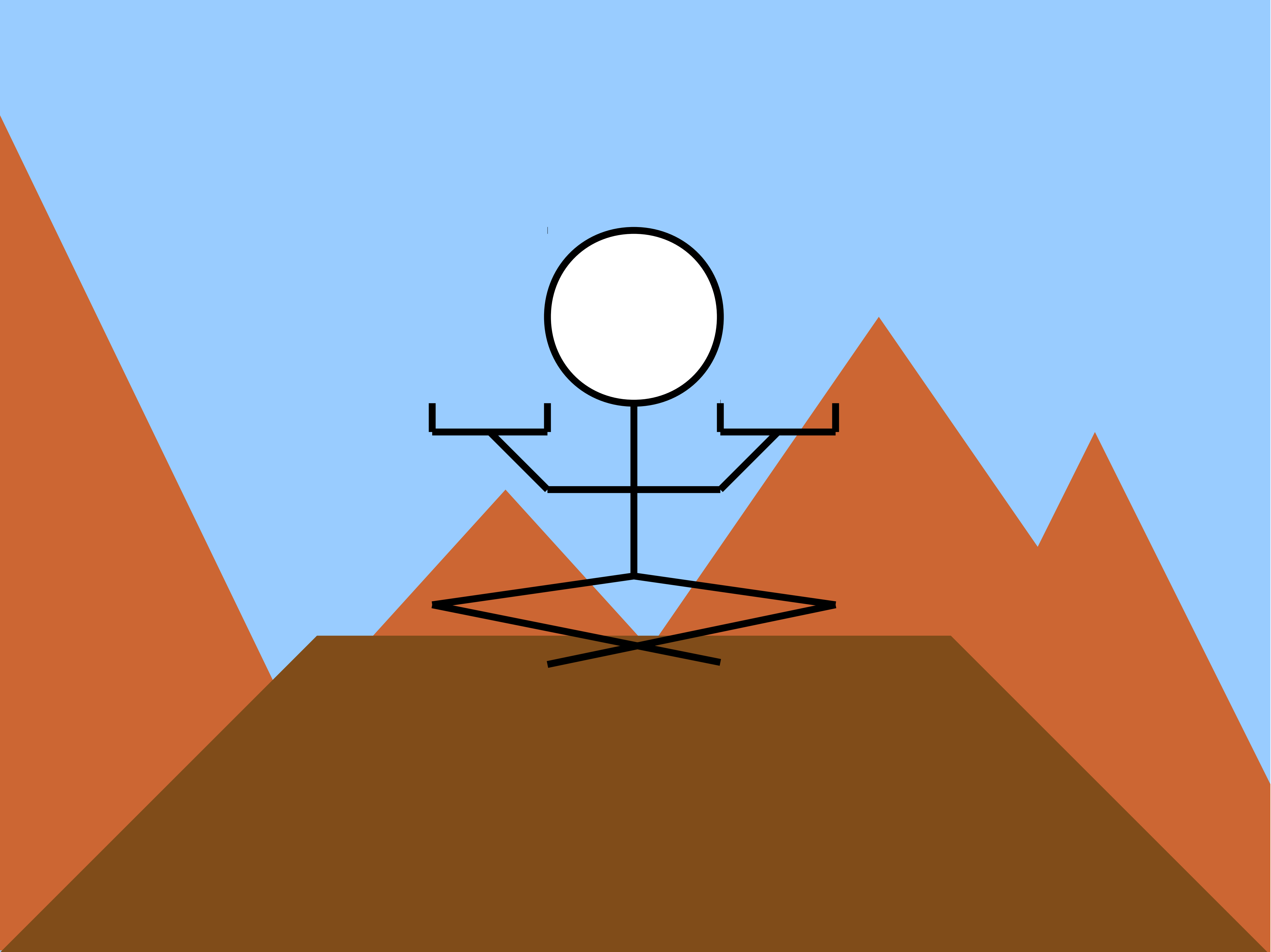
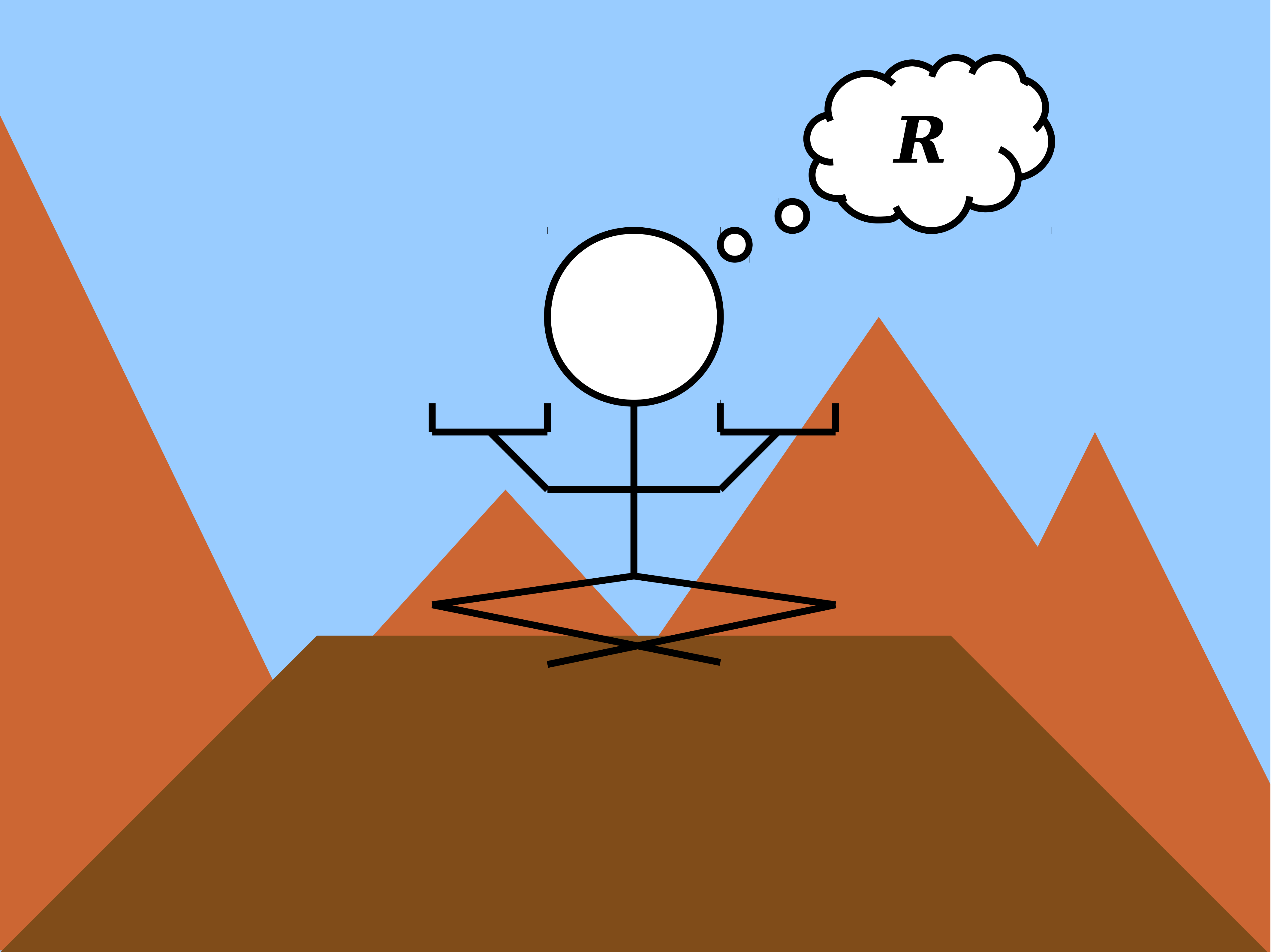
$6 \not\equiv_3 11$

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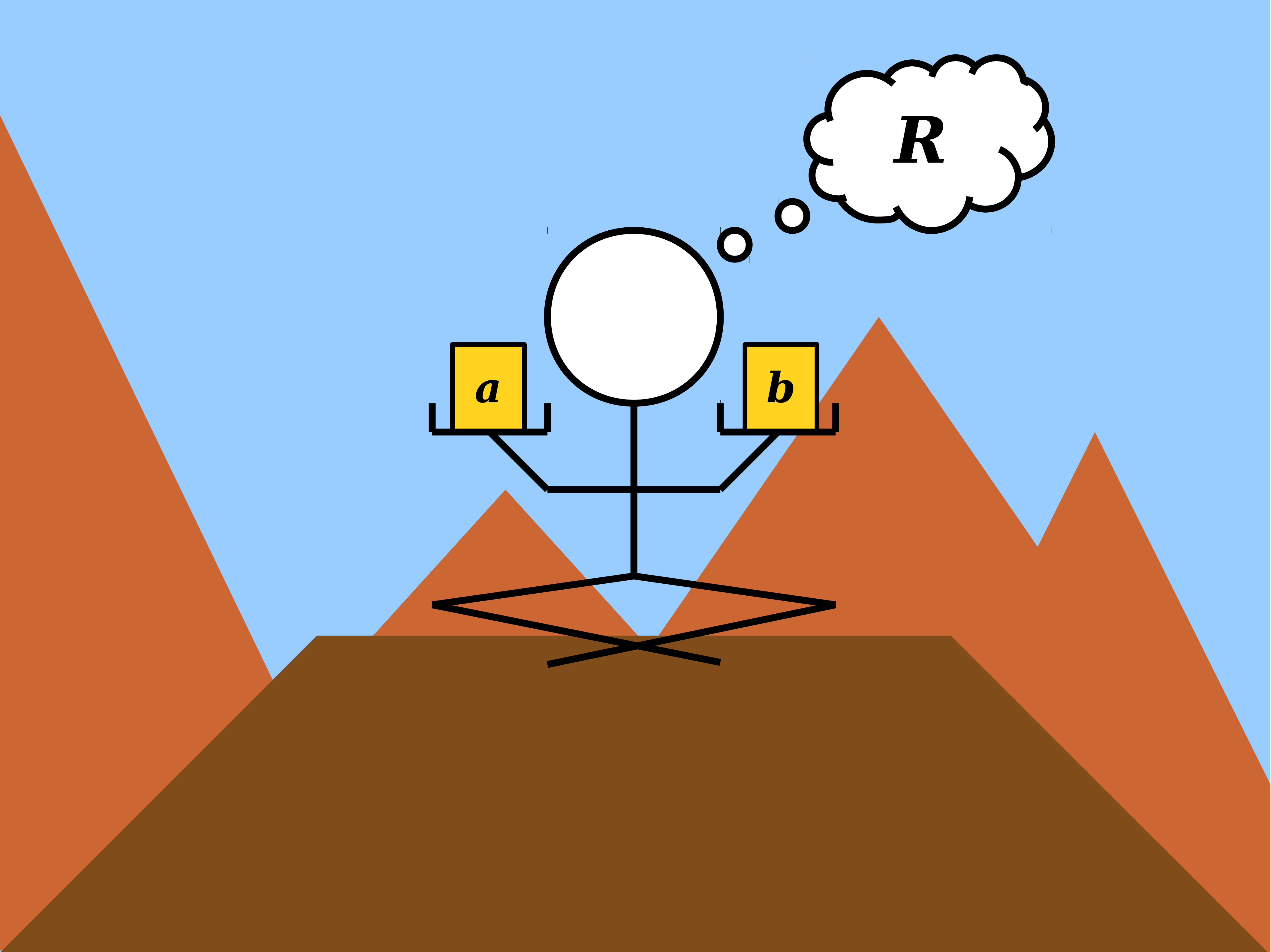
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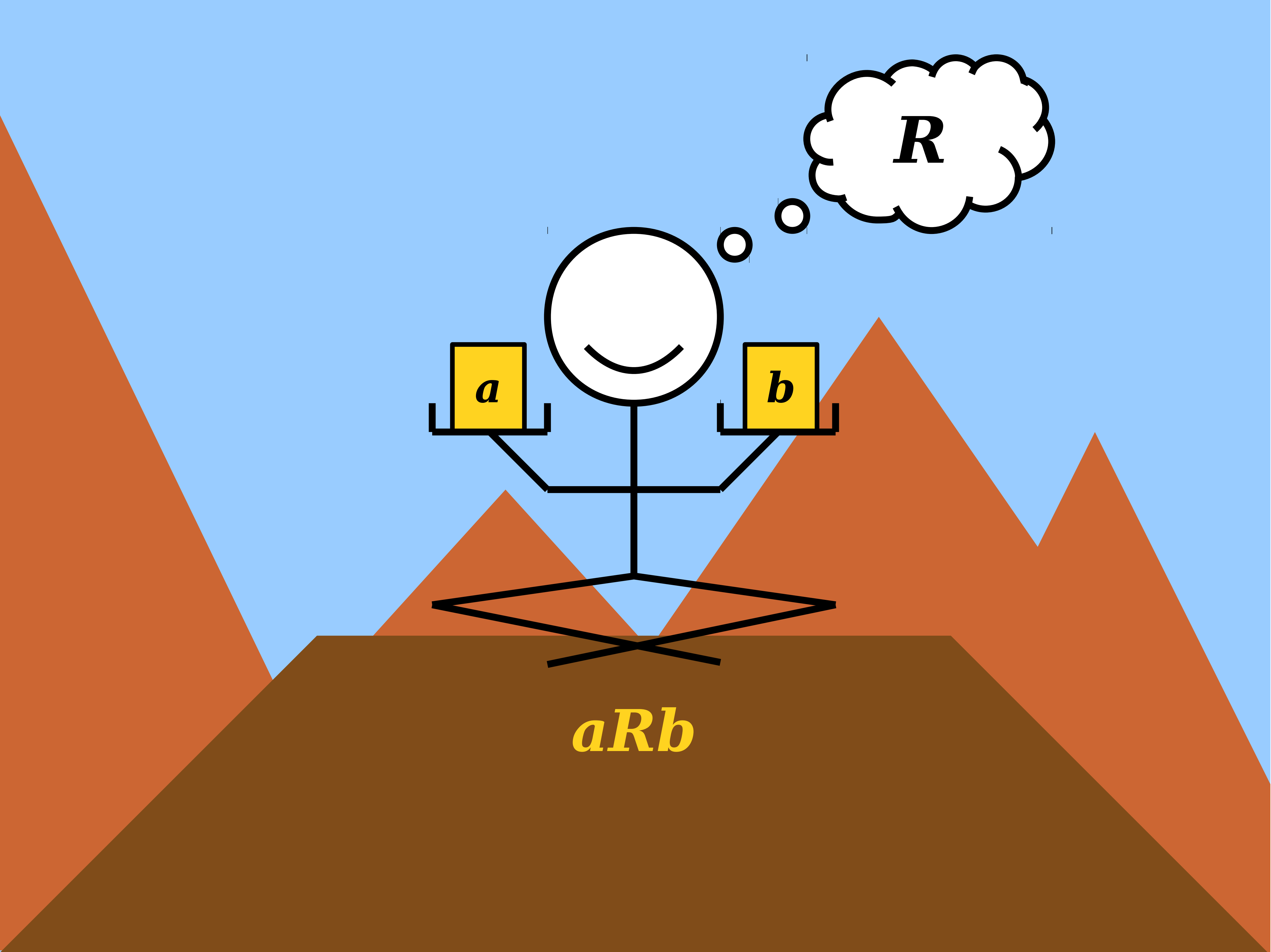
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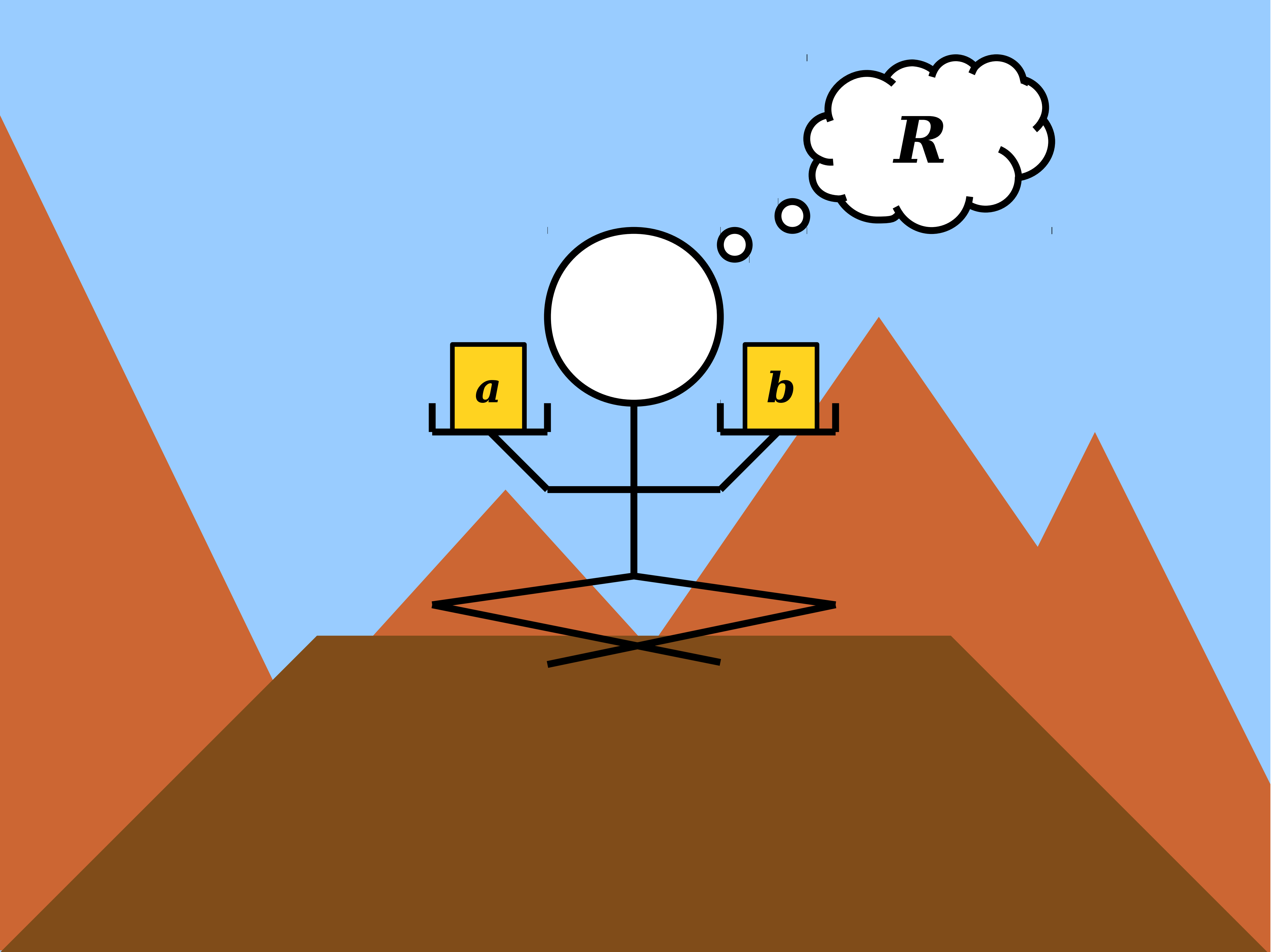
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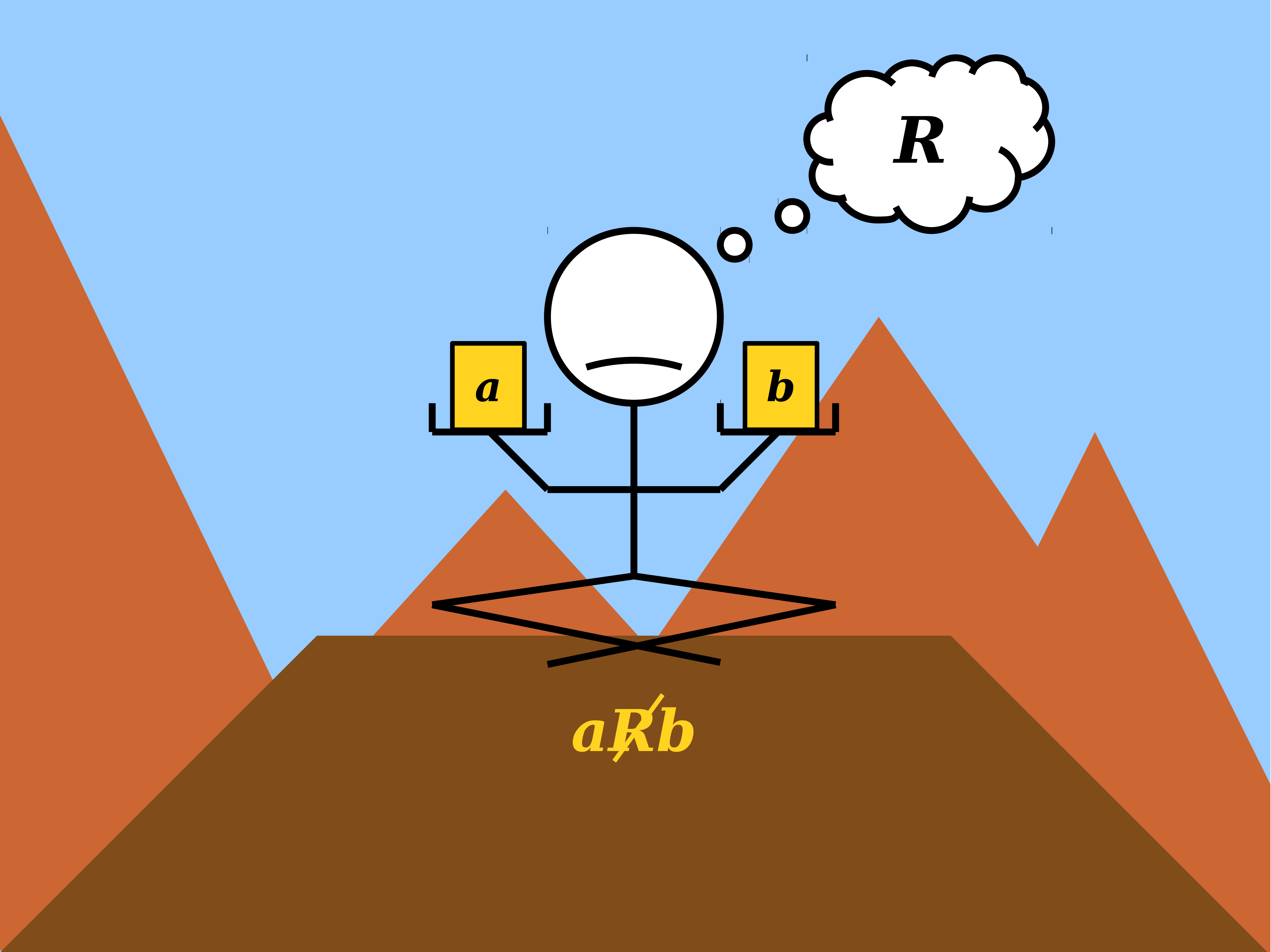
R
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Binary Relations

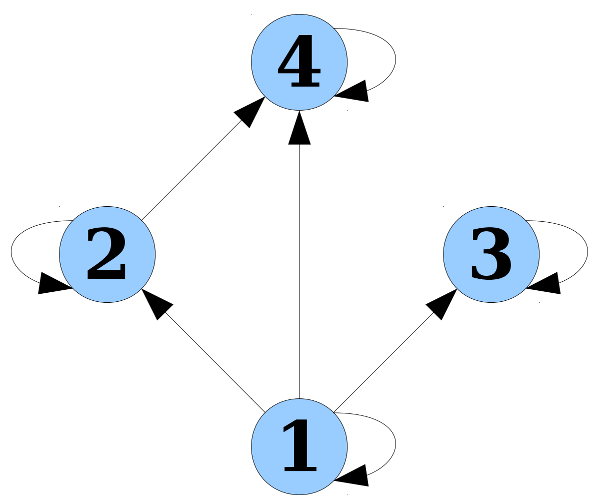
- A ***binary relation over a set A*** is a predicate R that can be applied to pairs of elements drawn from A.
- If R is a binary relation over A and it holds for the pair (a, b), we write ***aRb***.
  - For example: $3 = 3$, $5 < 7$, and $\emptyset \subseteq \mathbb{N}$.
- If R is a binary relation over A and it does not hold for the pair (a, b), we write $\boldsymbol{a \not R b}$.
  - For example: $4 \neq 3$, $4 \not< 3$, and $\mathbb{N} \not\subseteq \emptyset$.

Properties of Relations

- Generally speaking, if R is a binary relation over a set A, the order of the operands is significant.
  - For example, $3 < 5$, but $5 \not< 3$.
  - In some relations order is irrelevant; more on that later.
- Like functions, relations are defined relative to some underlying set.
  - It's not meaningful to ask whether ☺ $\subseteq$ 15, for example, since $\subseteq$ is defined over sets, not arbitrary objects.

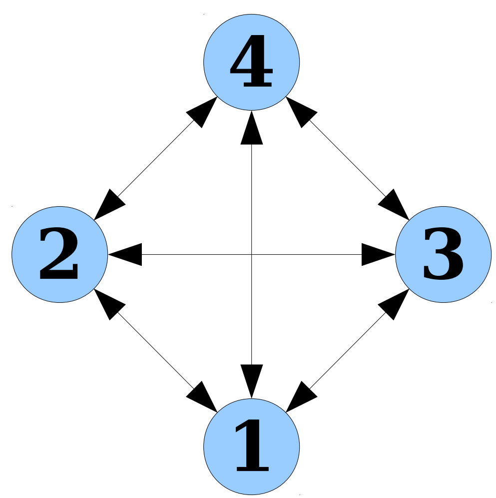
Visualizing Relations

- We can visualize a binary relation R over a set A by drawing the elements of A and drawing a line between an element a and an element b if aRb is true.
- Example: the relation $a \mid b$ (meaning "a divides b") over the set {1, 2, 3, 4} looks like this:



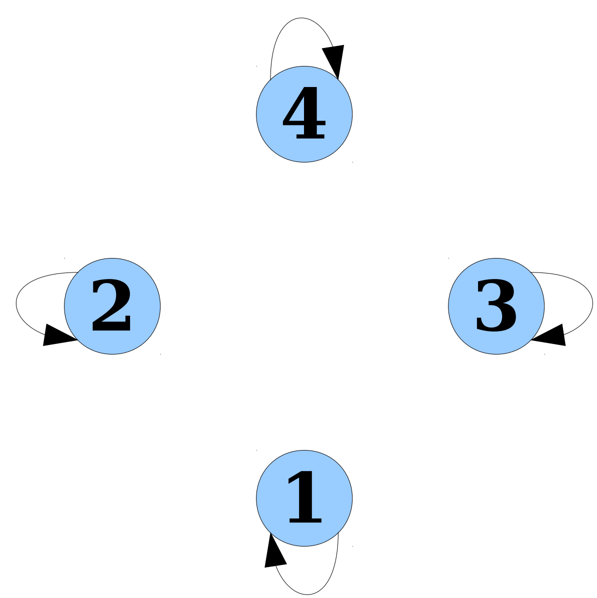
Visualizing Relations

- We can visualize a binary relation R over a set A by drawing the elements of A and drawing a line between an element a and an element b if aRb is true.
- Example: the relation $a \neq b$ over the set {1, 2, 3, 4} looks like this:



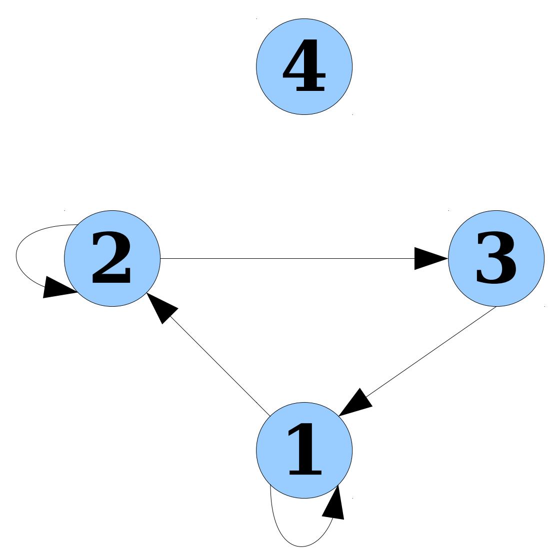
Visualizing Relations

- We can visualize a binary relation R over a set A by drawing the elements of A and drawing a line between an element a and an element b if aRb is true.
- Example: the relation $a = b$ over the set $\{1, 2, 3, 4\}$ looks like this:



Visualizing Relations

- We can visualize a binary relation R over a set A by drawing the elements of A and drawing a line between an element a and an element b if aRb is true.
- Example: below is some relation over {1, 2, 3, 4} that's a totally valid relation even though there doesn't appear to be a simple unifying rule.

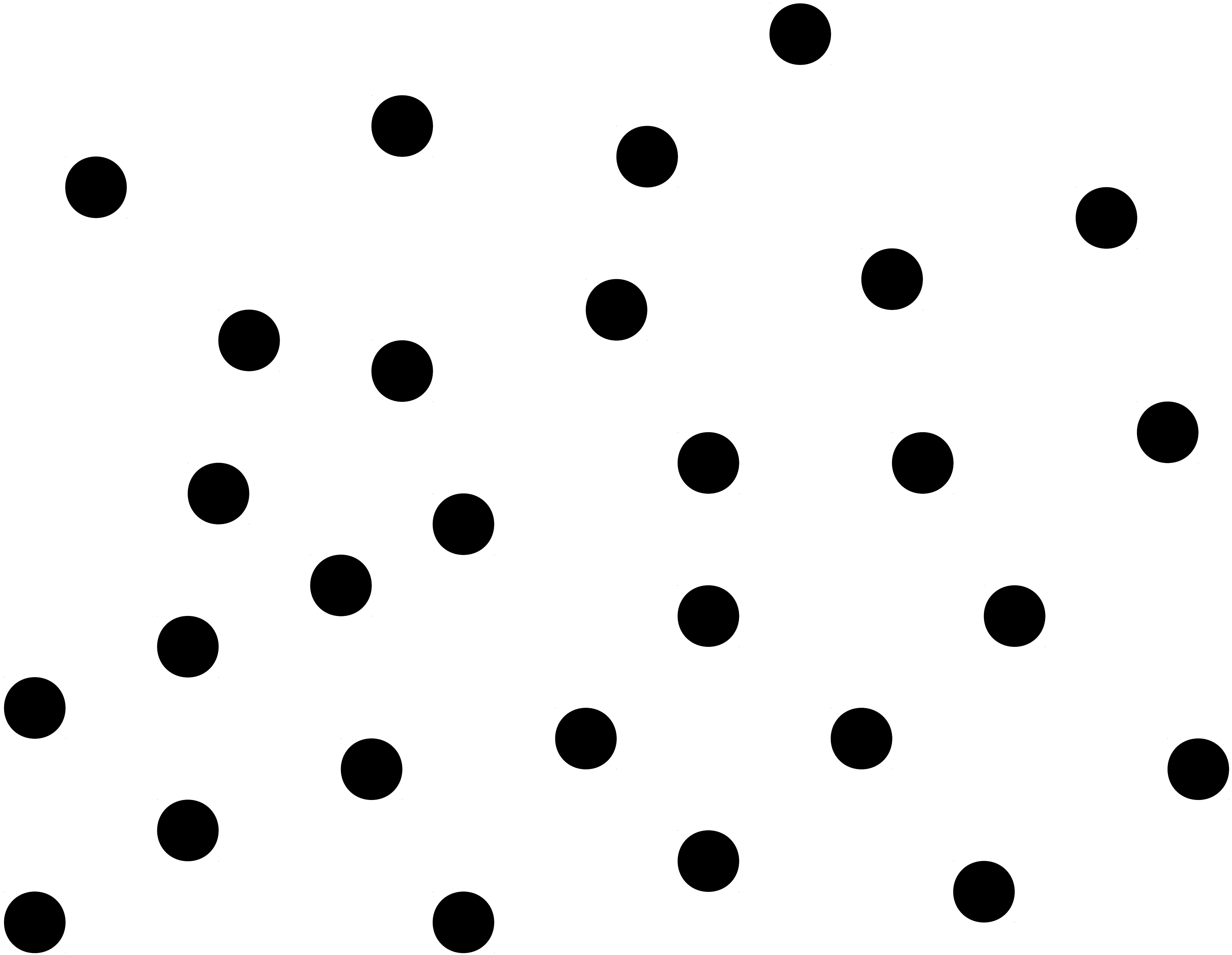


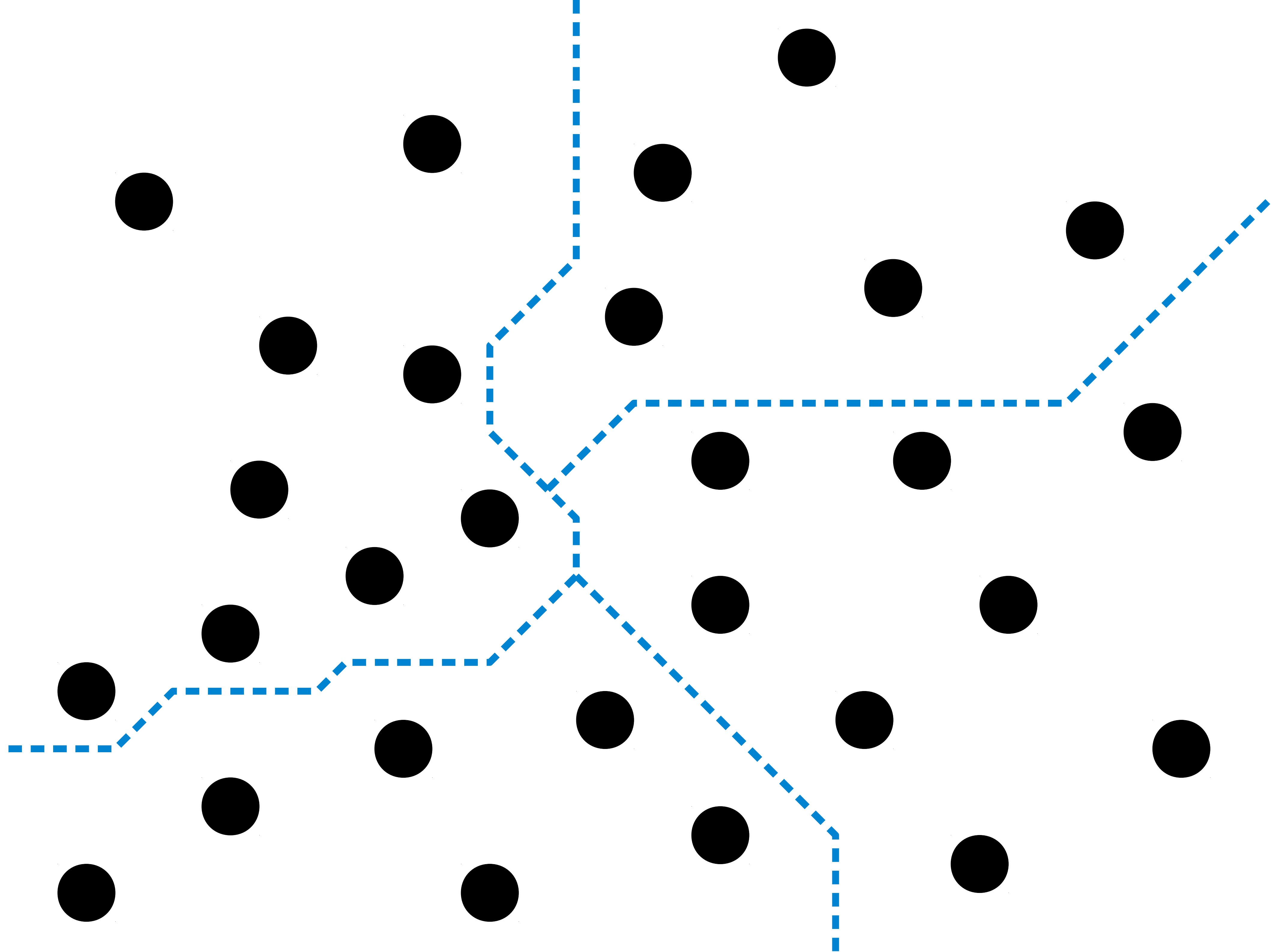
Capturing Structure

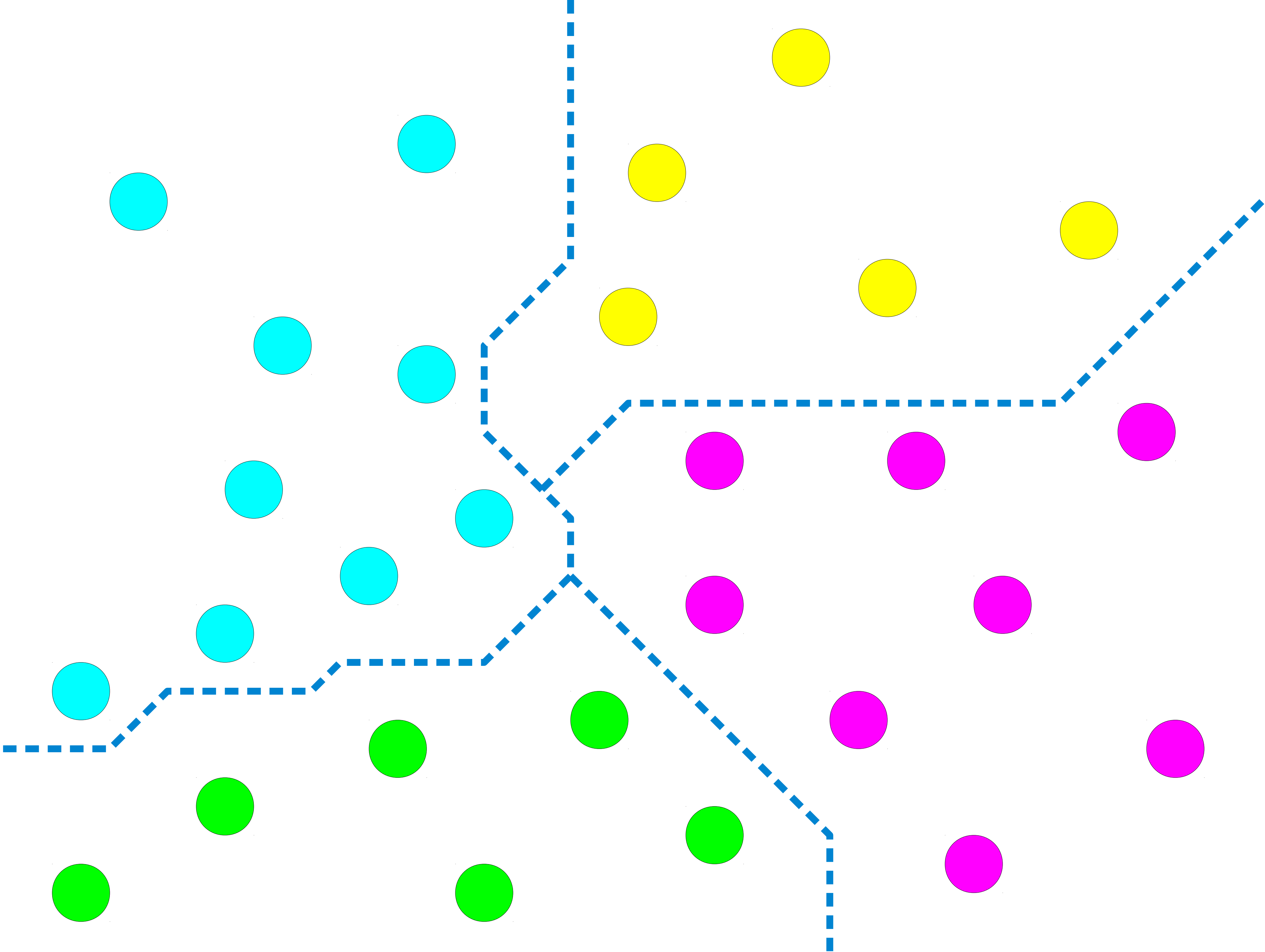
Capturing Structure

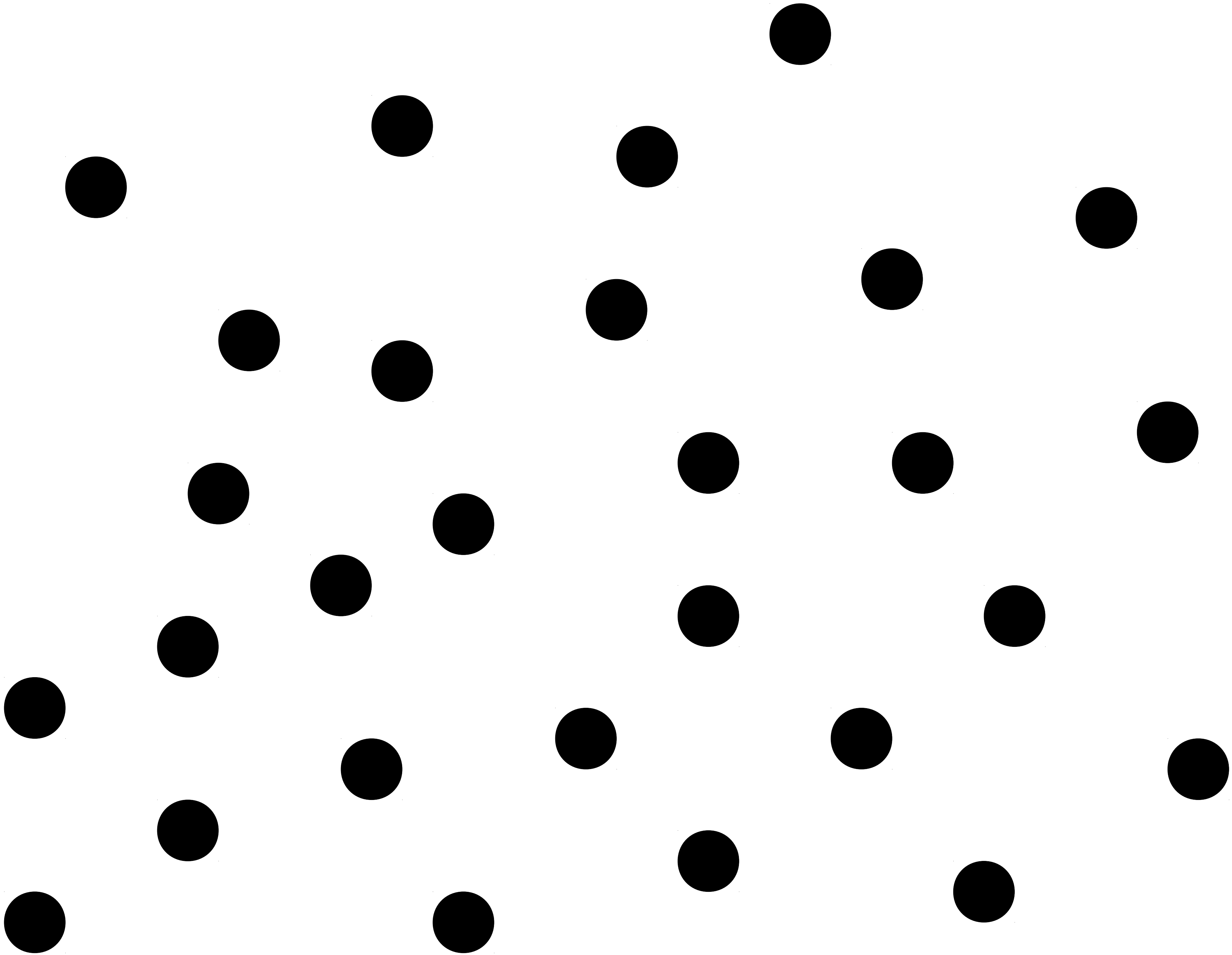
- Binary relations are an excellent way for capturing certain structures that appear in computer science.
- Today, we'll look at two of them (and possibly one more if we have time.)
  - **Partitions**
  - **Prerequisites**

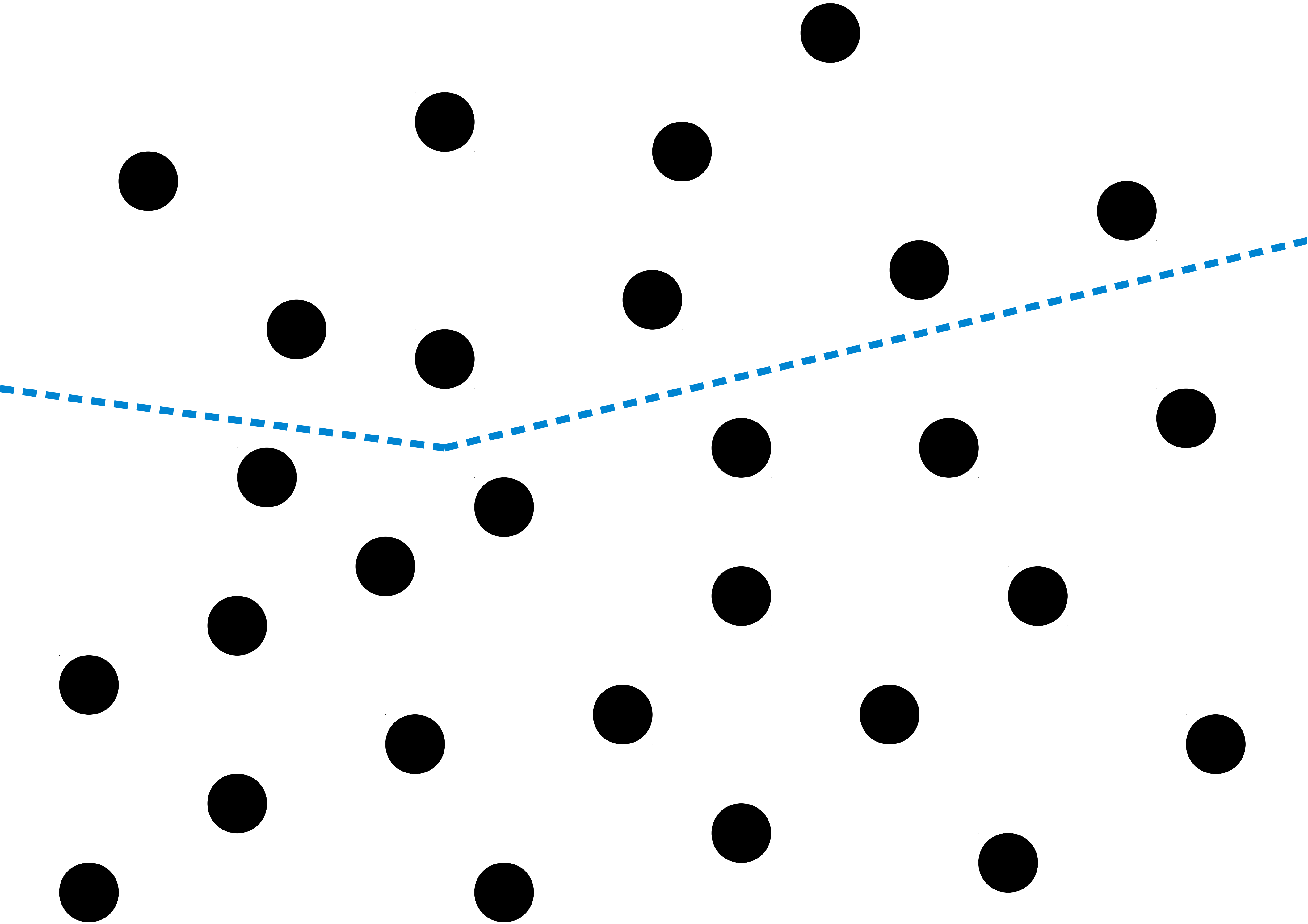
Partitions

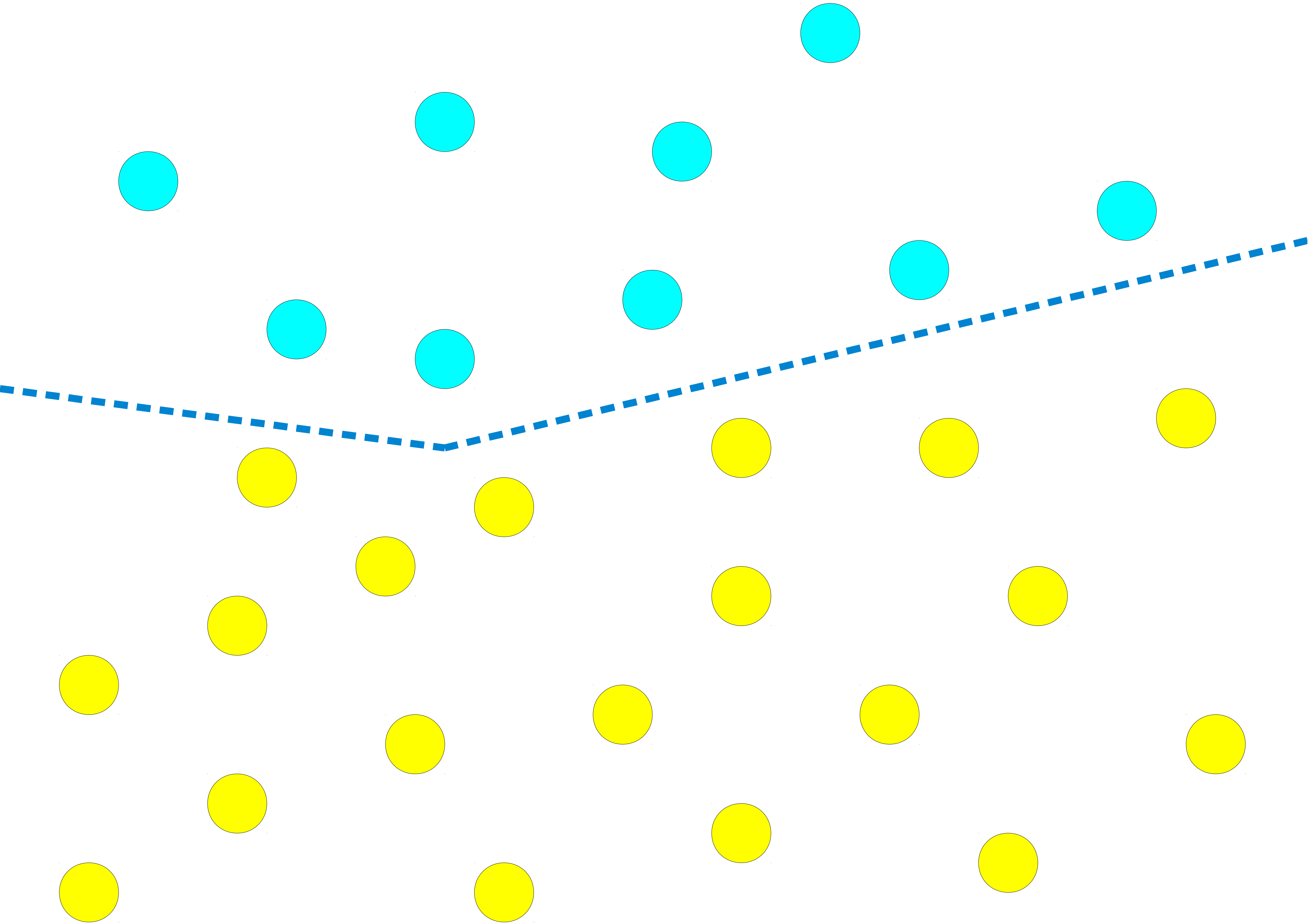


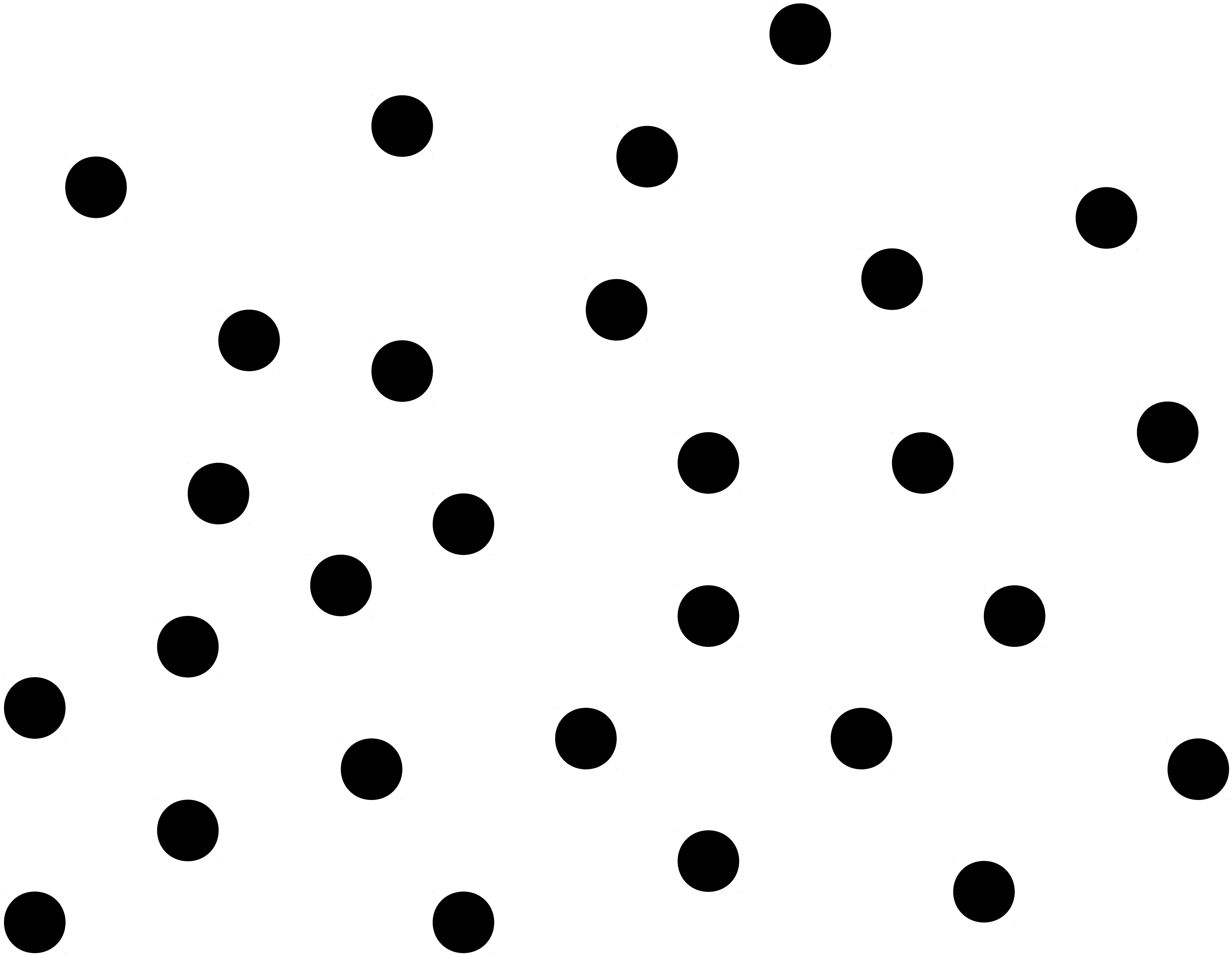


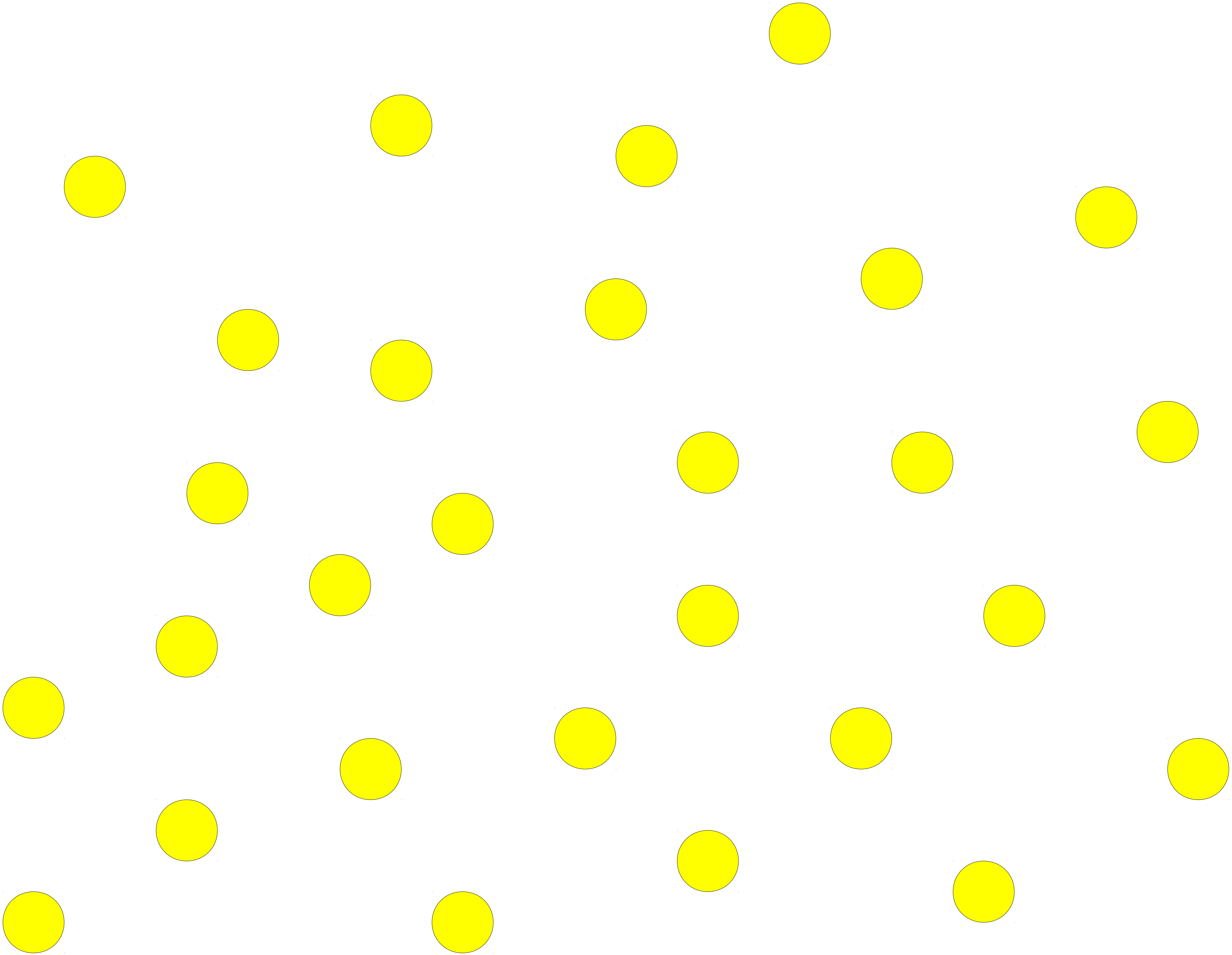


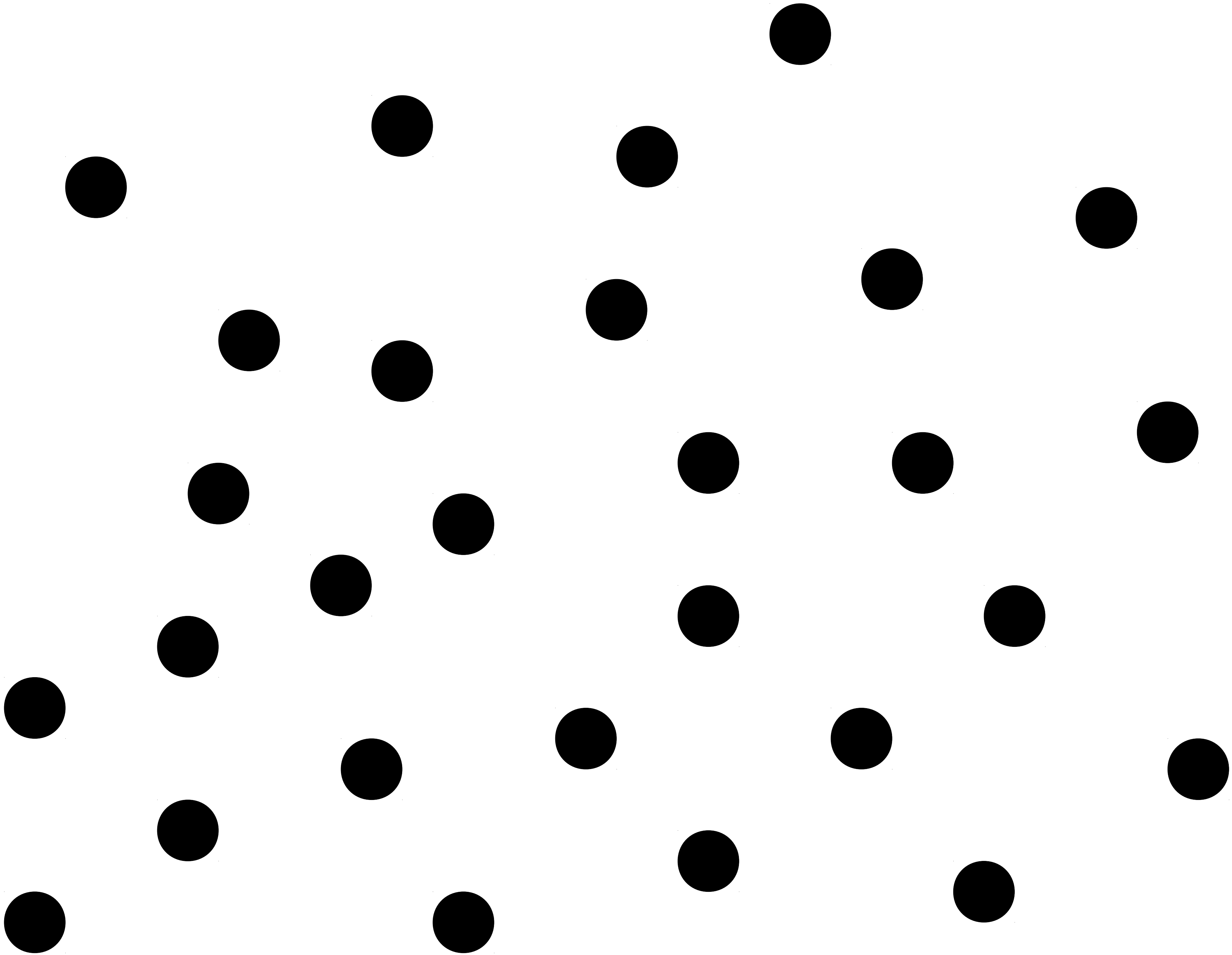


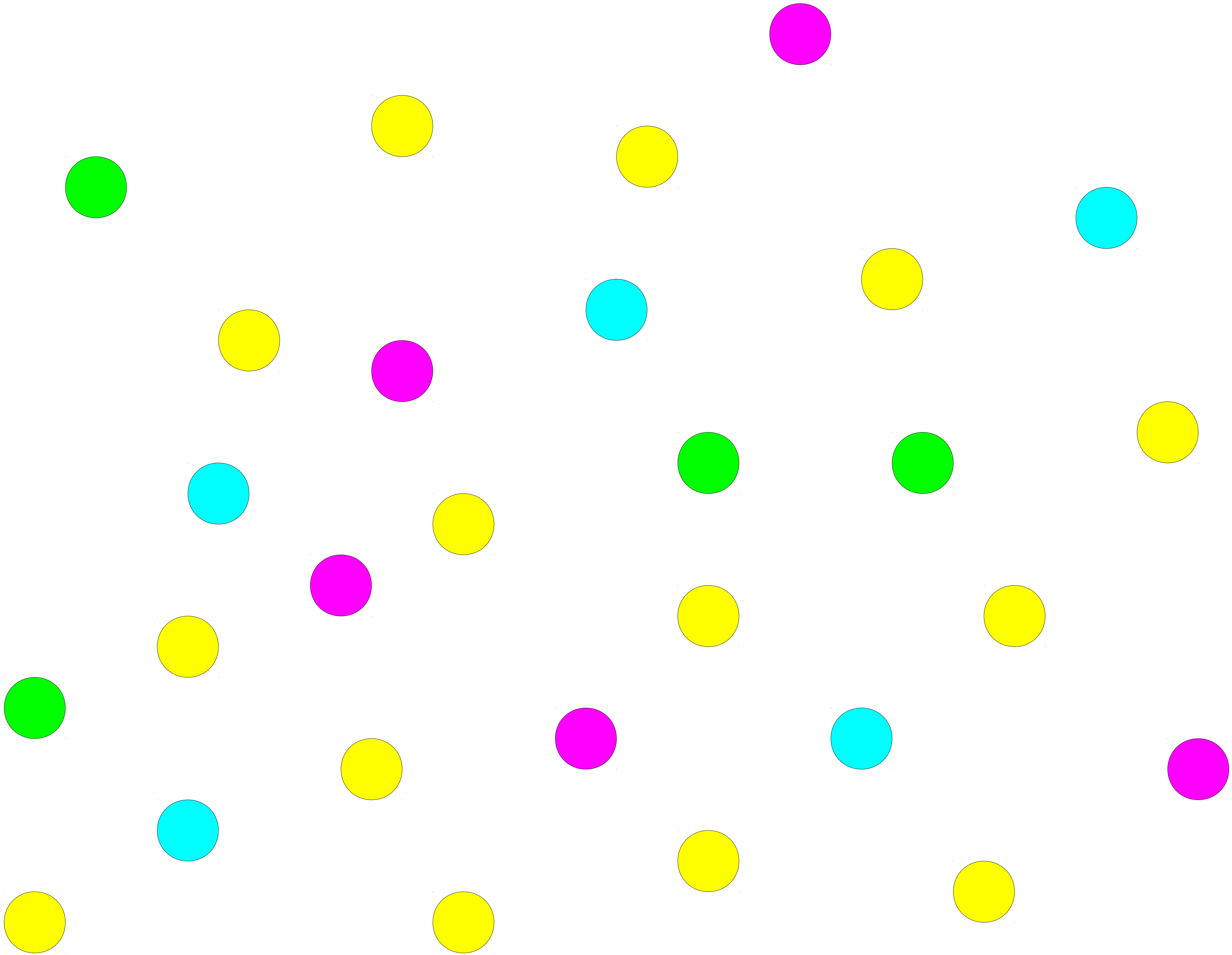












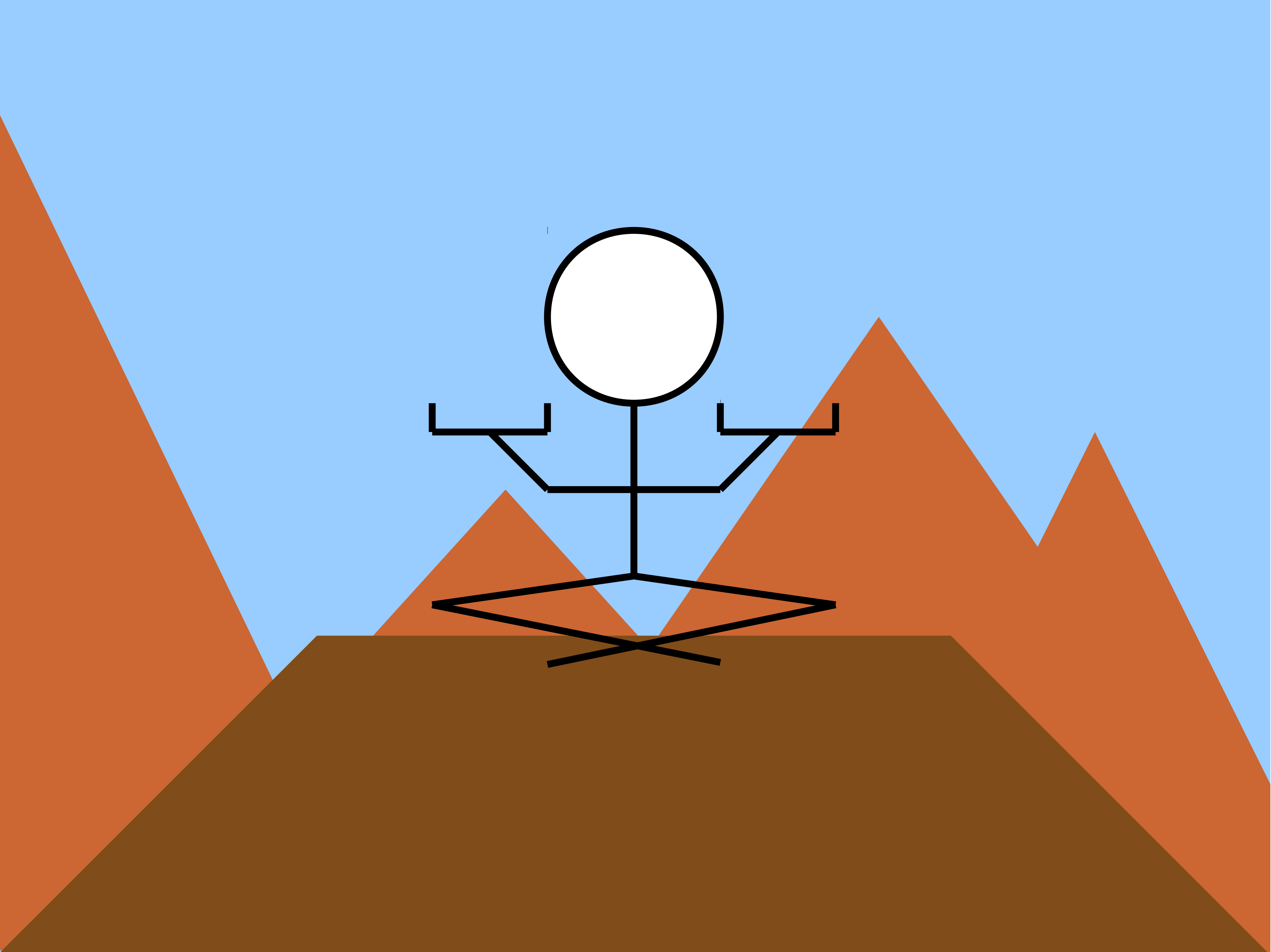
Partitions

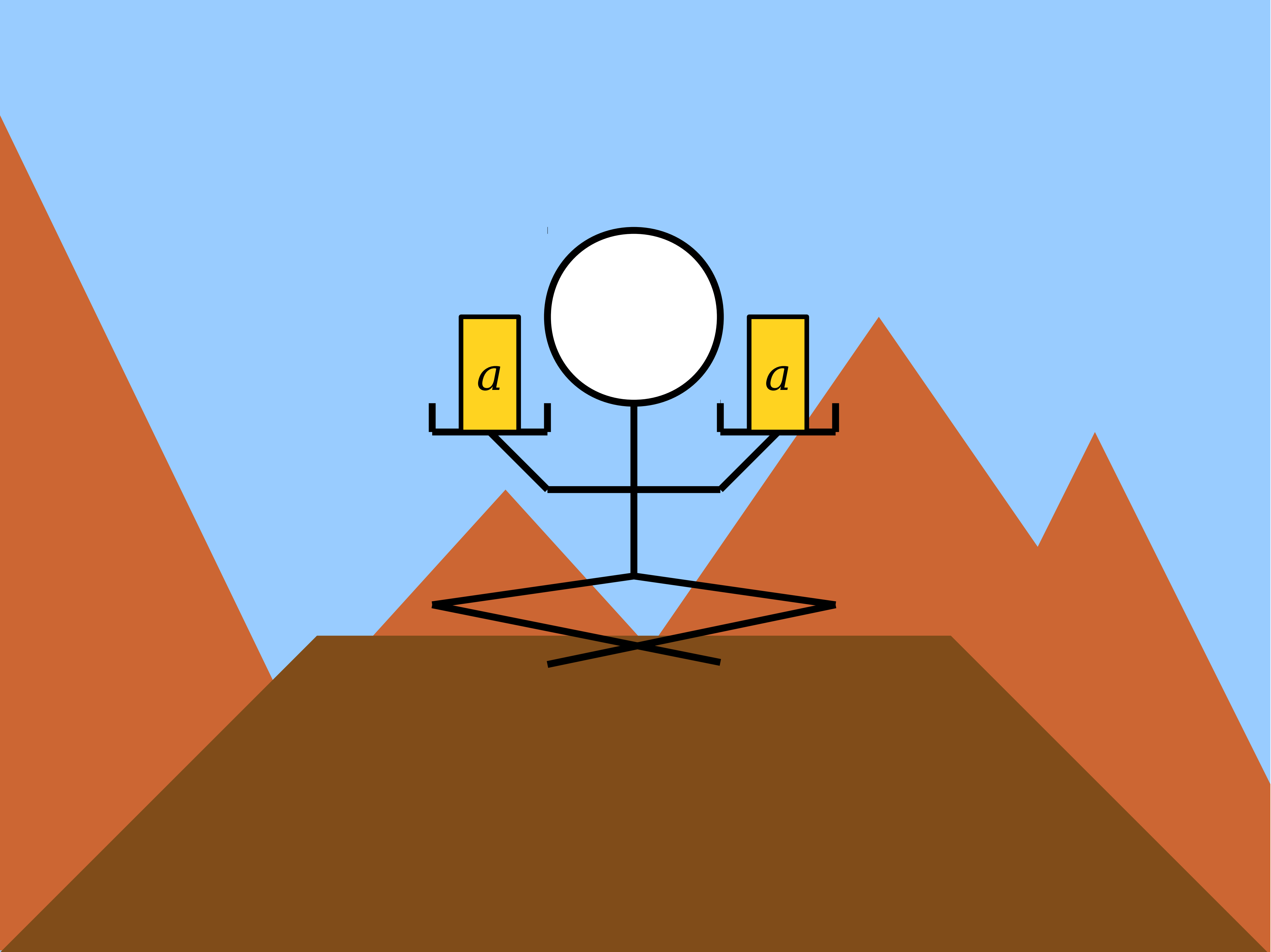
- A ***partition of a set*** is a way of splitting the set into disjoint, nonempty subsets so that every element belongs to exactly one subset.
- Intuitively, a partition of a set breaks the set apart into smaller pieces.
- There doesn't have to be any rhyme or reason to what those pieces are, though often there is one.

Partitions and Clustering

- If you have a set of data, you can often learn something from the data by finding a “good” partition of that data and inspecting the partitions.
  - Usually, the term ***clustering*** is used in data analysis rather than *partitioning*.
- Interested to learn more? Take CS161 or CS246!

What's the connection between partitions and binary relations?

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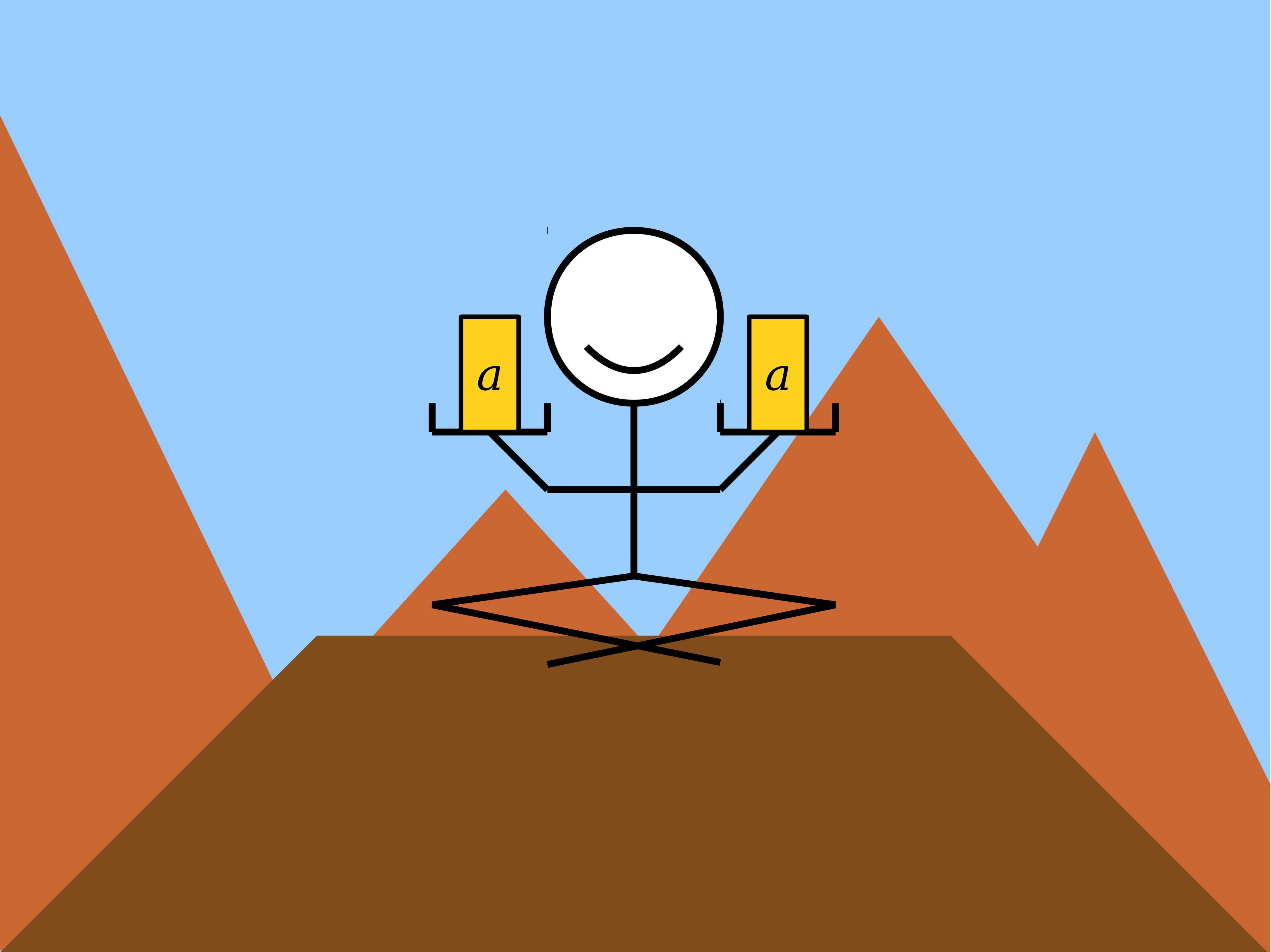
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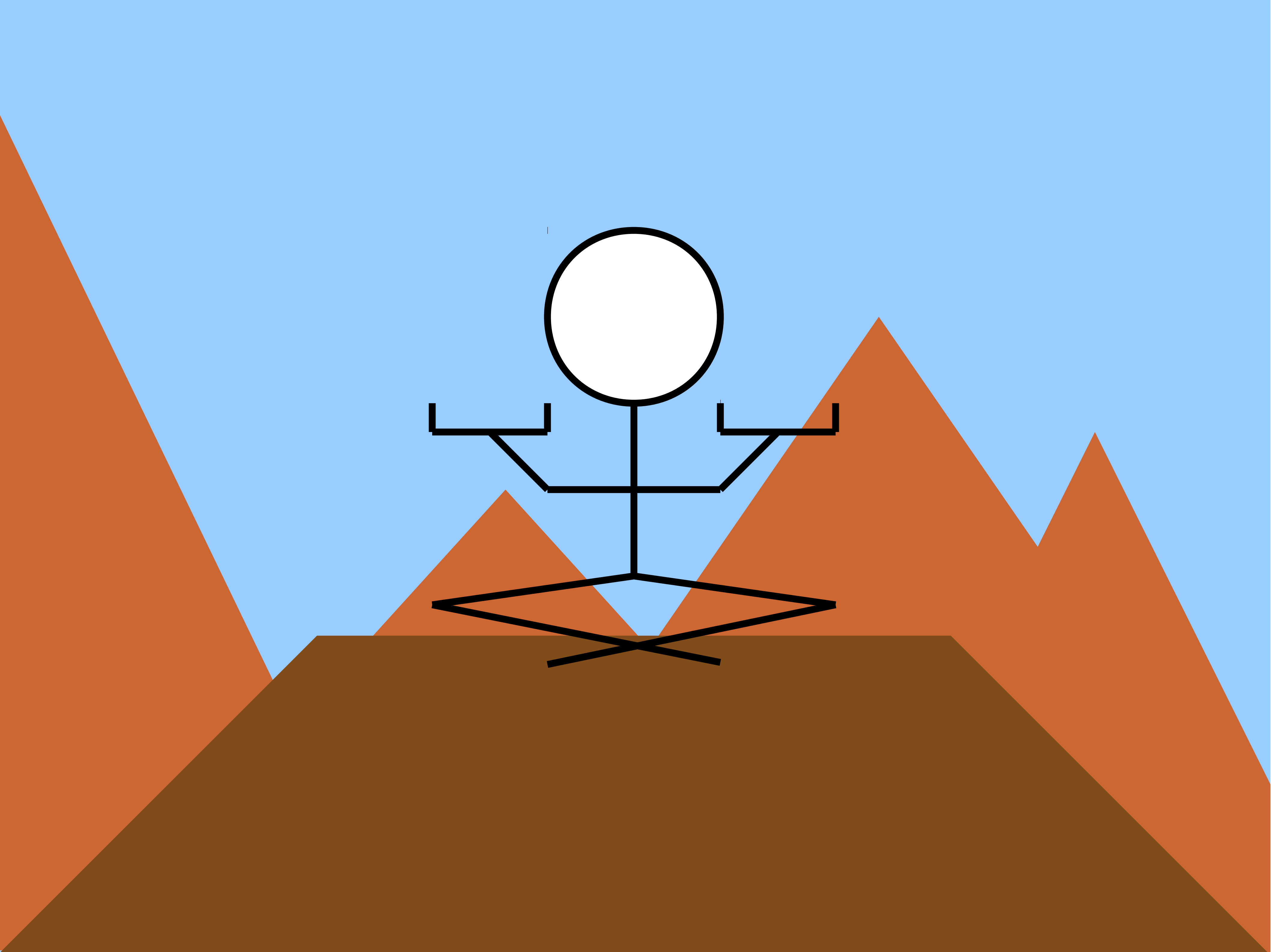

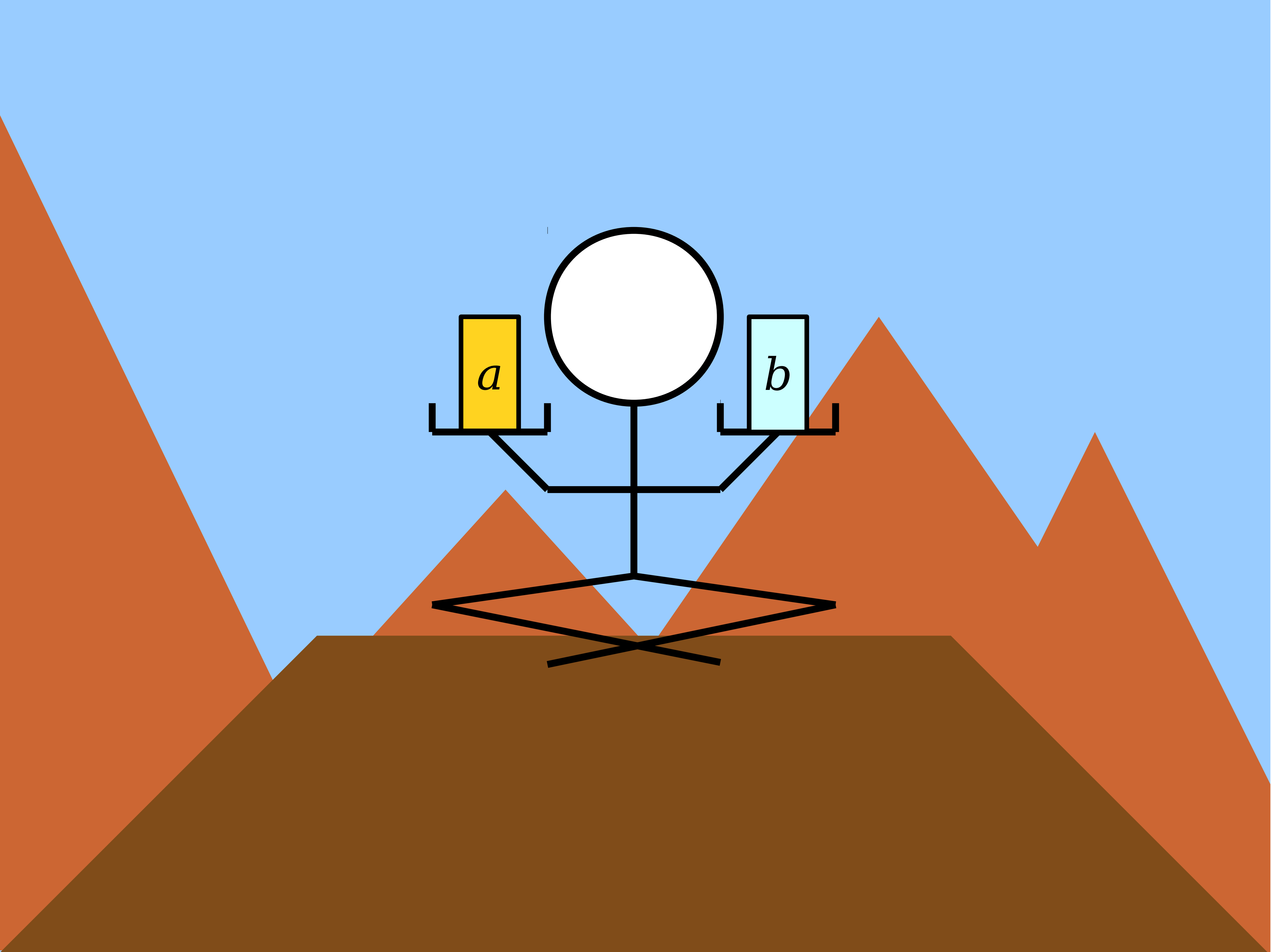
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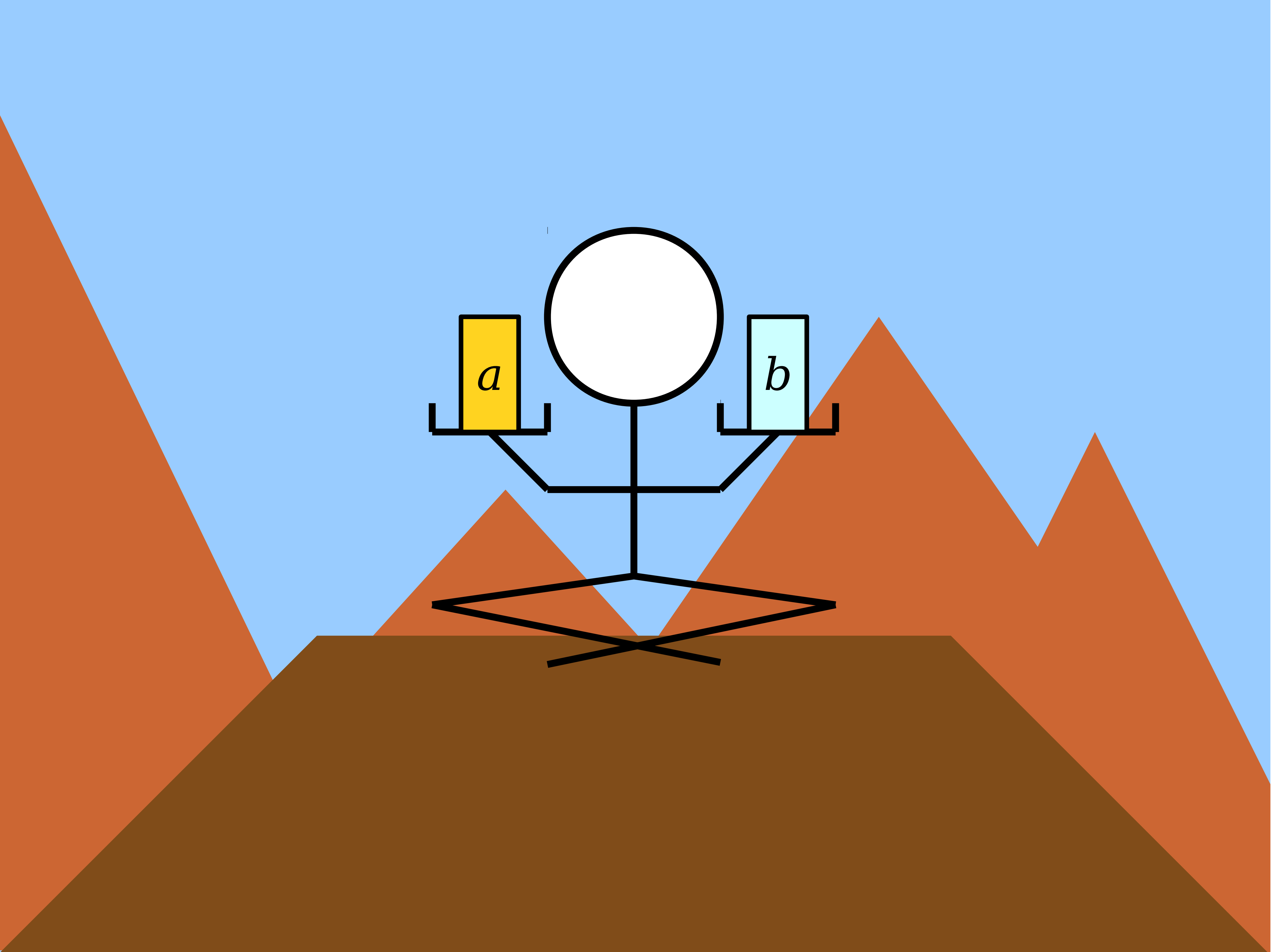
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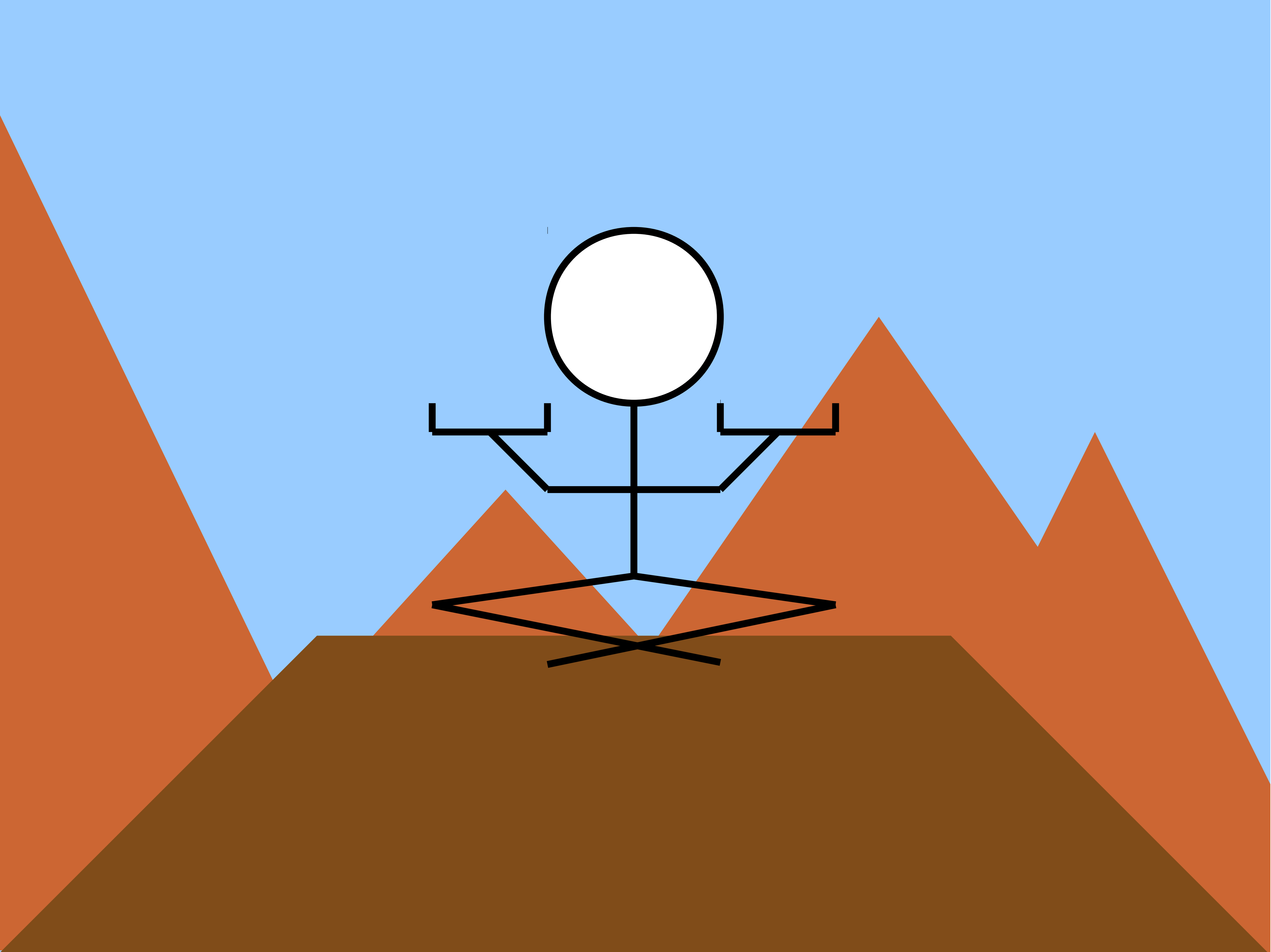

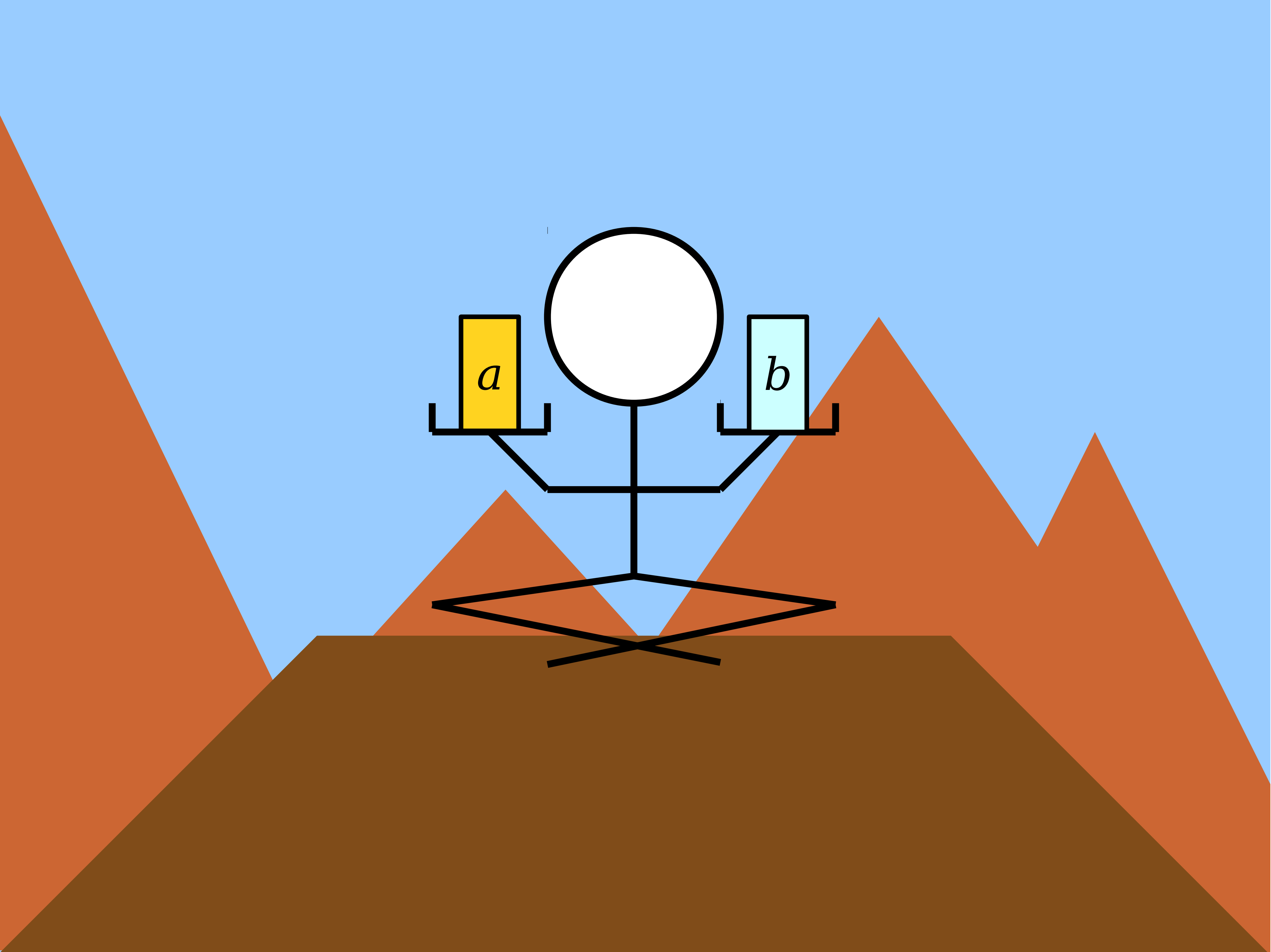
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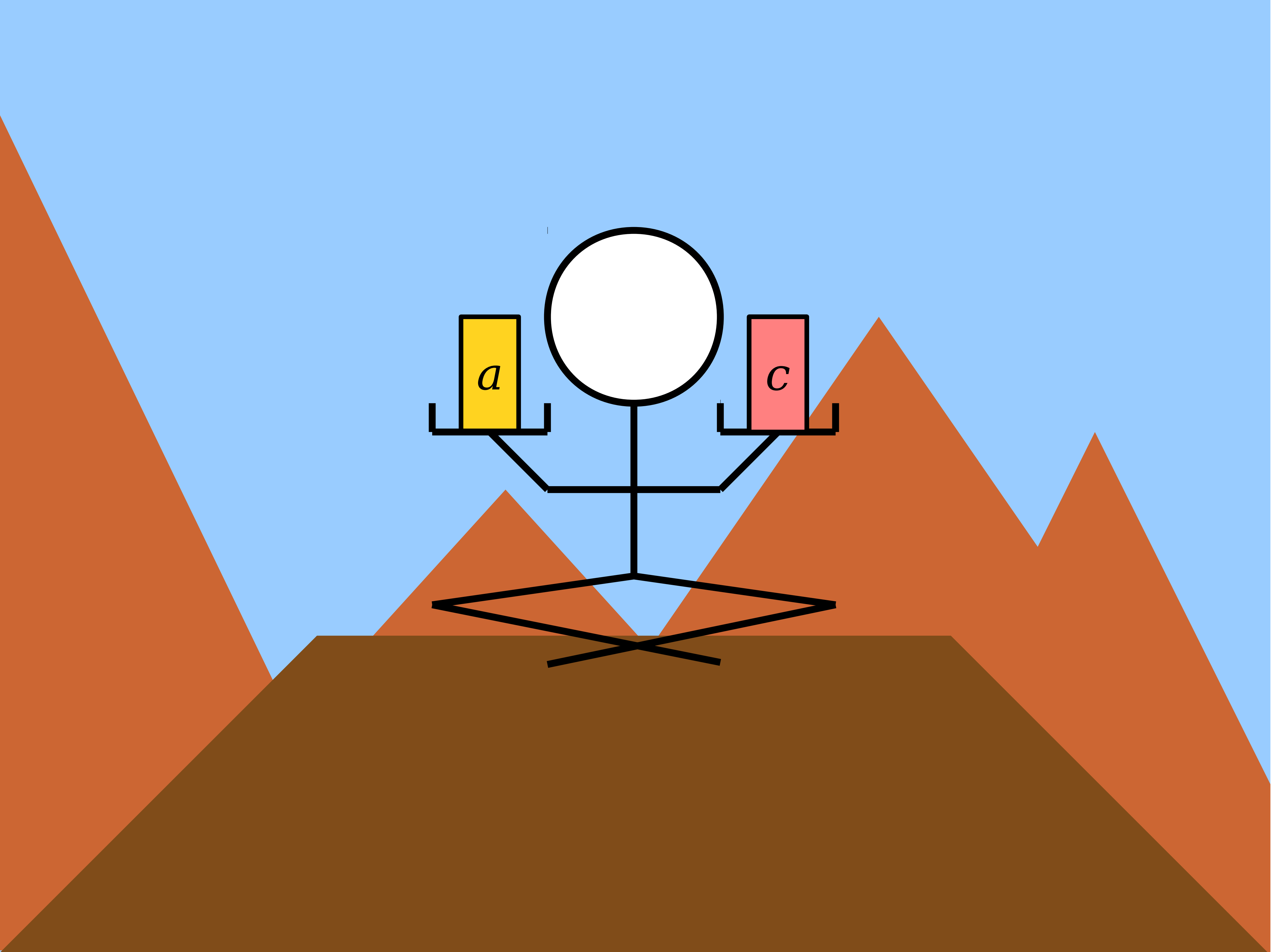
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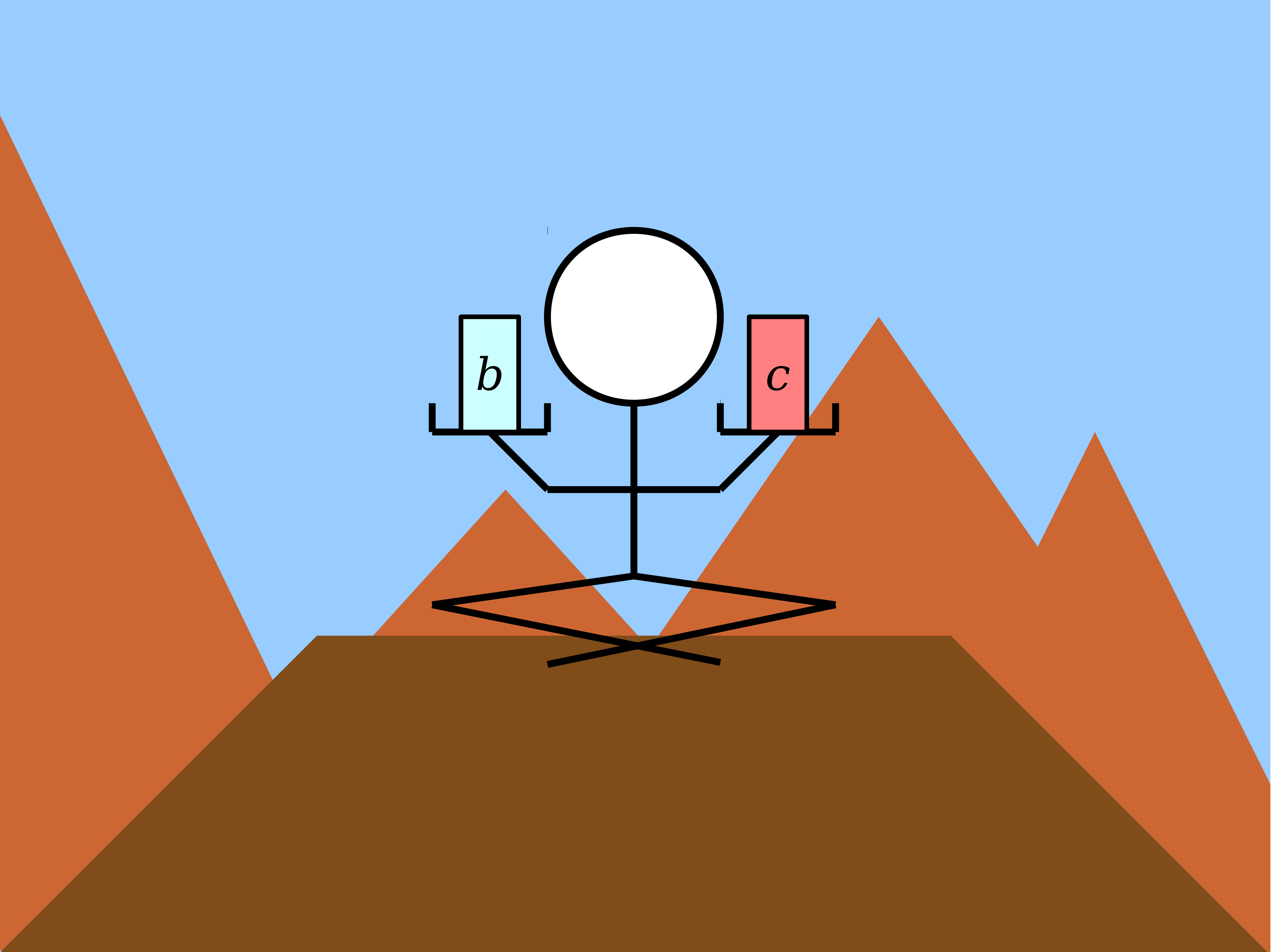
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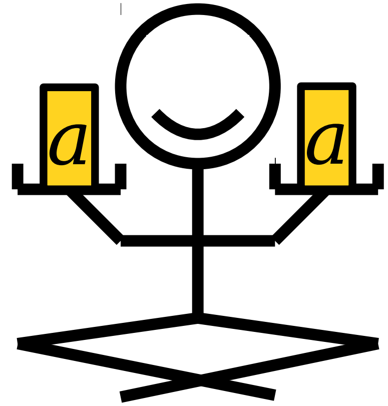
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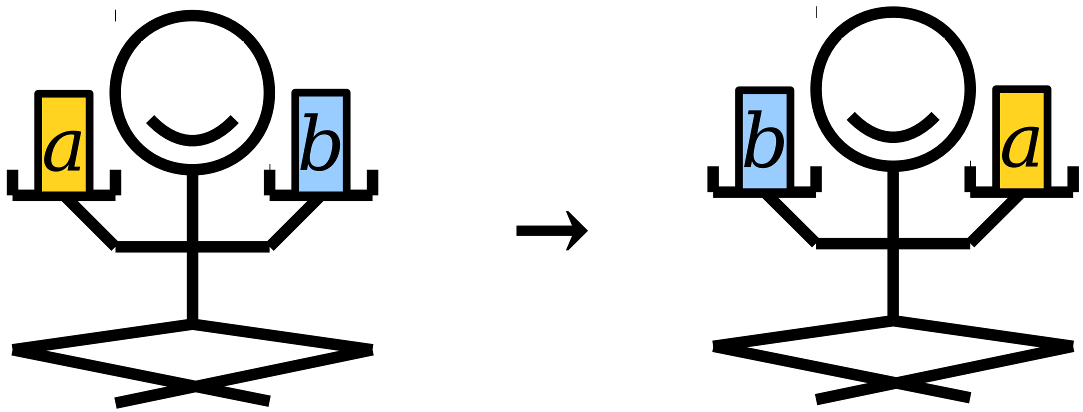
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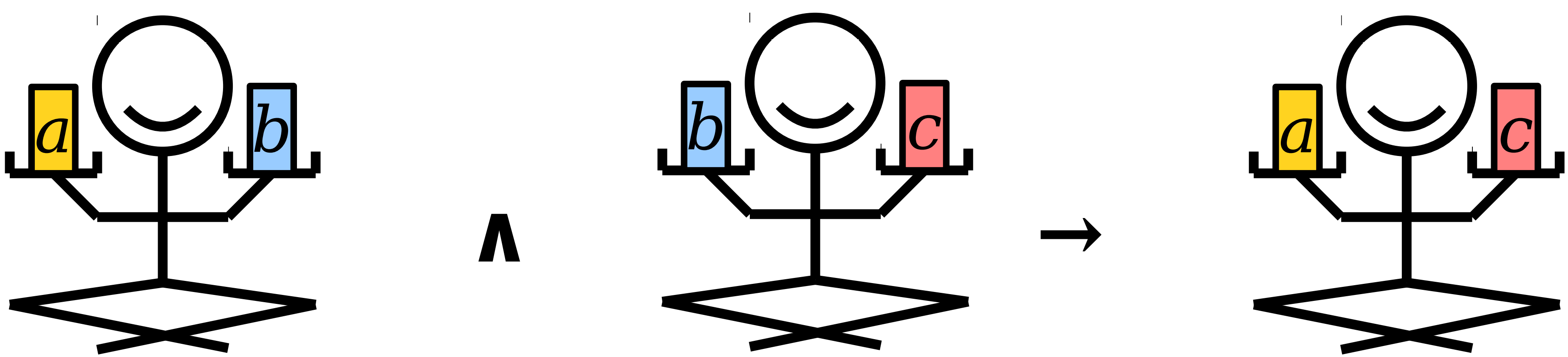

a
b
b
c
a
c

$$aRa$$

$$aRb \rightarrow bRa$$

$$aRb \wedge bRc \rightarrow aRc$$

$$\forall a \in A.\ aRa$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

$$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \to aRc)$$

$$\forall a \in A.\ aRa$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

$$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \to aRc)$$

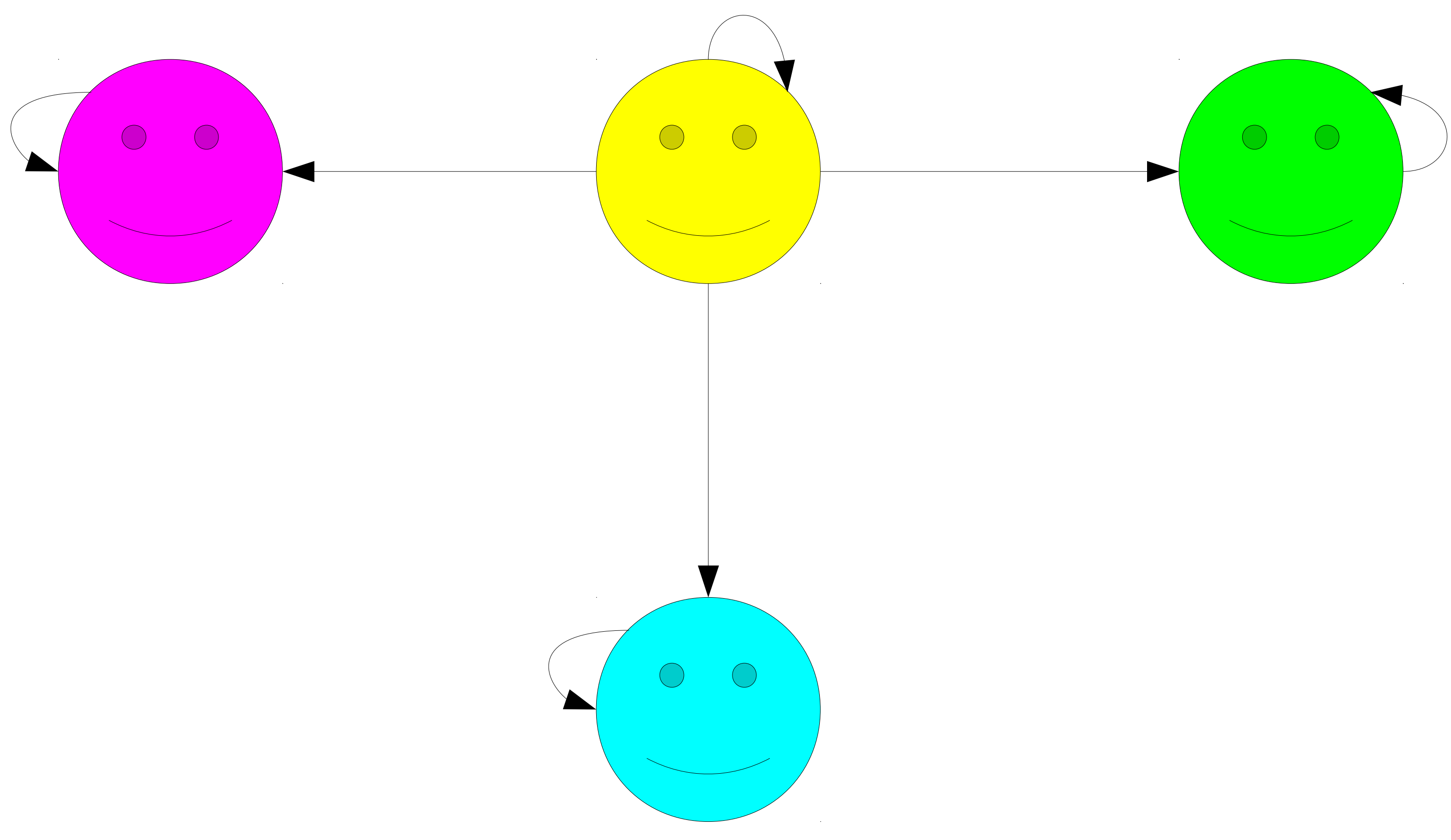
Reflexivity

- Some relations always hold from any element to itself.
- Examples:
  - $x = x$ for any x.
  - $A \subseteq A$ for any set A.
  - $x \equiv_k x$ for any x.
- Relations of this sort are called ***reflexive***.
- Formally speaking, a binary relation R over a set A is reflexive if the following is true:

$$\forall a \in A.\ aRa$$

("Every element is related to itself.")

Reflexivity Visualized



$$\forall a \in A.\ aRa$$

("Every element is related to itself.")

$$\forall a \in A.\ aRa$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

$$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \to aRc)$$

$$\forall a \in A.\ aRa$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow bRa)$$

$$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \rightarrow aRc)$$

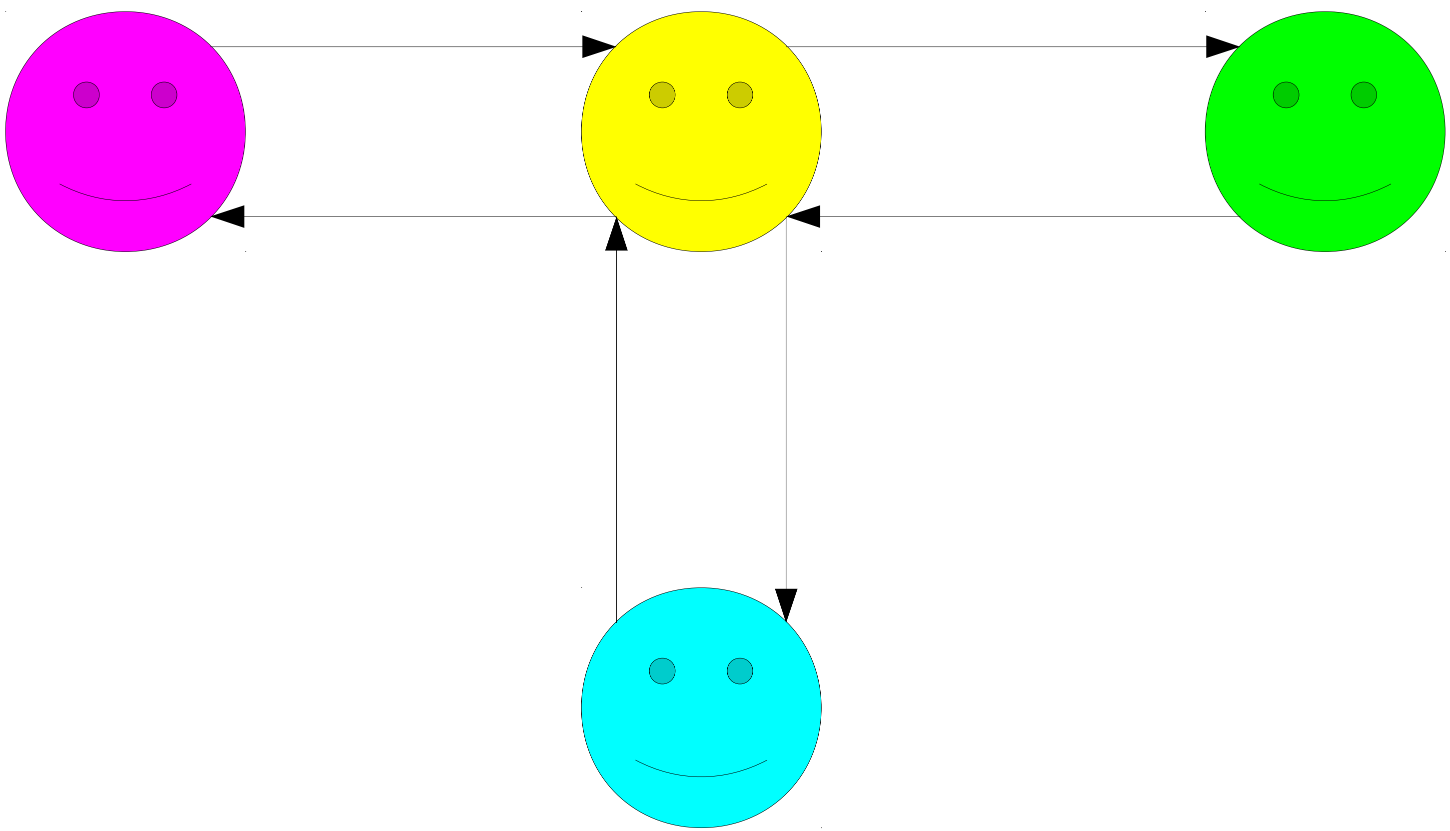
Symmetry

- In some relations, the relative order of the objects doesn't matter.
- Examples:
  - If $x = y$, then $y = x$.
  - If $x \equiv_k y$, then $y \equiv_k x$.
- These relations are called ***symmetric***.
- Formally: a binary relation R over a set A is called *symmetric* if

$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow bRa)$$

(*"If a is related to b, then b is related to a."*)

Symmetry Visualized



$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow bRa)$$

(*"If a is related to b, then b is related to a."*)

$$\forall a \in A.\ aRa$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

$$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \to aRc)$$

$$\forall a \in A.\ aRa$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow bRa)$$

$$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \rightarrow aRc)$$

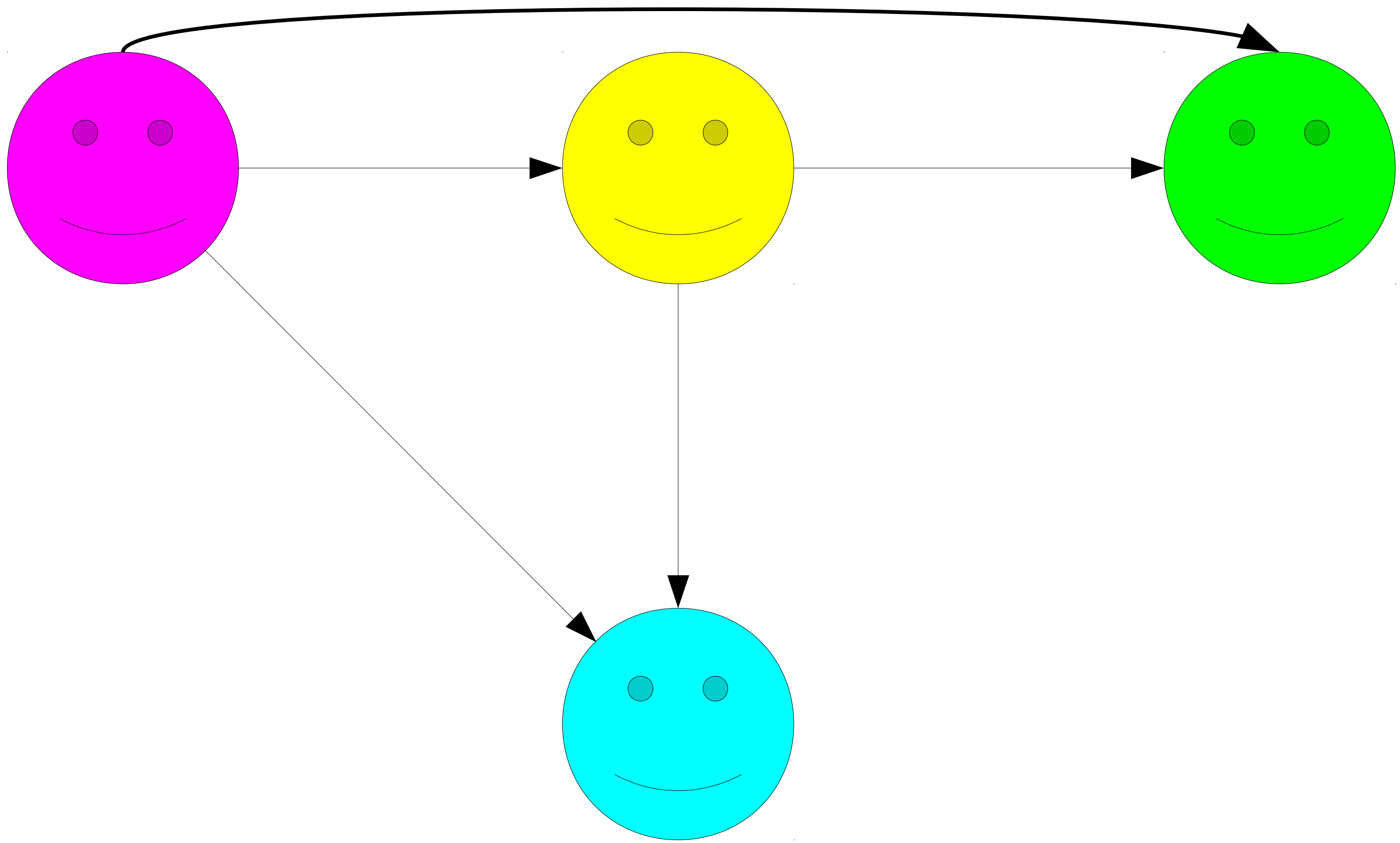
Transitivity

- Many relations can be chained together.
- Examples:
  - If $x = y$ and $y = z$, then $x = z$.
  - If $R \subseteq S$ and $S \subseteq T$, then $R \subseteq T$.
  - If $x \equiv_k y$ and $y \equiv_k z$, then $x \equiv_k z$.
- These relations are called ***transitive***.
- A binary relation R over a set A is called *transitive* if

$$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \rightarrow aRc)$$

(*"Whenever a is related to b and b is related to c, we know a is related to c.*)

Transitivity Visualized



$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$

(*"Whenever a is related to b and b is related to c, we know a is related to c.*)

Equivalence Relations

- An ***equivalence relation*** is a relation that is reflexive, symmetric and transitive.
- Some examples:
  - $x = y$
  - $x \equiv_k y$
  - x has the same color as y
  - x has the same shape as y.

Time-Out for Announcements!

Problem Set Three

- The Problem Set Three checkpoint problem was due at 3:00PM today.
  - We'll get back to you with feedback by Wednesday.
- The remaining problems are due on Friday.
  - Please feel free to stop by office hours with questions!
- Problem Set Two solutions are now available.
  - ***Please read over them!*** That was a tricky problem set and we want you to have a deep understanding of the problems on it.

Apply to Section Lead

- Applications are now open to section lead for CS106A/B/X.
- Incredible program: you'll meet a ton of cool people (students and faculty), get a rock-solid understanding of the fundamentals, and will learn what goes on on the other side of the desk.
- If you've already taken CS106B/X, apply online by Thursday at

http://cs198.stanford.edu

- If you're currently in CS106B/X, you can apply by the seventh week of class.
- ***Highly recommended!***

Your Questions

“What advice would you have for a student who feels like their entire humanity-loving academic existence is being invalidated by the CS/Engineering/Tech-focused rhetoric at Stanford? At what point do you think STEM would be overly praised on campus?”

It's unhealthy if any one discipline crowds out the others on campus. The techniques and perspectives from STEM are wonderful and impactful, but they only apply in certain domains. The ideas and frameworks from the humanities and social sciences are valuable and definitely shouldn't be trivialized.

STEM rhetoric shouldn't invalidate humanities and vice-versa. We should be searching for ways to combine perspectives from each to help make the world a better place.

“With all the CS hype, how do you know if you're sincerely interested in the subject? vs. just doing it to be employable/because it's "the thing" now”

I would take the "Seven Samurai" approach here. There are a ton of good reasons to get into CS and to want to continue on with it. You could study it because you really like the classes you're taking. You could study it because you want to use CS to change the world. You could study it for the financial stability (there are a lot of people for whom this is a totally legitimate reason). You could study it because it's something you and your friends do together.

If you are concerned about whether you're making the right call, please feel free to come talk to me. A good question to ask would be what the alternative is. CS versus "something else" is hard to evaluate. CS versus "specific other field X" is a good and reasonable question to ask.

Back to CS103!

Equivalence Relation Proofs

- Let's suppose you've found a binary relation R over a set A and want to prove that it's an equivalence relation.
- How exactly would you go about doing this?

An Example Relation

- Consider the binary relation ~ defined over the set ℤ:

$a \sim b$ if $a + b$ is even

- Some examples:

0~4 1~9 2~6 5~5

- Turns out, this is an equivalence relation! Let's see how to prove it.

We can binary relations by giving a rule, like this:

a~b* if *some property of a and b holds

This is the general template for defining a relation. Although we're using "if" rather than "iff" here, the two above statements are definitionally equivalent. For a variety of reasons, definitions are often introduced with "if" rather than "iff." Check the "Mathematical Vocabulary" handout for details.

What properties must ~ have to be an equivalence relation?

Reflexivity
Symmetry
Transitivity

Let's prove each property independently.

$a \sim b$ if $a+b$ is even

Lemma 1: The binary relation $\sim$ is reflexive.

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Proof:

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Lemma 1: The binary relation ~ is reflexive.

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What is the formal definition of reflexivity?

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What is the formal definition of reflexivity?

$$\forall a \in \mathbb{Z}.\ a \sim a$$

$a \sim b$ if $a+b$ is even

Lemma 1: The binary relation ~ is reflexive.

Proof:

What is the formal definition of reflexivity?

$$\forall a \in \mathbb{Z}.\ a \sim a$$

Therefore, we'll choose an arbitrary integer a, then go prove that $a \sim a$.

$a \sim b$ if $a+b$ is even

Lemma 1: The binary relation $\sim$ is reflexive.

Proof: Consider an arbitrary $a \in \mathbb{Z}$. We need to prove that $a \sim a$.

$a \sim b$ if $a+b$ is even

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Proof: Consider an arbitrary $a \in \mathbb{Z}$. We need to prove that $a \sim a$. From the definition of the ~ relation, this means that we need to prove that $a+a$ is even.

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To see this, notice that $a+a = 2a$, so the sum $a+a$ can be written as $2k$ for some integer k (namely, a), so $a+a$ is even.

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$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation ~ is symmetric.

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What is the formal definition of symmetry?

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Lemma 2: The binary relation ~ is symmetric.

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What is the formal definition of symmetry?

$$\forall a \in \mathbb{Z}.\ \forall b \in \mathbb{Z}.\ (a \sim b \rightarrow b \sim a)$$

$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation ~ is symmetric.

Proof:

What is the formal definition of symmetry?

$$\forall a \in \mathbb{Z}.\ \forall b \in \mathbb{Z}.\ (a \sim b \to b \sim a)$$

Therefore, we'll choose arbitrary integers a and b where $a \sim b$, then prove that $b \sim a$.

$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation $\sim$ is symmetric.

Proof: Consider any integers a and b where $a \sim b$. We need to show that $b \sim a$.

***$a \sim b$* if *$a+b$* is even**

Lemma 2: The binary relation ~ is symmetric.

Proof: Consider any integers a and b where $a \sim b$.
We need to show that $b \sim a$.

Since $a \sim b$, we know that $a+b$ is even.

$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation ~ is symmetric.

Proof: Consider any integers a and b where $a \sim b$. We need to show that $b \sim a$.

Since $a \sim b$, we know that $a+b$ is even. Because $a+b = b+a$, this means that $b+a$ is even.

***a~b* if *a+b* is even**

Lemma 2: The binary relation ~ is symmetric.

Proof: Consider any integers *a* and *b* where *a~b*. We need to show that *b~a*.

Since *a~b*, we know that *a+b* is even. Because *a+b* = *b+a*, this means that *b+a* is even. Since *b+a* is even, we know that *b~a*, as required.

$a \sim b$ if $a+b$ is even

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$a \sim b$ if $a + b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

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What is the formal definition of transitivity?

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Lemma 3: The binary relation ~ is transitive.

Proof:

What is the formal definition of transitivity?

$$\forall a \in \mathbb{Z}.\ \forall b \in \mathbb{Z}.\ \forall c \in \mathbb{Z}.\ (a \sim b \land b \sim c \rightarrow a \sim c)$$

$a \sim b$ **if** $a + b$ **is even**

Lemma 3: The binary relation ~ is transitive.

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What is the formal definition of transitivity?

$$\forall a \in \mathbb{Z}.\ \forall b \in \mathbb{Z}.\ \forall c \in \mathbb{Z}.\ (a \sim b \land b \sim c \to a \sim c)$$

Therefore, we'll choose arbitrary integers a, b, and c where $a \sim b$ and $b \sim c$, then prove that $a \sim c$.

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

Proof: Consider arbitrary integers a, b and c where $a \sim b$ and $b \sim c$.

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

Proof: Consider arbitrary integers a, b and c where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

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Proof: Consider arbitrary integers a, b and c where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even.

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

Proof: Consider arbitrary integers a, b and c where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even. This means there are integers k and m where $a+b = 2k$ and $b+c = 2m$.

***a*~*b* if *a*+*b* is even**

Lemma 3: The binary relation ~ is transitive.

Proof: Consider arbitrary integers a, b and c where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even. This means there are integers k and m where $a+b = 2k$ and $b+c = 2m$. Notice that

$$(a+b) + (b+c) = 2k + 2m.$$

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$$(a+b) + (b+c) = 2k + 2m.$$

Rearranging, we see that

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so

$$a+c = 2k + 2m - 2b = 2(k+m-b).$$

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So there is an integer r, namely $k+m-b$, such that $a+c = 2r$. Thus $a+c$ is even, so $a \sim c$, as required.

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L

P

Notice that these are grammatically complete sentences. In your own proofs, make sure to write in complete sentences and use appropriate punctuation. It looks really classy and makes your proofs easier to read.

Try the "mugga mugga test." If you read a proof and replace mathematical symbols with "mugga mugga," it should still be grammatically correct.

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Rearranging, we see that

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so

$$a+c = 2k + 2m - 2b = 2(k+m-b).$$

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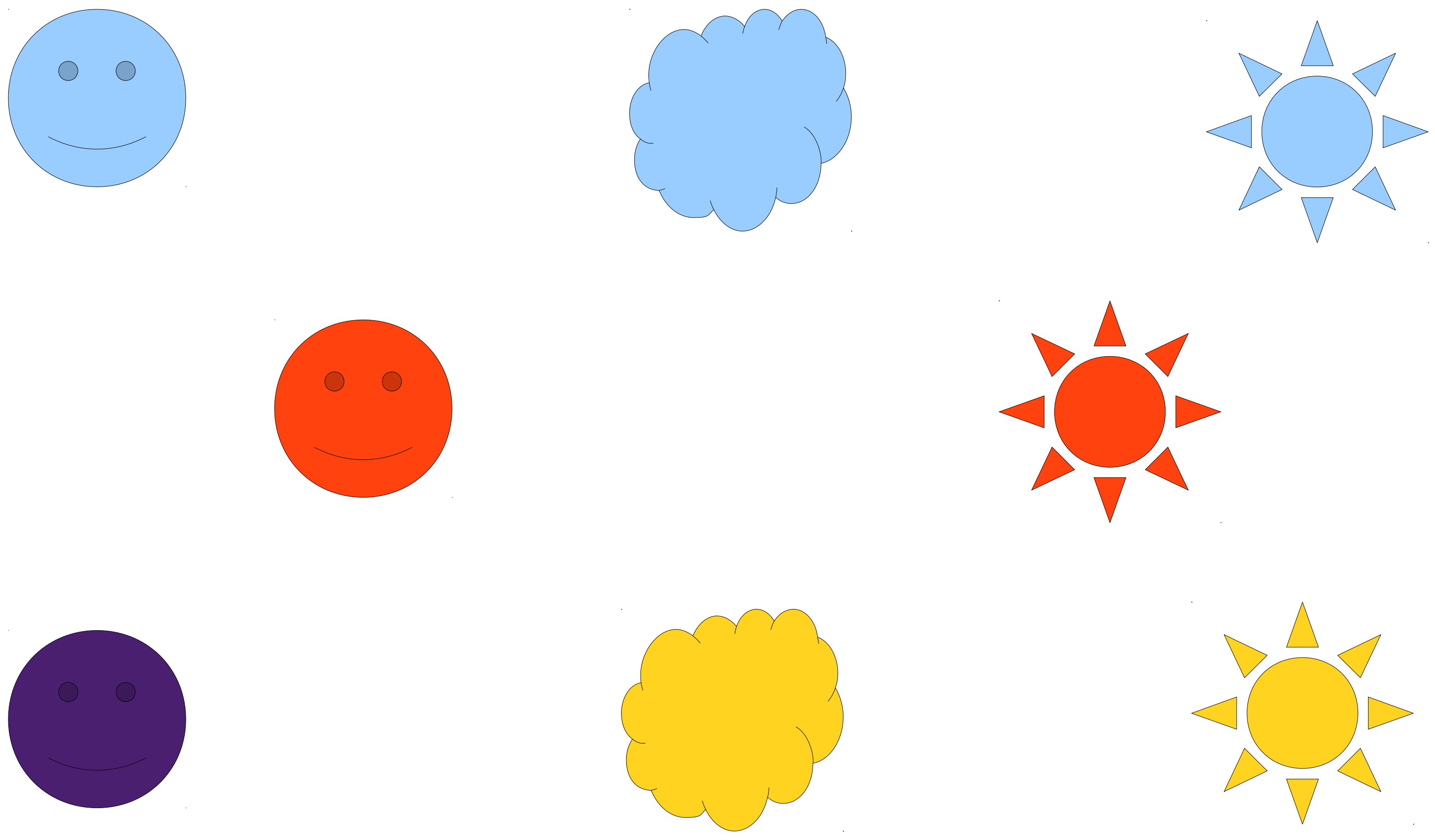
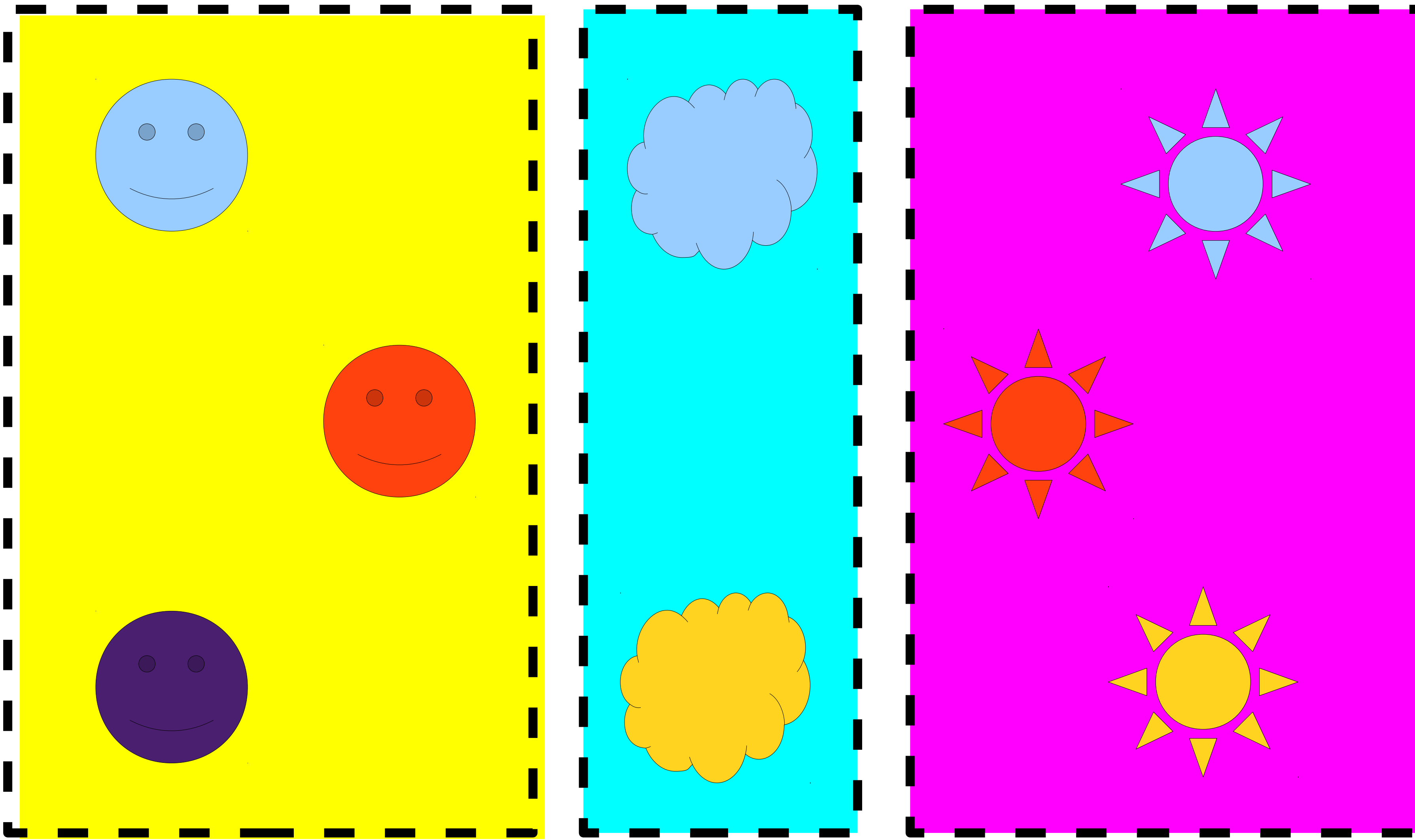
Properties of Equivalence Relations



Let R be “has the same shape as”



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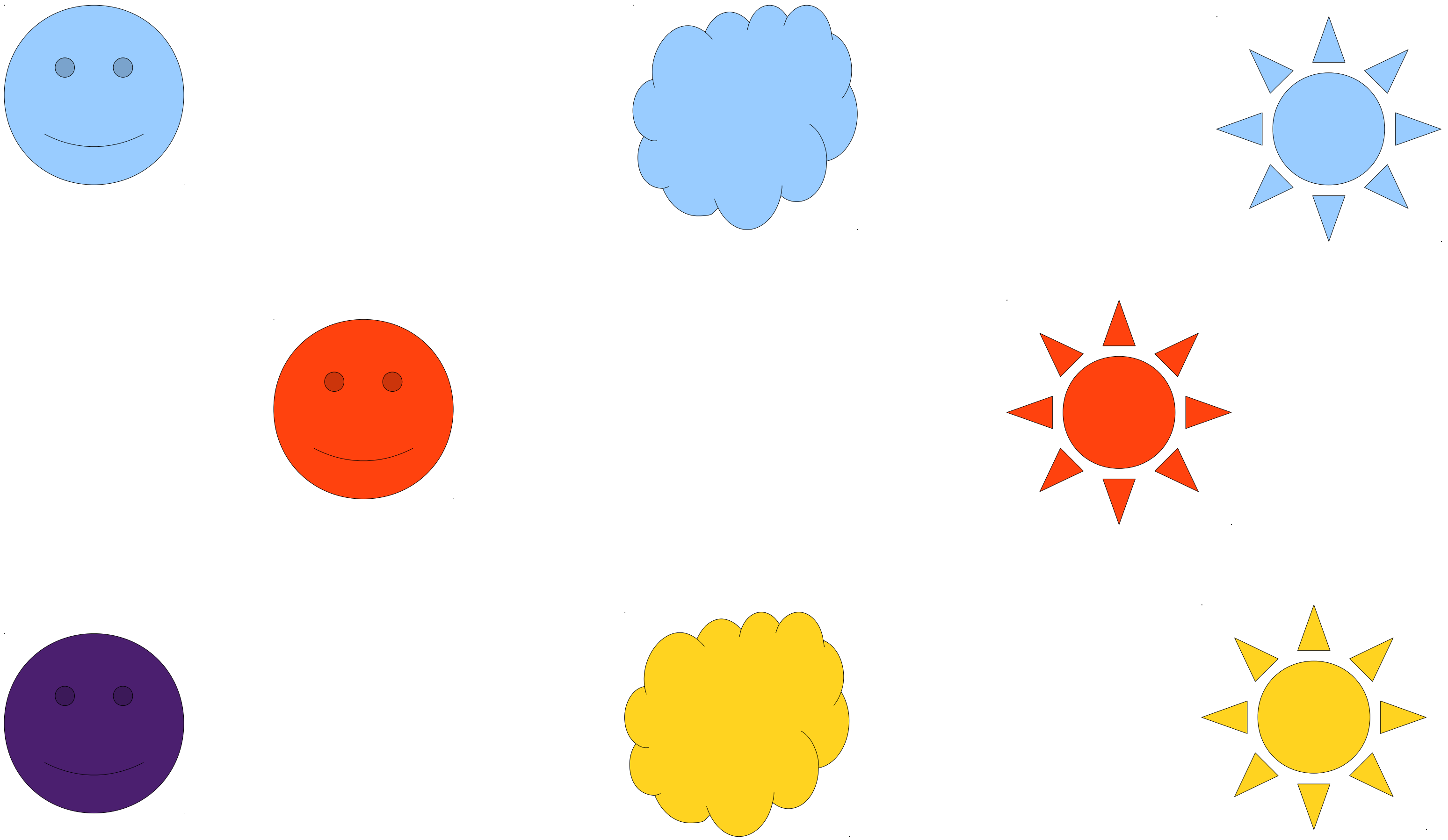
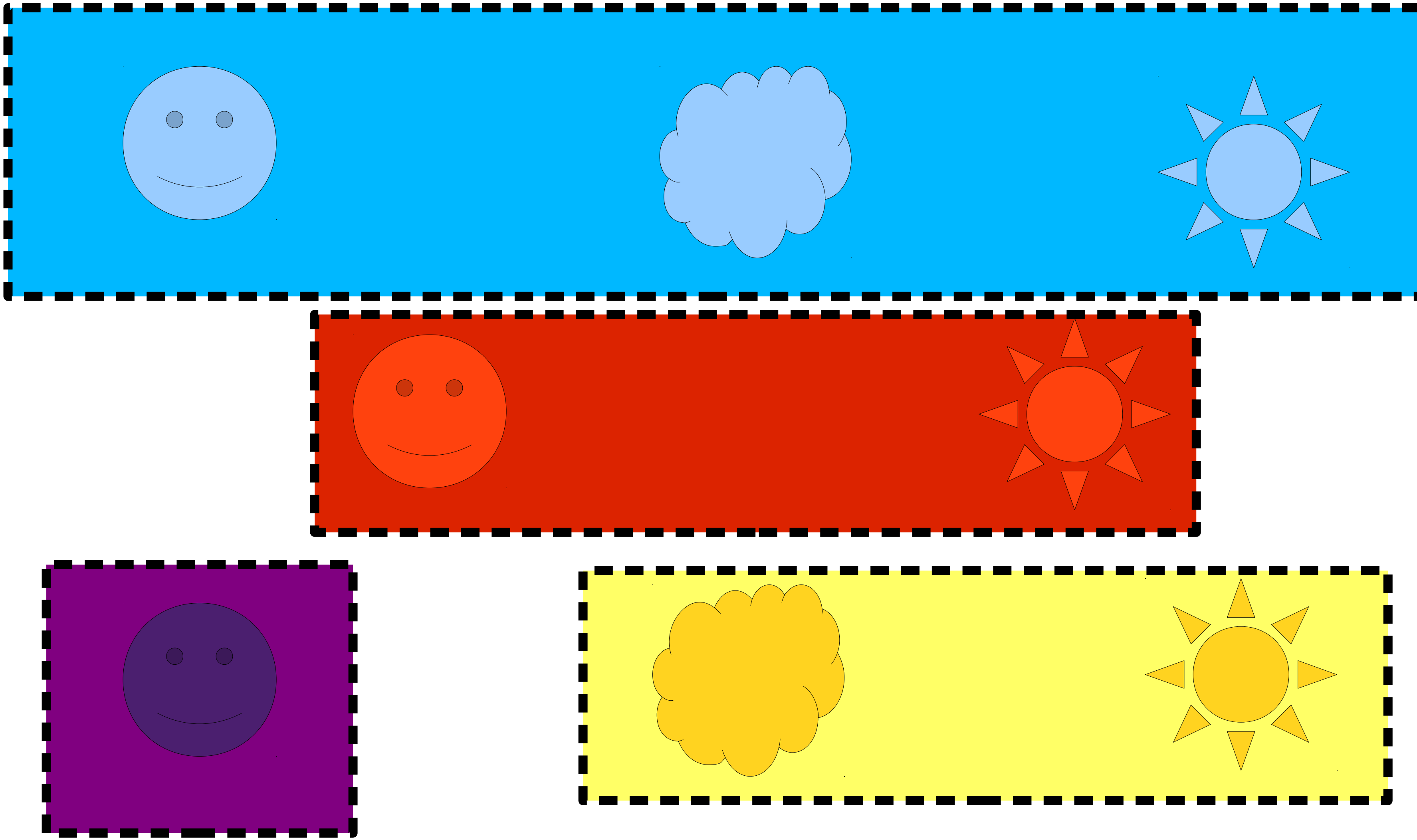


Let *T* be “is the same **color** as”



Let T be “is the same **color** as”

Equivalences and Partitions

- Our definition of equivalence relations was motivated by the idea of partitioning elements into groups.

Partition of Elements ⇒ Equivalence Relation

- In the previous examples, we've seen several equivalence relations that then give rise to a partition. It turns out that this is not a coincidence!

Equivalence Relation ⇒ Partition of Elements

Equivalence Classes

- Given an equivalence relation R over a set A, for any $x \in A$, the ***equivalence class of x*** is the set

$$[x]_R = \{ y \in A \mid xRy \}$$

- $[x]_R$ is the set of all elements of A that are related to x by relation R.
- ***Theorem:*** If R is an equivalence relation over A, then every $a \in A$ belongs to exactly one equivalence class.

Prerequisite Structures

Pancakes

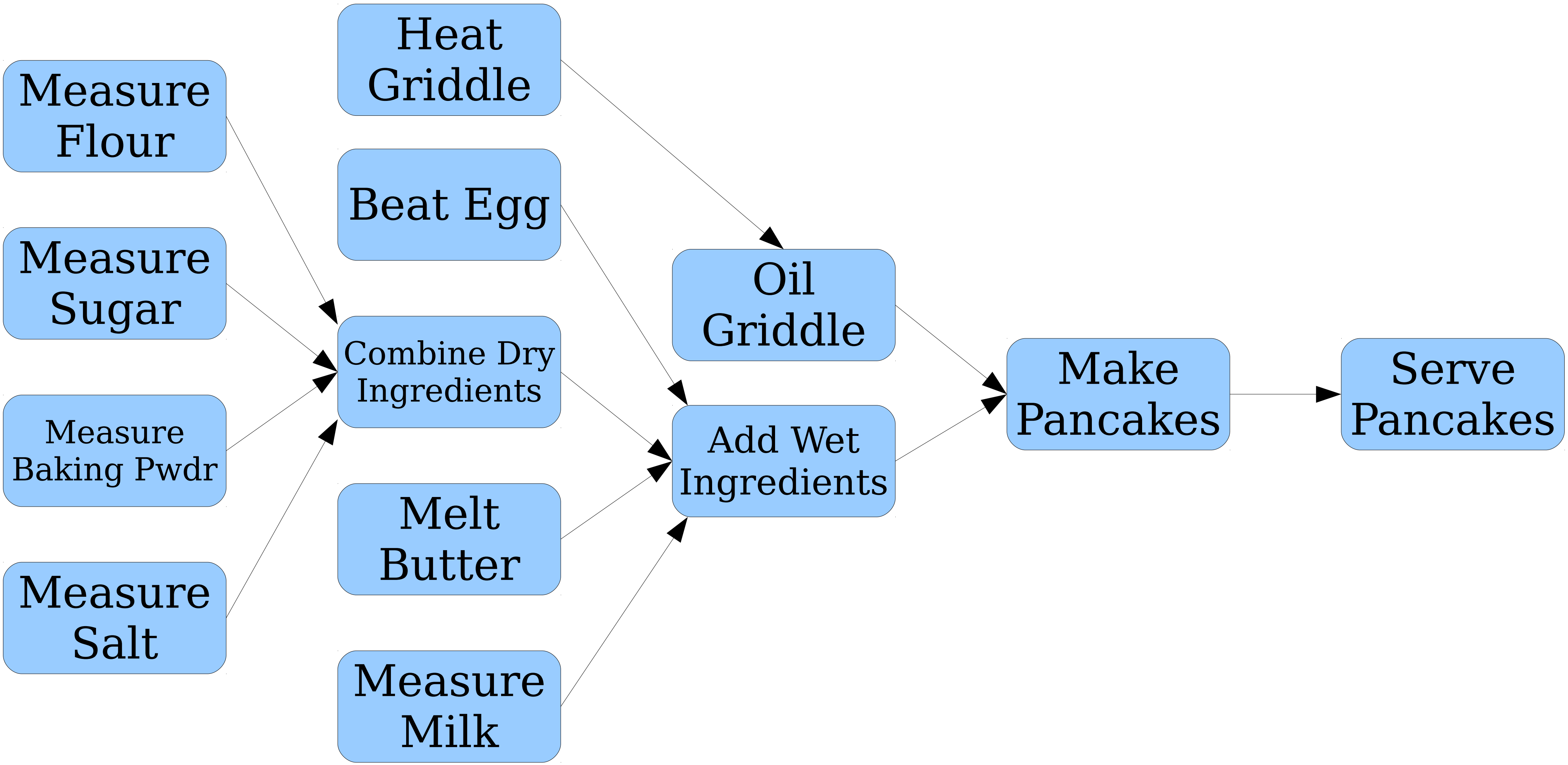
Everyone's got a pancake recipe. This one comes from Food Wishes (http://foodwishes.blogspot.com/2011/08/grandma-kellys-good-old-fashioned.html).

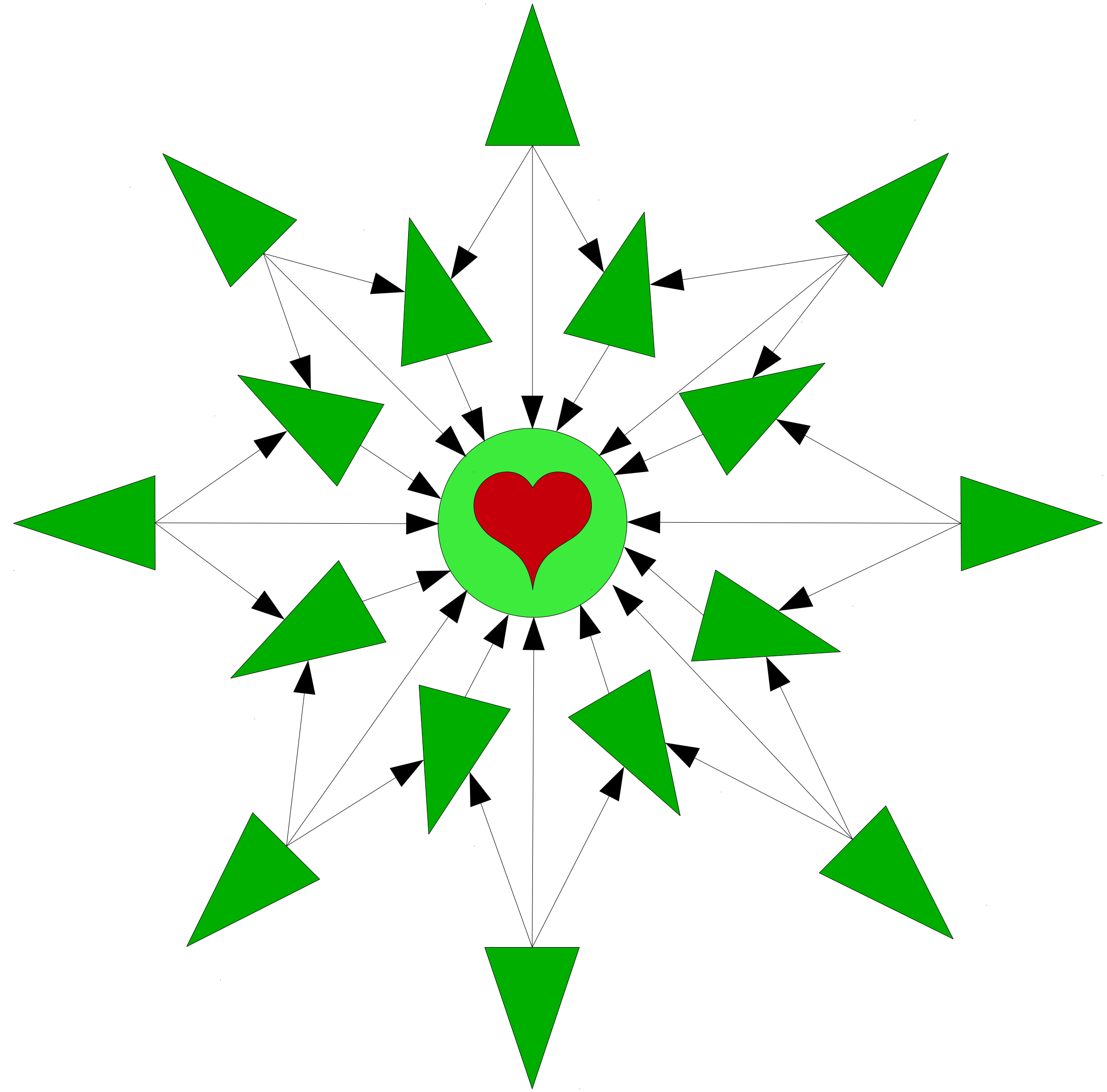
Ingredients

- 1 1/2 cups all-purpose flour
- 3 1/2 tsp baking powder
- 1 tsp salt
- 1 tbsp sugar
- 1 1/4 cup milk
- 1 egg
- 3 tbsp butter, melted

Directions

1. Sift the dry ingredients together.
2. Stir in the butter, egg, and milk. Whisk together to form the batter.
3. Heat a large pan or griddle on medium-high heat. Add some oil.
4. Make pancakes one at a time using 1/4 cup batter each. They're ready to flip when the centers of the pancakes start to bubble.


Measure Flour
Measure Sugar
Measure Baking Pwdr
Measure Salt
Heat Griddle
Beat Egg
Combine Dry Ingredients
Melt Butter
Measure Milk
Oil Griddle
Add Wet Ingredients
Make Pancakes
Serve Pancakes

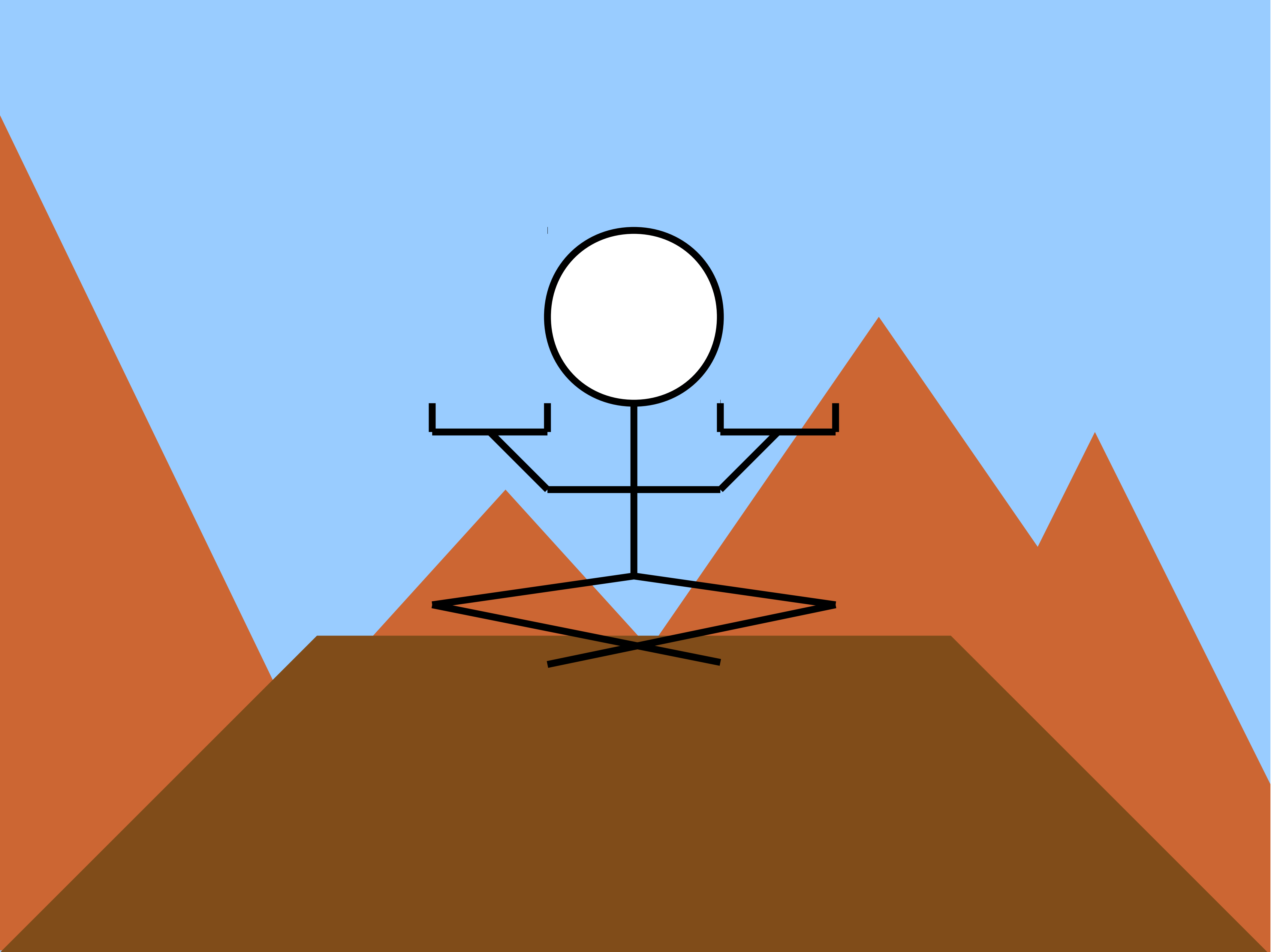


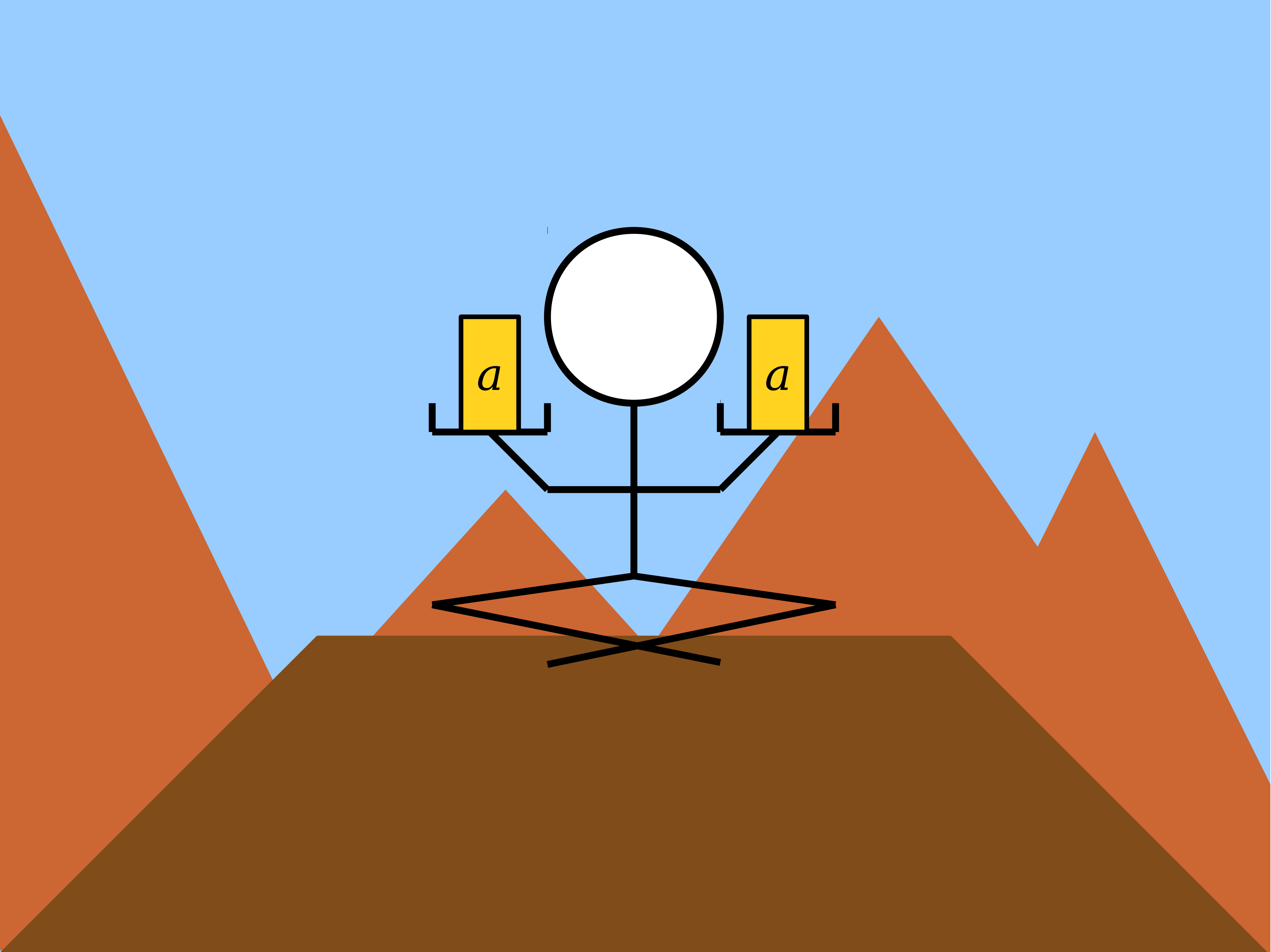
Relations and Prerequisites

- Let's imagine that we have a prerequisite structure with no circular dependencies.
- We can think about a binary relation R where aRb means

“***a* must happen before *b***”

- What properties of R could we deduce just from this?

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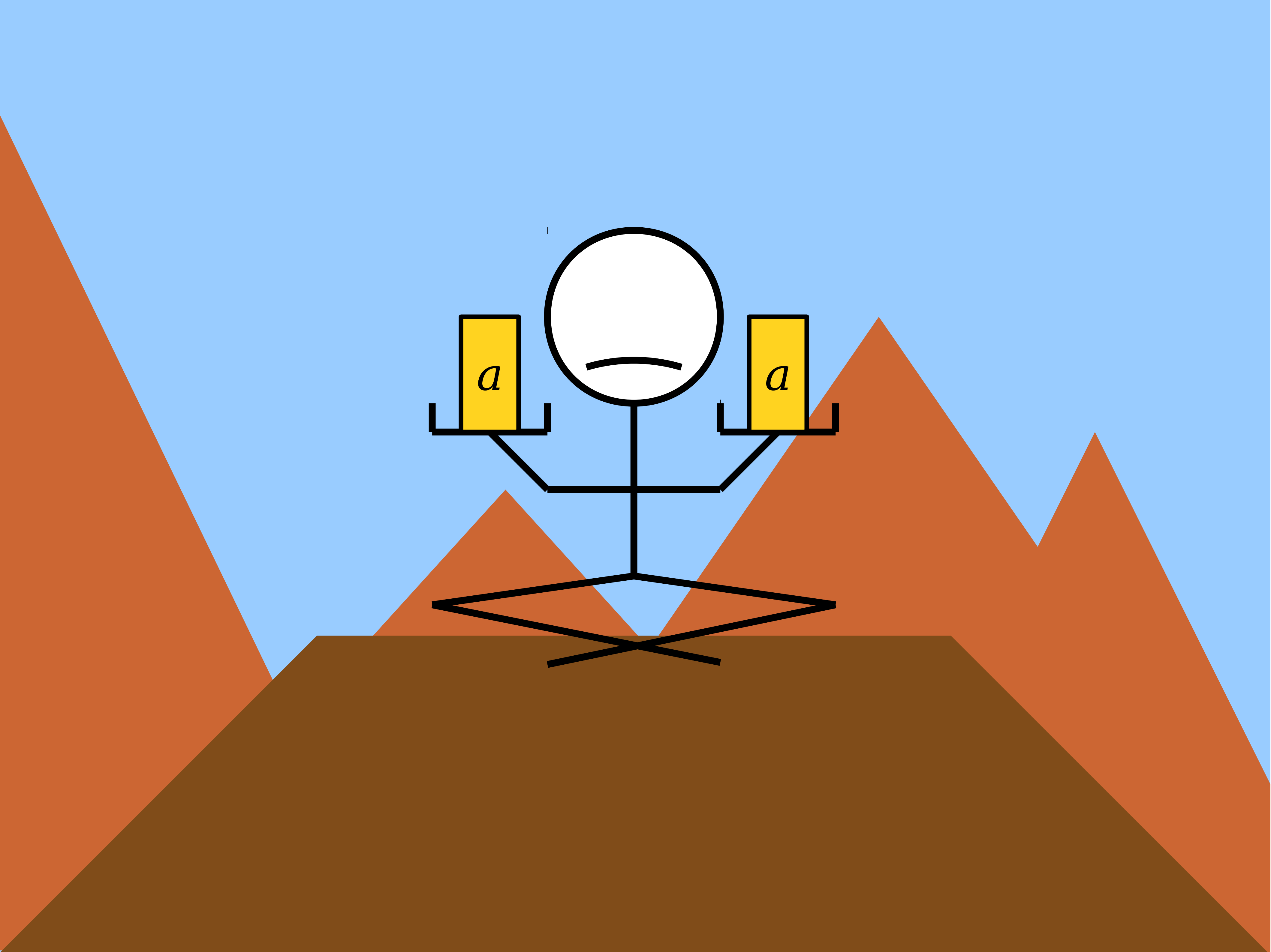

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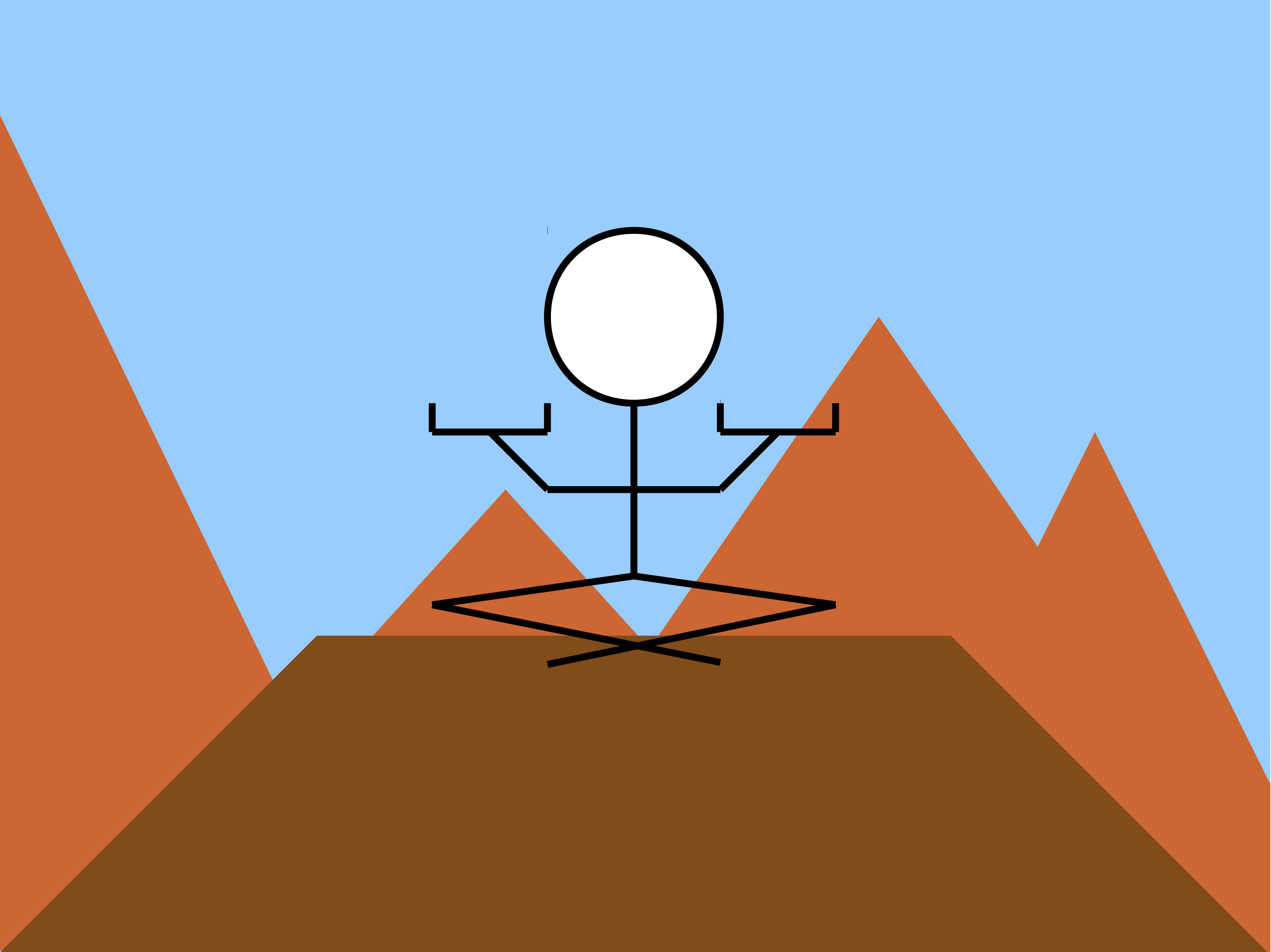

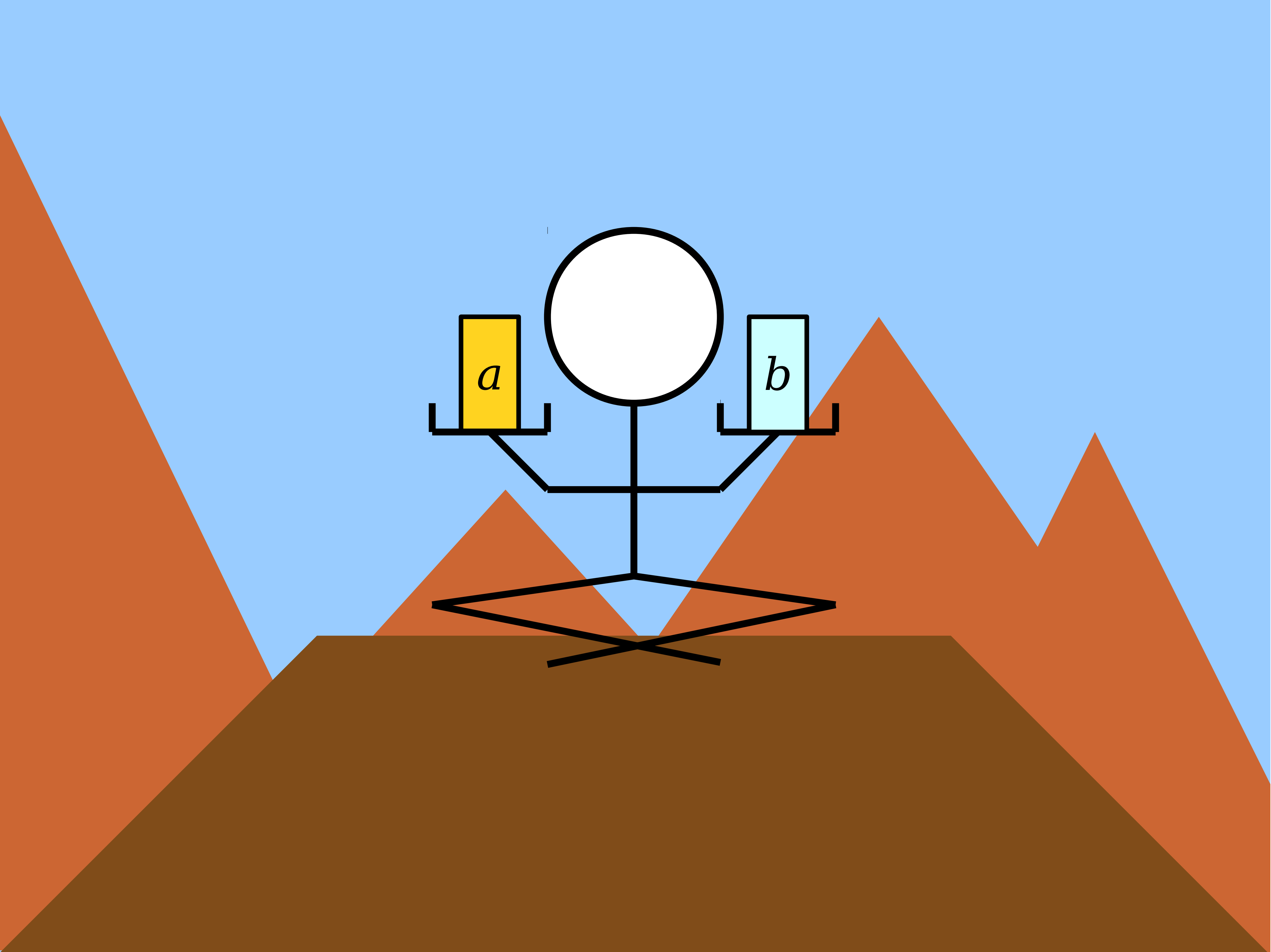
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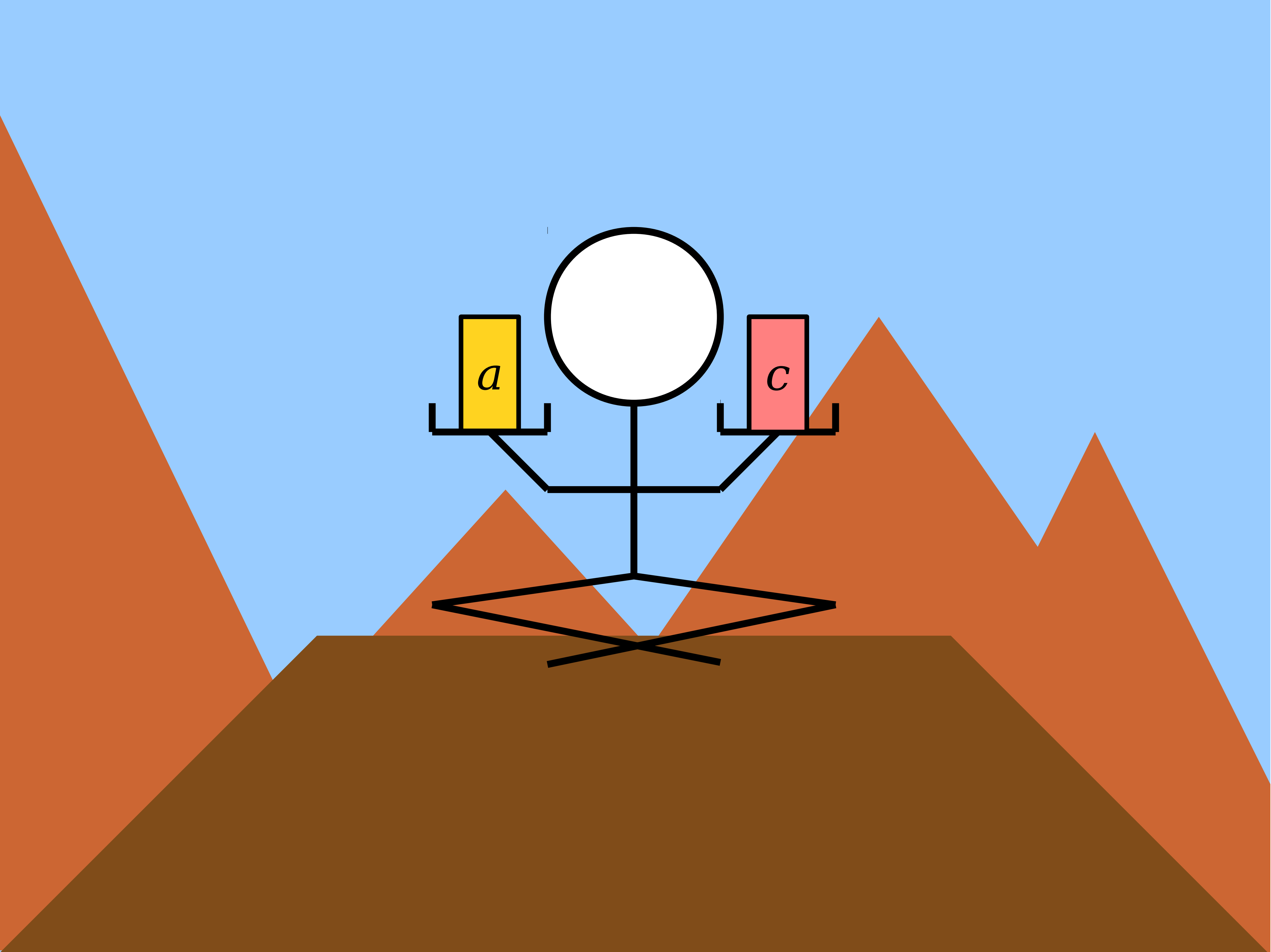
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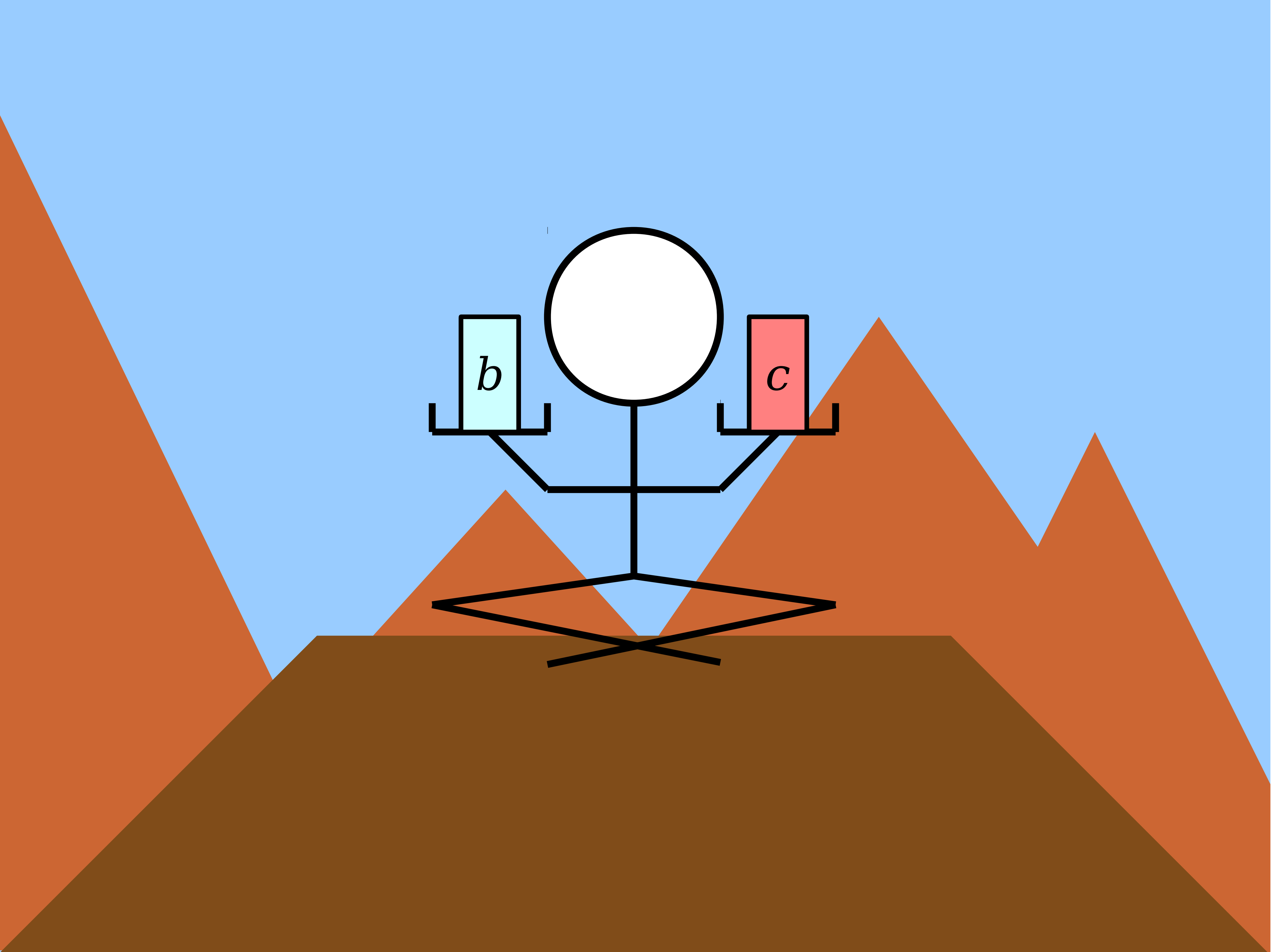
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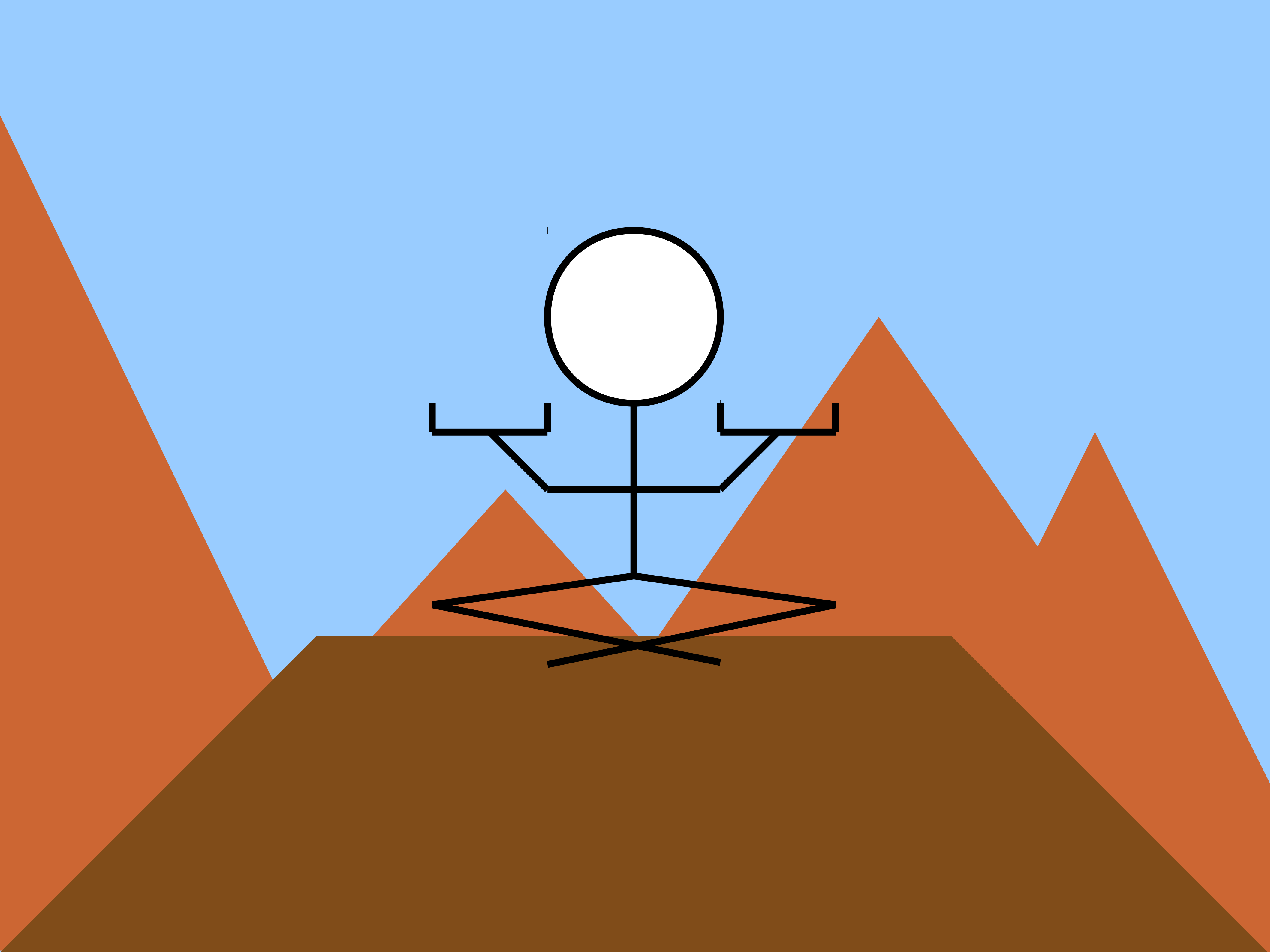

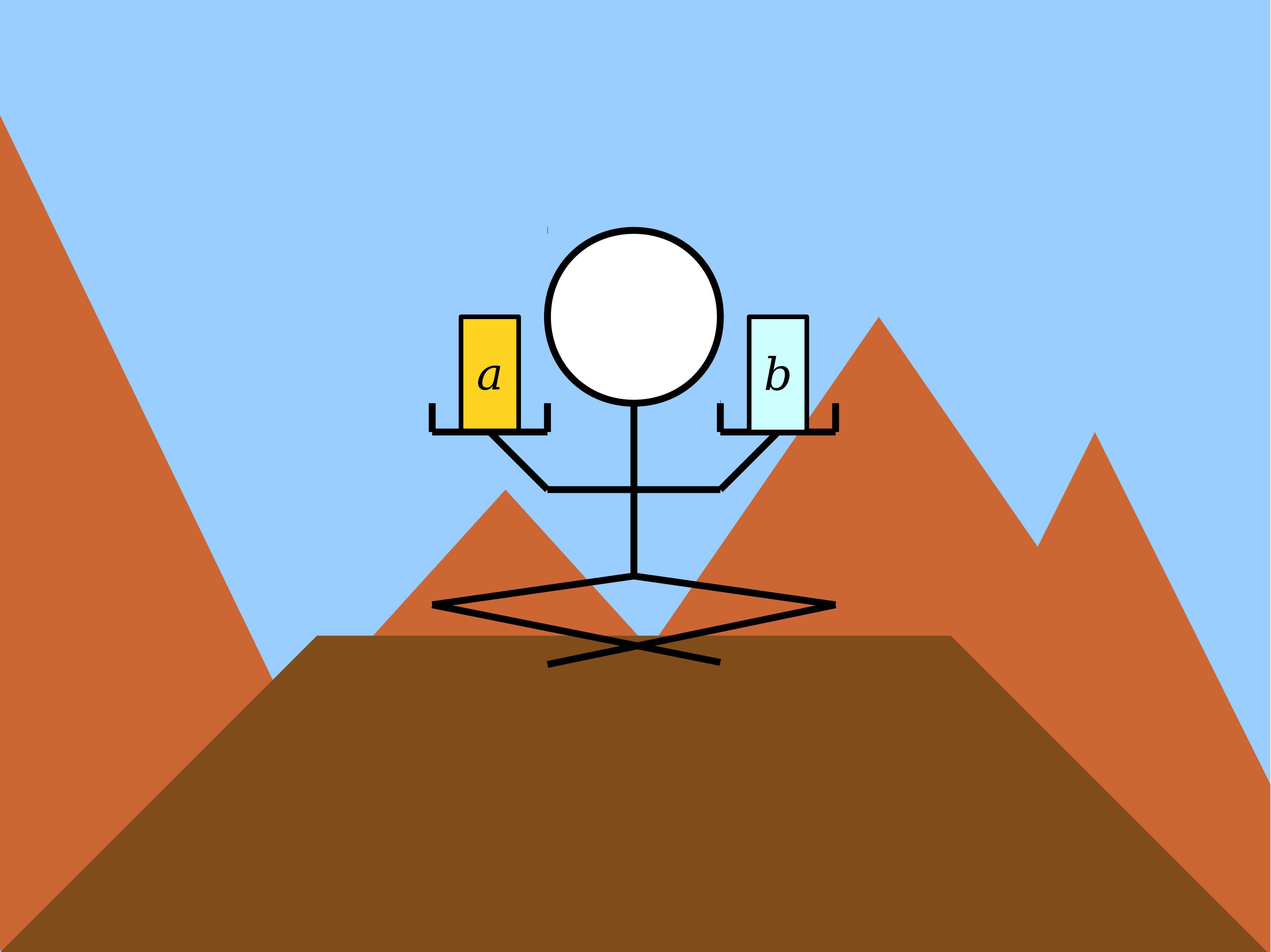
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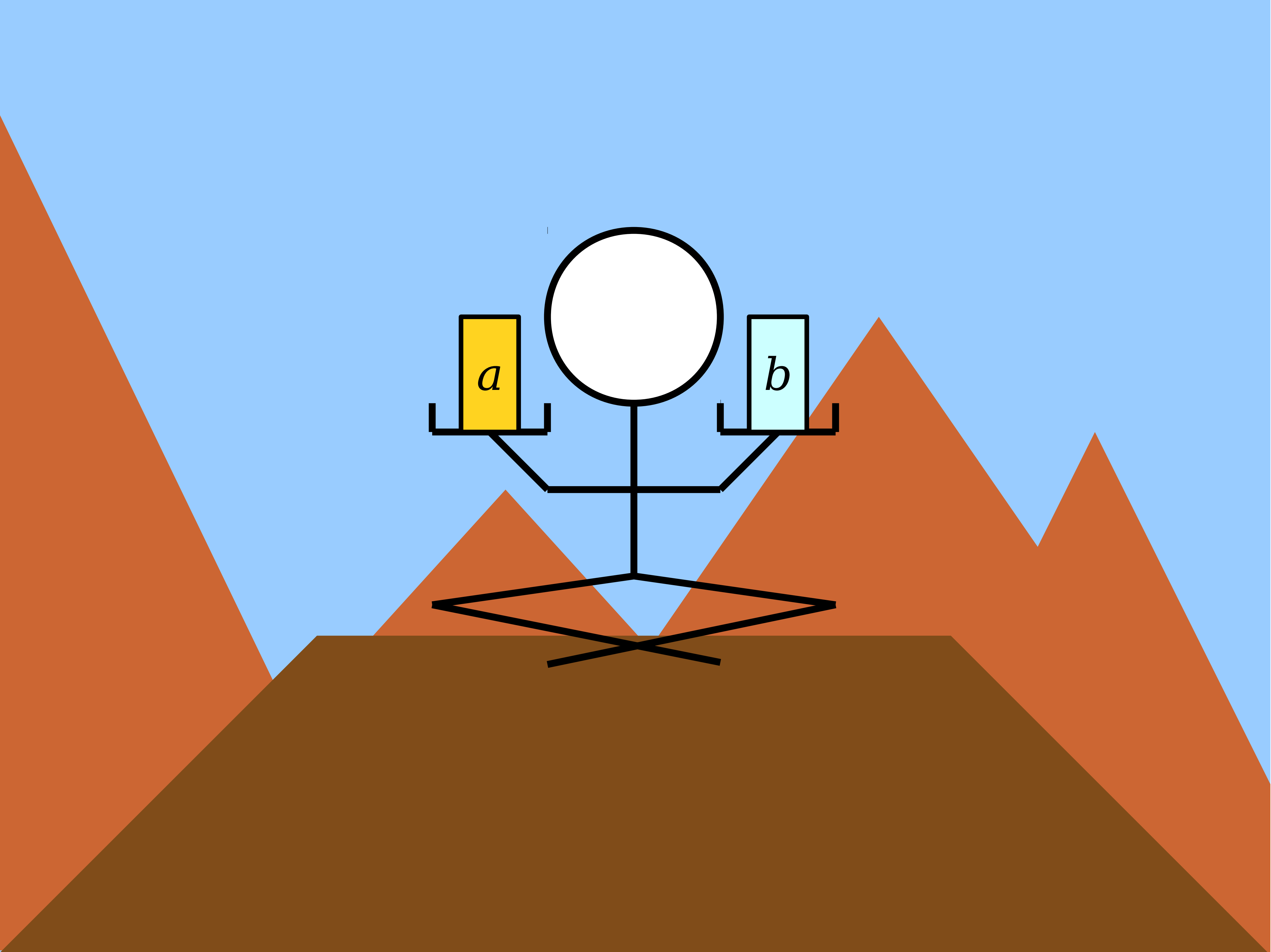
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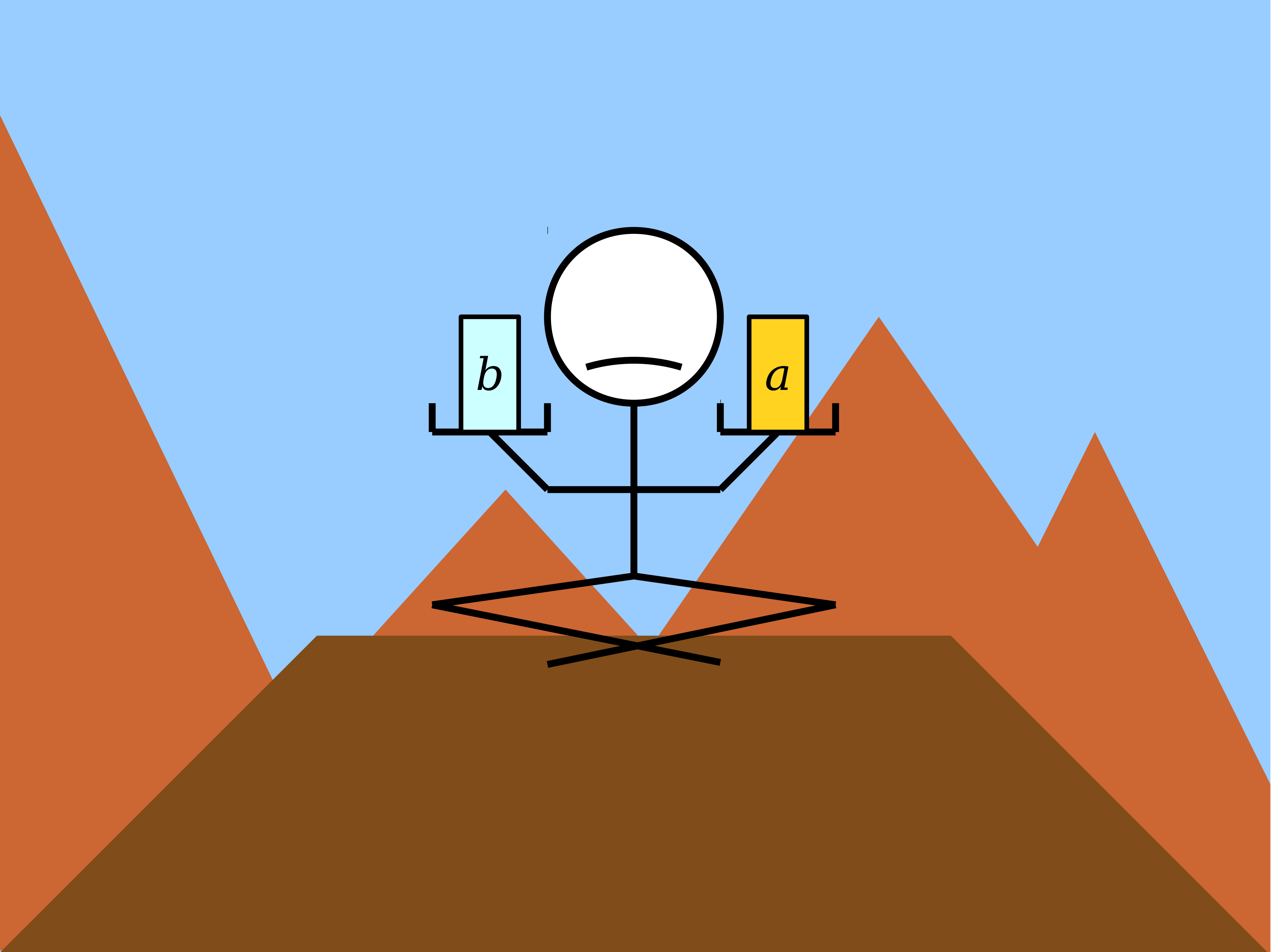
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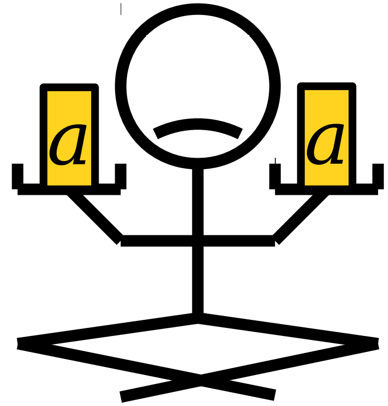
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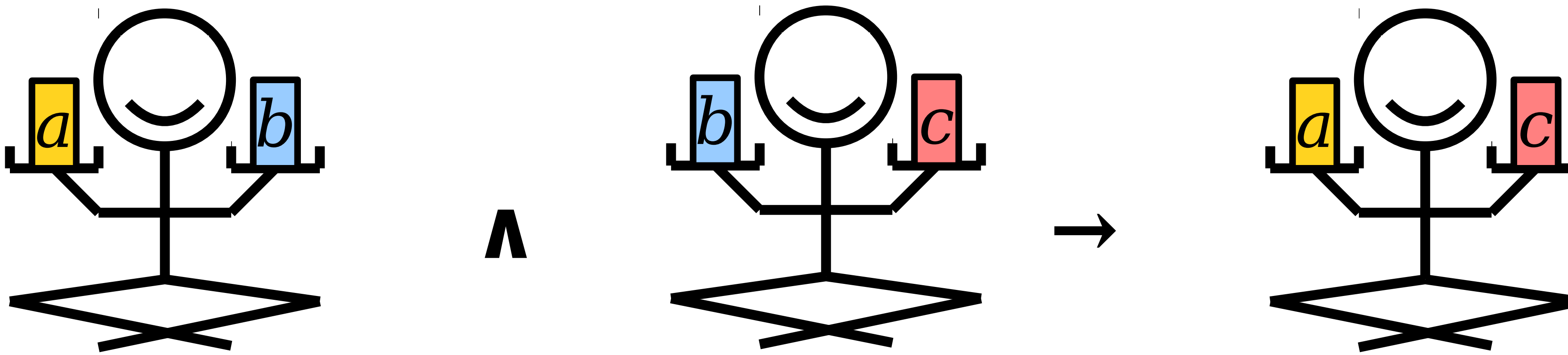
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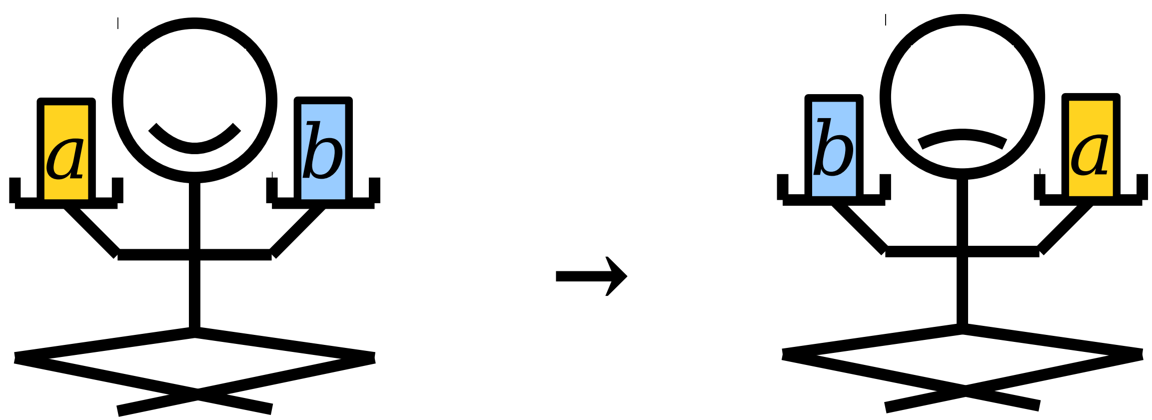
a
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$$a \not R a$$

$$aRb \wedge bRc \rightarrow aRc$$

$$aRb \rightarrow b \not R a$$

$$\forall a \in A.\ a \not{R} a$$

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \to b \not{R} a)$$

$$\forall a \in A.\ a \not{R} a$$

Transitivity

$$\forall a \in A.\ \forall b \in A.\ (aRb \to b \not{R} a)$$

$$\forall a \in A.\ a \not R a$$

Transitivity

$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow b \not R a)$$

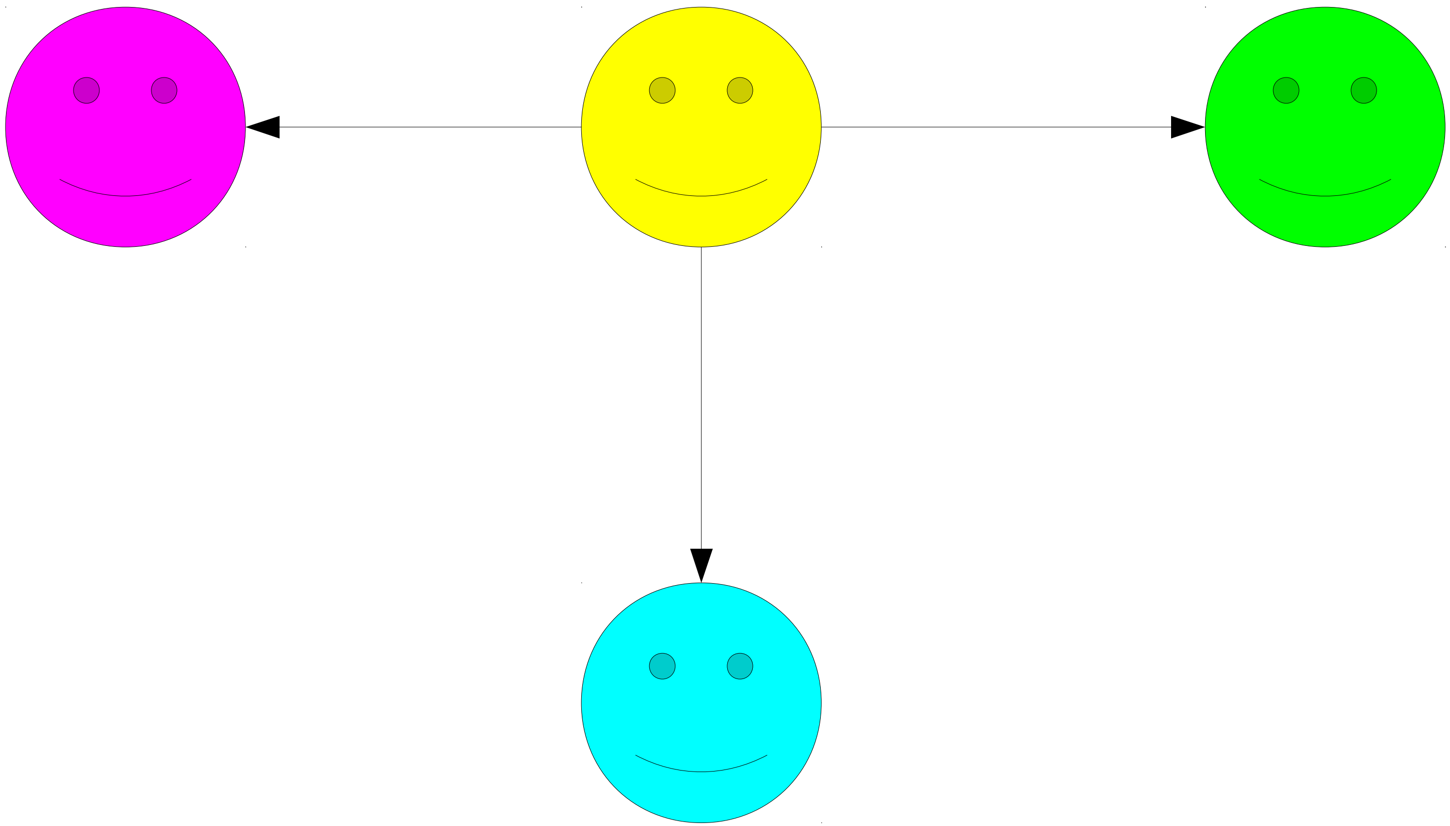
Irreflexivity

- Some relations *never* hold from any element to itself.
- As an example, $x \not< x$ for any x.
- Relations of this sort are called ***irreflexive***.
- Formally speaking, a binary relation R over a set A is ifreflexive if the following is true:

$$\forall a \in A.\ a \not R a$$

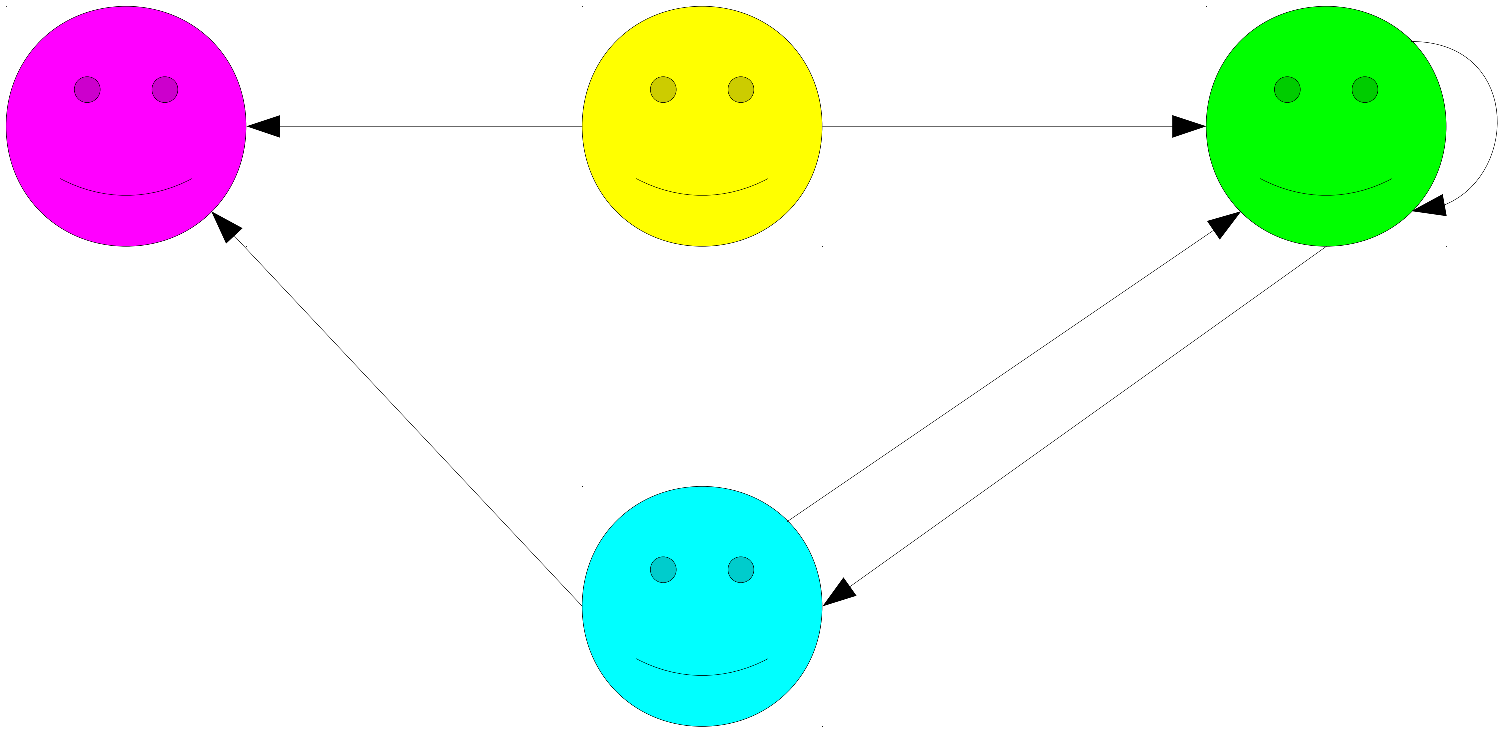
(*"No element is related to itself."*)

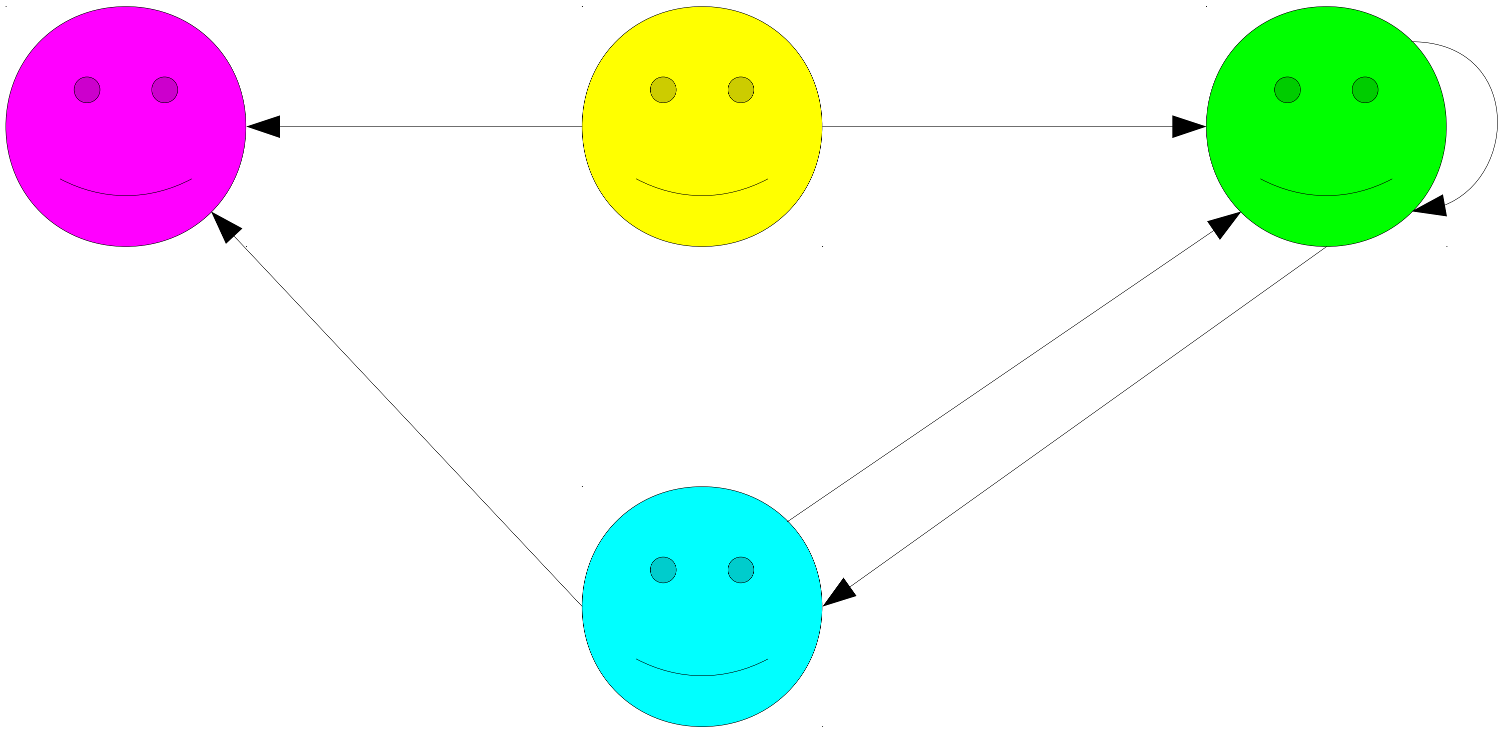
Irreflexivity Visualized



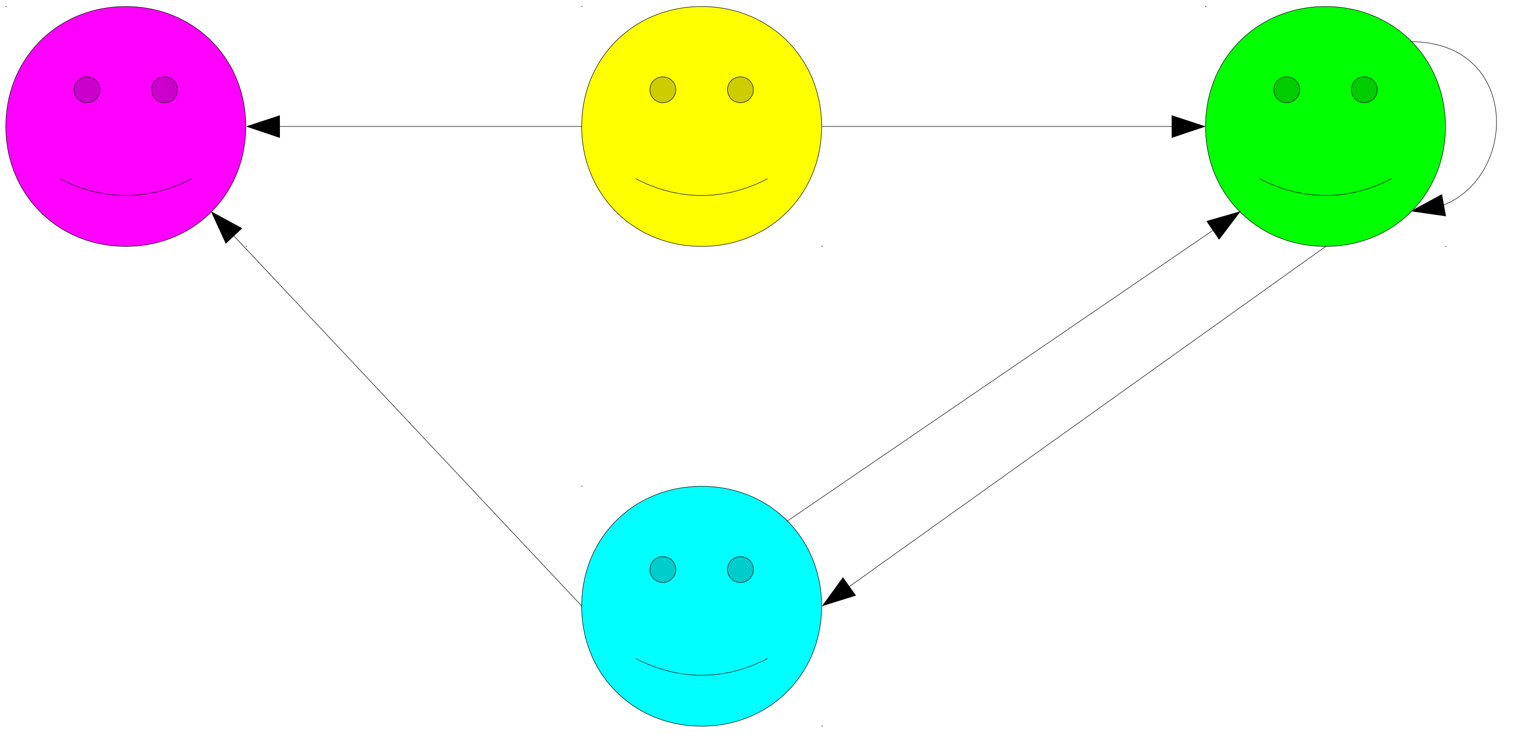
$$\forall a \in A.\ a \not R a$$

("No element is related to itself.")





Is this relation reflexive?



Is this relation reflexive?

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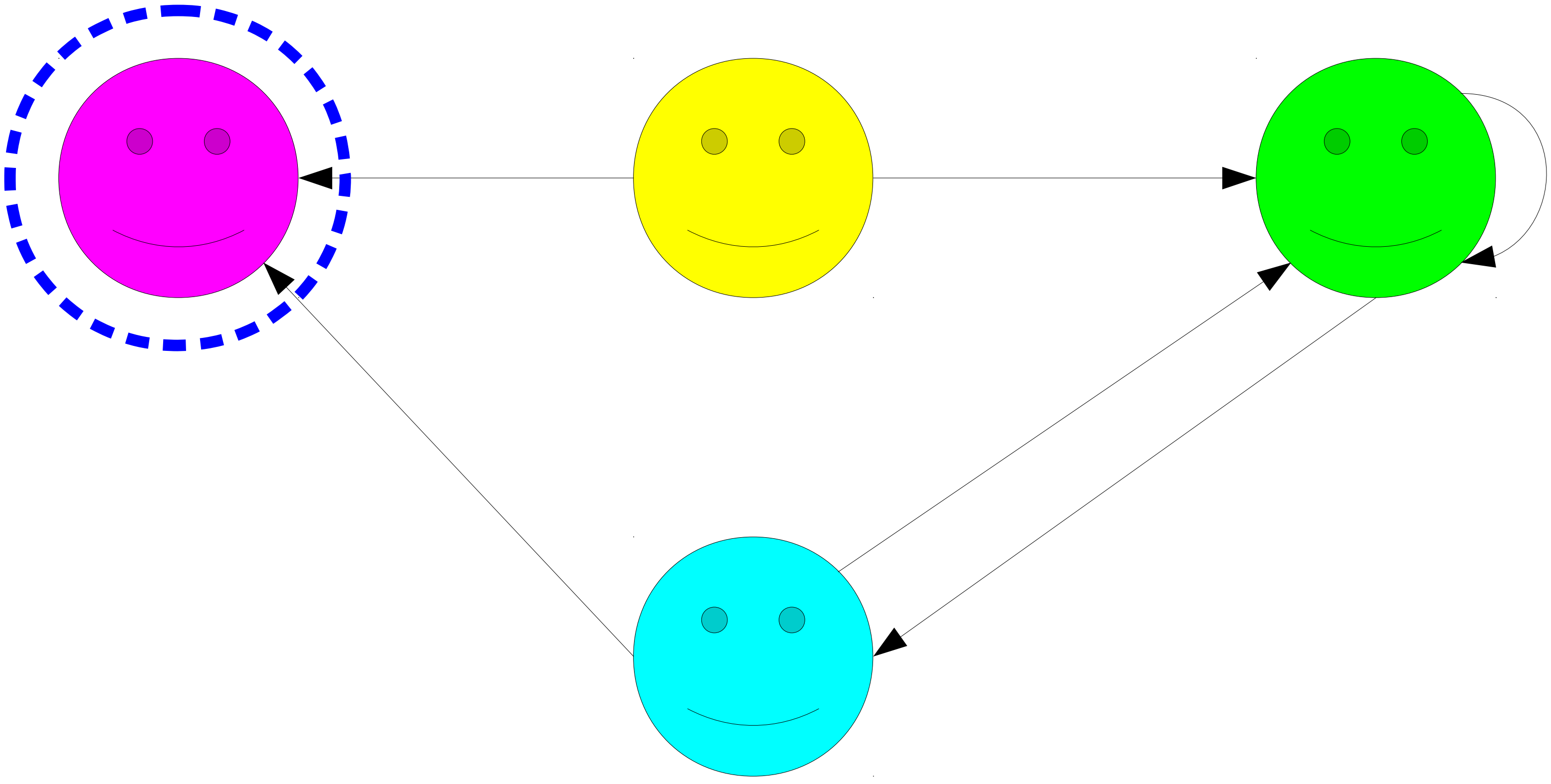
(*"Every element is related to itself."*)



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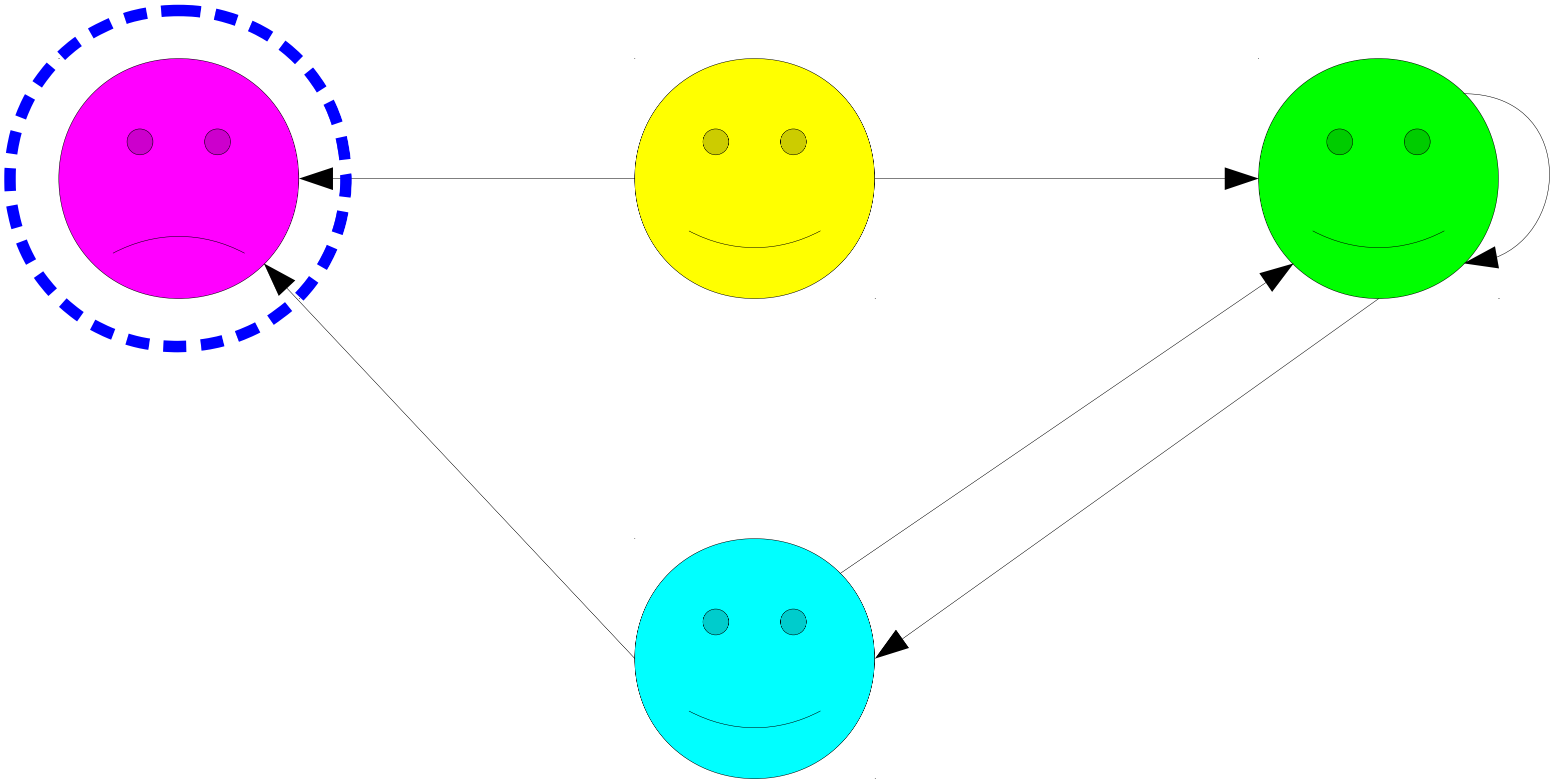
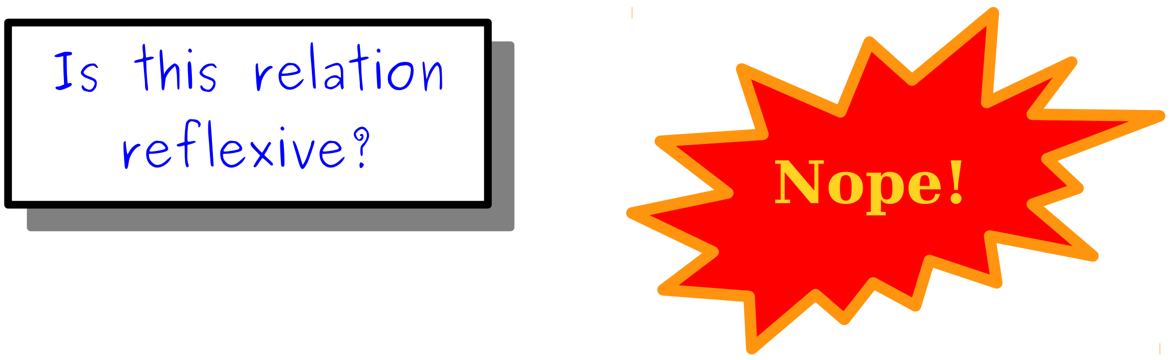
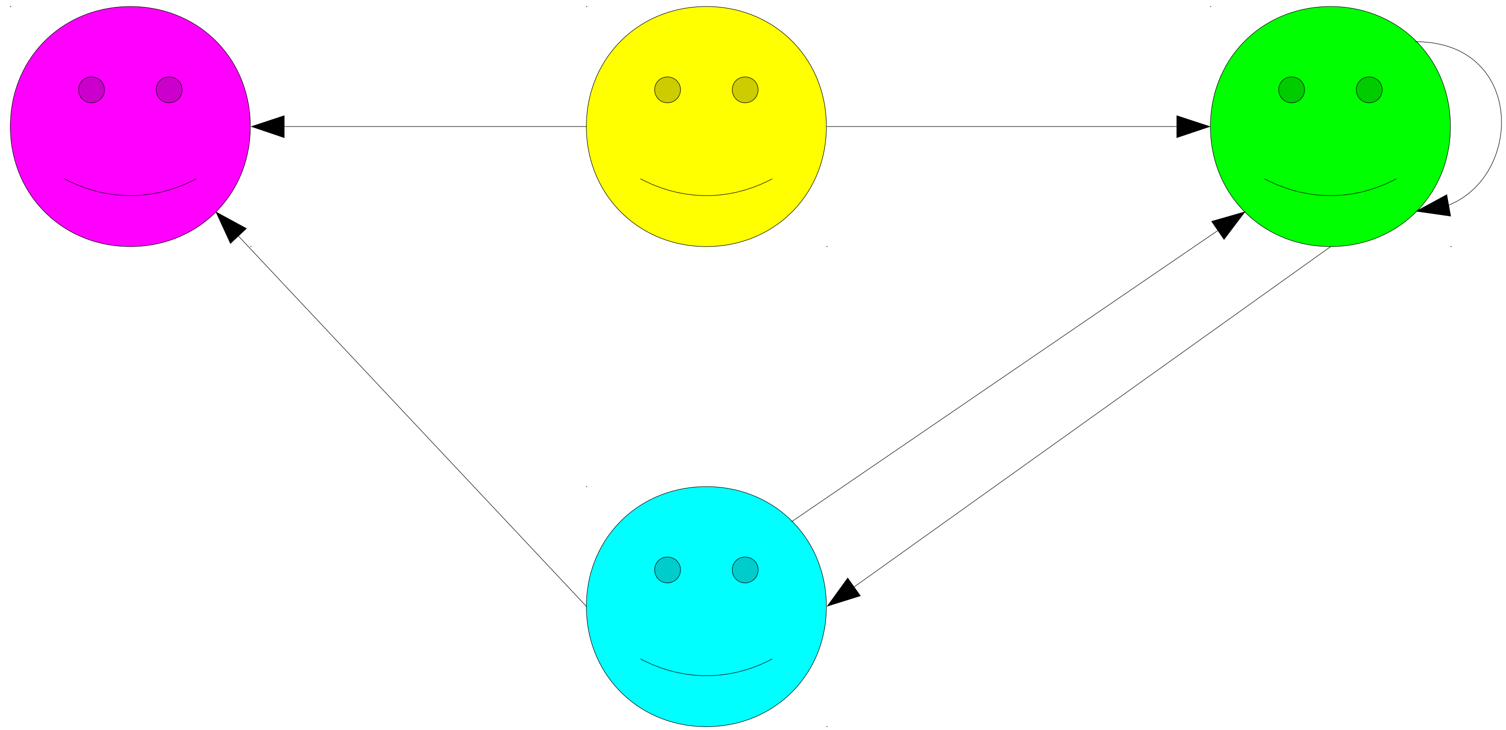
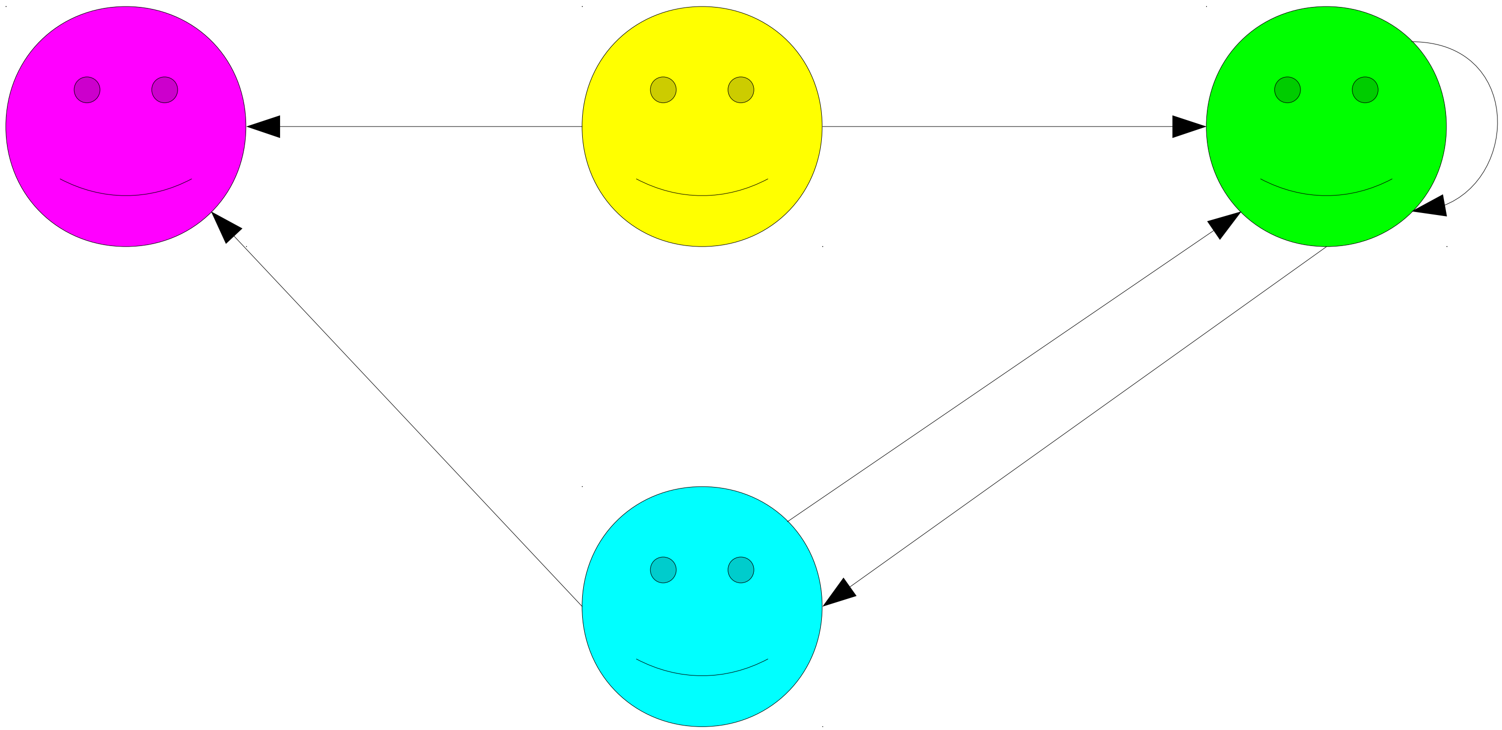




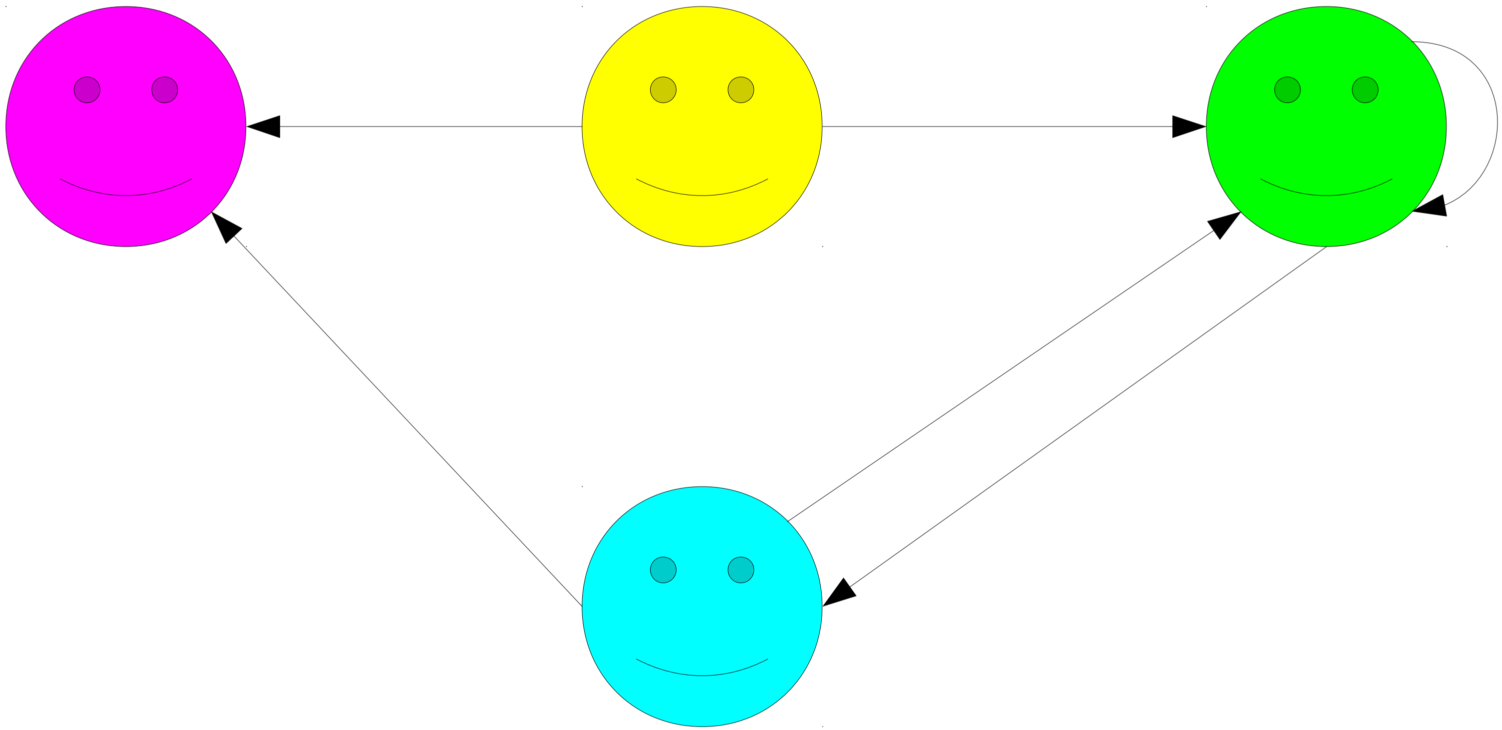
$$\forall a \in A.\ aRa$$

(*"Every element is related to itself."*)





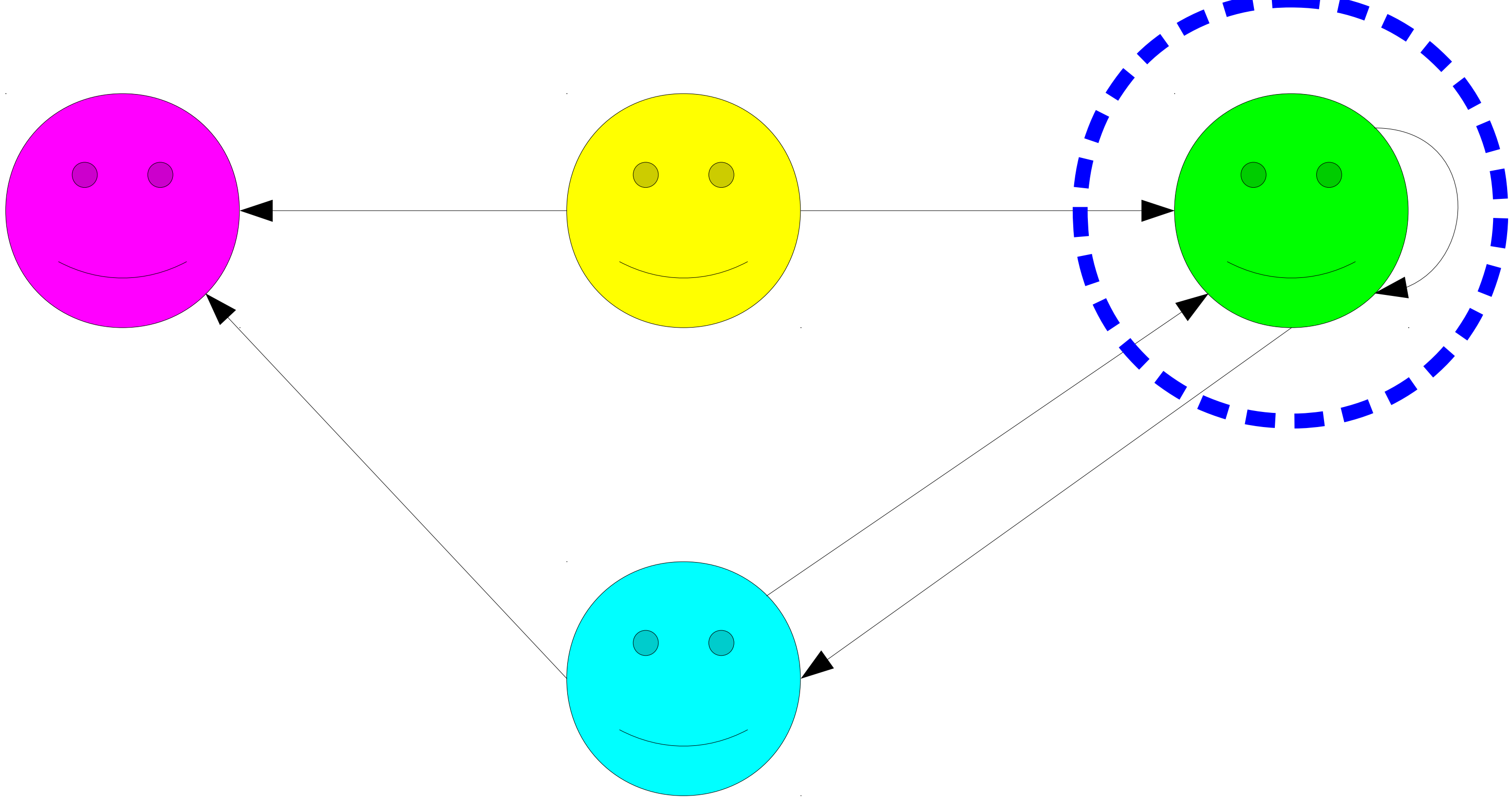
Is this relation irreflexive?



Is this relation irreflexive?

$$\forall a \in A.\ a \not R a$$

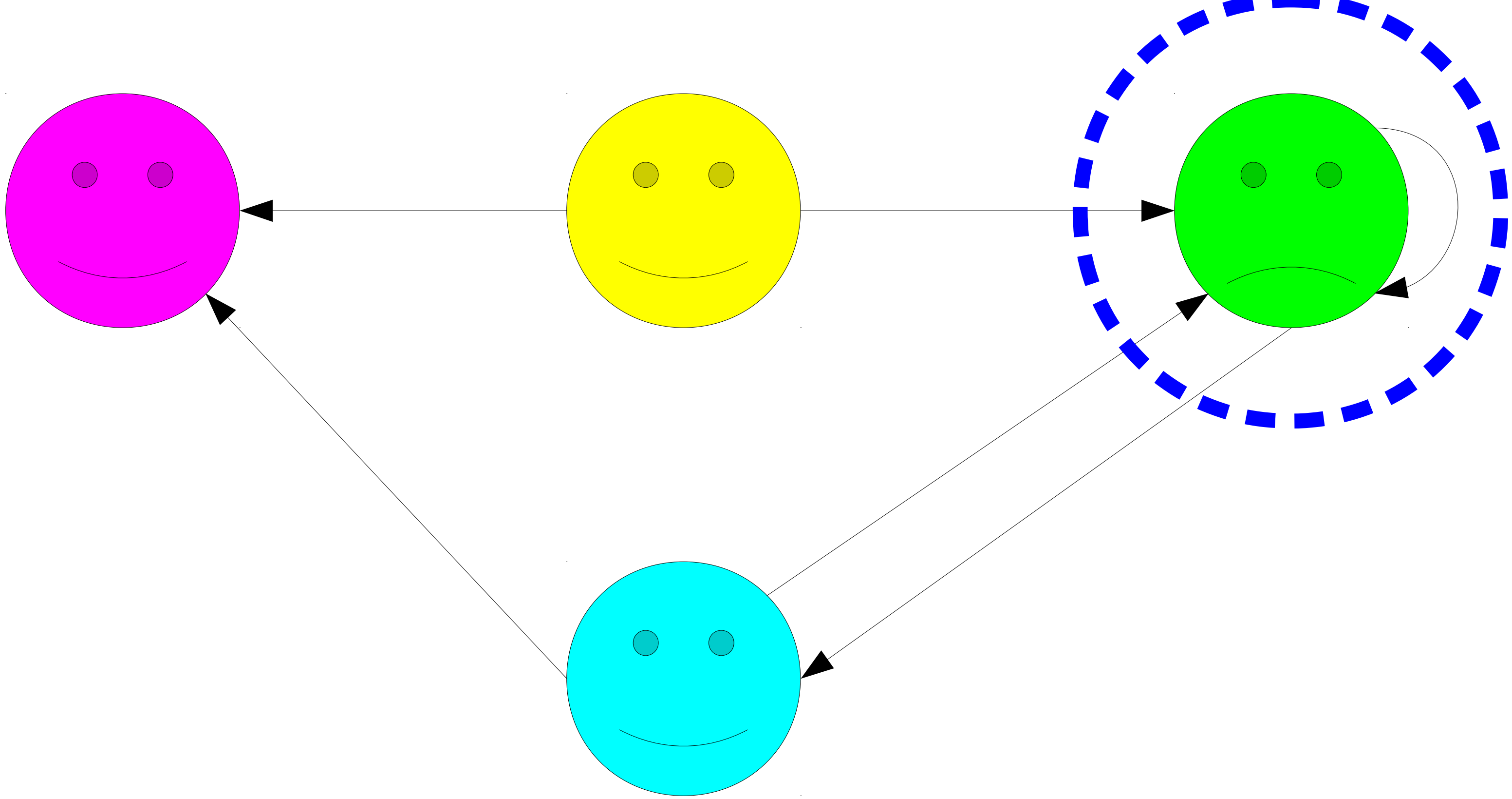
(*"No element is related to itself."*)



Is this relation irreflexive?

$$\forall a \in A.\ a \not R a$$

(*“No element is related to itself.”*)



Is this relation irreflexive?

$$\forall a \in A.\ a \not R a$$

(*“No element is related to itself.”*)

Reflexivity and Irreflexivity

- Reflexivity and irreflexivity are ***not*** opposites!
- Here's the definition of reflexivity:

$$\forall a \in A.\ aRa$$

- What is the negation of the above statement?

$$\exists a \in A.\ a\not{R}a$$

- What is the definition of irreflexivity?

$$\forall a \in A.\ a\not{R}a$$

$$\forall a \in A.\ a \not R a$$

Transitivity

$$\forall a \in A.\ \forall b \in A.\ (aRb \to b \not R a)$$

Irreflexivity

Transitivity

$$\forall a \in A.\ \forall b \in A.\ (aRb \to b \not R a)$$

Irreflexivity

Transitivity

$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow b\not{R}a)$$

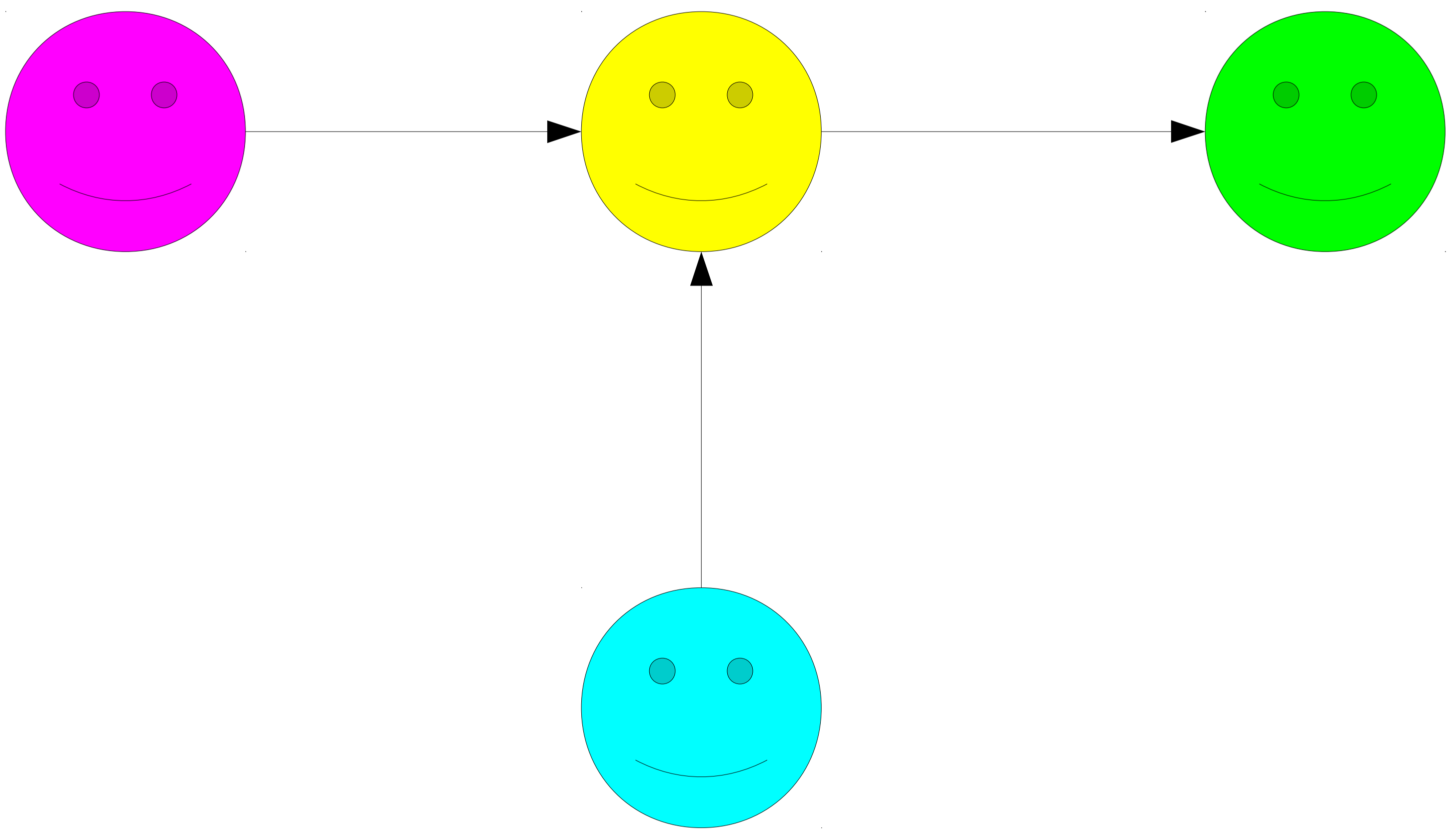
Asymmetry

- In some relations, the relative order of the objects can never be reversed.
- As an example, if $x < y$, then $y \not< x$.
- These relations are called ***asymmetric***.
- Formally: a binary relation R over a set A is called *asymmetric* if

$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow b\not R a)$$

(*"If a relates to b, then b does not relate to a."*)

Asymmetry Visualized



$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow b \not R a)$$

(*"If a relates to b, then b does not relate to a."*)

Question to Ponder: Are symmetry and asymmetry opposites of one another?

Irreflexivity

Transitivity

$$\forall a \in A. \forall b \in A. (aRb \to b \not R a)$$

Irreflexivity

Transitivity

Asymmetry

Strict Orders

- A ***strict order*** is a relation that is irreflexive, asymmetric and transitive.
- Some examples:
  - $x < y$.
  - a can run faster than b.
  - $A \subset B$ (that is, $A \subseteq B$ and $A \neq B$).
- Strict orders are useful for representing prerequisite structures and have applications in complexity theory (measuring notions of relative hardness) and algorithms (searching and sorting)