# Nonregular Languages

# Recap from Last Time

***Theorem:*** The following are all equivalent:

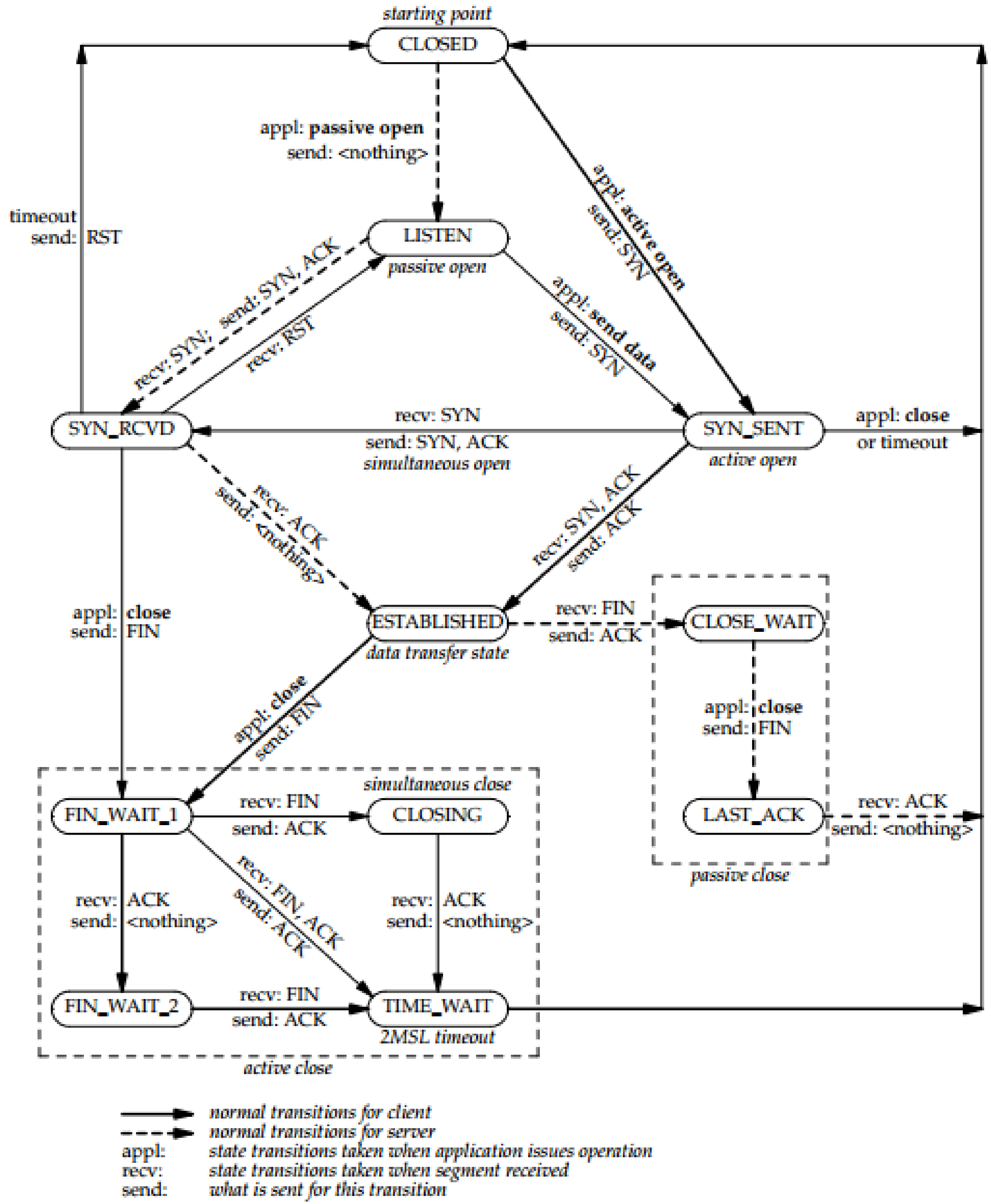
- $L$ is a regular language.
- There is a DFA $D$ such that $\mathscr{L}(D) = L$.
- There is an NFA $N$ such that $\mathscr{L}(N) = L$.
- There is a regular expression $R$ such that $\mathscr{L}(R) = L$.

# New Stuff!

Why does this matter?

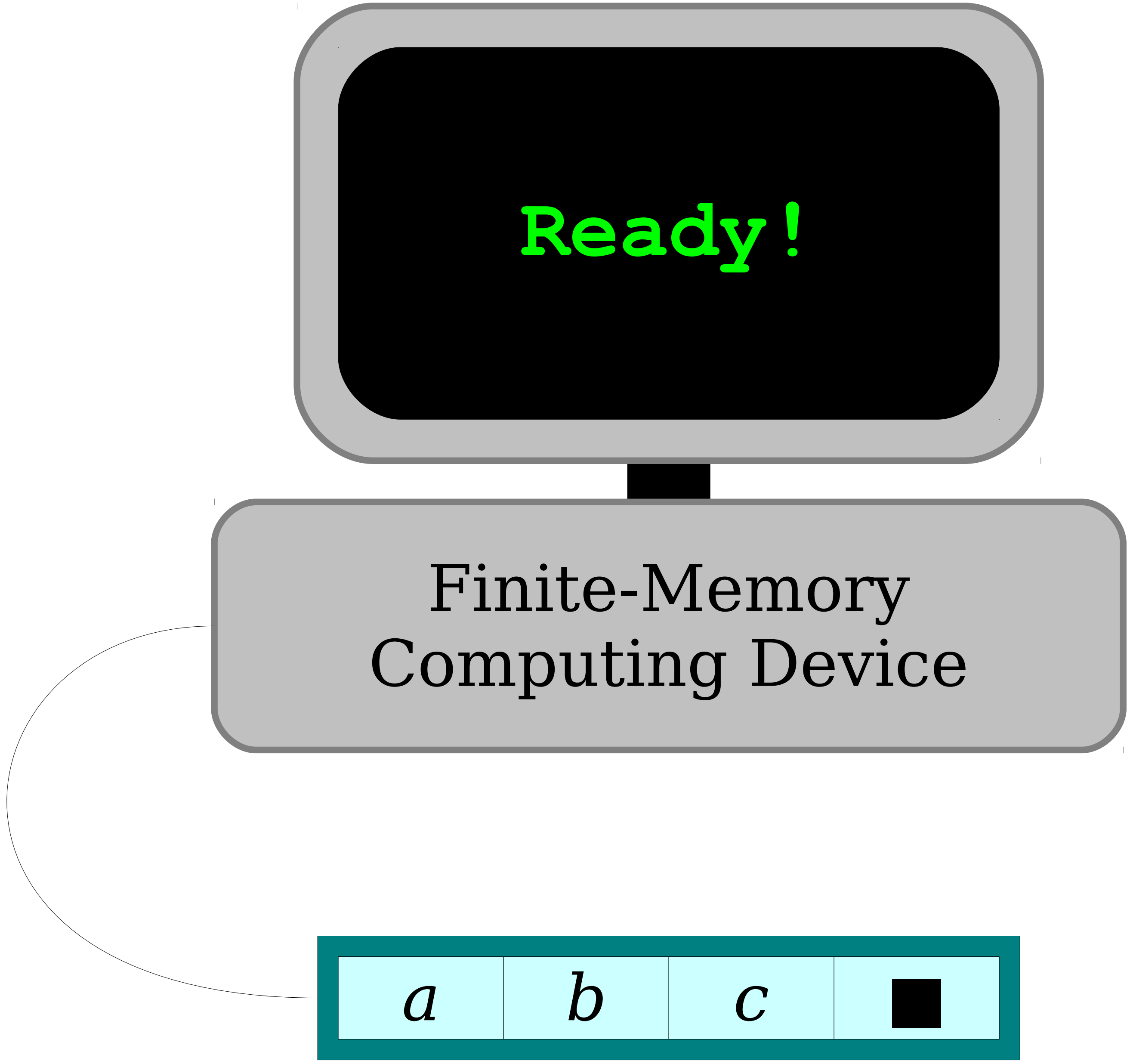
Buttons as Finite-State Machines:

http://cs103.stanford.edu/tools/button-fsm/

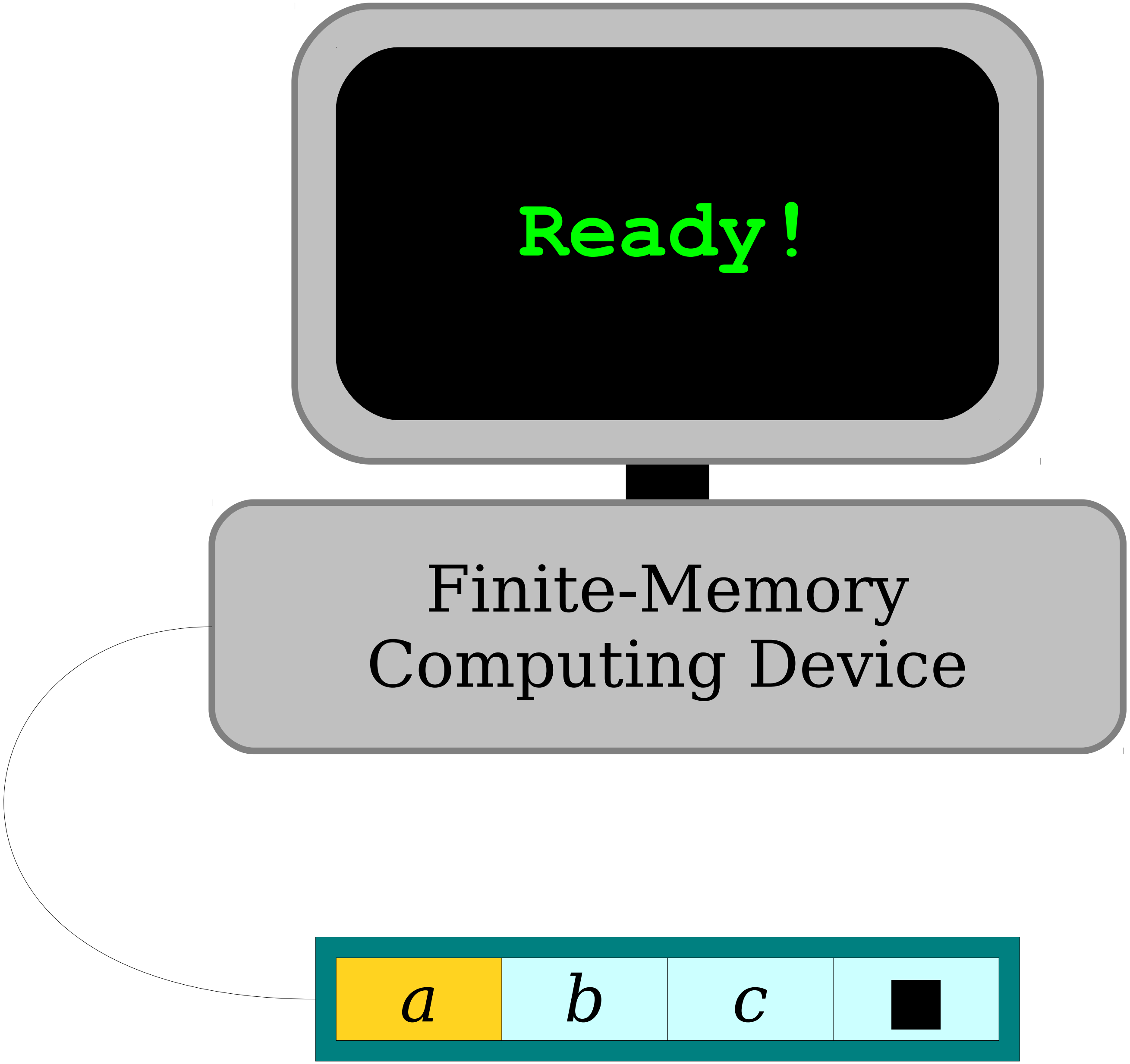



http://www.tti.unipa.it/~gneglia/ip_networks06/slides/TCPIP_State_Transition_Diagram.pdf

What exactly is a finite-state machine?



Ready!
Finite-Memory
Computing Device
a
b
c



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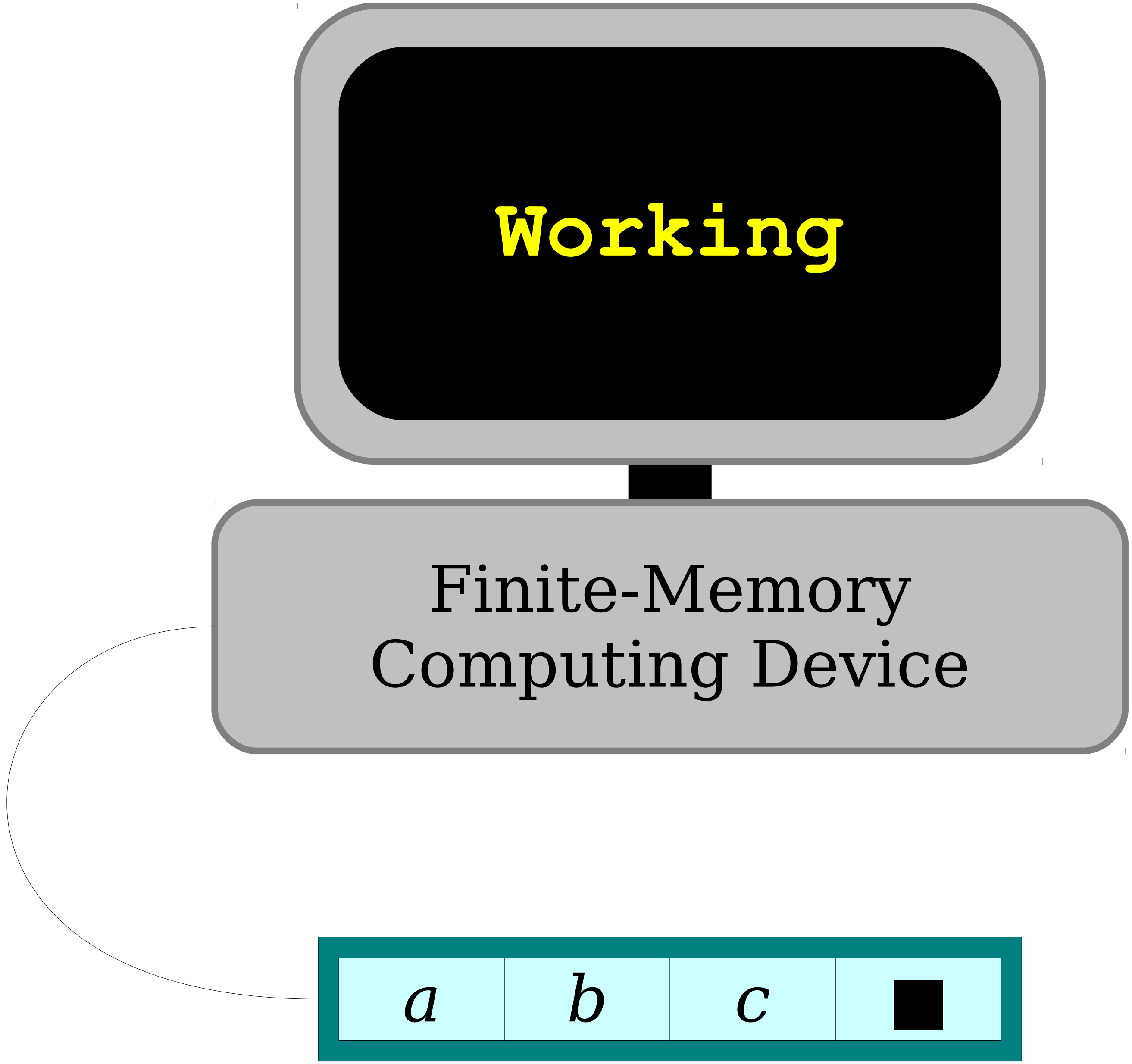
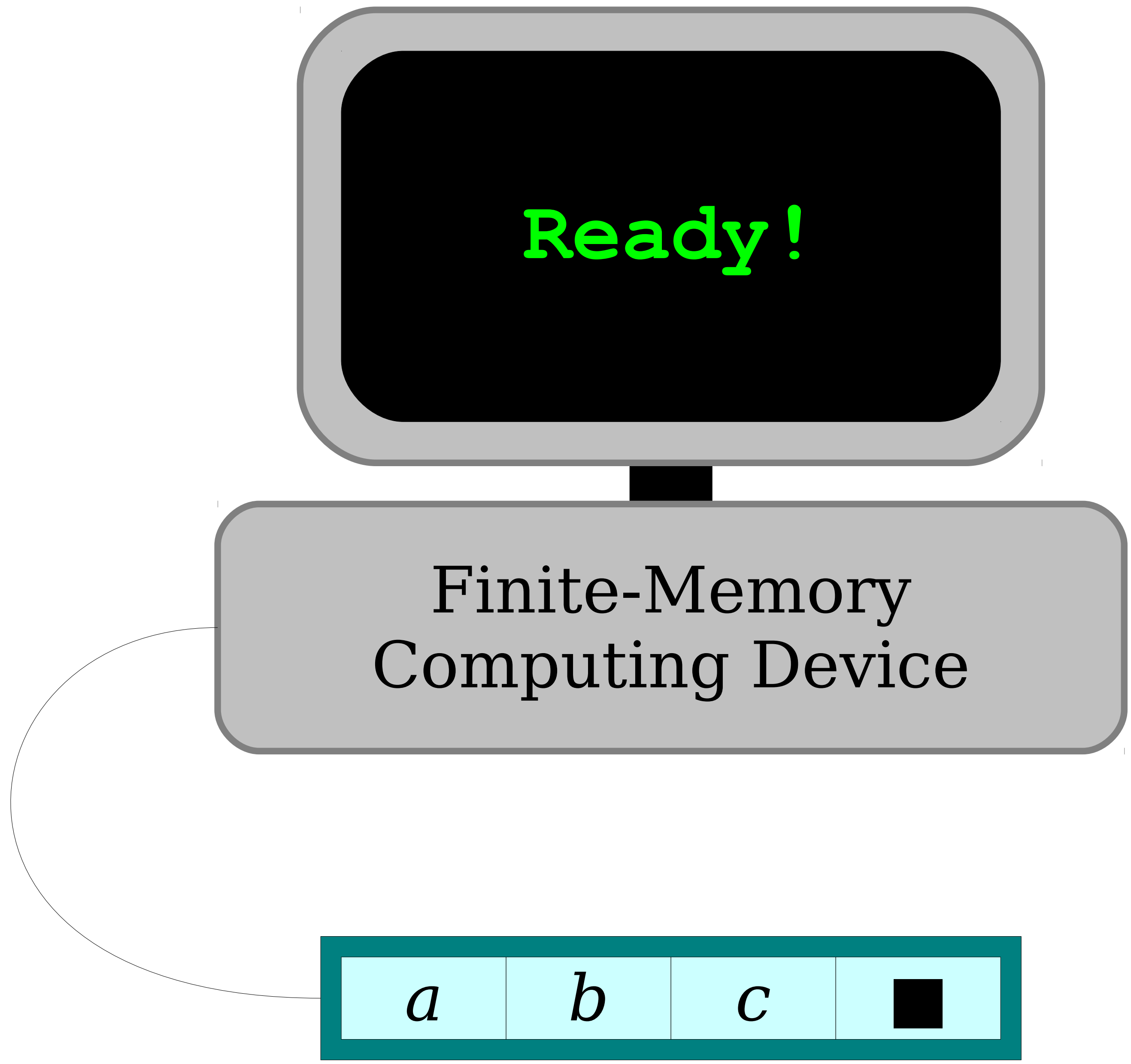


Working
Finite-Memory
Computing Device
a
b
c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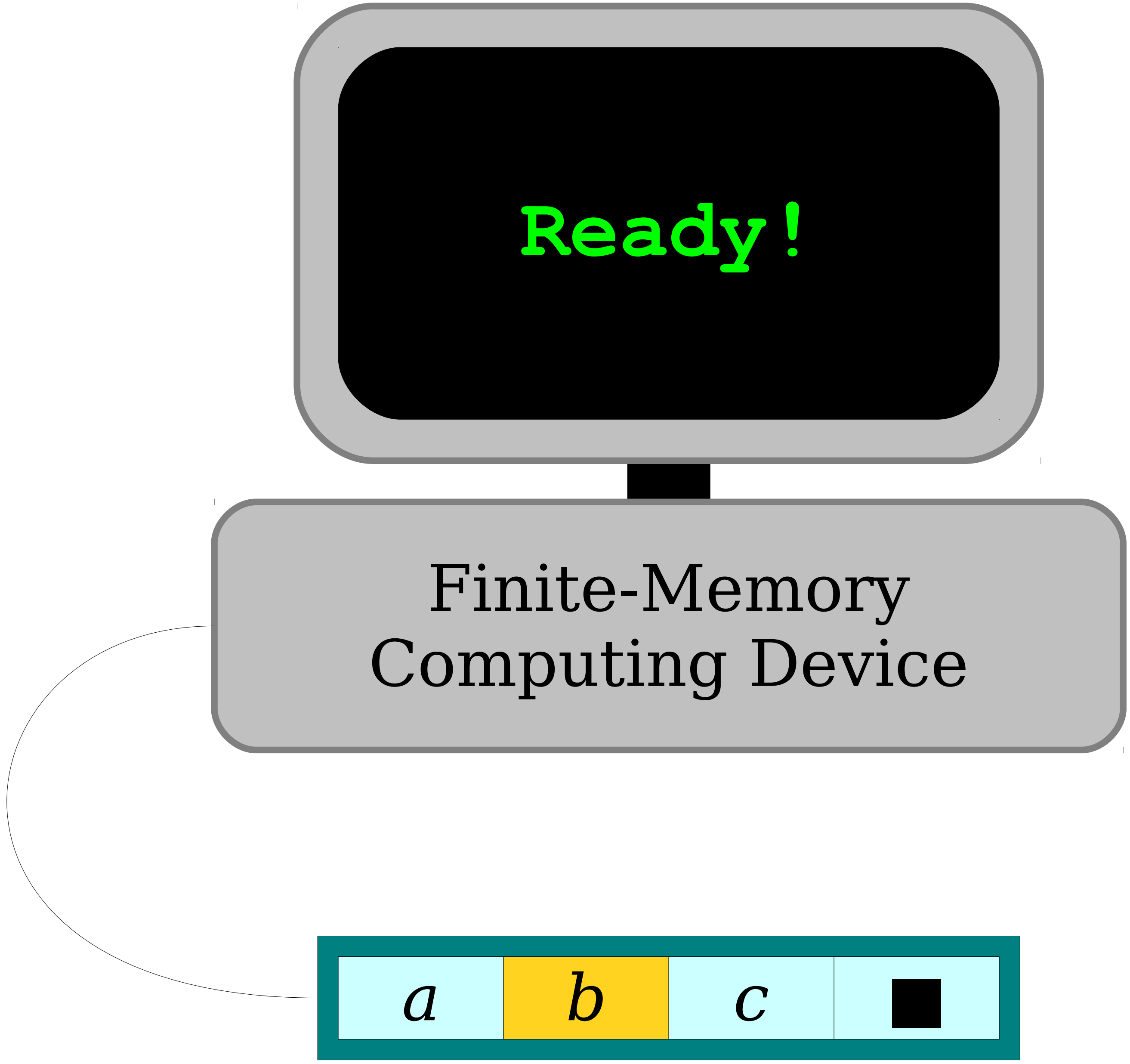

Ready!
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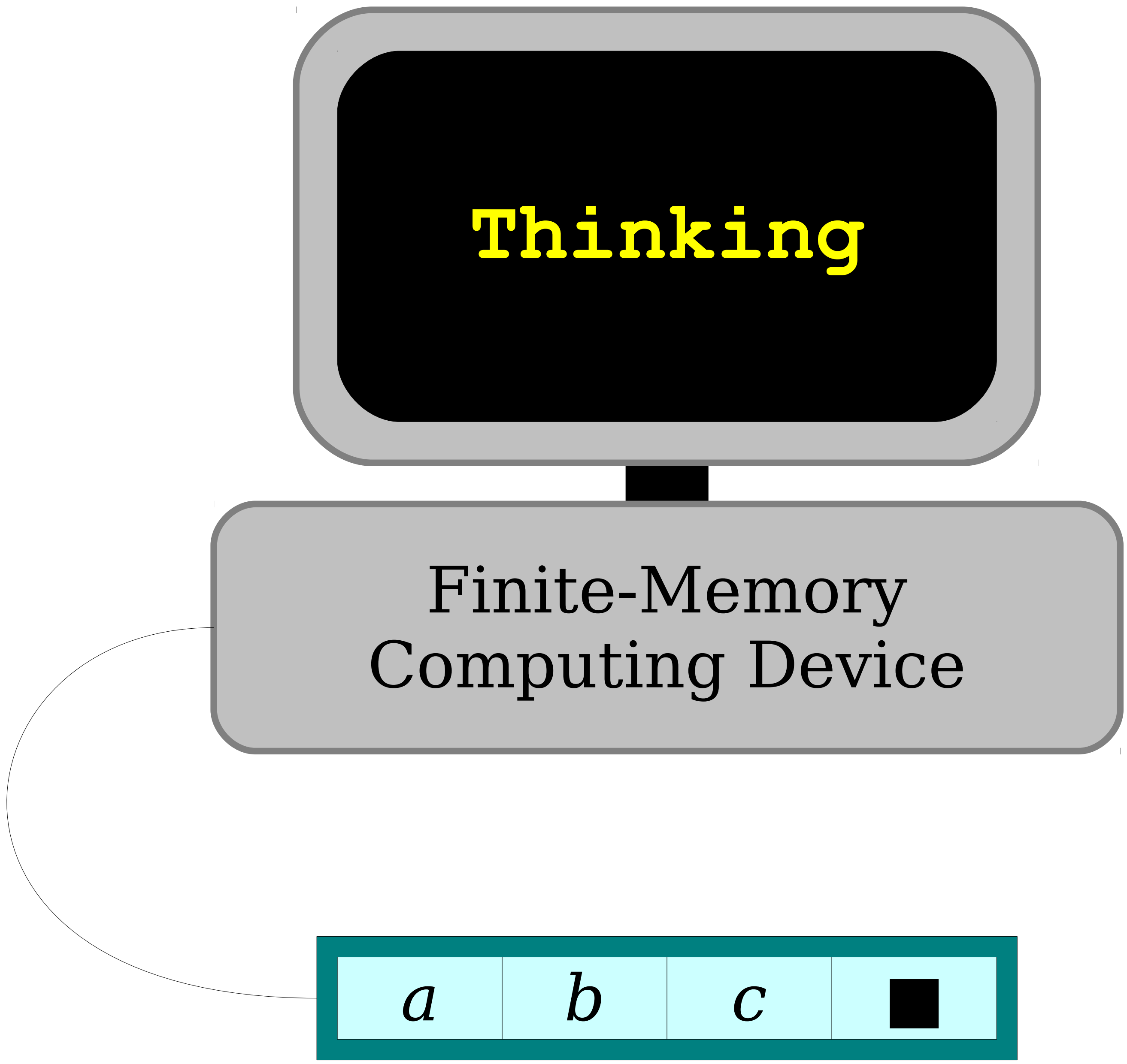
Thinking
Finite-Memory
Computing Device
a
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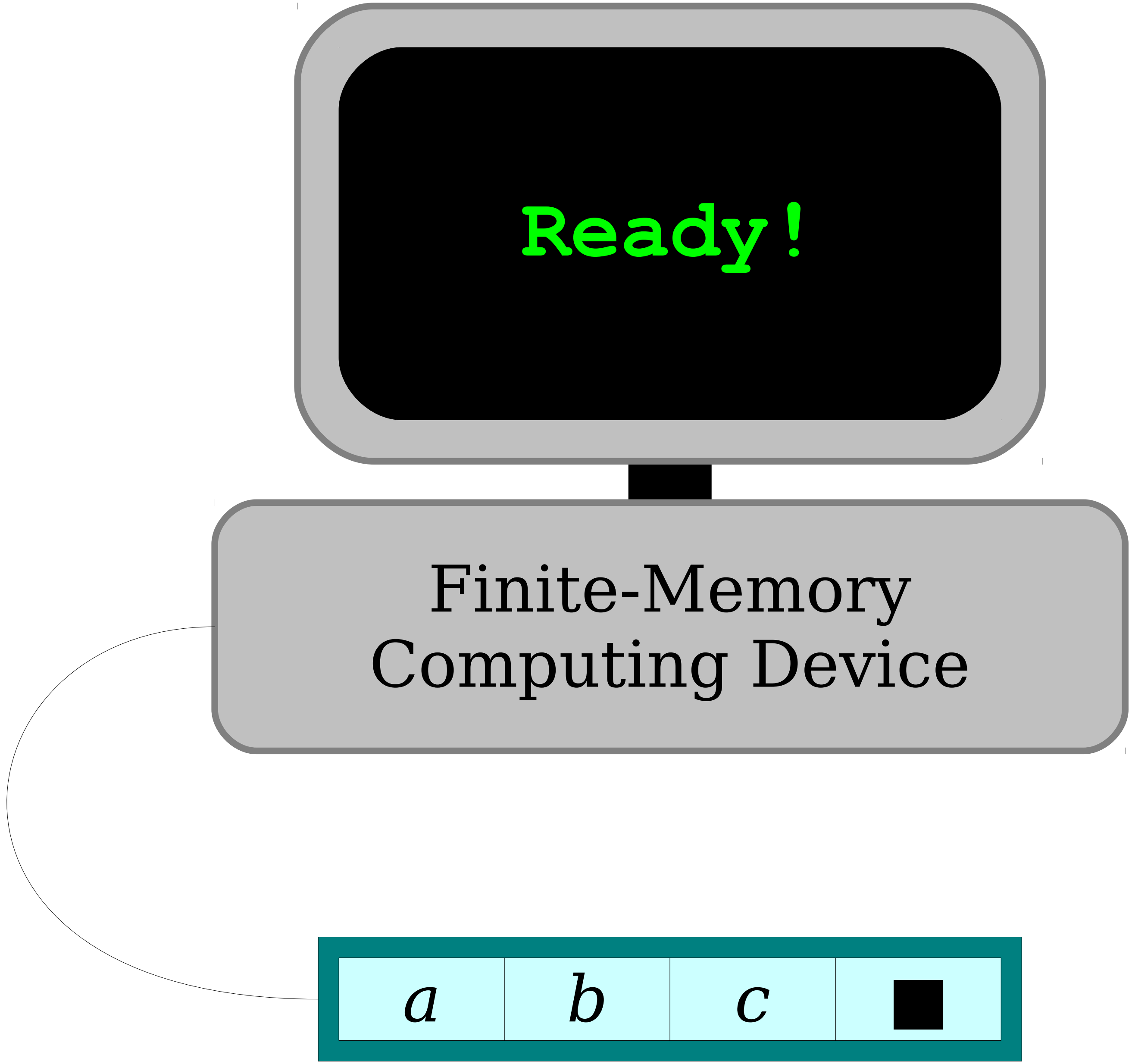
Ready!
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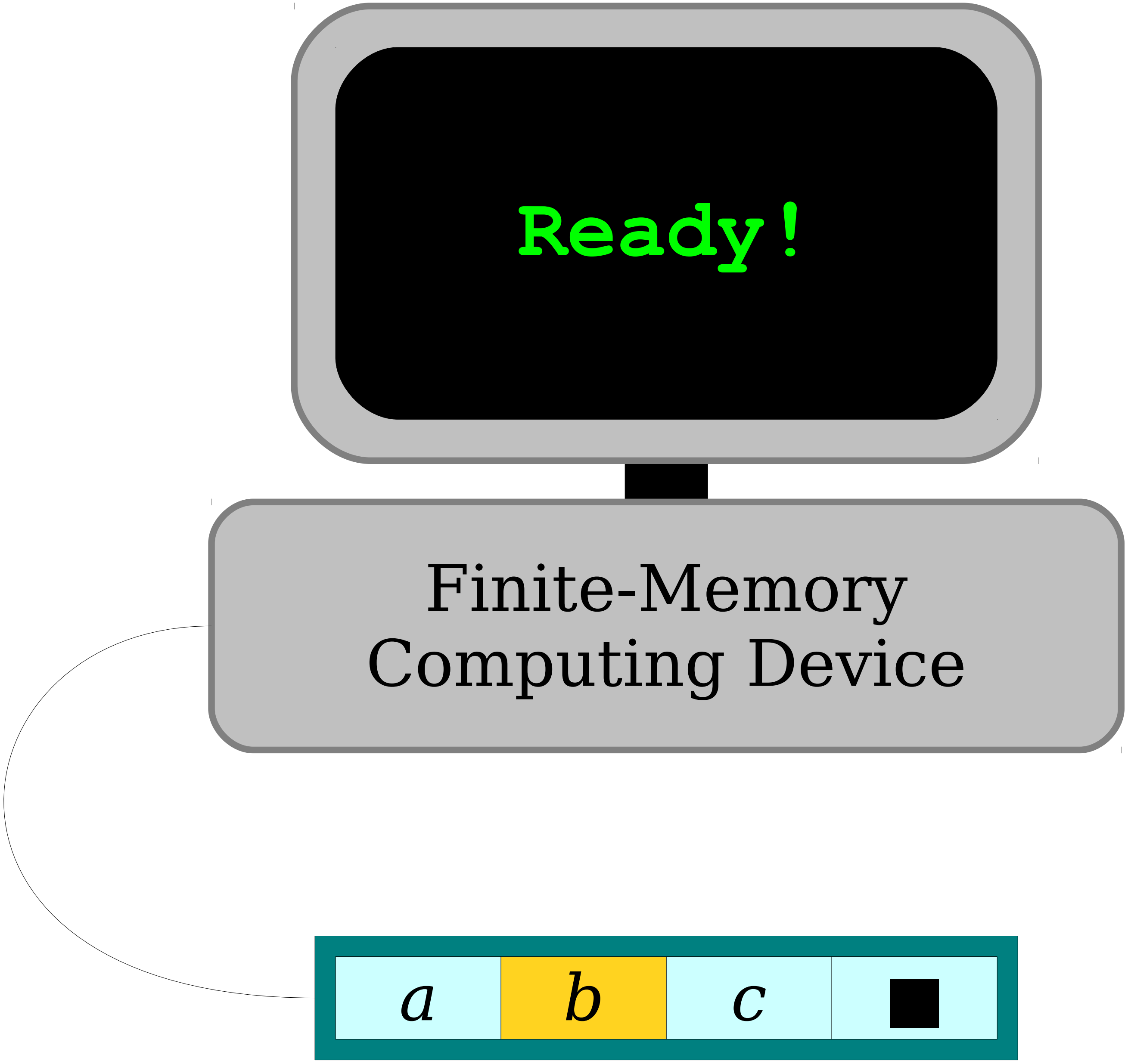
Ready!
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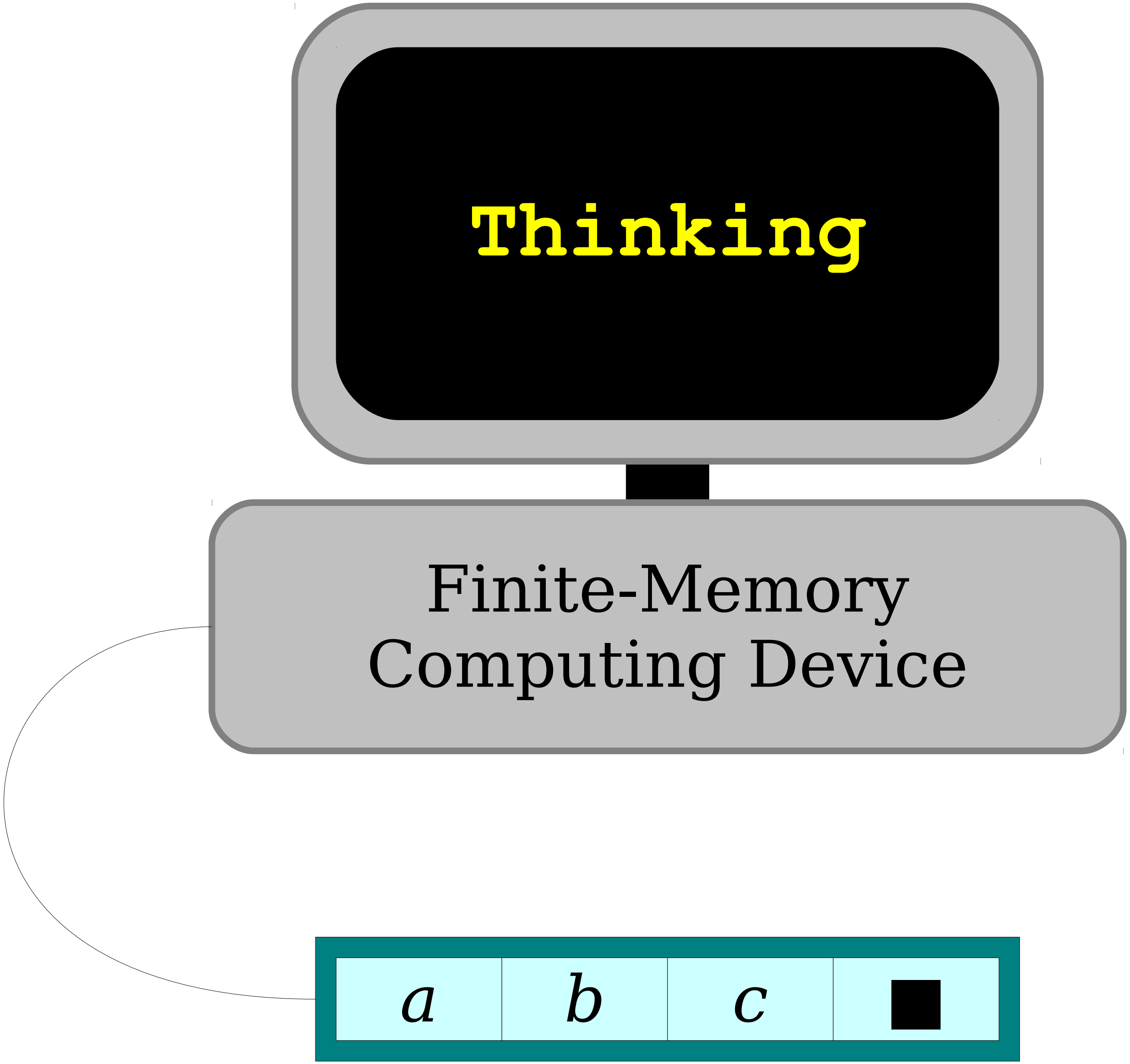
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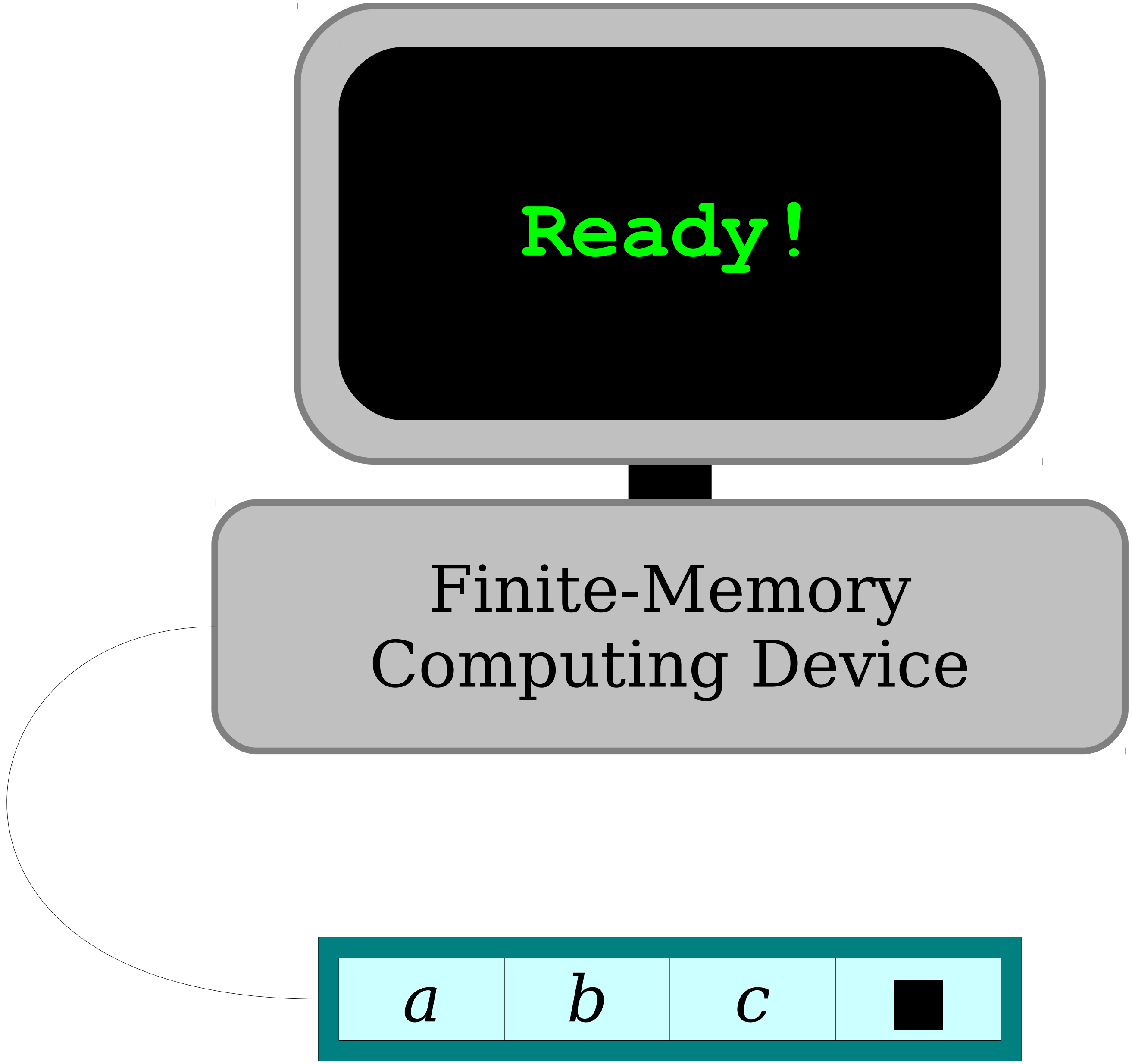
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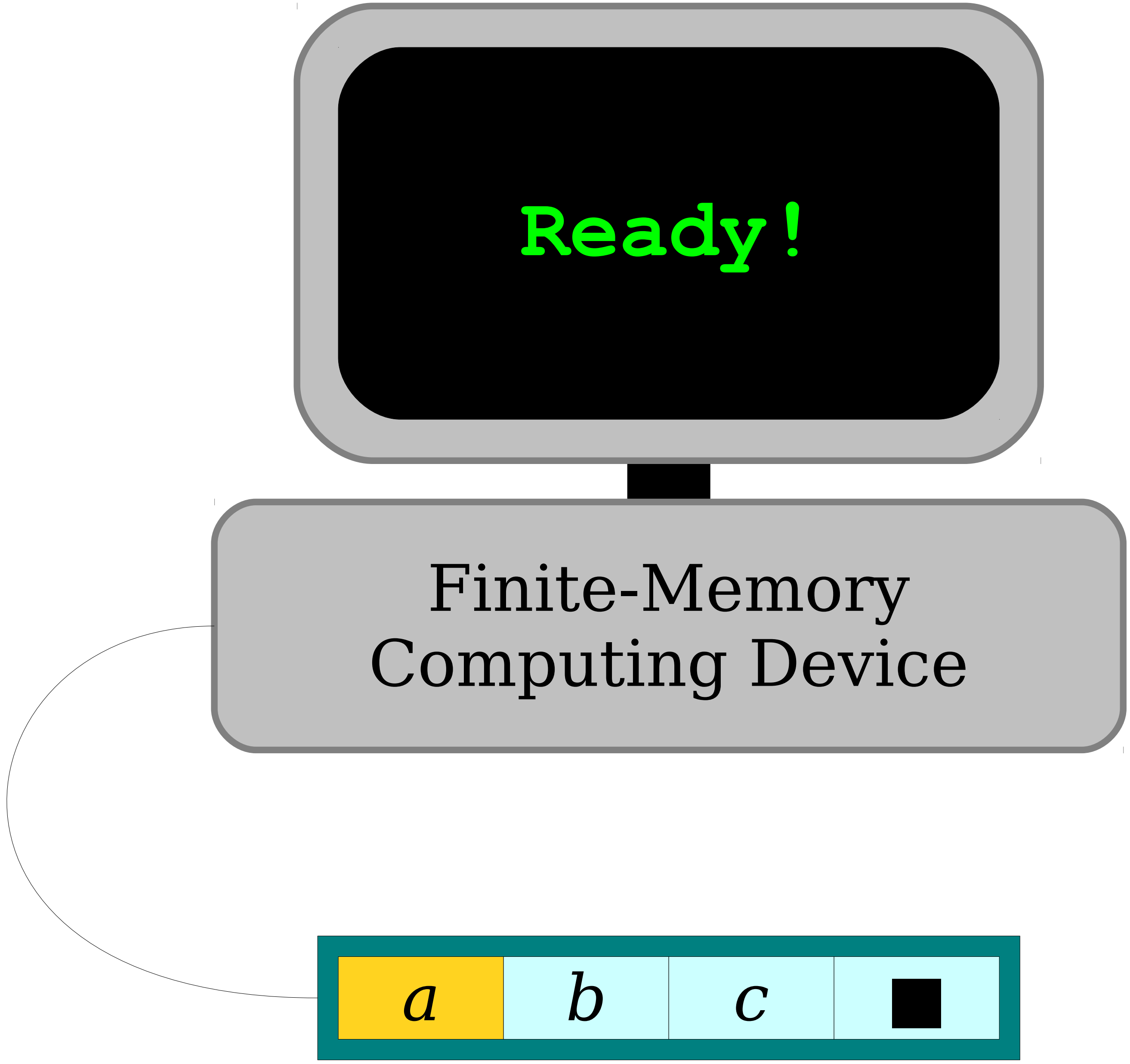
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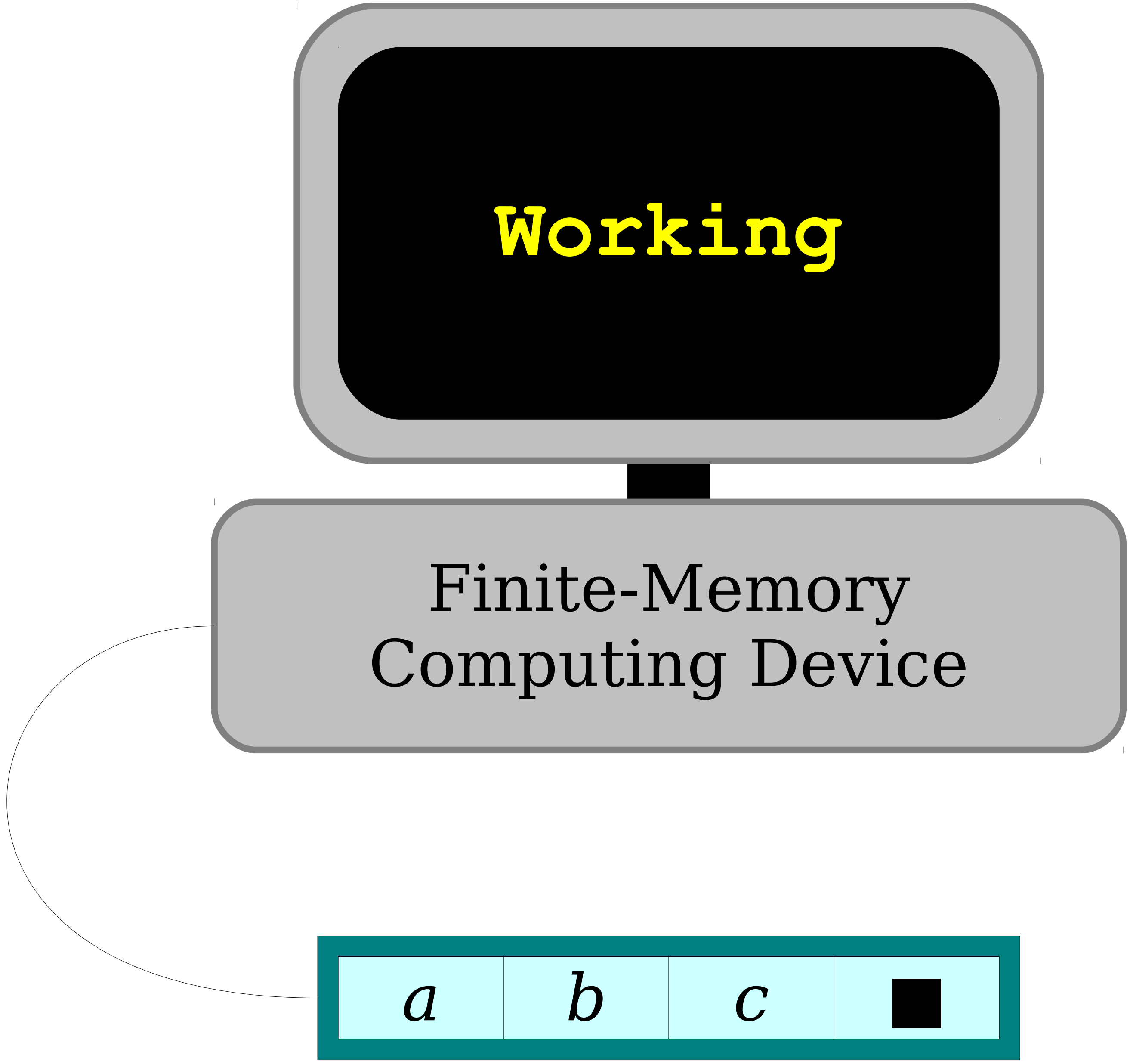
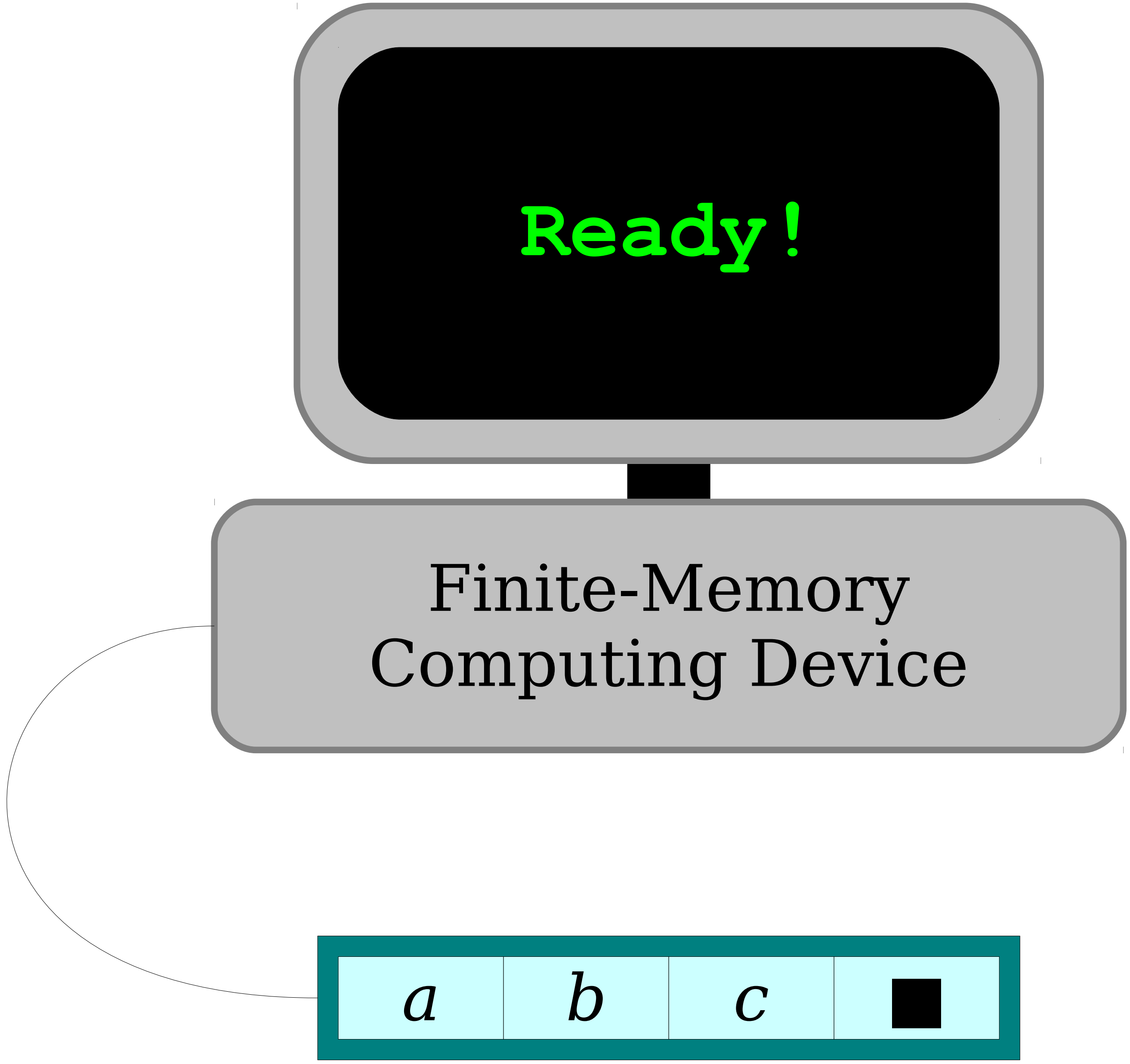
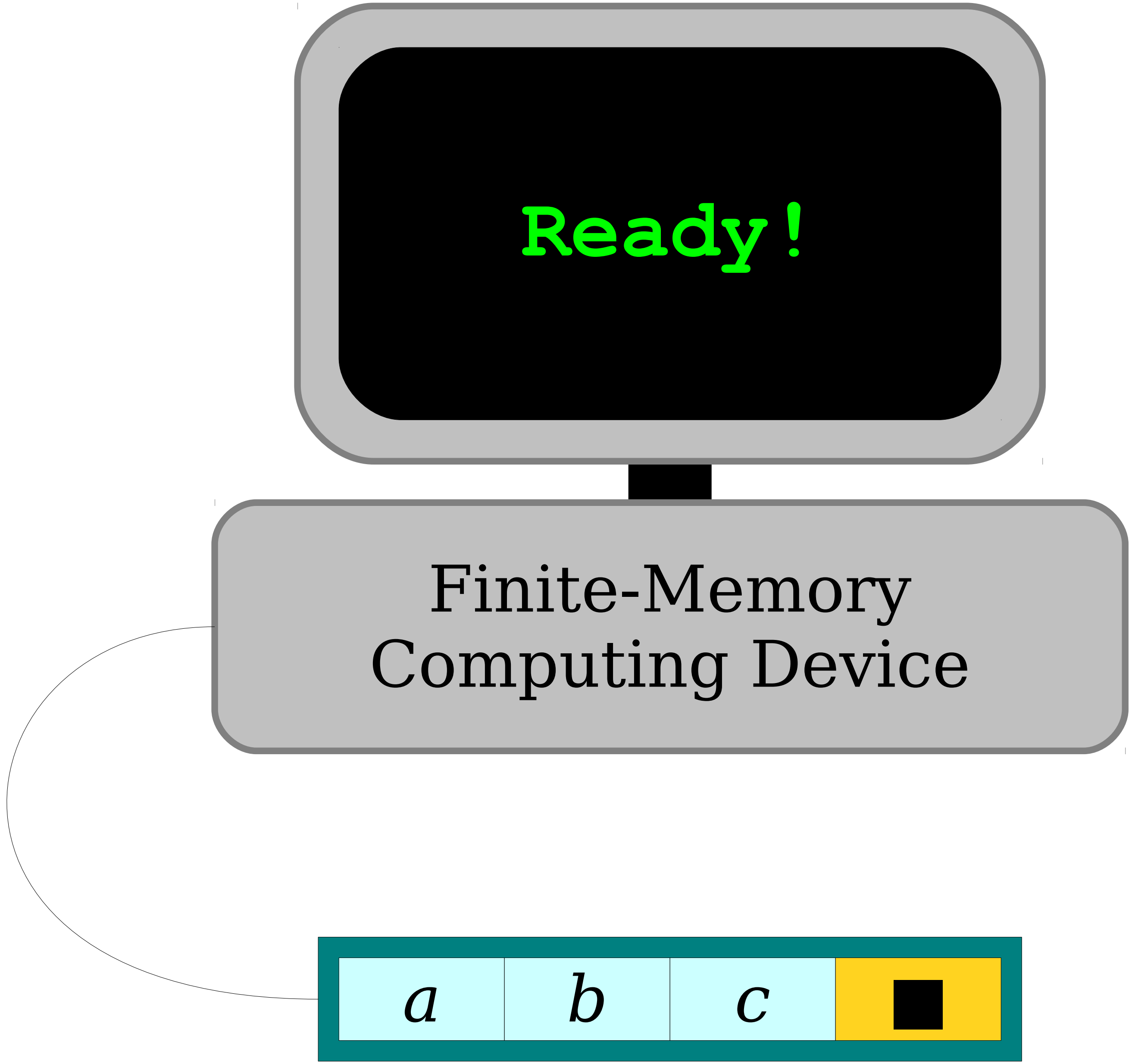


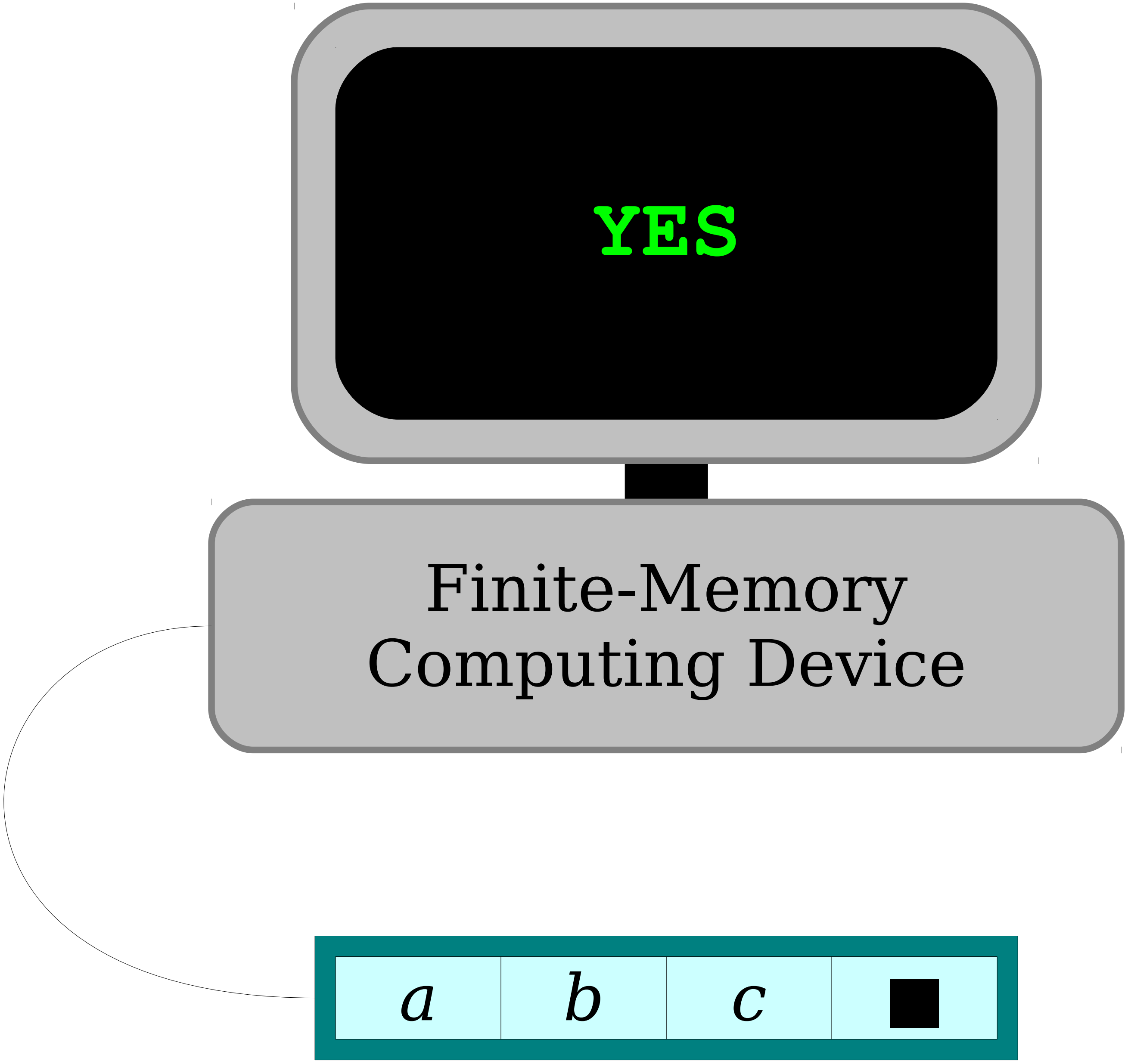
Working
Finite-Memory
Computing Device
a
b
c



Ready!
Finite-Memory
Computing Device
a
b
c



Ready!
Finite-Memory
Computing Device
a
b
c
■


YES
Finite-Memory Computing Device
a
b
c

# The Model

- The computing device has internal workings that can be in one of finitely many possible configurations.
  - Each ***state*** in a DFA corresponds to some possible configuration of the internal workings.
- After each button press, the computing device does some amount of processing, then gets to a configuration where it's ready to receive more input.
  - Each ***transition*** abstracts away how the computation is done and just indicates what the ultimate configuration looks like.
- After the user presses the “done” button, the computer outputs either YES or NO.
  - The ***accepting*** and ***rejecting*** states of the machine model what happens when that button is pressed.

# Computers as Finite Automata

- My computer has 12GB of RAM and 750GB of hard disk space.
- That's a total of 766GB of memory, which is 26,319,559,589,888 bits.
- There are “only” $2^{26,319,559,589,888}$ possible configurations of the memory in my computer.
- You could in principle build a DFA representing my computer, where there's one symbol per type of input the computer can receive.

# A Powerful Intuition

- ***Regular languages correspond to problems that can be solved with finite memory.***
  - At each point in time, we only need to store one of finitely many pieces of information.
- Nonregular languages, in a sense, correspond to problems that cannot be solved with finite memory.
- Since every computer ever built has finite memory, in a sense, nonregular languages correspond to problems that cannot be solved by physical computers!

# Finding Nonregular Languages

# Finding Nonregular Languages

- To prove that a language is regular, we can just find a DFA, NFA, or regex for it.
- To prove that a language is not regular, we need to prove that there are no possible DFAs, NFAs, or regexes for it.
  - ***Claim:*** We can actually just prove that there's no DFA for it. Why is this?
- ***This sort of argument will be challenging.*** Our arguments will be somewhat technical in nature, since we need to rigorously establish that no amount of creativity could produce a DFA for a given language.
- Let's see an example of how to do this.

# A Simple Language

- Let $\Sigma = \{\mathbf{a}, \mathbf{b}\}$ and consider the following language:

$$E = \{\mathbf{a}^n\mathbf{b}^n \mid n \in \mathbb{N}\ \}$$

- $E$ is the language of all strings of $n$ **a**'s followed by $n$ **b**'s:

$$\{\ \varepsilon, \mathbf{ab}, \mathbf{aabb}, \mathbf{aaabbb}, \mathbf{aaaabbbb}, \dots\ \}$$

- Is this language regular? Let's see!

# An Attempt

- Let's see if we can make a regular expression for the language

$$E = \{\mathbf{a}^n\mathbf{b}^n \mid n \in \mathbb{N}\ \}.$$

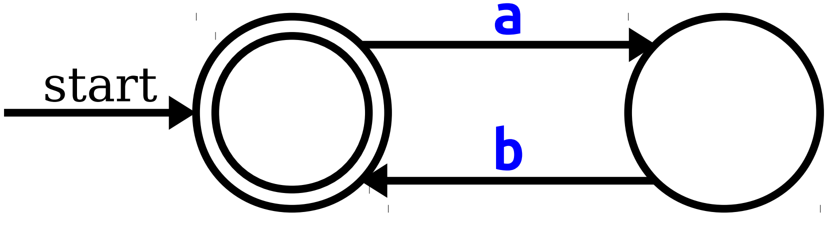
- Does `a*b*` work?
- How about `(ab)*`?
- How about $\varepsilon \cup \mathbf{ab} \cup \mathbf{a}^2\mathbf{b}^2 \cup \mathbf{a}^3\mathbf{b}^3$?

# Another Attempt

- Perhaps we can make an NFA for

$$E = \{ \mathtt{a}^n \mathtt{b}^n \mid n \in \mathbb{N} \}.$$

- Does this machine work?

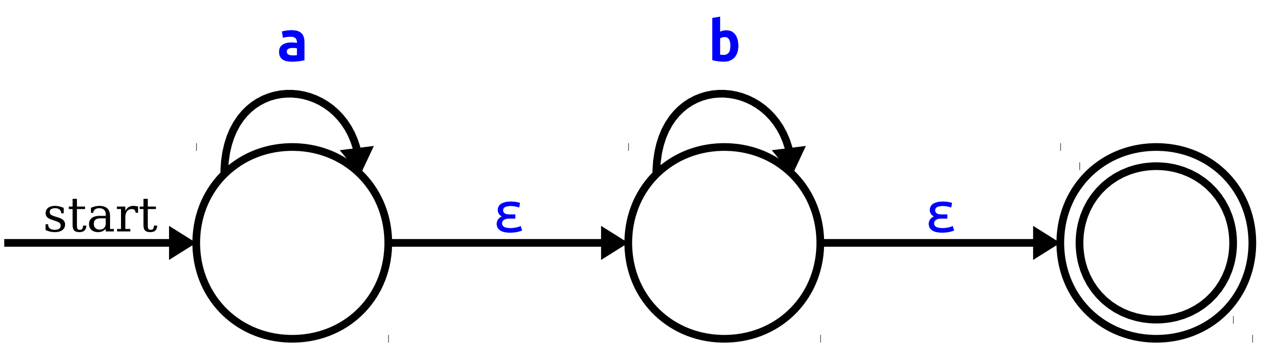


# Another Attempt

- Perhaps we can make an NFA for

$$E = \{ \mathtt{a}^n \mathtt{b}^n \mid n \in \mathbb{N} \}.$$

- How about this one?

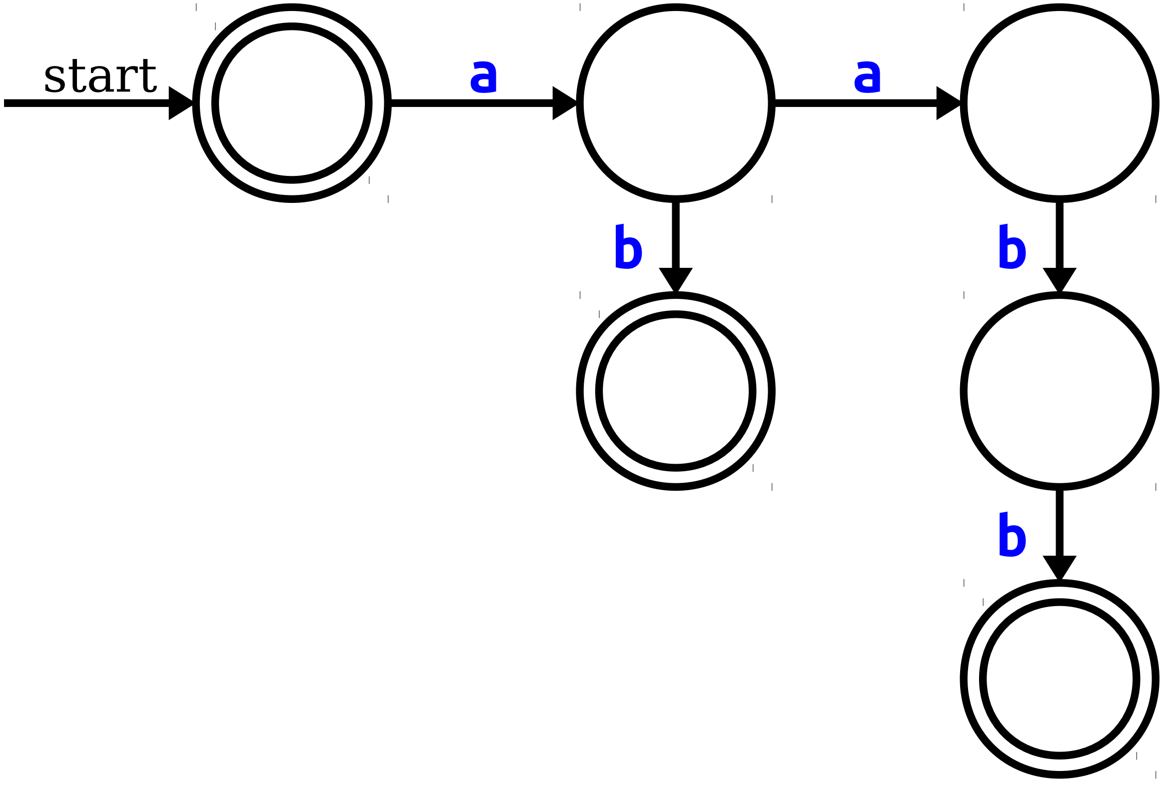


# Another Attempt

- Perhaps we can make an NFA for

$$E = \{ \mathbf{a}^n \mathbf{b}^n \mid n \in \mathbb{N} \}.$$

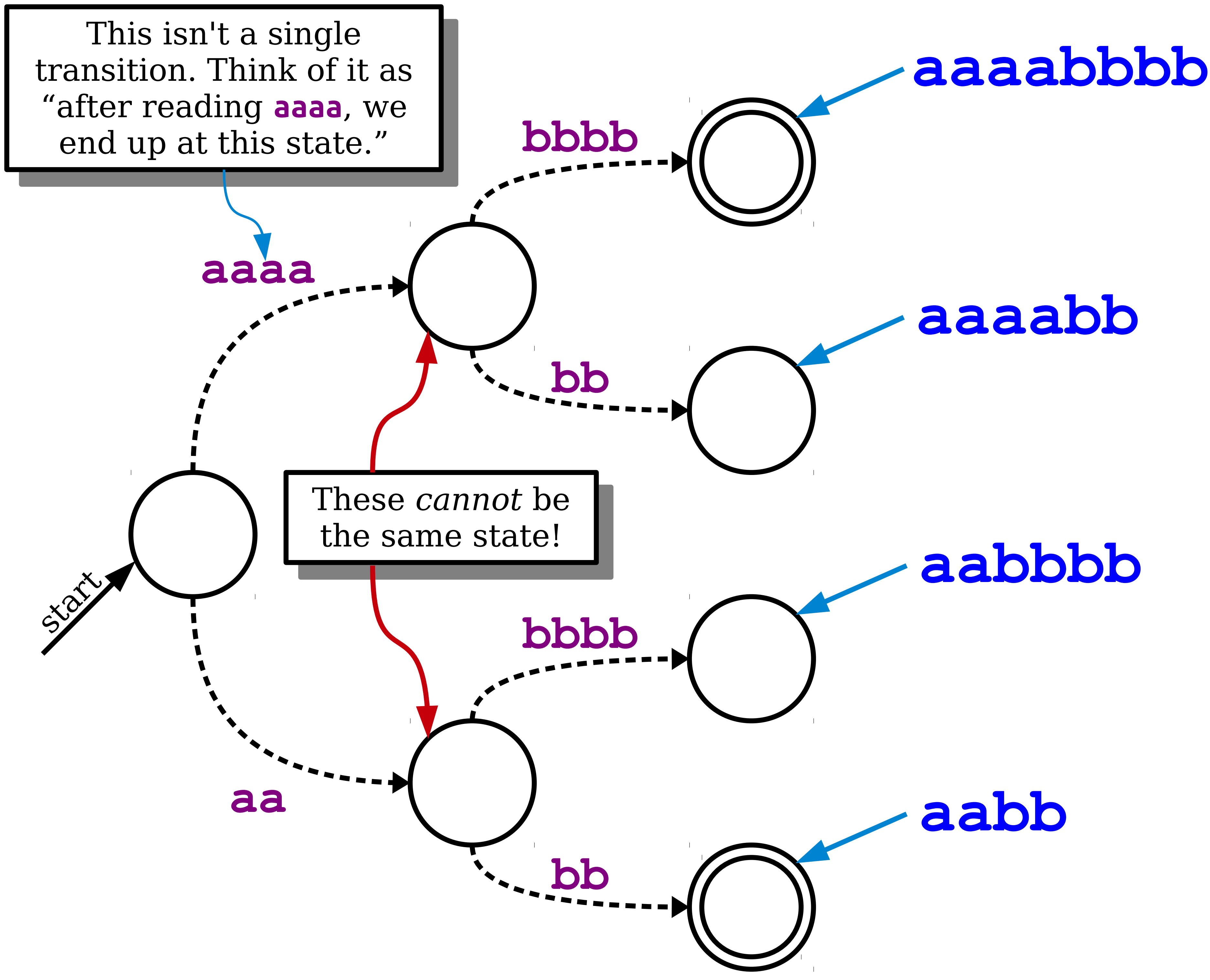
- What about this?



We seem to be running into some trouble.
Why is that?

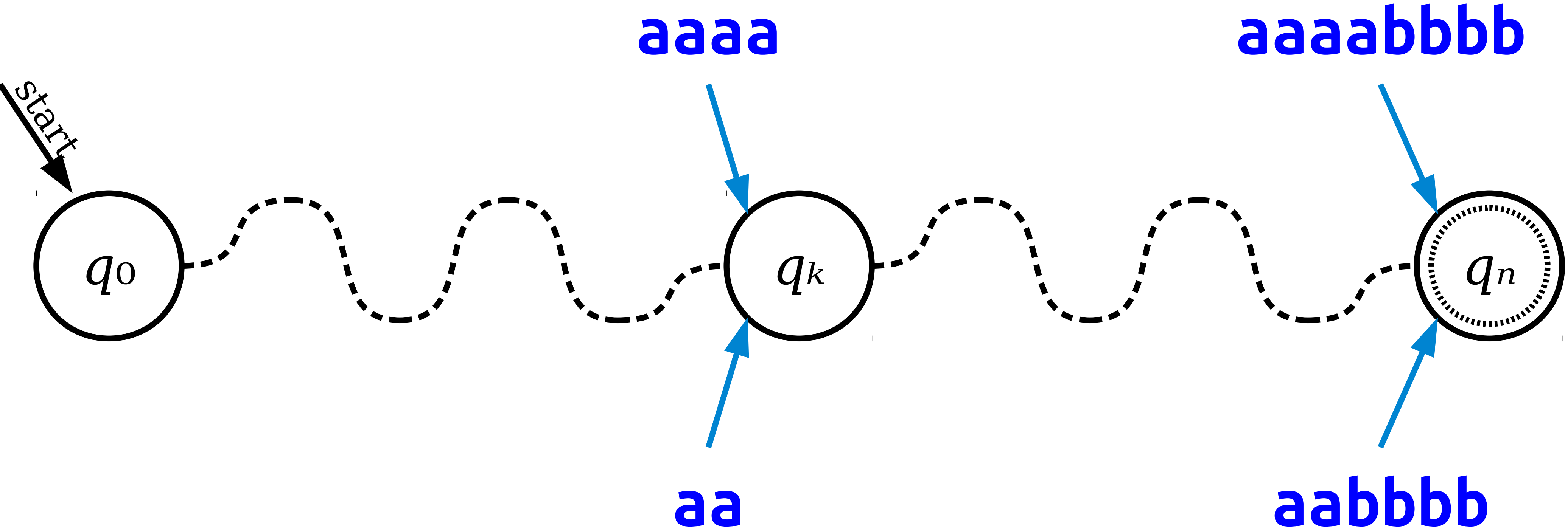
Let's imagine what a DFA for the language $\{ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N}\}$ would have to look like.

Can we say anything about it?



This isn't a single transition. Think of it as "after reading aaaa, we end up at this state."
aaaabbbb
bbbb
aaaa
aaaabb
bb
These *cannot* be the same state!
start
aabbbb
bbbb
aa
aabb
bb

# A Different Perspective



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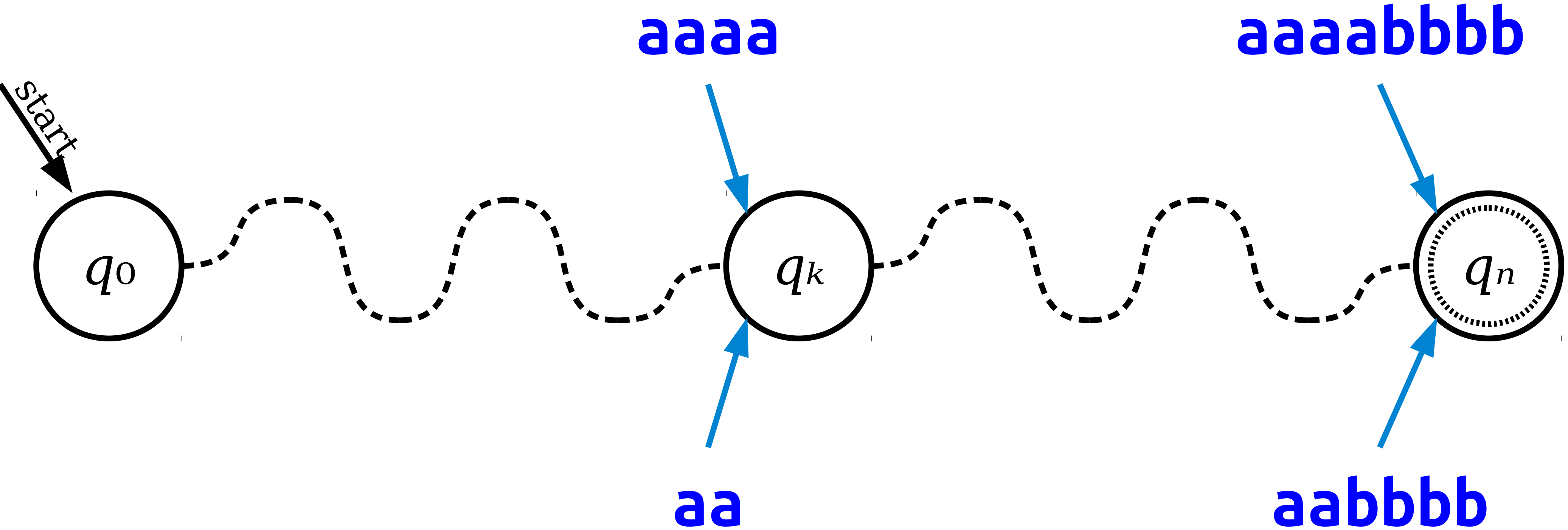
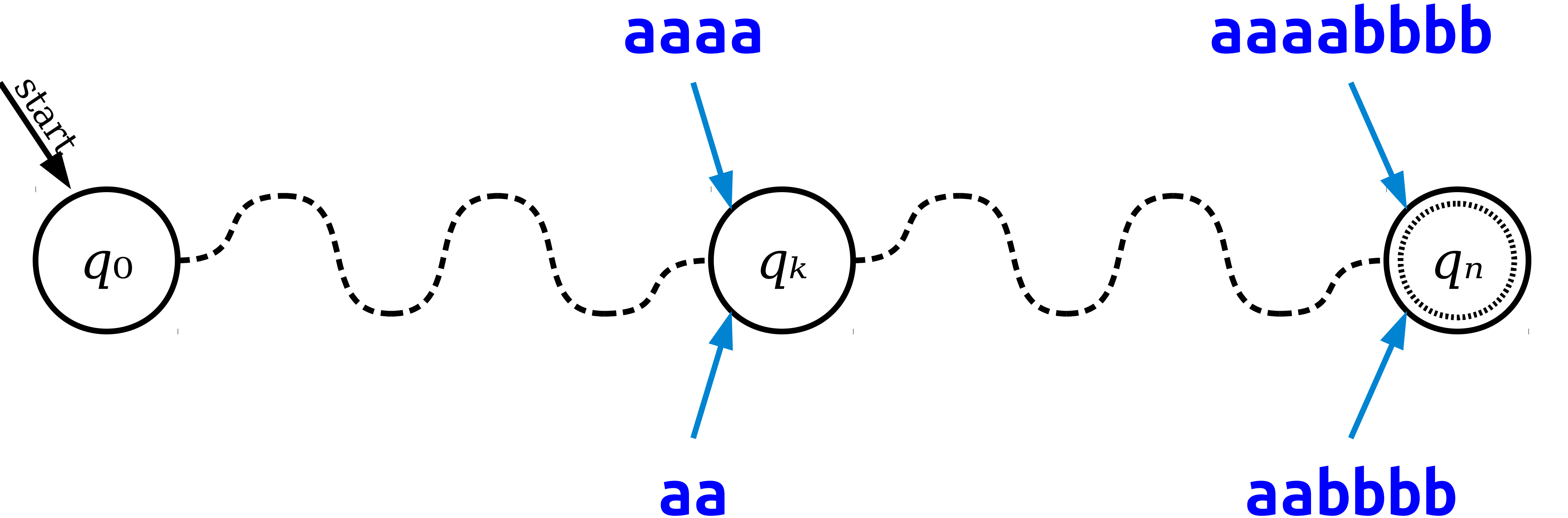



What happens if $q_n$ is…

…an accepting state?
…a rejecting state?

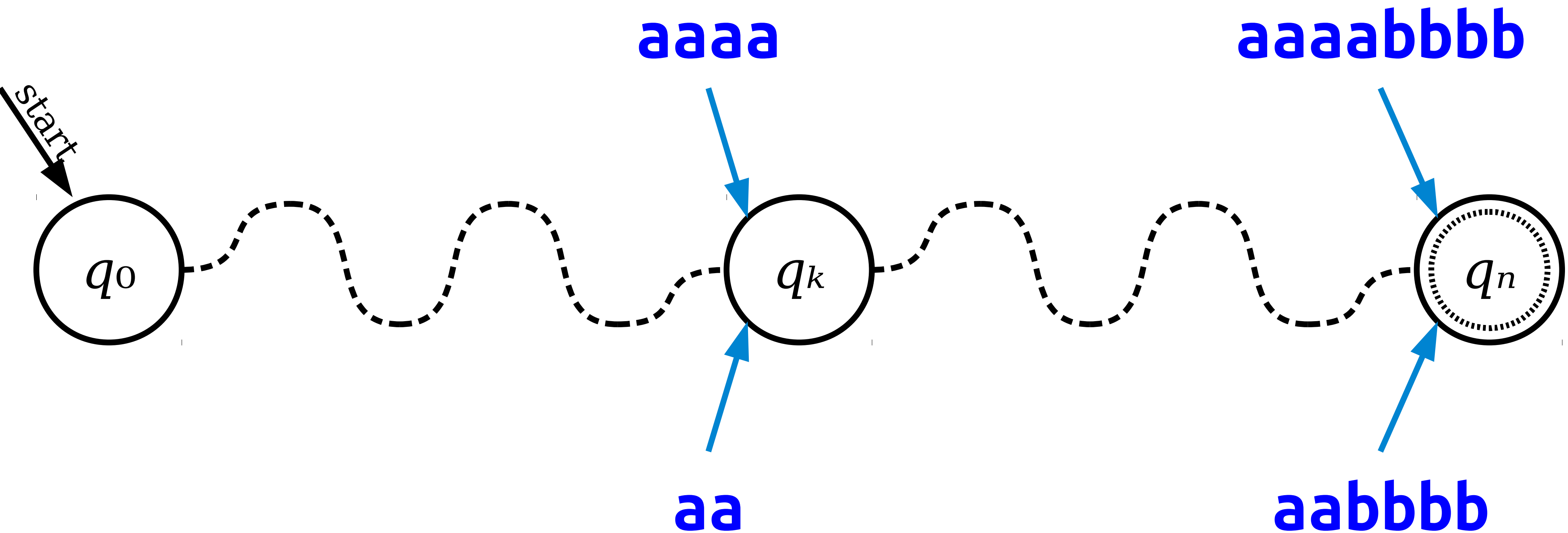
# A Different Perspective




What happens if $q_n$ is…

…an accepting state? We accept **aabbbb** $\notin E$!
…a rejecting state?

# A Different Perspective




What happens if $q_n$ is…

…an accepting state? We accept **aabbbb** $\notin E$!

…a rejecting state? We reject **aaaabbbb** $\in E$!

# The Intuition

- As you just saw, the strings $\mathbf{a}^4$ and $\mathbf{a}^2$ can't end up in the same state in *any* DFA for the language $E = \{\mathbf{a}^n\mathbf{b}^n \mid n \in \mathbb{N}\}$ .
- Two proof routes:
  - *Direct:* The states you reach for $\mathbf{a}^4$ and $\mathbf{a}^2$ have to behave differently when reading $\mathbf{b}^4$ – in one case it should lead to an accept state, in the other it should lead to a reject state. Therefore, they must be different states.
  - *Contradiction:* Suppose you do end up in the same state. Then $\mathbf{a}^4\mathbf{b}^4$ and $\mathbf{a}^2\mathbf{b}^4$ end up in the same state, so we either reject $\mathbf{a}^4\mathbf{b}^4$ (oops) or accept $\mathbf{a}^2\mathbf{b}^4$ (oops).

This idea – that two strings shouldn't end up in the same DFA state – is fundamental to discovering nonregular languages.

Let's go formalize this!

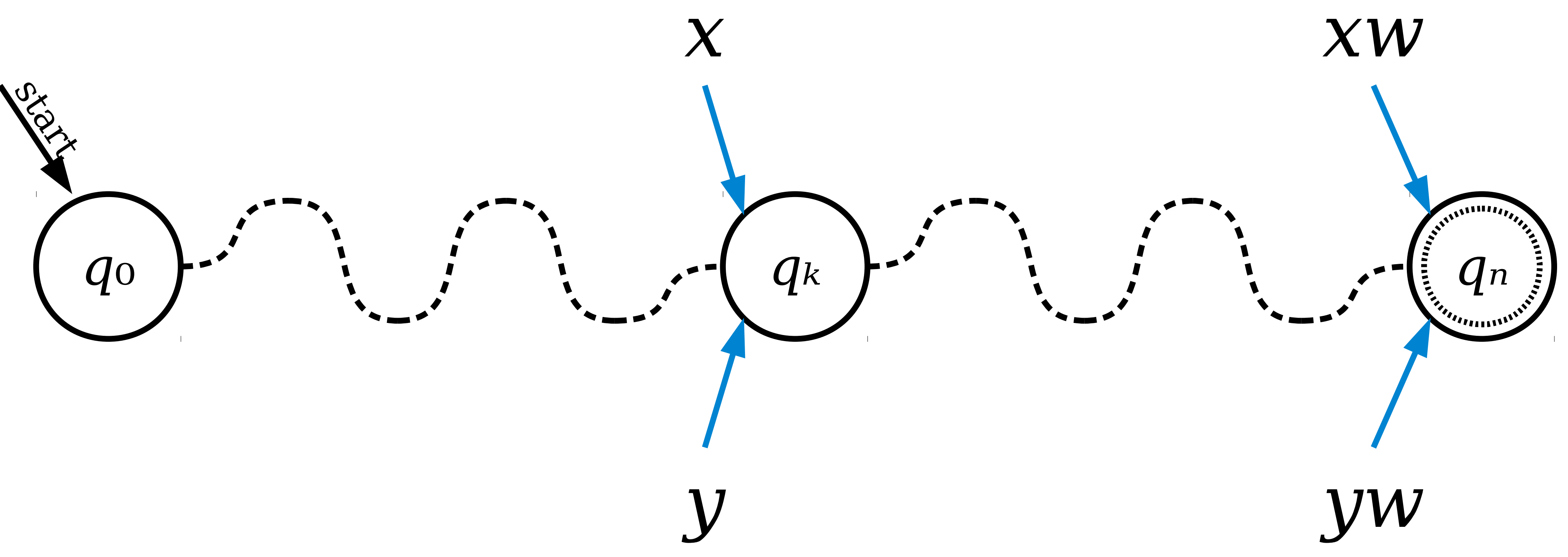
# Distinguishability

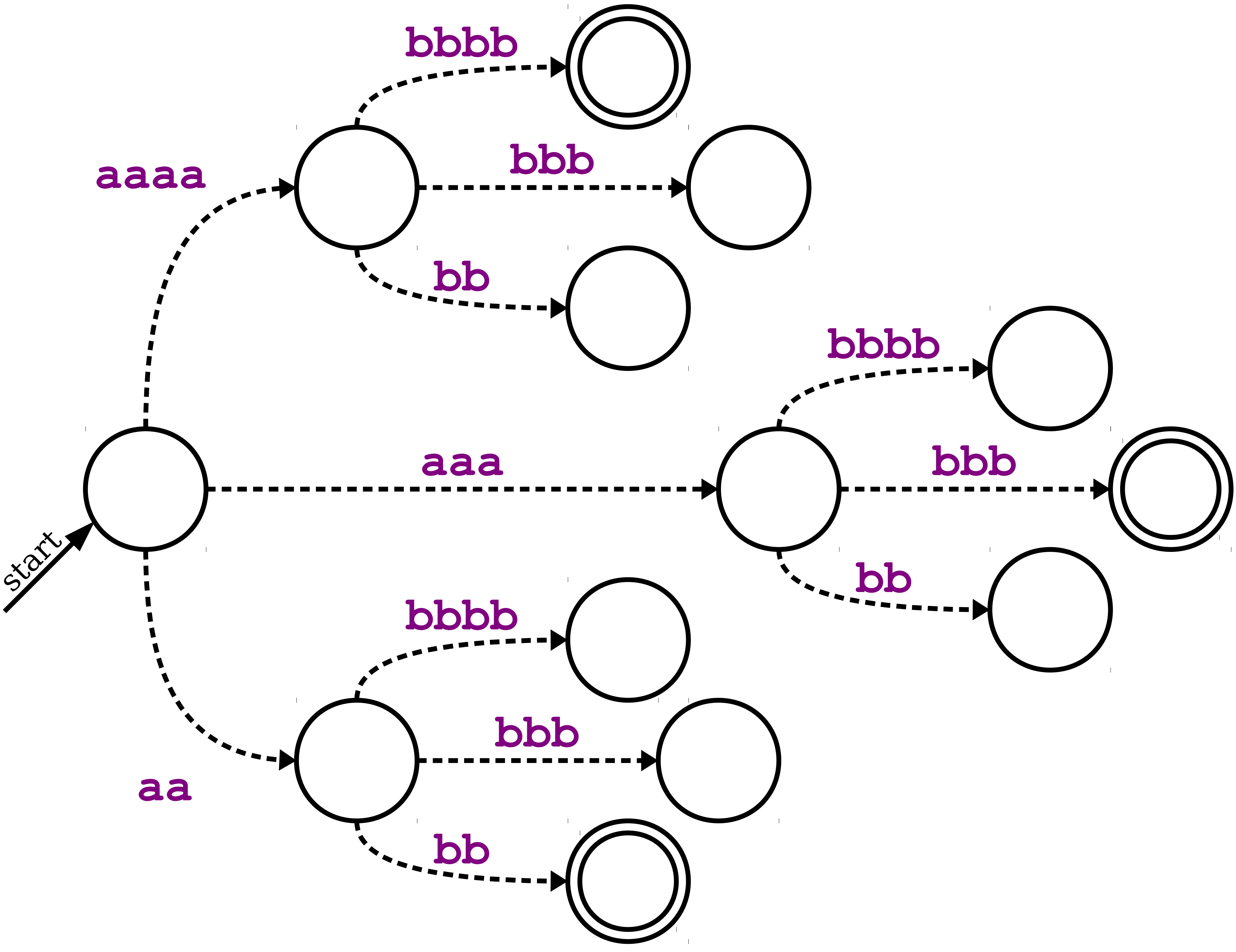
- Let $L$ be an arbitrary language over Σ.
- Two strings $x \in \Sigma^*$ and $y \in \Sigma^*$ are called ***distinguishable relative to L*** if there is a string $w \in \Sigma^*$ such that exactly one of $xw$ and $yw$ is in $L$.
- We denote this by writing $\boldsymbol{x \not\equiv_L y}$.
- In our previous example, we saw that $\mathtt{a}^2$ and $\mathtt{a}^4$ are distinguishable relative to $E = \{ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N} \}$.
  - Try appending $\mathtt{b}^4$ to both of them.
- Formally, we say that $x \not\equiv_L y$ if the following is true:

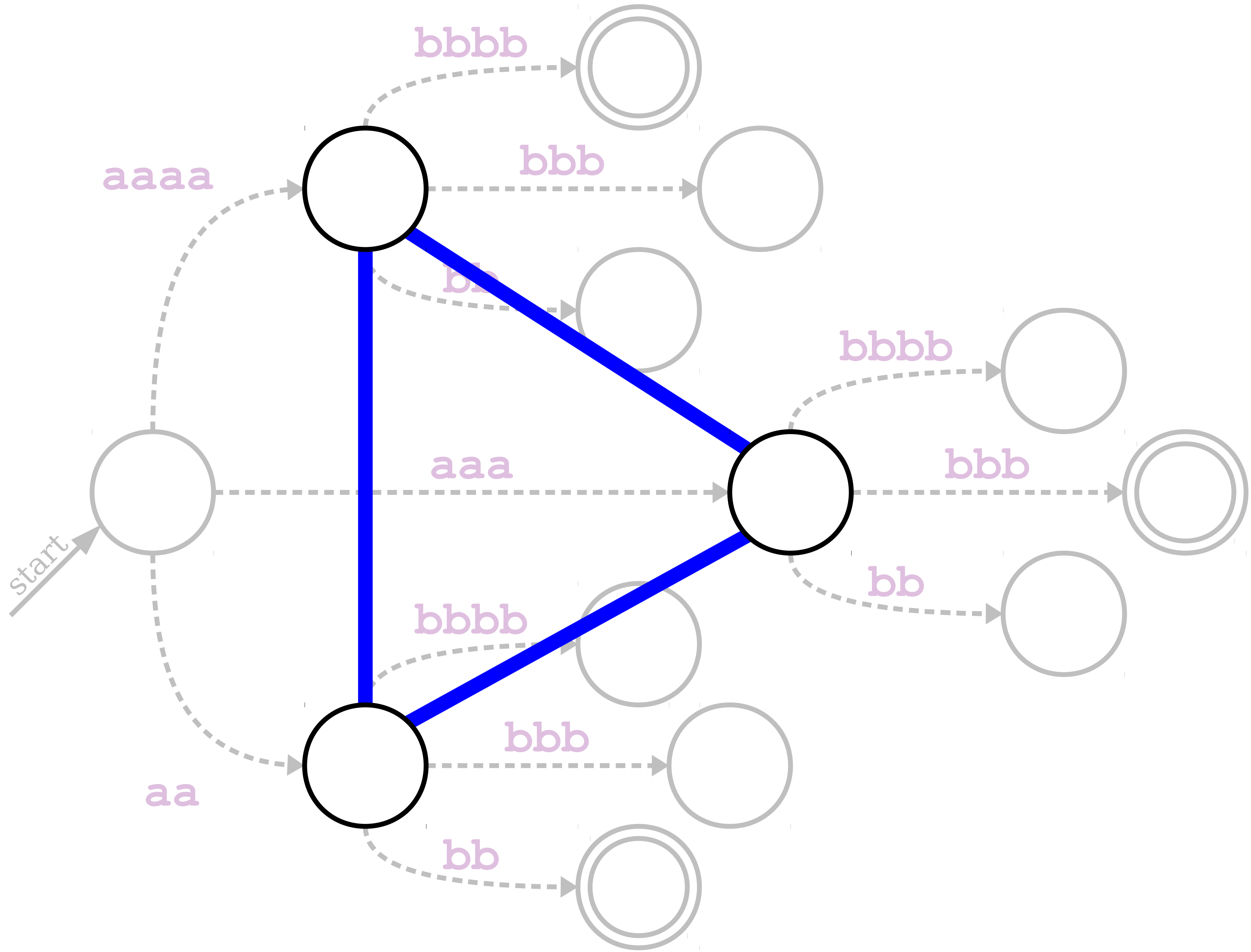
$$\exists w \in \Sigma^*. \ (xw \in L \leftrightarrow yw \notin L)$$

# Distinguishability

- ***Theorem:*** Let $L$ be an arbitrary language over $\Sigma$. Let $x \in \Sigma^*$ and $y \in \Sigma^*$ be strings where $x \not\equiv_L y$. Then if $D$ is ***any*** DFA for $L$, then $D$ must end in different states when run on inputs $x$ and $y$.
- ***Proof sketch:***




start
aaaa
aaa
aa
bbbb
bbb
bb
bbbb
bbb
bb
bbbb
bbb
bb



bbbb
aaaa
bbb
bb
bbbb
aaa
bbb
start
bb
bbbb
bbb
aa
bb

# Distinguishability

- Let's focus on this language for now:

$$E = \{ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N} \}$$

***Lemma:*** If $m, n \in \mathbb{N}$ and $m \neq n$, then $\mathtt{a}^m \not\equiv_E \mathtt{a}^n$.

***Proof:*** Let $\mathtt{a}^m$ and $\mathtt{a}^n$ be strings where $m \neq n$. Then $\mathtt{a}^m\mathtt{b}^m \in E$ and $\mathtt{a}^n\mathtt{b}^m \notin E$. Therefore, we see that $\mathtt{a}^m \not\equiv_E \mathtt{a}^n$, as required. ■

# A Bad Combination

- Suppose there is a DFA $D$ for the language $E = \{ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N} \}$.
- We know the following:
  - Any two strings of the form $\mathtt{a}^m$ and $\mathtt{a}^n$, where $m \neq n$, cannot end in the same state when run through $D$.
  - There are infinitely many pairs of strings of the form $\mathtt{a}^m$ and $\mathtt{a}^n$.
  - However, there are only *finitely many* states they can end up in, since $D$ is a deterministic ***finite*** automaton!
- If we put the pieces together, we see that…

The language $E = \{ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N} \}$ is not regular.

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We have reached a contradiction, so our assumption must have been wrong.

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*We're going to see a simpler proof of this result later on once we've built more machinery. If (hypothetically speaking) you want to prove something like this on the problem set, we'd recommend not using this proof as a template.*

# What Just Happened?

- ***We've just hit the limit of finite-memory computation.***
- To build a DFA for $E = \{ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N} \}$, we need to have different memory configurations (states) for all possible strings of the form $\mathtt{a}^n$.
- There's no way to do this with finitely many possible states!

# Where We're Going

- We just used the idea of *distinguishability* to show that no possible DFA can exist for some language.
- This technique turns out to be pretty powerful.
- We're going to see one more example of this technique in action, then generalize it to an extremely powerful theorem for finding nonregular languages.

# More Nonregular Languages

# Another Language

- Consider the following language $L$ over the alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}, \stackrel{?}{=}\}$:

$$EQ = \{\ w \stackrel{?}{=} w \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$$

- $EQ$ is the language all strings consisting of the same string of a's and b's twice, with a $\stackrel{?}{=}$ symbol in-between.
- Examples:

$\mathtt{ab}\stackrel{?}{=}\mathtt{ab} \in EQ$ $\quad$ $\mathtt{bbb}\stackrel{?}{=}\mathtt{bbb} \in EQ$ $\quad$ $\stackrel{?}{=} \in EQ$

$\mathtt{ab}\stackrel{?}{=}\mathtt{ba} \notin EQ$ $\quad$ $\mathtt{bbb}\stackrel{?}{=}\mathtt{aaa} \notin EQ$ $\quad$ $\mathtt{b}\stackrel{?}{=} \notin EQ$

# Another Language

$$EQ = \{\ w \overset{?}{=} w \mid w \in \{\mathtt{a}, \mathtt{b}\}^* \}$$

- This language corresponds to the following problem:

**Given strings *x* and *y*, does *x* = *y*?**

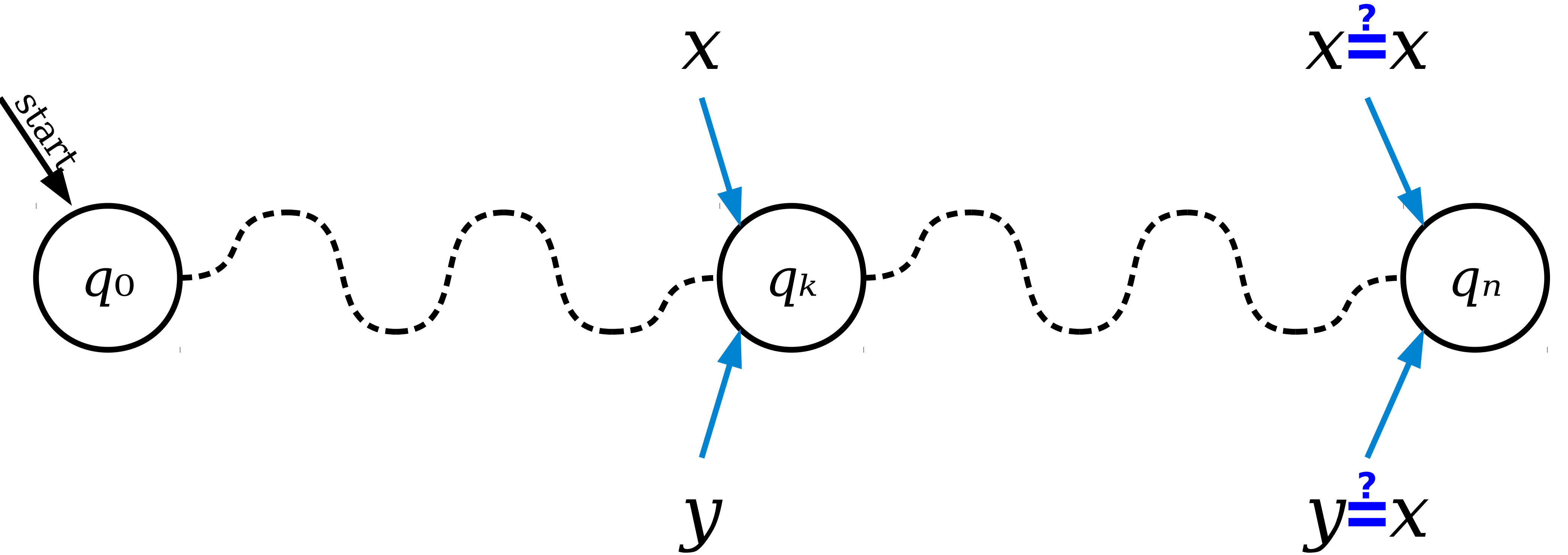
- Justification: $x = y$ iff $x \overset{?}{=} y \in EQ$.
- Is this language regular?

# The Intuition

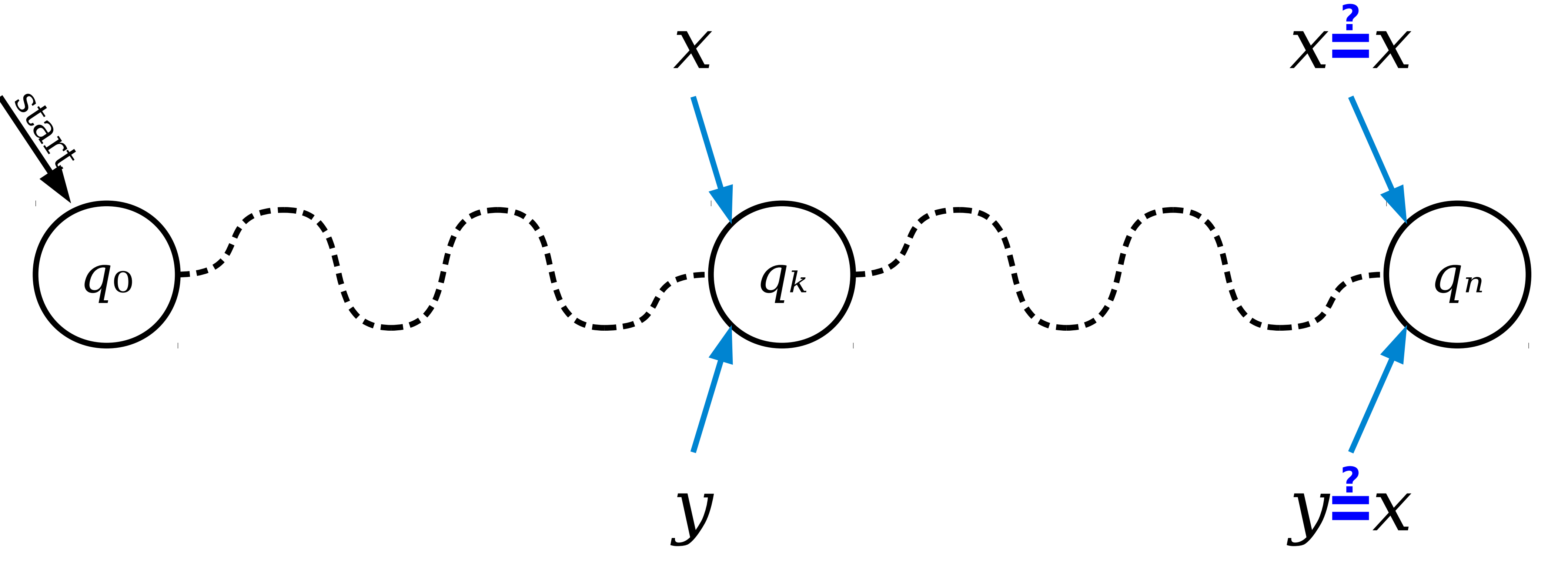
$$EQ = \{ \; w \stackrel{?}{=} w \mid w \in \{\texttt{a}, \texttt{b}\}^* \}$$

- Intuitively, any machine for $EQ$ has to be able to remember the contents of everything to the left of the $\stackrel{?}{=}$ so that it can match them against the contents of the string to the right of the $\stackrel{?}{=}$.
- There are infinitely many possible strings we can see, but we only have finite memory to store which string we saw.
- That's a problem… can we formalize this?

# The Intuition



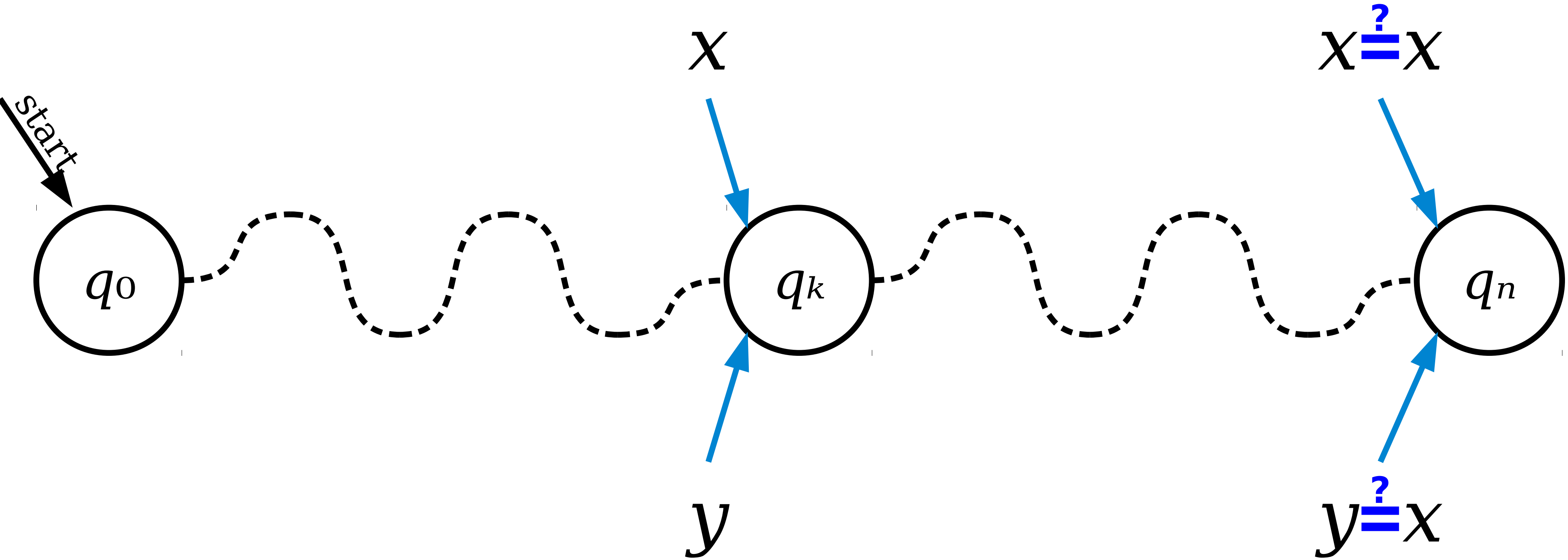
# The Intuition




What happens if $q_n$ is…

…an accepting state?
…a rejecting state?

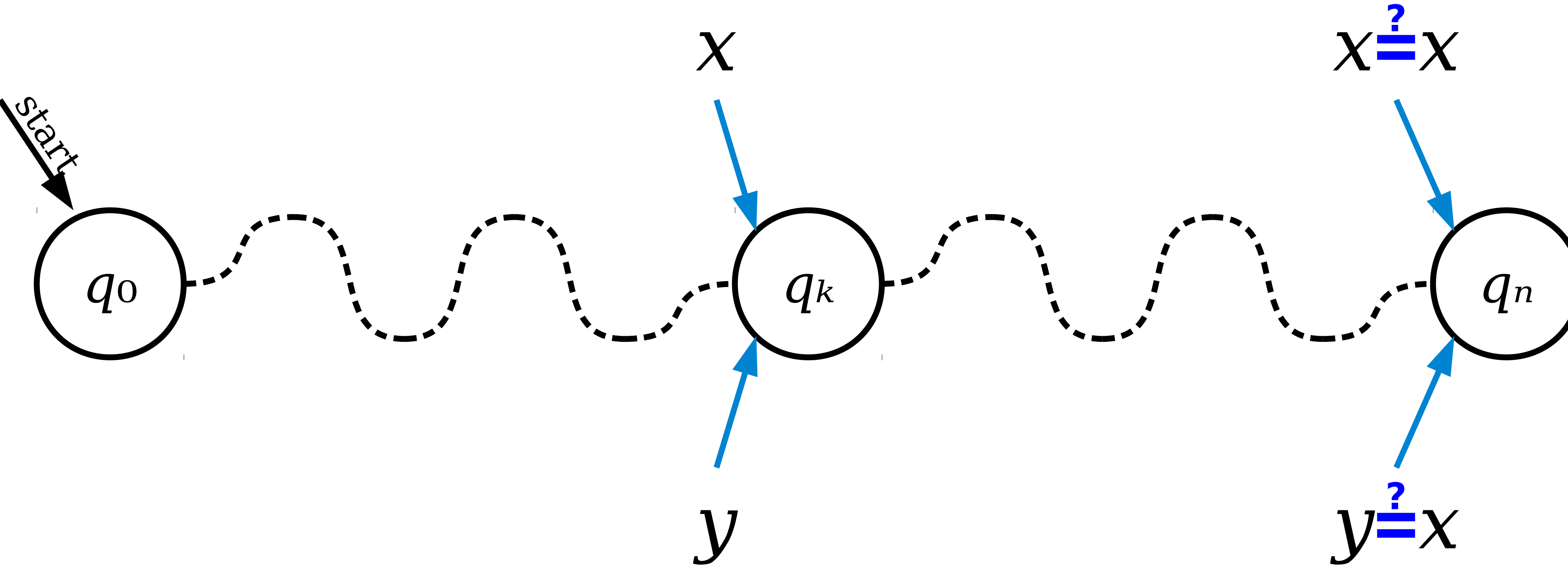
# The Intuition




What happens if $q_n$ is…

…an accepting state? We accept $y \stackrel{?}{=} x \notin EQ$!
…a rejecting state?

# The Intuition




What happens if $q_n$ is...

...an accepting state? We accept $y \stackrel{?}{=} x \notin EQ$!

...a rejecting state? We reject $x \stackrel{?}{=} x \in EQ$!

# Distinguishability

- Let's focus on this language for now:

$$EQ = \{\ w\overset{?}{=}w \mid w \in \{\texttt{a}, \texttt{b}\}^*\}$$

***Lemma:*** If $x, y \in \{\texttt{a}, \texttt{b}\}^*$ and $x \neq y$, then $x \not\equiv_{EQ} y$.

***Proof:*** Let $x$ and $y$ be two distinct, arbitrary strings from $\{\texttt{a}, \texttt{b}\}^*$. Then we see that $x\overset{?}{=}x \in L$ and $y\overset{?}{=}x \notin L$, so we conclude that $x \not\equiv_{EQ} y$, as required. ■

***Theorem:*** The language $EQ = \{\ w \stackrel{?}{=} w \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$ is not regular.

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***Proof:*** Suppose for the sake of contradiction that $EQ$ is regular. Let $D$ be a DFA for $EQ$ and let $k$ be the number of states in $D$.

***Theorem:*** The language $EQ = \{ w \stackrel{?}{=} w \mid w \in \{$**`a`**, **`b`**$\}^* \}$ is not regular.

***Proof:*** Suppose for the sake of contradiction that $EQ$ is regular. Let $D$ be a DFA for $EQ$ and let $k$ be the number of states in $D$. Consider any $k+1$ distinct strings in $\{$**`a`**, **`b`**$\}^*$.

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***Proof:*** Suppose for the sake of contradiction that $EQ$ is regular. Let $D$ be a DFA for $EQ$ and let $k$ be the number of states in $D$. Consider any $k+1$ distinct strings in $\{\texttt{a}, \texttt{b}\}^*$. Because $D$ only has $k$ states, by the pigeonhole principle there must be at least two strings $x$ and $y$ that, when run through $D$, end in the same state.

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Our lemma tells us that $x \not\equiv_{EQ} y$.

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Our lemma tells us that $x \not\equiv_{EQ} y$. By our earlier theorem, this means that $x$ and $y$ cannot end in the same state when run through $D$.

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Our lemma tells us that $x \not\equiv_{EQ} y$. By our earlier theorem, this means that $x$ and $y$ cannot end in the same state when run through $D$. But this is impossible, since specifically chose $x$ and $y$ to end in the same state when run through $D$.

We have reached a contradiction, so our assumption must have been wrong. Thus $EQ$ is not regular.

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Our lemma tells us that $x \not\equiv_{EQ} y$. By our earlier theorem, this means that $x$ and $y$ cannot end in the same state when run through $D$. But this is impossible, since specifically chose $x$ and $y$ to end in the same state when run through $D$.

We have reached a c

have been wrong. Th

*We're going to see a simpler proof of this result later on once we've built more machinery. If (hypothetically speaking) you want to prove something like this on the problem set, we'd recommend not using this proof as a template.*

Time-Out for Announcements!

# WiCS E**CS**plore

- WiCS is hosting E**CS**plore, a panel event where CS faculty members will talk about the different tracks in CS.
  - It's this Thursday from 5:30PM – 7:00PM in Gates 219.
- This was a huge success the last time it ran – highly recommended!
- Please RSVP using **this link** so they can estimate how much food to order.

# Your Questions

“So this question popped up in my head during the midterm: why do we have to write "we" in our proofs? Ex) "We will prove" or "we have shown"? I'm the one proving the theorem, not someone else. I feel like some third party is taking credit for my work.”

Part of the purpose of writing proofs is so that you can help other people follow the line of reasoning that you came up with. To make the reader feel included with what you're doing, it's good to use “we” to mean “me, plus you, the Most Excellent Reader.”

“What were some of your favorite extracurricular activities/experiences when you were an undergrad here?”

Section leading was easily the top one. I met a lot of my lasting friends through that community, got a chance to do silly things like ride tricycles down parking lots (until the po-po shut us down) and steam tunnel under the quad while playing Capture the Flag. Oh yeah, and it helped me get my current job. ☺

Back to CS103!

# Comparing Proofs

***Theorem:*** The language $E = \{ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N} \}$ is not a regular language.

***Proof:*** Suppose for the sake of contradiction that $E$ is regular. Let $D$ be a DFA for $E$ and let $k$ be the number of states in $D$.

Consider the strings $\mathtt{a}^0, \mathtt{a}^1, \mathtt{a}^2, \ldots, \mathtt{a}^k$. This is a collection of $k+1$ strings and there are only $k$ states in $D$. Therefore, by the pigeonhole principle, there must be two distinct strings $\mathtt{a}^m$ and $\mathtt{a}^n$ that end in the same state when run through $D$.

Our lemma tells us that $\mathtt{a}^m \not\equiv_E \mathtt{a}^n$. By our earlier theorem we know that $\mathtt{a}^m$ and $\mathtt{a}^n$ cannot end in the same state when run through $D$. But this is impossible, since we know that $\mathtt{a}^m$ and $\mathtt{a}^n$ do end in the same state when run through $D$.

We have reached a contradiction, so our assumption must have been wrong. Therefore, $E$ is not regular. ■

***Theorem:*** The language $EQ = \{ w \stackrel{?}{=} w \mid w \in \{\mathtt{a}, \mathtt{b}\}^* \}$ is not a regular language.

***Proof:*** Suppose for the sake of contradiction that $EQ$ is regular. Let $D$ be a DFA for $EQ$ and let $k$ be the number of states in $D$.

Consider any $k+1$ distinct strings in $\{\mathtt{a}, \mathtt{b}\}^*$.These are $k+1$ strings and there are only $k$ states in $D$. By the pigeonhole principle, there must be two distinct strings $x$ and $y$ from this group that end in the same state when run through $D$.

Our lemma tells us that $x \not\equiv_{EQ} y$. By our earlier theorem we know that $x$ and $y$ cannot end in the same state when run through $D$. But this is impossible, since specifically chose $x$ and $y$ to end in the same state when run through $D$.

We have reached a contradiction, so our assumption must have been wrong. Therefore, $EQ$ is not regular. ■

***Theorem:*** The language $L$ = **[ fill in the blank ]** is not a regular language.

***Proof:*** Suppose for the sake of contradiction that $L$ is regular. Let $D$ be a DFA for $L$ and let $k$ be the number of states in $D$.

Consider **[ some $k$+1 specific strings. ]** This is a collection of $k+1$ strings and there are only $k$ states in $D$. Therefore, by the pigeonhole principle, there must be two distinct strings $x$ and $y$ that end in the same state when run through $D$.

**[ Somehow we know ]** that $x \not\equiv_L y$. By our earlier theorem we know that $x$ and $y$ cannot end in the same state when run through $D$. But this is impossible, since we know that $x$ and $y$ must end in the same state when run through $D$.

We have reached a contradiction, so our assumption must have been wrong. Therefore, $L$ is not regular. ■

***Theorem:*** The language $L$ = **`[ fill in the blank ]`** is not a regular language.

***Proof:*** Suppose for the sake of contradiction that $L$ is regular. Let $D$ be a DFA for $L$ and let $k$ be the number of states in $D$.

Consider **`[ some k+1 specific strings. ]`** This is a collection of $k+1$ strings and there are only $k$ states in $D$. Therefore, by the pigeonhole principle, there must be two distinct strings $x$ and $y$ that end in the same state when run through $D$.

**`[ Somehow we know ]`** that $x \not\equiv_L y$. By our earlier theorem we know that $x$ and $y$ cannot end in the same state when run through $D$. But this is impossible, since we know that $x$ and $y$ must end in the same state when run through $D$.

We have reached a contradiction, so our assumption must have been wrong. Therefore, $L$ is not regular. ■

For any number of states $k$, we need a way to find $k+1$ strings so that two of them get into the same state…

***Theorem:*** The language $L$ = **[**

regular language.

***Proof:*** Suppose for the sake of contradiction that $L$ is regular. Let $D$ be a DFA for $L$ and let $k$ be the number of states in $D$.

Consider **[ `some k+1 specific strings.` ]** This is a collection of $k+1$ strings and there are only $k$ states in $D$. Therefore, by the pigeonhole principle, there must be two distinct strings $x$ and $y$ that end in the same state when run through $D$.

**[ `Somehow we know` ]** that $x \not\equiv_L y$. By our earlier theorem we know that $x$ and $y$ cannot end in the same state when run through $D$. But this is impossible, since we know that $x$ and $y$ must end in the same state when run through $D$.

We have reached a contradiction, so our assumption must have been wrong. Therefore, $L$ is not regular. ■

For any number of states $k$, we need a way to find $k$+1 strings so that two of them get into the same state…

***Theorem:*** The language $L$ = [

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***Proof:*** Suppose for the sake of contradiction that $L$ is regular. Let $D$ be a DFA for $L$ and let $k$ be the number of states in $D$.

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We have reached a contradiction, s

have been wrong. Therefore, $L$ is n

… and all those strings need to be distinguishable so that we get a contradiction.

Imagine we have an infinite set of strings $S$ with the following property:

$$\forall x \in S.\ \forall y \in S.\ (x \neq y \to x \not\equiv_L y)$$

What happens?

For any number of states $k$, we need a way to find $k$+1 strings so that two of them get into the same state…

… of contradiction that $L$ is regular. … et $k$ be the number of states in $D$.

Consider **[ some *k*+1 specific strings. ]** This is a collection of $k+1$ strings and there are only $k$ states in $D$. Therefore, by the pigeonhole principle, there must be two distinct strings $x$ and $y$ that end in the same state when run through $D$.

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We have reached a contradiction, s… have been wrong. Therefore, $L$ is n…

… and all those strings need to be distinguishable so that we get a contradiction.

# The Myhill-Nerode Theorem

***Theorem:*** Let $L$ be a language over $\Sigma$. If there is a set $S \subseteq \Sigma^*$ with the following properties, then $L$ is not regular:

- $S$ is infinite (that is, $S$ contains infinitely many strings).
- The strings in $S$ are *pairwise distinguishable relative to $L$*. That is,

$$\forall x \in S.\ \forall y \in S.\ (x \neq y \to x \not\equiv_L y).$$

***Proof:***

***Proof:*** Let $L$ be an arbitrary language over $\Sigma$. Let $S \subseteq \Sigma^*$ be an infinite set of strings with the following property: if $x, y \in S$ and $x \neq y$, then $x \not\equiv_L y$.

***Proof:*** Let $L$ be an arbitrary language over $\Sigma$. Let $S \subseteq \Sigma^*$ be an infinite set of strings with the following property: if $x, y \in S$ and $x \neq y$, then $x \not\equiv_L y$. We will show that $L$ is not regular.

***Proof:*** Let $L$ be an arbitrary language over $\Sigma$. Let $S \subseteq \Sigma^*$ be an infinite set of strings with the following property: if $x, y \in S$ and $x \neq y$, then $x \not\equiv_L y$. We will show that $L$ is not regular.

Suppose for the sake of contradiction that $L$ is regular. This means that there must be some DFA $D$ for $L$.

***Proof:*** Let $L$ be an arbitrary language over $\Sigma$. Let $S \subseteq \Sigma^*$ be an infinite set of strings with the following property: if $x, y \in S$ and $x \neq y$, then $x \not\equiv_L y$. We will show that $L$ is not regular.

Suppose for the sake of contradiction that $L$ is regular. This means that there must be some DFA $D$ for $L$. Let $k$ be the number of states in $D$.

***Proof:*** Let $L$ be an arbitrary language over $\Sigma$. Let $S \subseteq \Sigma^*$ be an infinite set of strings with the following property: if $x, y \in S$ and $x \neq y$, then $x \not\equiv_L y$. We will show that $L$ is not regular.

Suppose for the sake of contradiction that $L$ is regular. This means that there must be some DFA $D$ for $L$. Let $k$ be the number of states in $D$. Since there are infinitely many strings in $S$, we can choose $k+1$ distinct strings from $S$ and consider what happens when we run $D$ on all of those strings.

***Proof:*** Let $L$ be an arbitrary language over $\Sigma$. Let $S \subseteq \Sigma^*$ be an infinite set of strings with the following property: if $x, y \in S$ and $x \neq y$, then $x \not\equiv_L y$. We will show that $L$ is not regular.

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***Proof:*** Let $L$ be an arbitrary language over $\Sigma$. Let $S \subseteq \Sigma^*$ be an infinite set of strings with the following property: if $x, y \in S$ and $x \neq y$, then $x \not\equiv_L y$. We will show that $L$ is not regular.

Suppose for the sake of contradiction that $L$ is regular. This means that there must be some DFA $D$ for $L$. Let $k$ be the number of states in $D$. Since there are infinitely many strings in $S$, we can choose $k+1$ distinct strings from $S$ and consider what happens when we run $D$ on all of those strings. Because there are only $k$ states in $D$ and we've chosen $k+1$ strings from $S$, by the pigeonhole principle we know that at least two strings from $S$ must end in the same state in $D$. Choose any two such strings and call them $x$ and $y$.

***Proof:*** Let $L$ be an arbitrary language over $\Sigma$. Let $S \subseteq \Sigma^*$ be an infinite set of strings with the following property: if $x, y \in S$ and $x \neq y$, then $x \not\equiv_L y$. We will show that $L$ is not regular.

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Because $x$ and $y$ are distinct strings in $S$, we know that $x \not\equiv_L y$.

***Proof:*** Let $L$ be an arbitrary language over $\Sigma$. Let $S \subseteq \Sigma^*$ be an infinite set of strings with the following property: if $x, y \in S$ and $x \neq y$, then $x \not\equiv_L y$. We will show that $L$ is not regular.

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Because $x$ and $y$ are distinct strings in $S$, we know that $x \not\equiv_L y$. Our earlier theorem therefore tells us that when we run $D$ on inputs $x$ and $y$, they must end up in different states.

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Because $x$ and $y$ are distinct strings in $S$, we know that $x \not\equiv_L y$. Our earlier theorem therefore tells us that when we run $D$ on inputs $x$ and $y$, they must end up in different states. But this is impossible – we chose $x$ and $y$ precisely because they end in the same state when run through $D$.

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Because $x$ and $y$ are distinct strings in $S$, we know that $x \not\equiv_L y$. Our earlier theorem therefore tells us that when we run $D$ on inputs $x$ and $y$, they must end up in different states. But this is impossible – we chose $x$ and $y$ precisely because they end in the same state when run through $D$.

We have reached a contradiction, so our assumption must have been wrong. Thus $L$ is not a regular language.

***Proof:*** Let $L$ be an arbitrary language over $\Sigma$. Let $S \subseteq \Sigma^*$ be an infinite set of strings with the following property: if $x, y \in S$ and $x \neq y$, then $x \not\equiv_L y$. We will show that $L$ is not regular.

Suppose for the sake of contradiction that $L$ is regular. This means that there must be some DFA $D$ for $L$. Let $k$ be the number of states in $D$. Since there are infinitely many strings in $S$, we can choose $k+1$ distinct strings from $S$ and consider what happens when we run $D$ on all of those strings. Because there are only $k$ states in $D$ and we've chosen $k+1$ strings from $S$, by the pigeonhole principle we know that at least two strings from $S$ must end in the same state in $D$. Choose any two such strings and call them $x$ and $y$.

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# Using the Myhill-Nerode Theorem

***Theorem:*** The language $E = \{ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N} \}$ is not regular.

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***Theorem:*** The language $E = \{ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N} \}$ is not regular.

***Proof:*** Let $S = \{ \mathtt{a}^n \mid n \in \mathbb{N} \}$. This set is infinite because it contains one string for each natural number. Now, consider any strings $\mathtt{a}^n, \mathtt{a}^m \in S$ where $\mathtt{a}^n \neq \mathtt{a}^m$. Then $\mathtt{a}^n\mathtt{b}^n \in E$ and $\mathtt{a}^m\mathtt{b}^n \notin E$. Consequently, $\mathtt{a}^n \not\equiv_E \mathtt{a}^m$. Therefore, by the Myhill-Nerode theorem, $L$ is not regular. ■

***Theorem:*** The language $EQ$ = { $w \stackrel{?}{=} w$ | $w \in$ {**a**, **b**}*} is not regular.

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So pick the set $S = \{\texttt{a}, \texttt{b}\}^*$.

Notice that $S$ isn't a subset of $EQ$. That's okay: we never said that $S$ needs to be a subset of $EQ$!

***Theorem:*** The language $EQ = \{ w \stackrel{?}{=} w \mid w \in \{\mathtt{a}, \mathtt{b}\}^* \}$ is not regular.

***Proof:*** Let $S = \{\mathtt{a}, \mathtt{b}\}^*$. This set contains infinitely many strings. Now, consider any $x, y \in S$ where $x \neq y$. Then $x \stackrel{?}{=} x \in EQ$ and $y \stackrel{?}{=} x \notin EQ$. Consequently, $x \not\equiv_{EQ} y$. Therefore, by the Myhill-Nerode theorem, $EQ$ is not regular. ■

# Approaching Myhill-Nerode

- The challenge in using the Myhill-Nerode theorem is finding the right set of strings to use.
- ***General intuition:***
  - Start by thinking about what information a computer "must" remember in order to answer correctly.
  - Choose a group of strings that all require different information.
  - Prove that those strings are distinguishable relative to the language in question.

# Tying Everything Together

- One of the intuitions we hope you develop for DFAs is to have each state in a DFA represent some key piece of information the automaton has to remember.
- If you only need to remember one of finitely many pieces of information, that gives you a DFA.
  - ***You can formalize this!*** Take CS154 for details.
- If you need to remember one of infinitely many pieces of information, you can use the Myhill-Nerode theorem to prove that the language has no DFA.

# Where We Stand

# Where We Stand

- We've ended up where we are now by trying to answer the question “what problems can you solve with a computer?”
- We defined a computer to be DFA, which means that the problems we can solve are precisely the regular languages.
- We've discovered several equivalent ways to think about regular languages (DFAs, NFAs, and regular expressions) and used that to reason about the regular languages.
- We now have a powerful intuition for where we ended up: DFAs are finite-memory computers, and regular languages correspond to problems solvable with finite memory.
- Putting all of this together, we have a much deeper sense for what finite memory computation looks like – *and what it doesn't look like!*

# Where We're Going

- What does computation look like with unbounded memory?
- What problems can you solve with unbounded-memory computers?
- What does it even mean to “solve” such a problem?
- And how do we know the answers to any of these questions?

# Next Time

- **Context-Free Languages**
  - Context-Free Grammars
  - Generating Languages from Scratch