

# Indirect Proofs

# Outline for Today

- ***What is an Implication?***
  - Understanding a key type of mathematical statement.
- ***Proof by Contrapositive***
  - What's a contrapositive?
  - Applications to bird storage.
- ***Proof by Contradiction***
  - The basic method.
  - Applications to geometry.
  - A famous, ancient proof.

# Fundamentals: Logical Implication

# Implications

- An ***implication*** is a statement of the form

**If  $P$  is true, then  $Q$  is true.**

- Some examples:
  - If  $n$  is an even integer, then  $n^2$  is an even integer.
  - If  $A \subseteq B$  and  $B \subseteq A$ , then  $A = B$ .
  - If you like the way you look that much, (ohhh baby) then you should go and love yourself.

# Implications

- An ***implication*** is a statement of the form

**If  $P$  is true, then  $Q$  is true.**

- In the above statement, the term “ $P$  is true” is called the ***antecedent*** and the term “ $Q$  is true” is called the ***consequent***.

# What Implications Mean

- Consider this statement:

***If I put fire near cotton, it will burn.***

- Some questions to consider:
  - Does this apply to all fire and all cotton, or just some types of fire and some types of cotton? (*Scope*)
  - Does the fire cause the cotton to burn, or does the cotton burn for another reason? (*Causality*)
- These are significantly deeper questions than they might seem.
- To mathematically study implications, we need to formalize what implications really mean.

# Understanding Implications

**“If there's a rainbow,  
then it's raining somewhere.”**

- In mathematics, implication is *directional*.
  - The above statement doesn't mean that if it's raining somewhere, there has to be a rainbow.
- In mathematics, implication only cares about cases where the antecedent is true.
  - If there's no rainbow, it doesn't mean that there's no rain.
- In mathematics, implication says nothing about *causality*.
  - Rainbows do not cause rain. ☺

# Scoping Implications

- Consider the following statements:
  - If  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .**
  - If  $n$  is even, then  $n^2$  is even.**
  - If  $A \subseteq B$  and  $B \subseteq A$ , then  $A = B$ .**
- In the above statements, what are  $A$ ,  $B$ ,  $C$ , and  $n$ ? Are they *specific* objects? Or do these claims hold for all objects?

# Implications and Universals

- In discrete math, most\* implications involving unknown quantities are, implicitly, universal statements.
- For example, the statement

**If  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$**

actually means

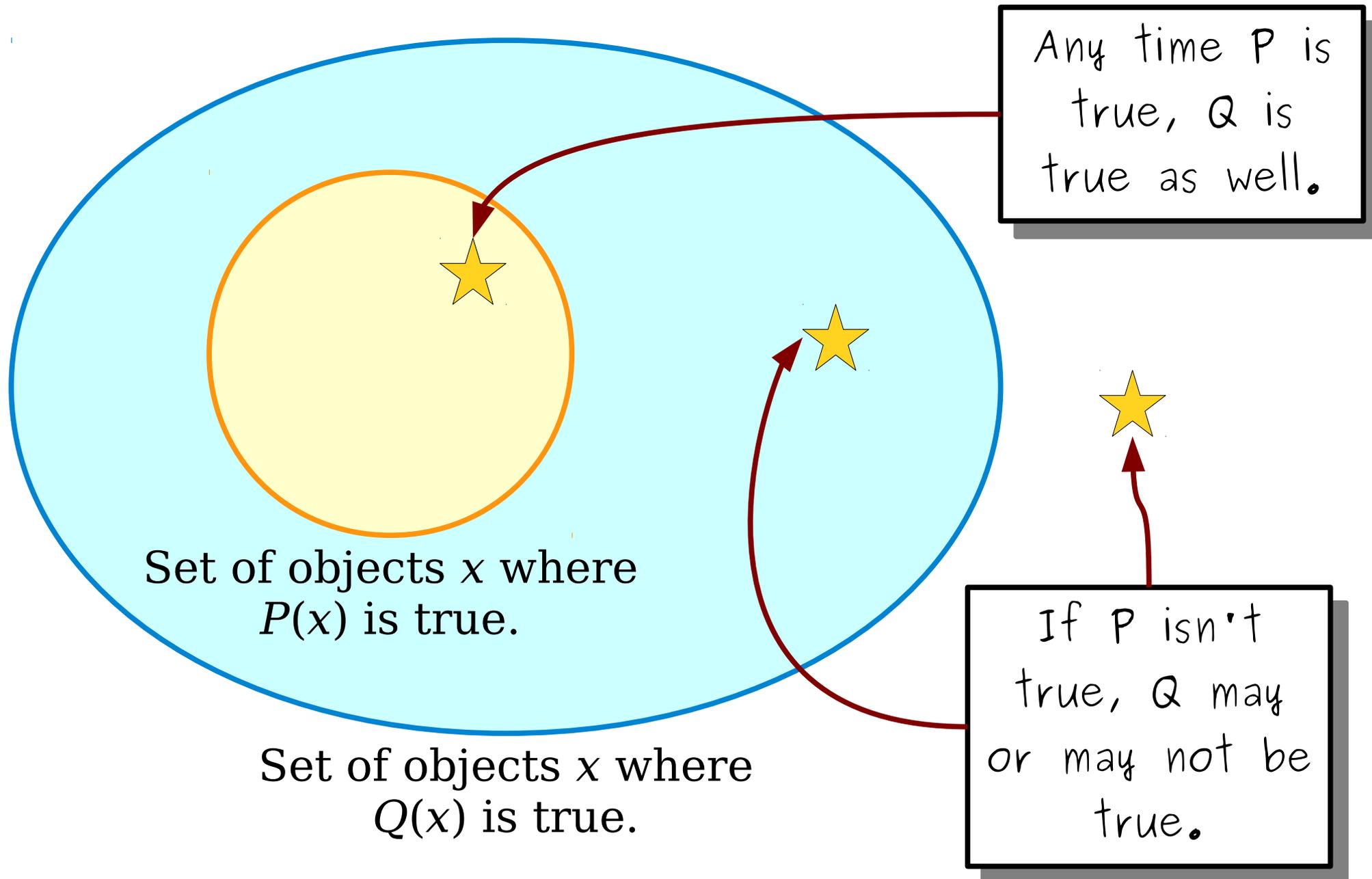
**For any sets  $A$ ,  $B$ , and  $C$ ,  
if  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .**

*\* this will become clearer on Wednesday.*

# What Implications Mean

- In mathematics, a statement of the form  
**For any  $x$ , if  $P(x)$  is true, then  $Q(x)$  is true**  
means that any time you find an object  $x$  where  $P(x)$  is true, you will find that  $Q(x)$  is also true.
- There is no discussion of correlation or causation here. It simply means that if you find that  $P(x)$  is true, you'll find that  $Q(x)$  is true.

# Implication, Diagrammatically



# Negations

- The ***negation*** of a statement  $X$  is a statement meaning the exact opposite of  $X$ .
- Examples:
  - The negation of “ $1 + 1 = 2$ ” is “ $1 + 1 \neq 2$ .”
  - The negation of “everything is awesome” is “something is not awesome.”
  - The negation of “there's something happenin' round here” is “nothing is happenin' round here.”
- Let's talk about the interplay between negations and implications.

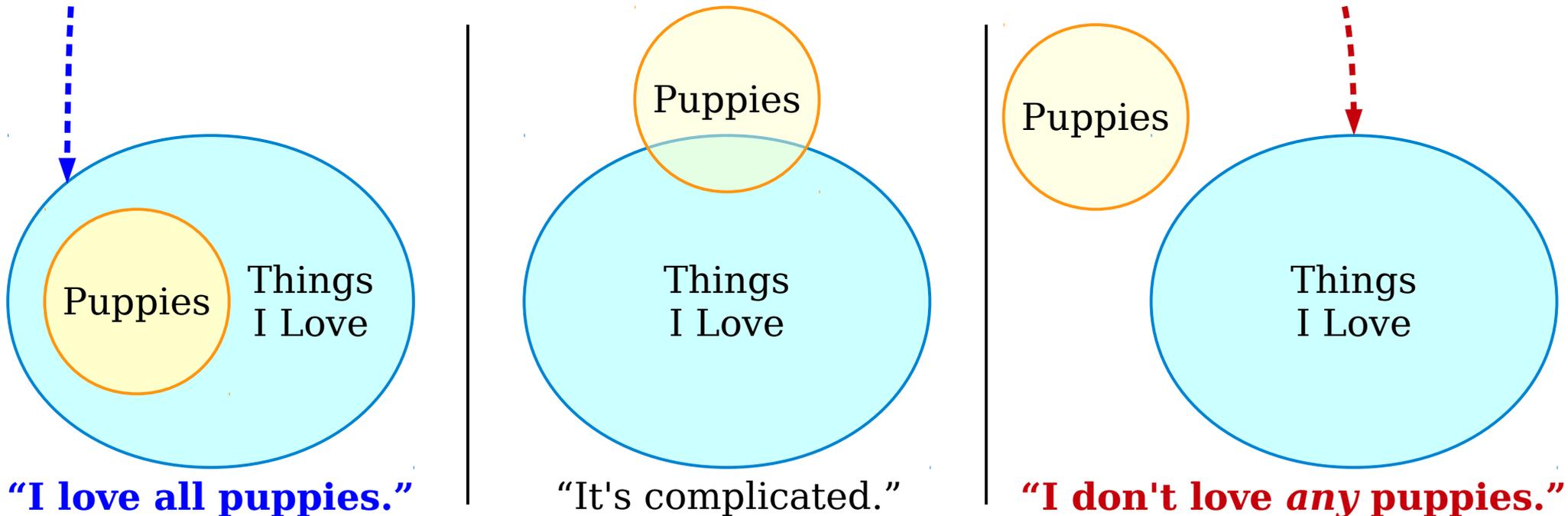
# Puppy Logic

- Consider the statement

**If  $x$  is a puppy, then I love  $x$ .**

- The following statement is **not** the negation of the original statement:

**⚠ If  $x$  is a puppy, then I don't love  $x$ . ⚠**



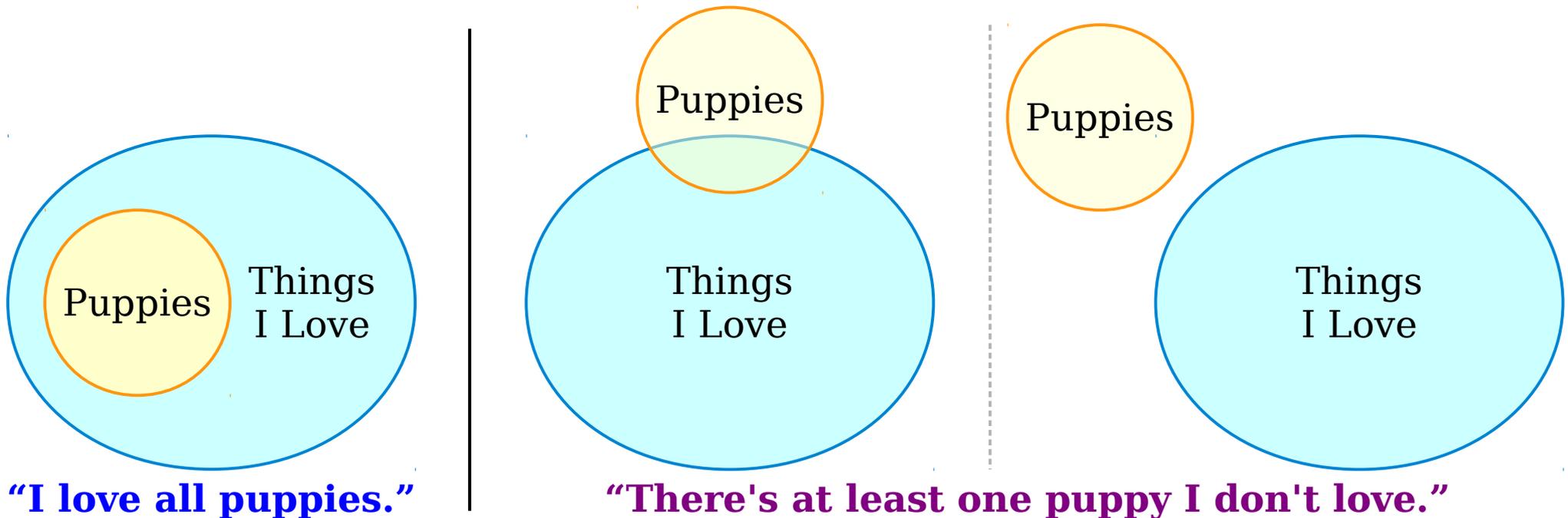
# Puppy Logic

- Consider the statement

**If  $x$  is a puppy, then I love  $x$ .**

- Here's the proper negation of our initial statement about puppies:

**There's at least one puppy I don't love.**



The negation of the statement

**“For any  $x$ , if  $P(x)$  is true,  
then  $Q(x)$  is true”**

is the statement

**“There is at least one  $x$  where  
 $P(x)$  is true and  $Q(x)$  is false.”**

***The negation of an implication  
is not an implication!***

Techniques: ***Proof by Contrapositive***

# The Contrapositive

- Given an implication like this one:

“If *P is true*, then *Q is true*”

its ***contrapositive*** is the new implication

“If *Q is false*, then *P is false*.”

- The contrapositive of an implication has the same meaning as the implication itself.
- An example:
  - “If *there's a rainbow*, then *it's raining*.”
  - Contrapositive: “If *it's not raining*, then *there is no rainbow*.”
- Another example:
  - “If *I store cat food inside*, then *raccoons will not steal my cat food*.”
  - Contrapositive: “If *raccoons stole my cat food*, then *I didn't store it inside*.”

To prove the statement

**“If  $P$  is true, then  $Q$  is true,”**

you may instead prove the statement

**“If  $Q$  is false, then  $P$  is false.”**

This is called a ***proof by contrapositive***.

**Theorem:** For any  $n \in \mathbb{Z}$ , if  $n^2$  is even, then  $n$  is even.

**Proof:** By contrapositive;

We're starting this proof by telling the reader that it's a proof by contrapositive. This helps cue the reader into what's about to come next.

**Theorem:** For any  $n \in \mathbb{Z}$ , if  $n^2$  is even, then  $n$  is even.

**Proof:** By contrapositive; we prove that if  $n$  is odd, then  $n^2$  is odd.

Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove. It also acts as a sanity check by forcing us to write out what we think the contrapositive is.

**Theorem:** For any  $n \in \mathbb{Z}$ , if  $n^2$  is even, then  $n$  is even.

**Proof:** By contrapositive; we prove that if  $n$  is odd, then  $n^2$  is odd.

We've said that we're going to prove this new implication, so let's go do it! The rest of this proof will look a lot like a standard direct proof.

**Theorem:** For any  $n \in \mathbb{Z}$ , if  $n^2$  is even, then  $n$  is even.

**Proof:** By contrapositive; we prove that if  $n$  is odd, then  $n^2$  is odd.

Let  $n$  be an arbitrary odd integer. Since  $n$  is odd, there is some integer  $k$  such that  $n = 2k + 1$ .

Squaring both sides of this equality and simplifying gives the following:

$$\begin{aligned}n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1.\end{aligned}$$

From this, we see that there is an integer  $m$  (namely,  $2k^2 + 2k$ ) such that  $n^2 = 2m + 1$ . Therefore,  $n^2$  is odd, as required. ■

**Theorem:** For any  $n \in \mathbb{Z}$ , if  $n^2$  is even, then  $n$  is even.

**Proof:** By contrapositive; we prove that if  $n$  is odd, then  $n^2$  is odd.

Let  $n$  be an arbitrary odd integer. Since  $n$  is odd, there exist integers  $k$  and  $l$  such that  $n = 2k + 1$ . Squaring both sides and simplifying, we get  $n^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$ .

The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.

2. Explicitly state the contrapositive of what we want to prove.

3. Go prove the contrapositive.

From  
(nam  
There

# Biconditionals

- Combined with what we saw on Wednesday, we have proven that, if  $n$  is an integer:

**If  $n$  is even, then  $n^2$  is even.**

**If  $n^2$  is even, then  $n$  is even.**

- Therefore, if  $n$  is an integer:

**$n$  is even if and only if  $n^2$  is even.**

- “If and only if” is often abbreviated *iff*:

**$n$  is even iff  $n^2$  is even.**

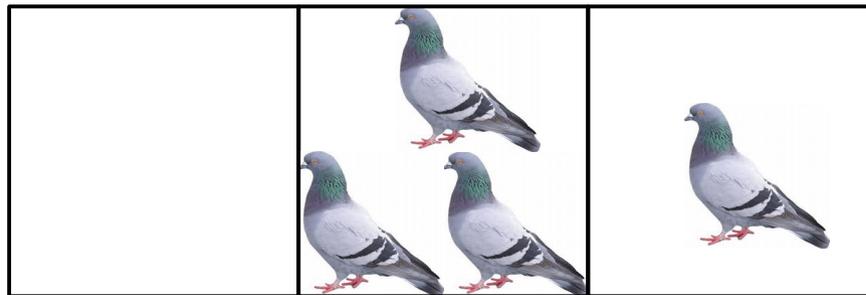
# Proving Biconditionals

- To prove  **$P$  iff  $Q$** , you need to prove that  $P$  implies  $Q$  and that  $Q$  implies  $P$ .
- You can use any proof techniques you'd like to show each of these statements.
  - In our case, we used a direct proof and a proof by contrapositive.
- Just make sure to cover both directions.

Application: ***The Pigeonhole Principle***

# The Pigeonhole Principle

- Suppose that you have  $n$  pigeonholes.
- Suppose that you have  $m > n$  pigeons.
- If you put the  $m$  pigeons into the  $n$  pigeonholes, some pigeonhole must have more than one pigeon in it.



# The Pigeonhole Principle

- Suppose that  $m$  objects are distributed into  $n$  bins.
- We want to prove the statement  
**If  $m > n$ , then some bin contains at least two objects.**
- What is the contrapositive of this statement?  
**If “some bin contains at least two objects” is false,  
then “ $m > n$ ” is false.**

# The Pigeonhole Principle

Suppose that  $m$  objects are distributed into  $n$  bins.

We want to prove the statement

**If  $m > n$ , then some bin contains at least two objects.**

What is the contrapositive of this statement?

**If “some bin contains at least two objects” is false,  
then  $m \leq n$ .**

# The Pigeonhole Principle

Suppose that  $m$  objects are distributed into  $n$  bins.

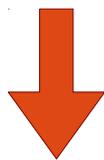
We want to prove the statement

**If  $m > n$ , then some bin contains at least two objects.**

What is the contrapositive of this statement?

**If “some bin contains at least two objects” is false,  
then  $m \leq n$ .**

There is a bin that doesn't have 0 or 1 objects in it



Every bin has at most one object in it.

# The Pigeonhole Principle

- Suppose that  $m$  objects are distributed into  $n$  bins.
- We want to prove the statement  
**If  $m > n$ , then some bin contains at least two objects.**
- What is the contrapositive of this statement?  
**If every bin contains at most one object, then  $m \leq n$ .**
- Look at the definitions of  $m$  and  $n$ . Does this make sense?

**Theorem:** Let  $m$  objects be distributed into  $n$  bins. If  $m > n$ , then some bin contains at least two objects.

**Proof:** By contrapositive; we prove that if every bin contains at most one object, then  $m \leq n$ .

Let  $x_i$  denote the number of objects in bin  $i$ . Since  $m$  is the number of total objects, we see that

$$m = x_1 + x_2 + \dots + x_n.$$

We're assuming every bin has at most one object. In our notation, this means that  $x_i \leq 1$  for all  $i$ . Using this inequality, we get the following:

$$\begin{aligned} m &= x_1 + x_2 + \dots + x_n \\ &\leq 1 + 1 + \dots + 1 \quad (n \text{ times}) \\ &= n. \end{aligned}$$

So  $m \leq n$ , as required. ■

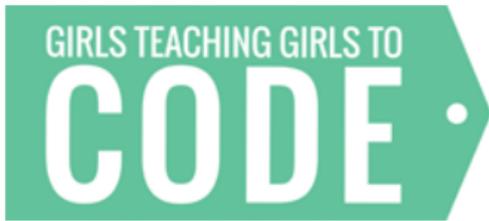
# Some Simple Applications

- Any group of 367 people must have a pair of people that share a birthday.
  - 366 possible birthdays (pigeonholes)
  - 367 people (pigeons)
- Two people in San Francisco have the exact same number of hairs on their head.
  - Maximum number of hairs ever found on a human head is no greater than 500,000.
  - There are over 800,000 people in San Francisco.
- Each day, two people in New York City drink the same amount of water, to the thousandth of a fluid ounce.
  - No one can drink more than 50 gallons of water each day.
  - That's 6,400 fluid ounces. This gives 6,400,001 possible numbers of thousands of fluid ounces.
  - There are about 8,000,000 people in New York City proper.

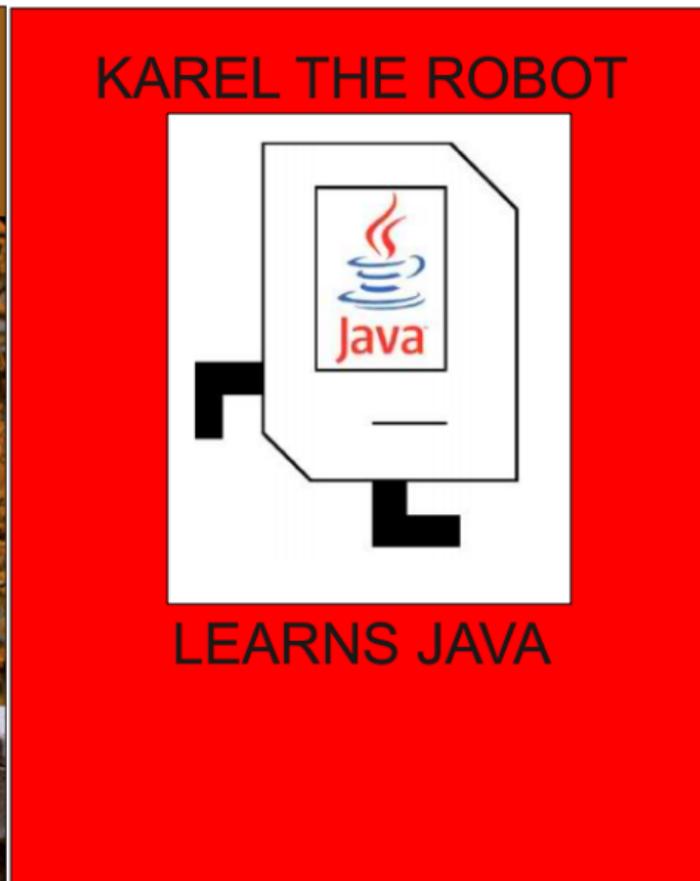
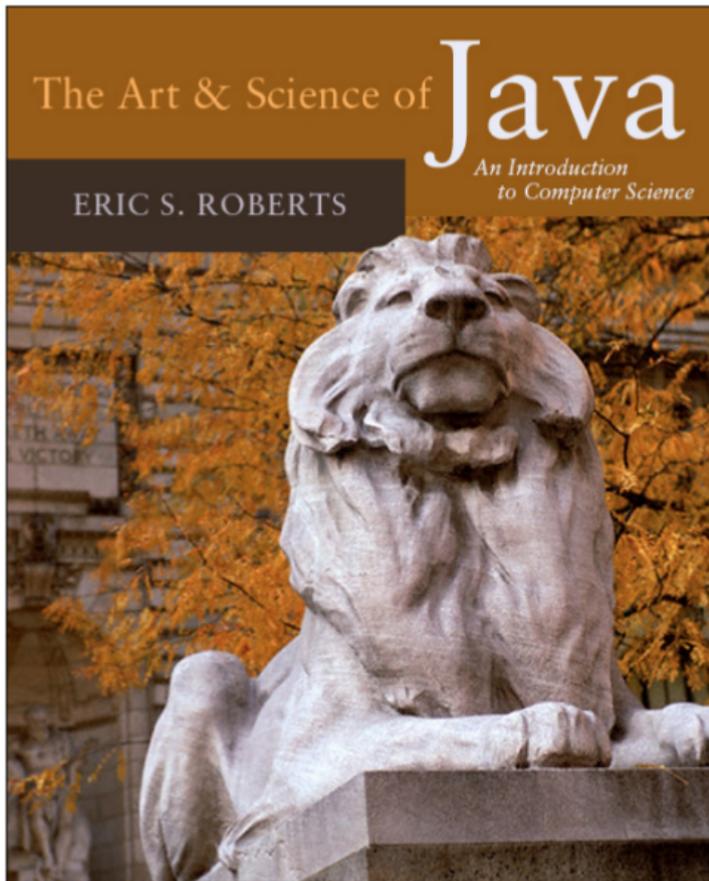
Time-Out for Announcements!

# Stanford oSTEM

- Stanford's chapter of oSTEM (Out in STEM) is holding two mixer events next week:
  - Undergrads: Wednesday, October 5th, 6PM at the LGBT-CRC (2nd floor of the Firetruck house.)
  - Grads/Post-docs: Thursday, October 6th, 6PM at the LGBT-CRC (2nd floor of the Firetruck house.)
- Highly recommended!



# CS 106A BOOKS FOR EAST PALO ALTO SCHOOLS



CONTACT: [cho19@stanford.edu](mailto:cho19@stanford.edu)



**GET READY TO BUILD A  
WEEKEND NO ONE WILL FORGET**

**OCTOBER 4<sup>th</sup>**  
6:00 PM - 7:00 PM

**INFO SESSION**  
all grades welcome

**LATHROP**  
298

# **JOIN THE TREEHACKS ORGANIZING TEAM**

**STANFORD'S  
NATIONAL HACKATHON**

**750 HACKERS**  
February 17-19, 2017

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APPLICATIONS @ <http://tinyurl.com/treehacks> DUE ON OCTOBER 07 AT MIDNIGHT

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# Handouts

- There are *six* (!) total handouts for today:
  - Handout 05: Problem Set Policies
  - Handout 06: Honor Code Policies
  - Handout 07: Guide to Proofs
  - Handout 08: Mathematical Vocabulary
  - Handout 09: Guide to Indirect Proofs
  - Handout 10: Problem Set One
- Be sure to read over Handouts 05 – 09; there's a lot of really important information in there!

# Announcements

- Problem Set 1 goes out today!
- **Checkpoint** due Monday, October 3 at the start of class.
  - Grade determined by *attempt* rather than *accuracy*. It's okay to make mistakes – we want you to give it your best effort, even if you're not completely sure what you have is correct.
  - We will get feedback back to you with comments on your proof technique and style.
  - The more effort you put in, the more you'll get out.
- **Remaining problems** due Friday, October 7.
  - Feel free to email us with questions, stop by office hours, or ask questions on Piazza!

# Submitting Assignments

- This quarter, we will be using GradeScope to handle assignment submissions. Visit [www.gradescope.com](http://www.gradescope.com) and enter code **9GK649**.
- Summary of the late policy:
  - Everyone has *three* 24-hour late days.
  - Late days can't be used on checkpoints.
  - Nothing may be submitted more than three days past the due date.
- Because submission times are recorded automatically, we're strict about the submission deadlines.
- **Very good idea:** Leave at least two hours buffer time for your first assignment submission, just in case something goes wrong.
- **Very bad idea:** Wait until the last minute to submit.

# Working in Groups

- You can work on the problem sets individually, in a pair, or in a group of three.
- Each group should only submit a single problem set, and should submit it only once.
- Full details about the problem sets, collaboration policy, and Honor Code can be found in Handouts 05 and 06.
- Our recommendations:
  - **Work in a pair or a group.** It is *extremely helpful* to have people to talk to about the problem set questions!
  - **Do all the problems individually.** You will only learn part of the material if you only do part of the problem set.
  - **Come together to write a final version.** The act of talking over your approaches and writing a single, final version will help solidify your understanding (and prevent mistakes!)

# A Note on the Honor Code

Office hours start tonight!

Schedule is available  
on the course website.

Back to CS103!

Techniques: ***Proof by Contradiction***

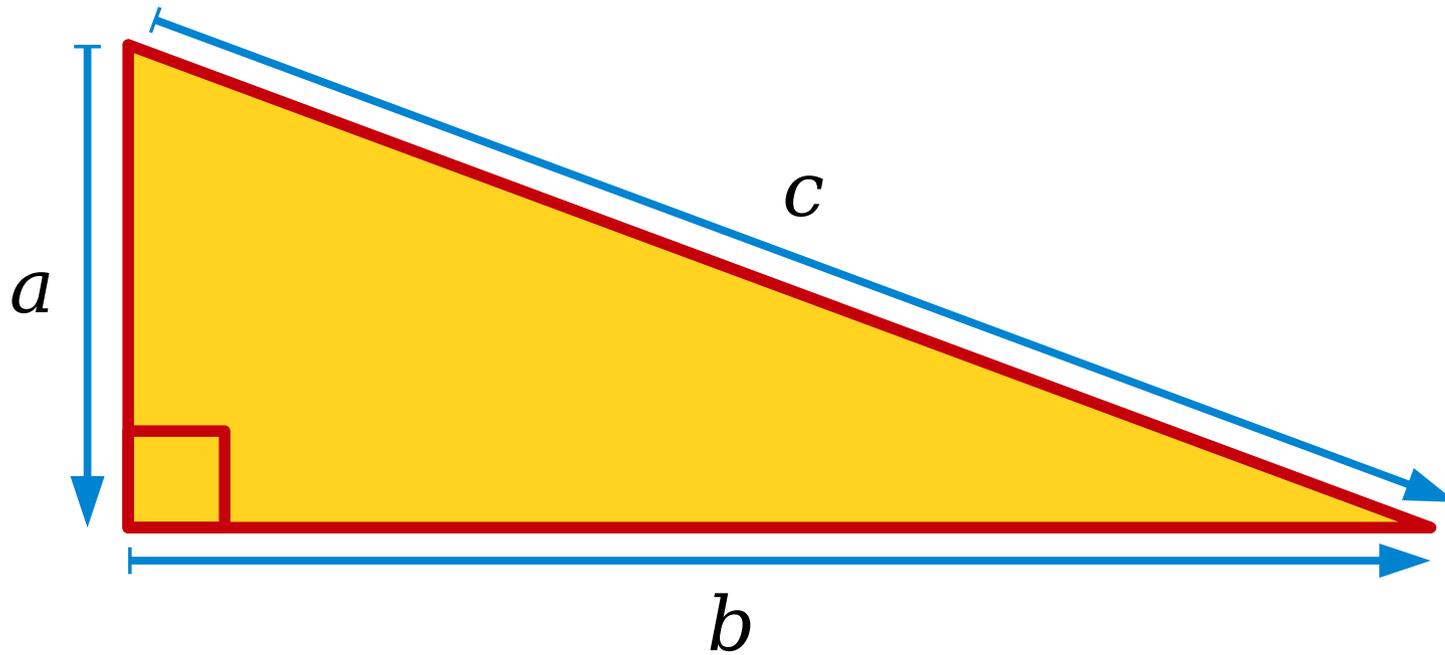
“When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth.”

- Sir Arthur Conan Doyle, *The Adventure of the Blanched Soldier*

# Proof by Contradiction

- A ***proof by contradiction*** is a proof that works as follows:
  - To prove that  $P$  is true, start by assuming that  $P$  is *not* true.
  - Beginning with this assumption, use logical reasoning to conclude something that is clearly impossible.
    - For example, that  $1 = 0$ , that a number is both even and odd, that both  $x \in S$  and  $x \notin S$ , etc.
  - This means that if  $P$  is false, something that cannot possibly happen must happen.
  - Therefore,  $P$  can't be false, so it must be true.

# Right Triangles



$$a^2 + b^2 = c^2$$

**Claim:**  $a + b \geq c$

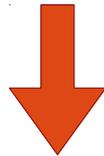
**Theorem:** For all natural numbers  $a$ ,  $b$ , and  $c$  where  $a^2 + b^2 = c^2$ , we have  $a + b \geq c$ .

**Proof:** Assume for the sake of contradiction that there are natural numbers  $a$ ,  $b$ , and  $c$  where  $a^2 + b^2 = c^2$ , but  $a + b < c$ .

To prove this by contradiction, we're going to assume the negation of the above statement.

What is its negation?

“For all natural numbers  $a$ ,  $b$ , and  $c$  where  $a^2 + b^2 = c^2$ , we have  $a + b \geq c$ ”



“There exist natural numbers  $a$ ,  $b$  and  $c$  where  $a^2 + b^2 = c^2$ , but  $a + b < c$ ”

**Theorem:** For all natural numbers  $a$ ,  $b$ , and  $c$  where  $a^2 + b^2 = c^2$ , we have  $a + b \geq c$ .

**Proof:** Assume for the sake of contradiction that there are natural numbers  $a$ ,  $b$ , and  $c$  where  $a^2 + b^2 = c^2$ , but  $a + b < c$ .

Notice that we're announcing

1. that this is a proof by contradiction, and
2. what, specifically, we're assuming.

This helps the reader understand where we're going. Remember – proofs are meant to be read by other people!

**Theorem:** For all natural numbers  $a$ ,  $b$ , and  $c$  where  $a^2 + b^2 = c^2$ , we have  $a + b \geq c$ .

**Proof:** Assume for the sake of contradiction that there are natural numbers  $a$ ,  $b$ , and  $c$  where  $a^2 + b^2 = c^2$ , but  $a + b < c$ .

Since both sides of the preceding inequality are nonnegative, we can square both sides to see that

$$(a + b)^2 < c^2. \quad (1)$$

We can expand **the left-hand side of inequality (1)** to see that

$$a^2 + 2ab + b^2 < c^2. \quad (2)$$

We're numbering our intermediate stages to make it easier to refer to them later. If you have a calculation-heavy proof, we recommend structuring it like this.

**Theorem:** For all natural numbers  $a$ ,  $b$ , and  $c$  where  $a^2 + b^2 = c^2$ , we have  $a + b \geq c$ .

**Proof:** Assume for the sake of contradiction that there are natural numbers  $a$ ,  $b$ , and  $c$  where  $a^2 + b^2 = c^2$ , but  $a + b < c$ .

Since both sides of the preceding inequality are nonnegative, we can square both sides to see that

$$(a + b)^2 < c^2. \quad (1)$$

We can expand the left-hand side of inequality (1) to see that

$$a^2 + 2ab + b^2 < c^2. \quad (2)$$

Since  $a^2 + b^2 = c^2$ , we can rewrite the left-hand side of inequality (2) as

$$c^2 + 2ab < c^2, \quad (3)$$

and subtracting  $c^2$  from both sides of inequality (3) yields

$$2ab < 0. \quad (4)$$

But this is impossible: we know that  $2ab \geq 0$ , since  $2ab$  is a natural number and all natural numbers are nonnegative.

We have reached a contradiction, so our assumption must have been wrong. Therefore,  $a + b \geq c$ , as required. ■

**Theorem:** For all natural numbers  $a$ ,  $b$ , and  $c$  where  $a^2 + b^2 = c^2$ , we have  $a + b \geq c$ .

**Proof:** Assume for the sake of contradiction that there are natural numbers  $a$ ,  $b$ , and  $c$  where  $a^2 + b^2 = c^2$ , but  $a + b < c$ .

Since both sides of the preceding inequality are nonnegative, we can square both sides to see that

The three key pieces:

1. State that the proof is by contradiction.
2. State what you are assuming is the negation of the statement to prove.
3. State you have reached a contradiction and what the contradiction entails.

In CS103, please include all these steps in your proofs!

But this is impossible: we know that  $2ab \geq 0$ , since  $2ab$  is a natural number and all natural numbers are nonnegative.

We have reached a contradiction, so our assumption must have been wrong. Therefore,  $a + b \geq c$ , as required. ■

# Proving Implications

- To prove the implication

**“If  $P$  is true, then  $Q$  is true.”**

you can use these three techniques:

- ***direct proof***
  - (assume  $P$  is true and prove  $Q$  is true),
- ***proof by contrapositive***
  - (assume  $Q$  is false and prove  $P$  is false), and
- ***proof by contradiction***
  - (what does that look like?)

# Negating Implications

- To prove the statement

**“For any  $x$ , if  $P(x)$ , then  $Q(x)$ ”**

by contradiction, we do the following:

- Assume this statement is false.
  - Derive a contradiction.
  - Conclude that the statement is true.
- What is the negation of the above statement?

**“There is an  $x$  where  
 $P(x)$  is true and  $Q(x)$  is false”**

# Contradictions and Implications

- To prove the statement

**“If  $P$  is true, then  $Q$  is true”**

using a proof by contradiction, do the following:

- Assume that  $P$  is true and that  $Q$  is false.
- Derive a contradiction.
- Conclude that if  $P$  is true,  $Q$  must be as well.

**Theorem:** If  $n$  is an integer and  $n^2$  is even, then  $n$  is even.

**Proof:** Assume for the sake of contradiction that  $n$  is an integer and that  $n^2$  is even, but that  $n$  is odd.

Since  $n$  is odd we know that there is an integer  $k$  such that

$$n = 2k + 1. \quad (1)$$

Squaring both sides of equation (1) and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned} \quad (2)$$

Equation (2) tells us that  $n^2$  is odd, which is impossible; by assumption,  $n^2$  is even.

We have reached a contradiction, so our assumption must have been incorrect. Thus if  $n$  is an integer and  $n^2$  is even,  $n$  is even as well. ■

Applications: ***Rational and Irrational Numbers***

# Rational and Irrational Numbers

- A number  $r$  is called a **rational number** if it can be written as

$$r = \frac{p}{q}$$

where  $p$  and  $q$  are integers and  $q \neq 0$ .

- A number that is not rational is called **irrational**.

# Simplest Forms

- *By definition*, if  $r$  is a rational number, then  $r$  can be written as  $p / q$  where  $p$  and  $q$  are integers and  $q \neq 0$ .
- ***Theorem:*** If  $r$  is a rational number, then  $r$  can be written as  $p / q$  where  $p$  and  $q$  are integers,  $q \neq 0$ , and  $p$  and  $q$  have no common factors other than 1 and -1.
  - That is,  $r$  can be written as a fraction in simplest form.
- We're just going to take this for granted for now, though with the techniques you'll see later in the quarter you'll be able to prove it!

***Question:*** Are all real numbers rational?

**Theorem:**  $\sqrt{2}$  is irrational.

**Proof:** Assume for the sake of contradiction that  $\sqrt{2}$  is rational. This means that there must be integers  $p$  and  $q$  where  $q \neq 0$ , where  $p$  and  $q$  have no common divisors other than 1 and -1, and where

$$p / q = \sqrt{2}. \quad (1)$$

Multiplying both sides of equation (1) by  $q$  and squaring both sides shows us that

$$p^2 = 2q^2. \quad (2)$$

From equation (2), we see that  $p^2$  is even. Earlier, we proved that if  $p^2$  is even, then  $p$  must also be even. Therefore, we know that there is some integer  $k$  such that  $p = 2k$ . Substituting this into equation (2) and simplifying gives us the following:

$$\begin{aligned} p^2 &= 2q^2 \\ (2k)^2 &= 2q^2 \\ 4k^2 &= 2q^2 \\ 2k^2 &= q^2 \end{aligned} \quad (3)$$

Equation (3) shows that  $q^2$  is even. Our earlier theorem tells us that, because  $q^2$  is even,  $q$  must also be even. But this is not possible - we know that  $p$  and  $q$  have no common factors other than 1 and -1, but we've shown that  $p$  and  $q$  must have two as a common factor.

We have reached a contradiction, so our original assumption must have been wrong. Therefore,  $\sqrt{2}$  is irrational. ■

**Theorem:**  $\sqrt{2}$  is irrational.

**Proof:** Assume for the sake of contradiction that  $\sqrt{2}$  is rational. This means that there must be integers  $p$  and  $q$  where  $q \neq 0$ , where  $p$  and  $q$  have no common divisors other than 1 and -1, and where

$$p / q = \sqrt{2}. \quad (1)$$

Multiplying both sides of equation (1) by  $q$  and squaring both sides

The three key pieces:

1. State that the proof is by contradiction.
2. State what you are assuming is the negation of the statement to prove.
3. State you have reached a contradiction and what the contradiction entails.

In CS103, please include all these steps in your proofs!

Equation (3) shows that  $q^2$  is even. Our earlier theorem tells us that, because  $q^2$  is even,  $q$  must also be even. But this is not possible - we know that  $p$  and  $q$  have no common factors other than 1 and -1, but we've shown that  $p$  and  $q$  must have two as a common factor.

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Vi Hart on Pythagoras and  
the Square Root of Two:

**[http://www.youtube.com/watch?v=X1E7I7\\_r3Cw](http://www.youtube.com/watch?v=X1E7I7_r3Cw)**

# What We Learned

- ***What's an implication?***
  - It's a statement of the form “if  $P$ , then  $Q$ ,” and states that if  $P$  is true, then  $Q$  is true.
- ***What is a proof by contrapositive?***
  - It's a proof of an implication that instead proves its contrapositive.
  - (The contrapositive of “if  $P$ , then  $Q$ ” is “if not  $Q$ , then not  $P$ .”)
- ***What's a proof by contradiction?***
  - It's a proof of a statement  $P$  that works by showing that  $P$  cannot be false.

# Next Time

- ***Mathematical Logic***
  - How do we formalize the reasoning from our proofs?
- ***Propositional Logic***
  - Reasoning about simple statements.
- ***Propositional Equivalences***
  - Simplifying complex statements.

# Appendix: Negating Statements

## Negating Universal Statements

**“For all  $x$ ,  $P(x)$  is true”**

becomes

**“There is an  $x$  where  $P(x)$  is false.”**

## Negating Existential Statements

**“There exists an  $x$  where  $P(x)$  is true”**

becomes

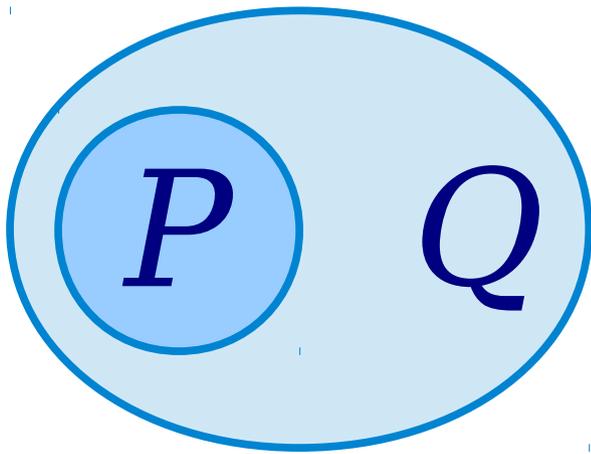
**“For all  $x$ ,  $P(x)$  is false.”**

## Negating Implications

**“For every  $x$ , if  $P(x)$  is true, then  $Q(x)$  is true”**

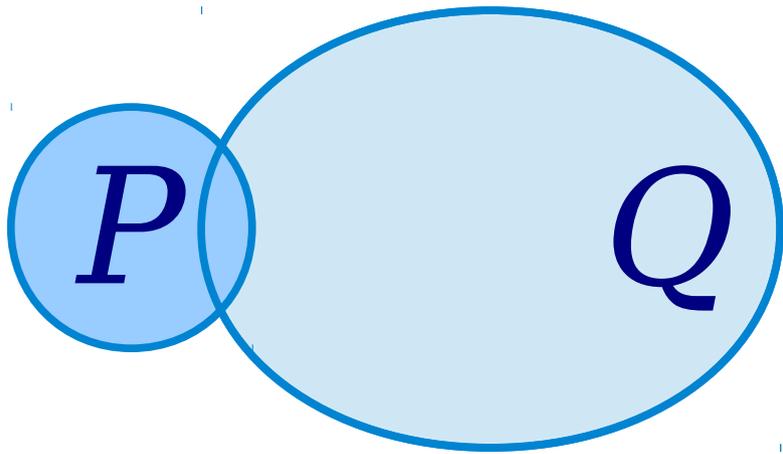
becomes

**“There is an  $x$  where  $P(x)$  is true and  $Q(x)$  is false”**



**$P(x)$  implies  $Q(x)$**

“If  $P(x)$  is true, then  $Q(x)$  is true.”



**$P(x)$  does not imply  $Q(x)$**

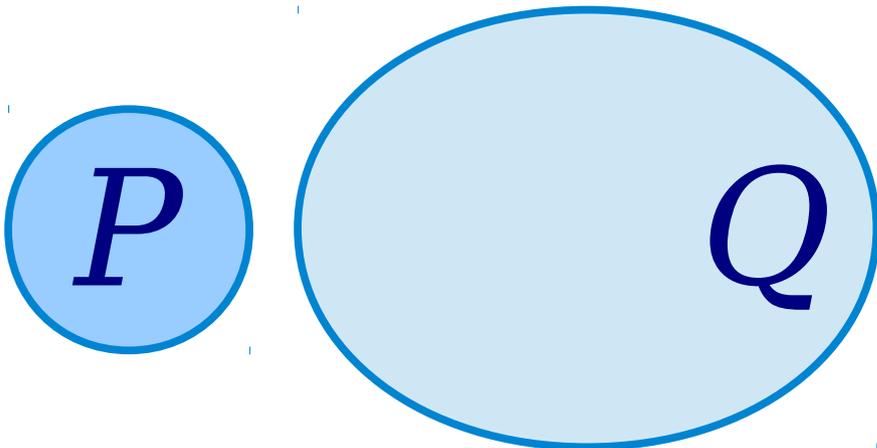
-and-

**$P(x)$  does not imply not  $Q(x)$**

“Sometimes  $P(x)$  is true and  $Q(x)$  is true,

-and-

sometimes  $P(x)$  is true and  $Q(x)$  is false.”



**$P(x)$  implies not  $Q(x)$**

“If  $P(x)$  is true, then  $Q(x)$  is false.”