Complexity Theory

Part One

Complexity Theory

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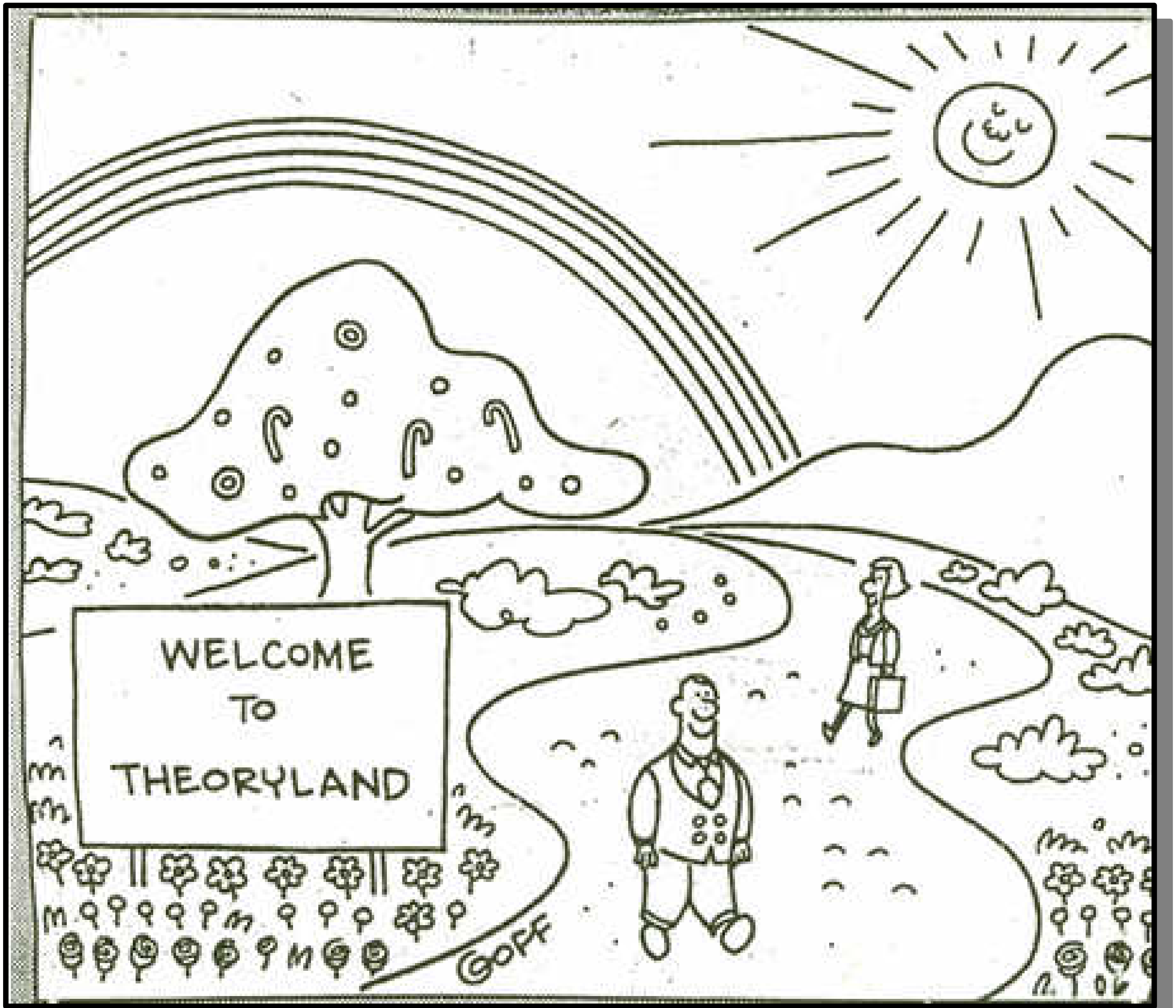
- Jack Edmonds, "Paths, Trees, and Flowers"


WELCOME
TO
THEORYLAND
GOFF

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It may be that since one is customarily concerned with existence, […] decidability, and so forth, one is not inclined to take seriously the question of the existence of a *better-than-decidable* algorithm.

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A Decidable Problem

- ***Presburger arithmetic*** is a logical system for reasoning about arithmetic.
    - $\forall x.\ x + 1 \neq 0$
    - $\forall x.\ \forall y.\ (x + 1 = y + 1 \rightarrow x = y)$
    - $\forall x.\ x + 0 = x$
    - $\forall x.\ \forall y.\ (x + y) + 1 = x + (y + 1)$
    - $(P(0) \wedge \forall y.\ (P(y) \rightarrow P(y + 1)) \rightarrow \forall x.\ P(x)$
- Given a statement, it is decidable whether that statement can be proven from the laws of Presburger arithmetic.
- Any Turing machine that decides whether a statement in Presburger arithmetic is true or false has to move the tape head at least $2^{2^{cn}}$ times on some inputs of length n (for some fixed constant c).

For Reference

- Assume c = 1.

$$2^{2^0} = 2$$

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$$2^{2^6}=340282366920938463463374607431768211456$$

The Limits of Decidability

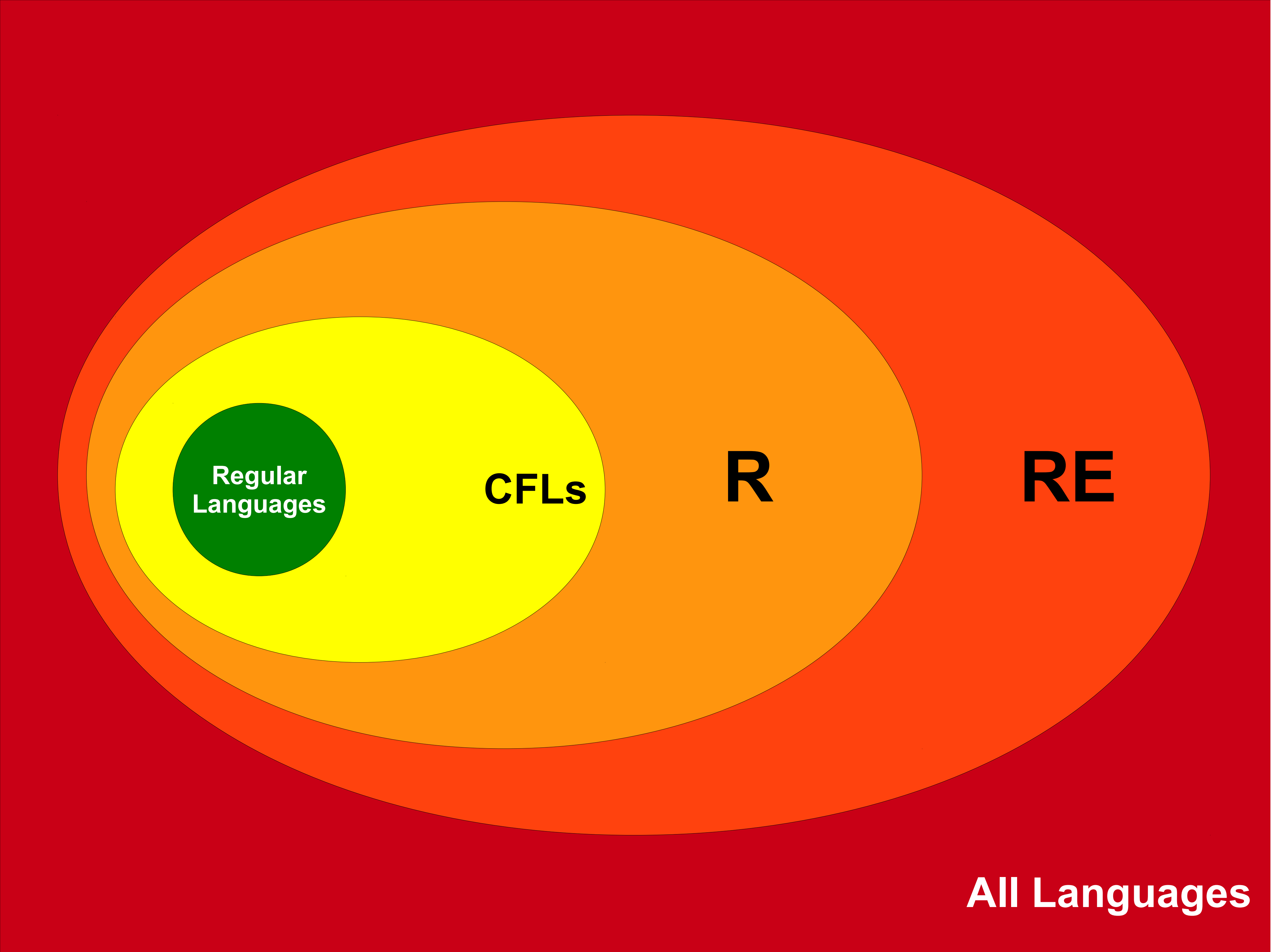
- The fact that a problem is decidable does not mean that it is *feasibly* decidable.
- In ***computability theory***, we ask the question

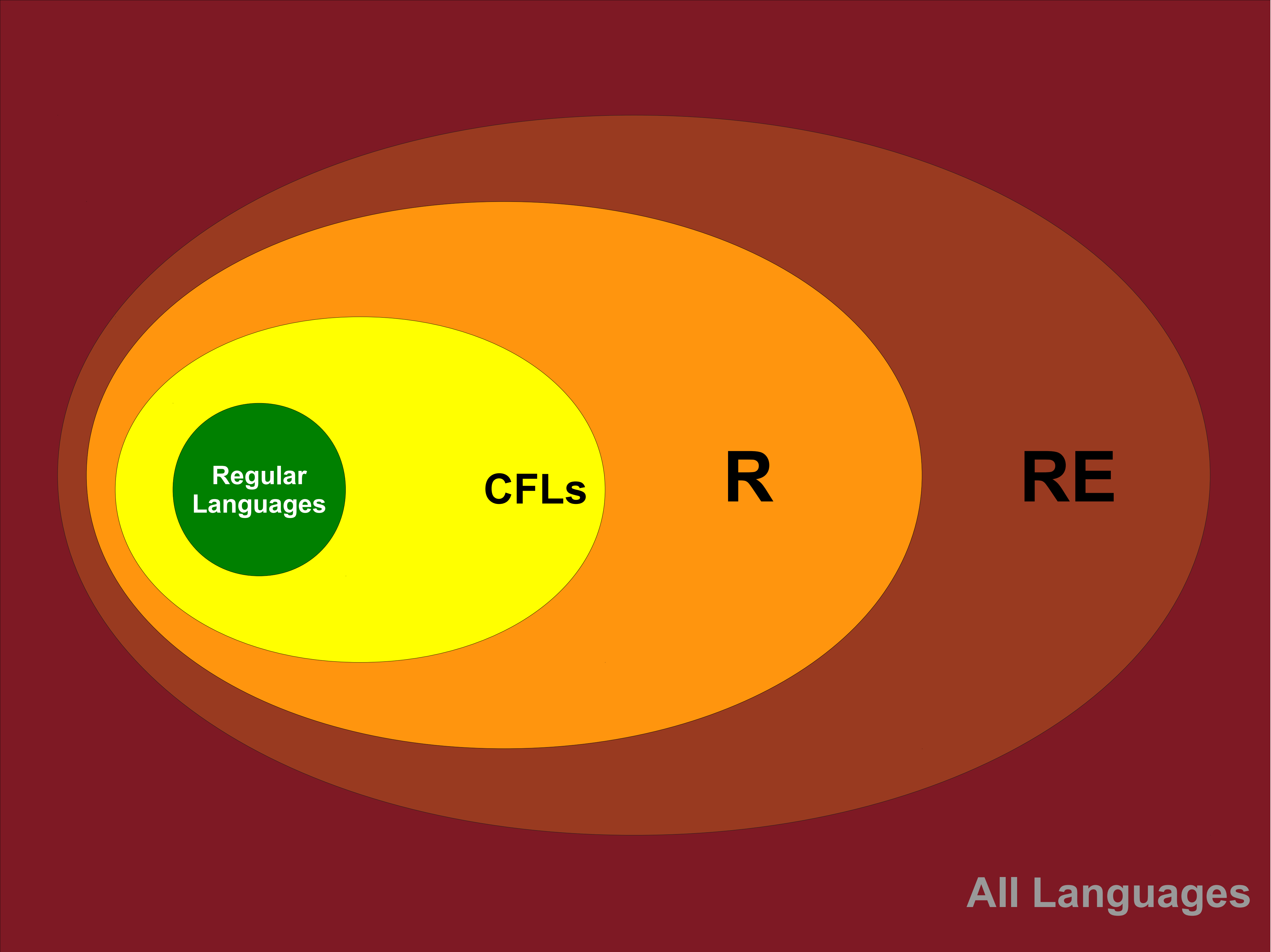
  What problems can be solved by a computer?

- In ***complexity theory***, we ask the question

  What problems can be solved ***efficiently*** by a computer?

- In the remainder of this course, we will explore this question in more detail.


Regular Languages
CFLs
R
RE
All Languages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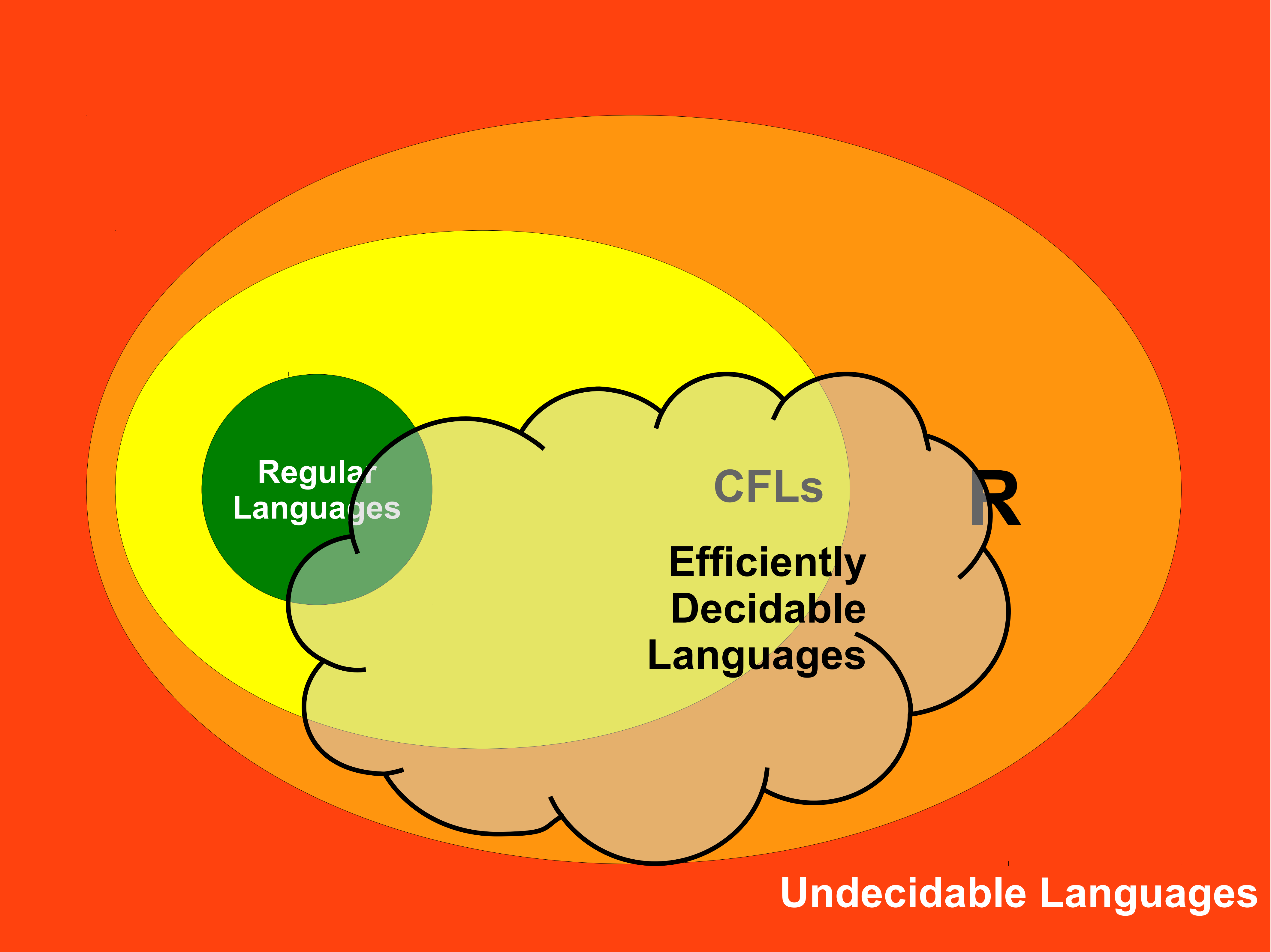
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The Setup

- In order to study computability, we needed to answer these questions:
  - What is “computation?”
  - What is a “problem?”
  - What does it mean to “solve” a problem?
- To study complexity, we need to answer these questions:
  - What does “complexity” even mean?
  - What is an “efficient” solution to a problem?

Measuring Complexity

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  - Number of states.
  - Size of tape alphabet.
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  - Amount of time required.
  - Number of times a given state is entered.
  - Number of times a given symbol is printed.
  - Number of times a given transition is taken.
  - (Plus a whole lot more...)

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What is an efficient algorithm?

Searching Finite Spaces

- Many decidable problems can be solved by searching over a large but finite space of possible options.
- Searching this space might take a staggeringly long time, but only finite time.
- From a decidability perspective, this is totally fine.
- From a complexity perspective, this is totally unacceptable.

A Sample Problem

4	3	11	9	7	13	5	6	1	12	2	8	0	10

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Goal: Find the length of the longest increasing subsequence of this sequence.

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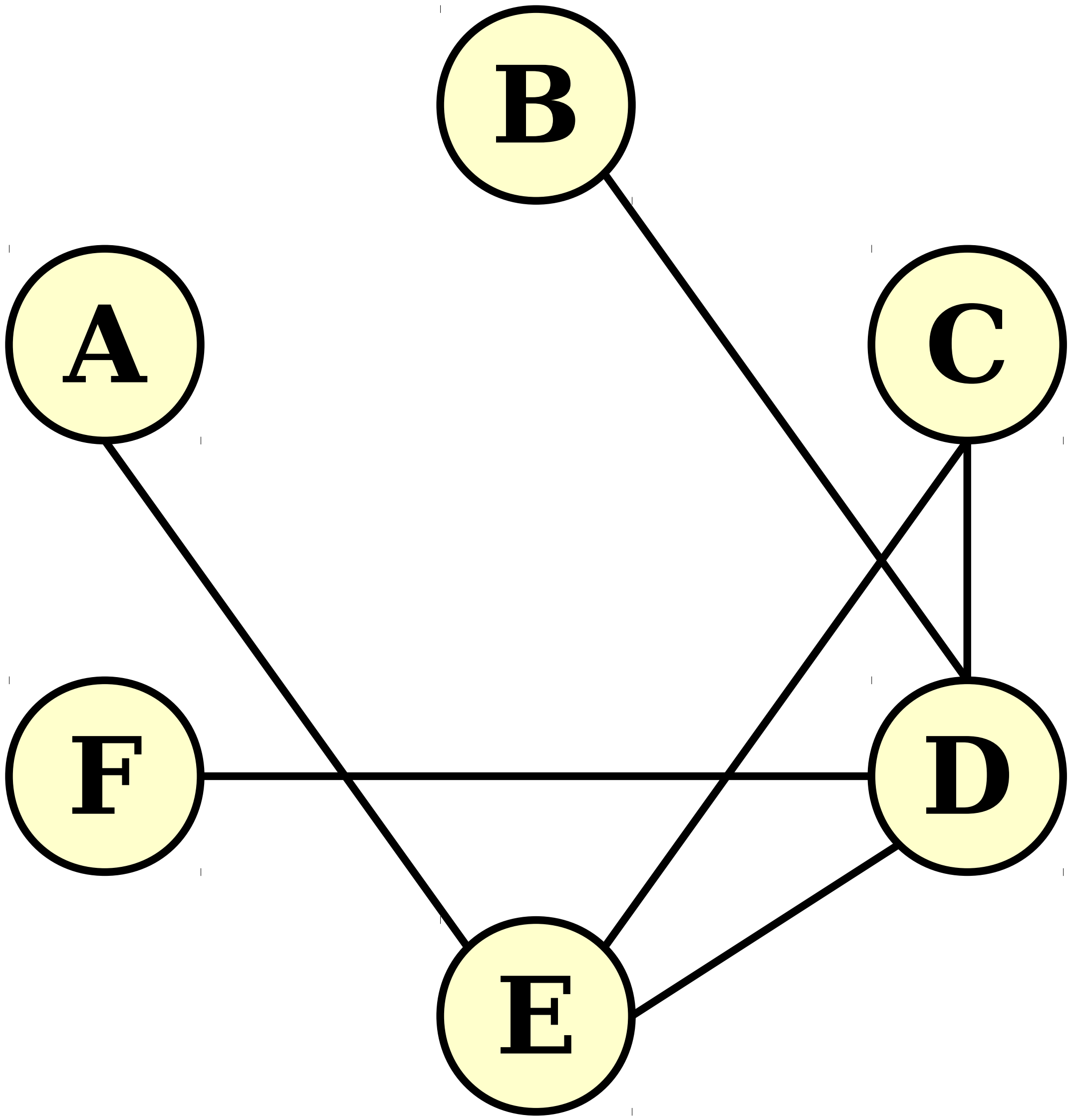
Longest Increasing Subsequences

- ***One possible algorithm:*** try all subsequences, find the longest one that's increasing, and return that.
- There are 2^n subsequences of an array of length n.
  - (Each subset of the elements gives back a subsequence.)
- Checking all of them to find the longest increasing subsequence will take time $O(n \cdot 2^n)$.
- Nifty fact: the age of the universe is about 4.3×10^{26} nanoseconds old. That's about 2^{85} nanoseconds.
- Practically speaking, this algorithm doesn't terminate if you give it an input of size 100 or more.

Longest Increasing Subsequences

- ***Theorem:*** There is an algorithm that can find the longest increasing subsequence of an array in time O(*n* log *n*).
- The algorithm is *beautiful* and surprisingly elegant. Look up ***patience sorting*** if you're curious.
- This algorithm works by exploiting particular aspects of how longest increasing subsequences are constructed. It's not immediately obvious that it works correctly.

Another Problem



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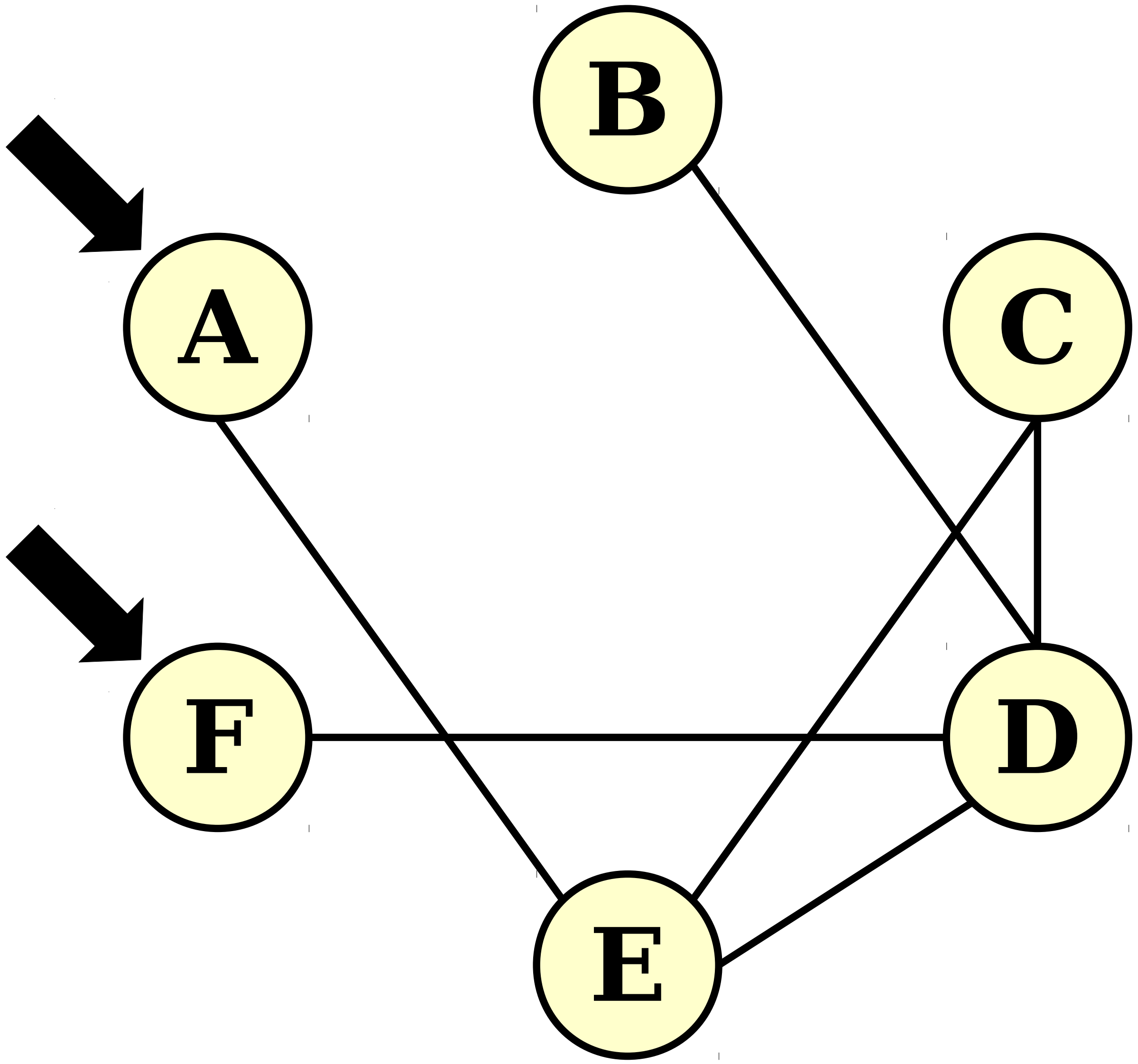

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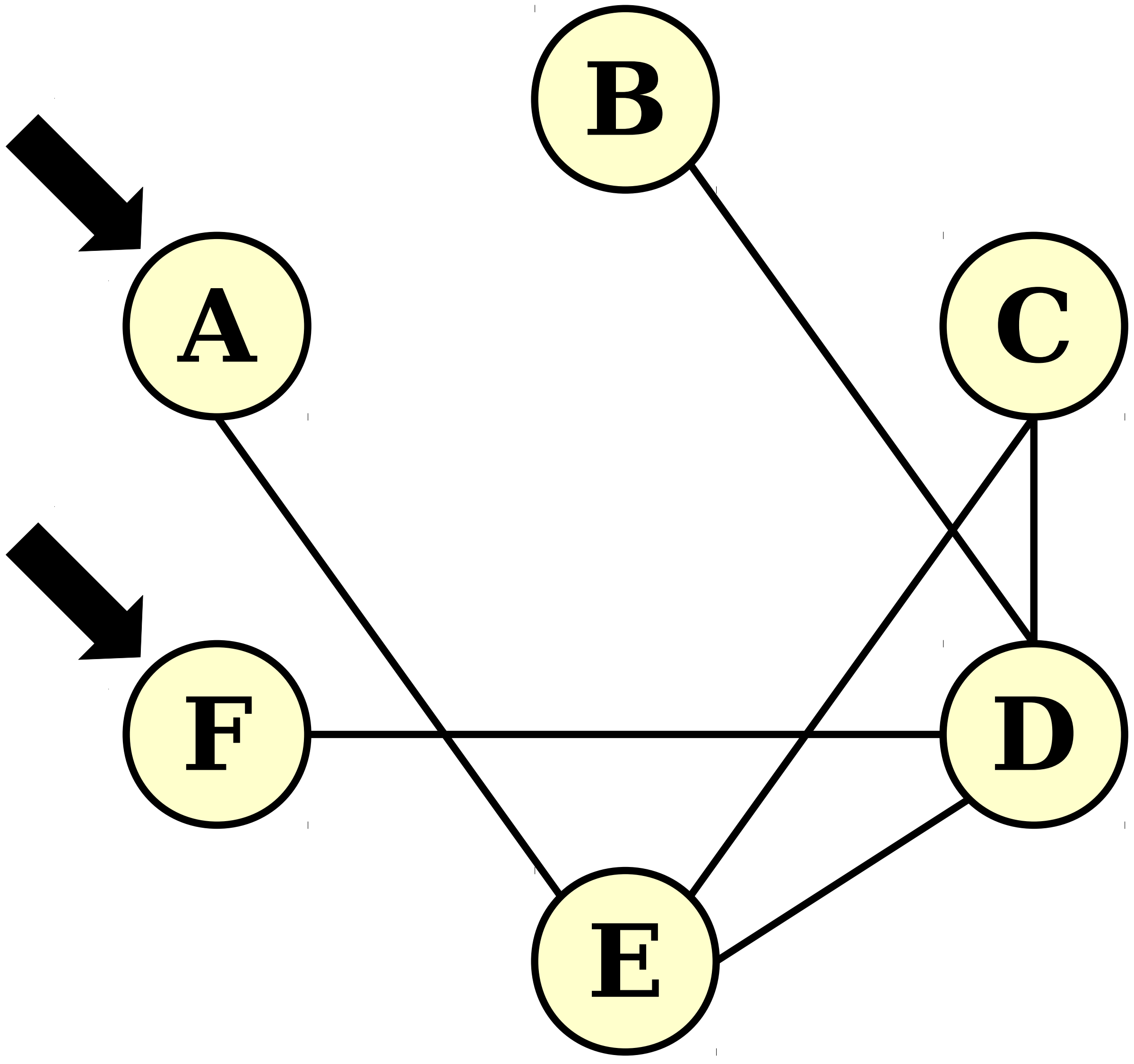


Goal: Determine the length of the shortest path from **A** to **F** in this graph.

Shortest Paths

- It is possible to find the shortest path in a graph by listing off all sequences of nodes in the graph in ascending order of length and finding the first that's a path.
- This takes time $O(n \cdot n!)$ in an n-node graph.
- For reference: 29! nanoseconds is longer than the lifetime of the universe.

Shortest Paths

- ***Theorem:*** It's possible to find the shortest path between two nodes in an n-node, m-edge graph in time $O(m + n)$.
- This is the breadth-first search algorithm. Take CS106B/X or CS161 for more details!
- The algorithm is a bit nuanced. It uses some specific properties of shortest paths and the proof of correctness is actually nontrivial.

For Comparison

- ***Longest increasing subsequence:***
  - Naïve: O($n \cdot 2^n$)
  - Fast: O($n \log n$)
- ***Shortest path problem:***
  - Naïve: O($n \cdot n!$)
  - Fast: O($n + m$).

Defining Efficiency

- When dealing with problems that search for the “best” object of some sort, there are often at least exponentially many possible options.
- Brute-force solutions tend to take at least exponential time to complete.
- Clever algorithms often run in time $O(n)$, or $O(n^2)$, or $O(n^3)$, etc.

Polynomials and Exponentials

- An algorithm runs in ***polynomial time*** if its runtime is some polynomial in n.
  - That is, time $O(n^k)$ for some constant k.
- Polynomial functions “scale well.”
  - Small changes to the size of the input do not typically induce enormous changes to the overall runtime.
- Exponential functions scale terribly.
  - Small changes to the size of the input induce huge changes in the overall runtime.

The Cobham-Edmonds Thesis

A language L can be ***decided efficiently*** if there is a TM that decides it in polynomial time.

Equivalently, L can be decided efficiently if it can be decided in time $O(n^k)$ for some $k \in \mathbb{N}$.

Like the Church-Turing thesis, this is ***not*** a theorem!

It's an assumption about the nature of efficient computation, and it is somewhat controversial.

The Cobham-Edmonds Thesis

- Efficient runtimes:
  - $4n + 13$
  - $n^3 - 2n^2 + 4n$
  - $n \log \log n$
- "Efficient" runtimes:
  - $n^{1,000,000,000,000}$
  - 10^{500}

- Inefficient runtimes:
  - 2^n
  - $n!$
  - n^n
- "Inefficient" runtimes:
  - $n^{0.0001 \log n}$
  - 1.000000001^n

Why Polynomials?

- Polynomial time *somewhat* captures efficient computation, but has a few edge cases.
- However, polynomials have very nice mathematical properties:
  - The sum of two polynomials is a polynomial. (Running one efficient algorithm after the other gives an efficient algorithm.)
  - The product of two polynomials is a polynomial. (Running one efficient algorithm a “reasonable” number of times gives an efficient algorithm.)
  - The *composition* of two polynomials is a polynomial. (Using the output of one efficient algorithm as the input to another efficient algorithm gives an efficient algorithm.)

The Complexity Class **P**

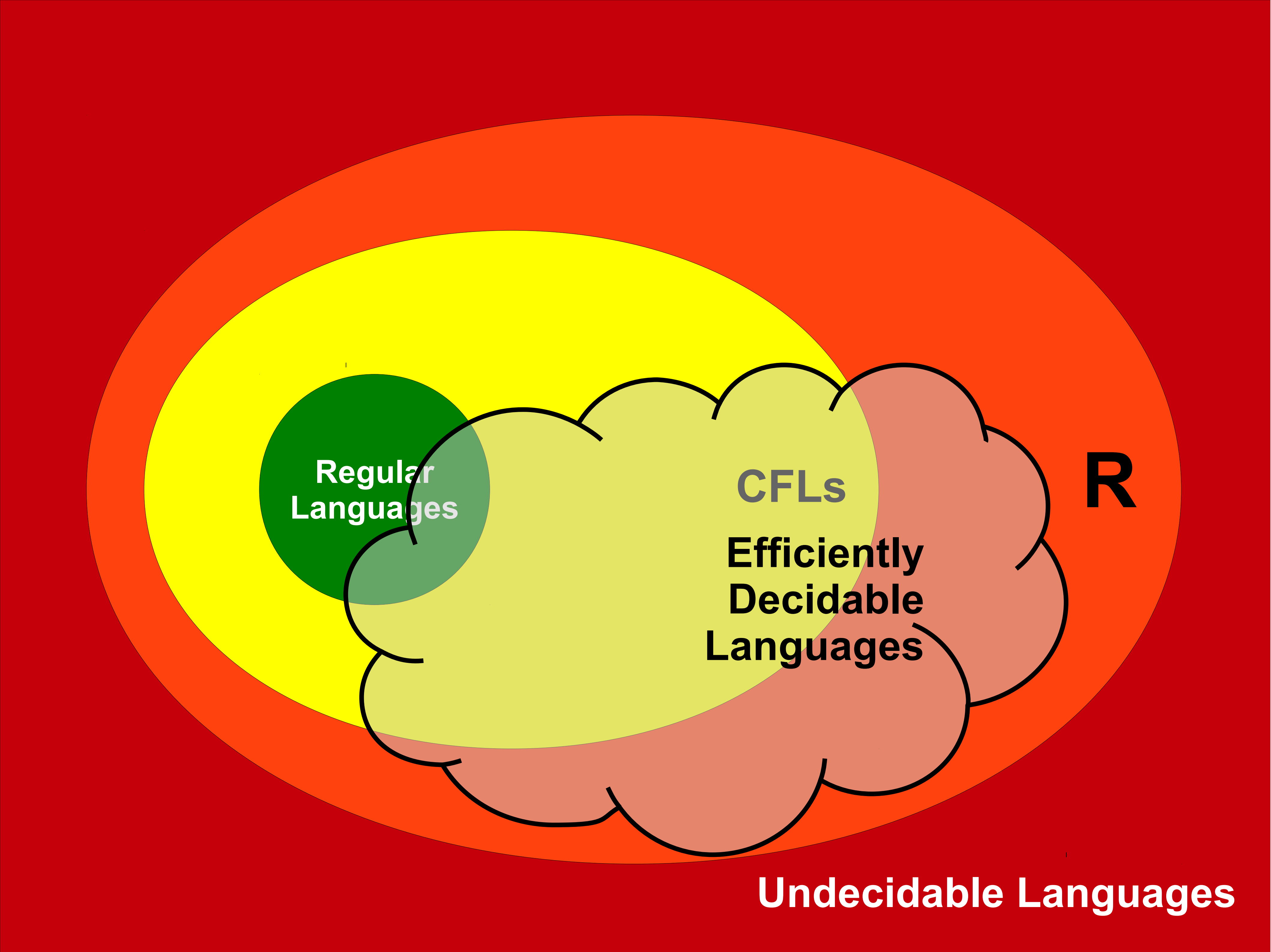
- The ***complexity class*** **P** (for ***p***olynomial time) contains all problems that can be solved in polynomial time.
- Formally:

  P = { *L* | There is a polynomial-time decider for *L* }

- Assuming the Cobham-Edmonds thesis, a language is in **P** if it can be decided efficiently.

Examples of Problems in **P**

- All regular languages are in **P**.
  - All have linear-time TMs.
- All CFLs are in **P**.
  - Requires a more nuanced argument (the *CYK algorithm* or *Earley's algorithm.*)
- And a *ton* of other problems are in **P** as well.
  - Curious? Take CS161!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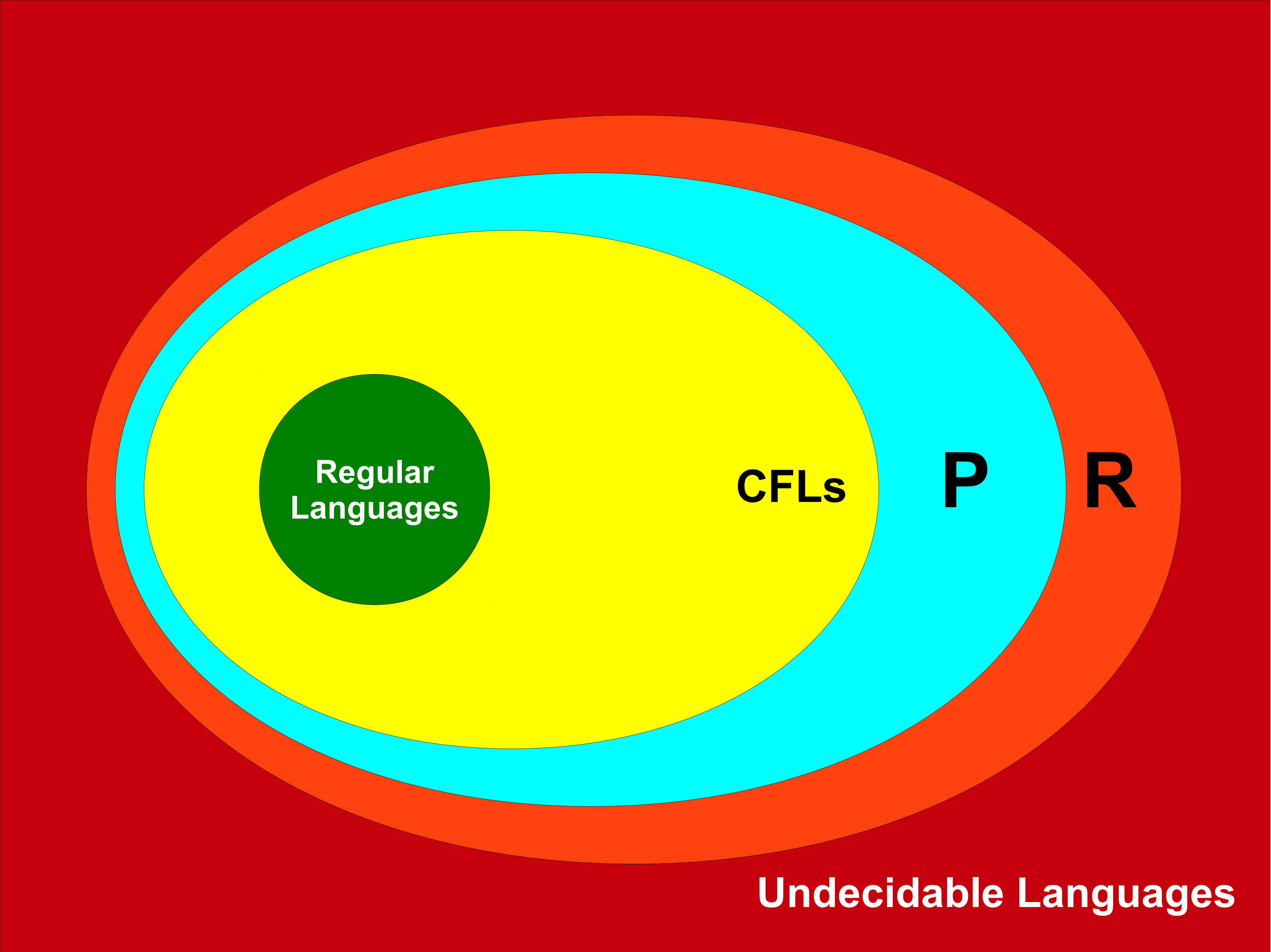
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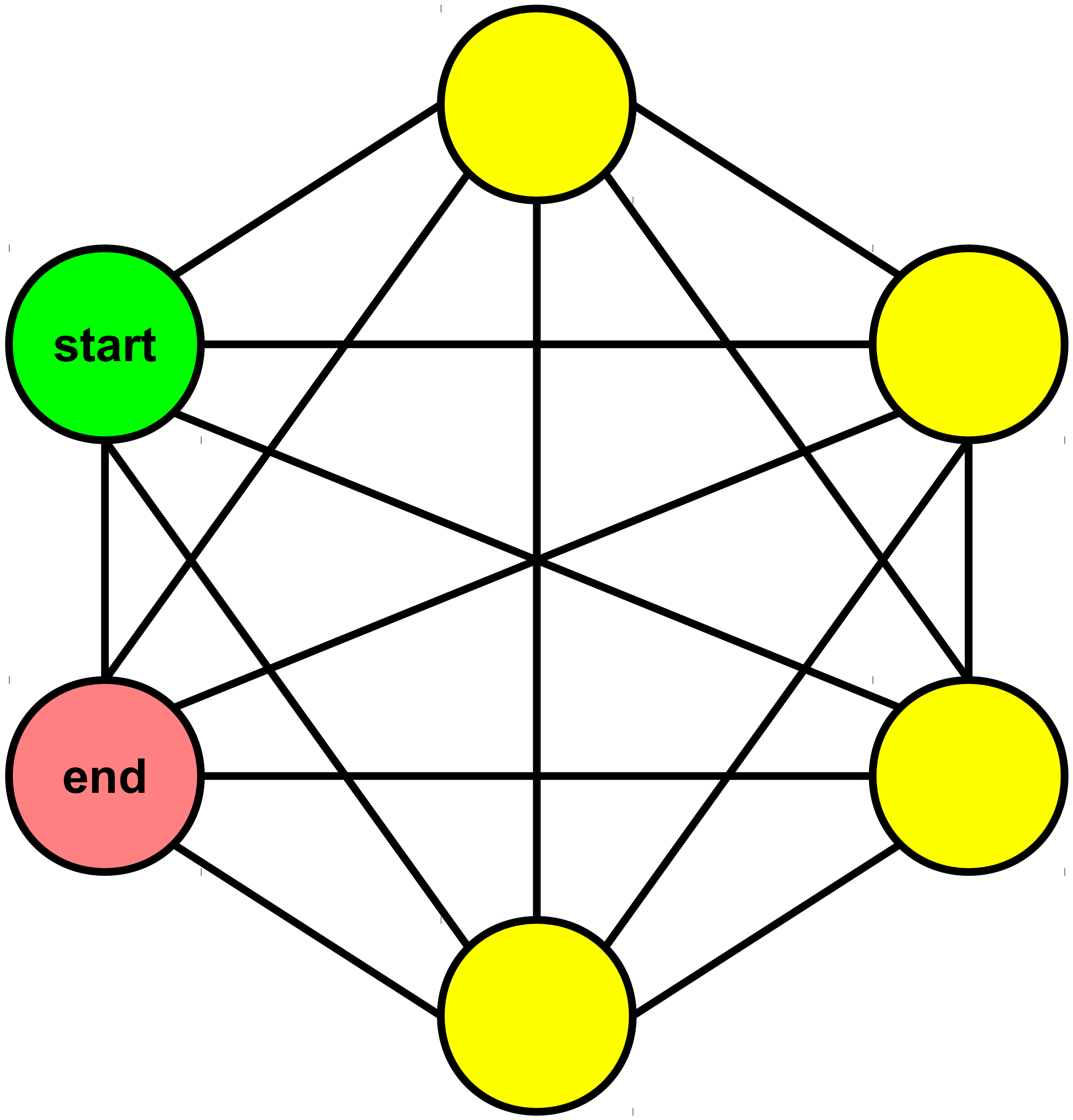
What *can't* you do in polynomial time?



How many simple paths are there from the start node to the end node?

{

,

,

,

}

How many subsets of this set are there?

An Interesting Observation

- There are (at least) exponentially many objects of each of the preceding types.
- However, each of those objects is not very large.
  - Each simple path has length no longer than the number of nodes in the graph.
  - Each subset of a set has no more elements than the original set.
- This brings us to our next topic…

N P

What if you need to search a large space for a single object?

Verifiers – Again

		7		6		1		
					3		5	2
3			1		5	9		7
6		5		3		8		9
	1						2	
8		2		1		5		4
1		3	2		7			8
5	7		4					
		4		8		7		

Does this Sudoku problem
have a solution?

Verifiers – Again

2	5	**7**	9	**6**	4	**1**	8	3
4	9	1	8	7	**3**	6	**5**	**2**
3	8	6	**1**	2	**5**	**9**	4	**7**
6	4	**5**	7	**3**	2	**8**	1	**9**
7	**1**	9	5	4	8	3	**2**	6
8	3	**2**	6	**1**	9	**5**	7	**4**
1	6	**3**	**2**	5	**7**	4	9	**8**
5	**7**	8	**4**	9	6	2	3	1
9	2	**4**	3	**8**	1	**7**	6	5

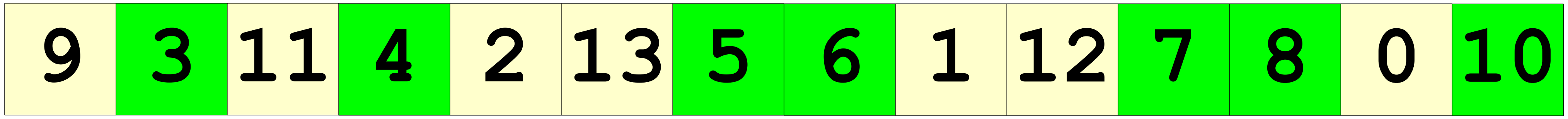
Does this Sudoku problem
have a solution?

Verifiers – Again

9	3	11	4	2	13	5	6	1	12	7	8	0	10

Is there an ascending subsequence of length at least 7?

Verifiers – Again



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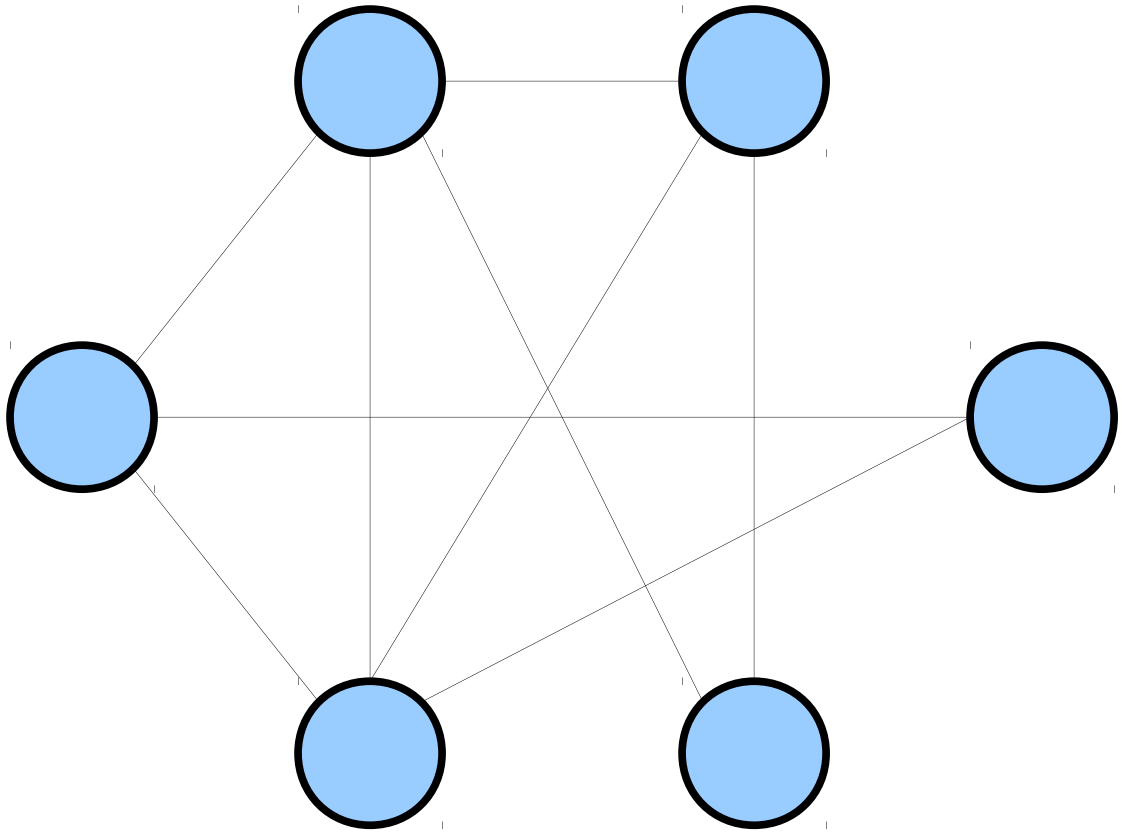
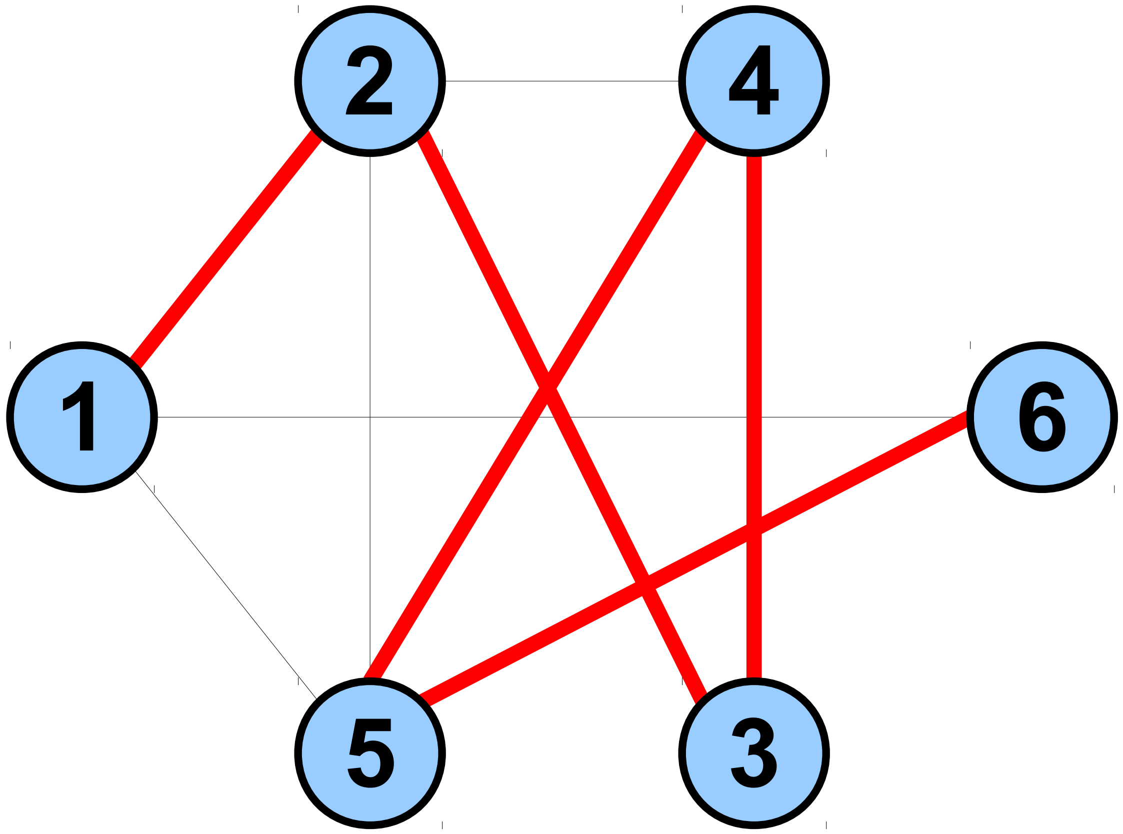


Is there a simple path that goes through every node exactly once?

Verifiers – Again



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Verifiers

- Recall that a ***verifier*** for L is a TM V such that
  - V halts on all inputs.
  - $w \in L \quad \text{iff} \quad \exists c \in \Sigma^*.\ V$ accepts $\langle w, c \rangle$.

Polynomial-Time Verifiers

- A ***polynomial-time verifier*** for L is a TM V such that
  - V halts on all inputs.
  - $w \in L \quad \text{iff} \quad \exists c \in \Sigma^*.\ V$ accepts $\langle w, c\rangle$.
  - V's runtime is a polynomial in $|w|$ (that is, V's runtime is $O(|w|^k)$ for some integer k)

The Complexity Class **NP**

- The complexity class **NP** (***nondeterministic polynomial time***) contains all problems that can be verified in polynomial time.
- Formally:

  NP = { *L* | There is a polynomial-time verifier for *L* }

- The name **NP** comes from another way of characterizing **NP**. If you introduce *nondeterministic Turing machines* and appropriately define "polynomial time," then **NP** is the set of problems that an NTM can solve in polynomial time.

And now…

The

Most Important Question

in

Theoretical Computer Science

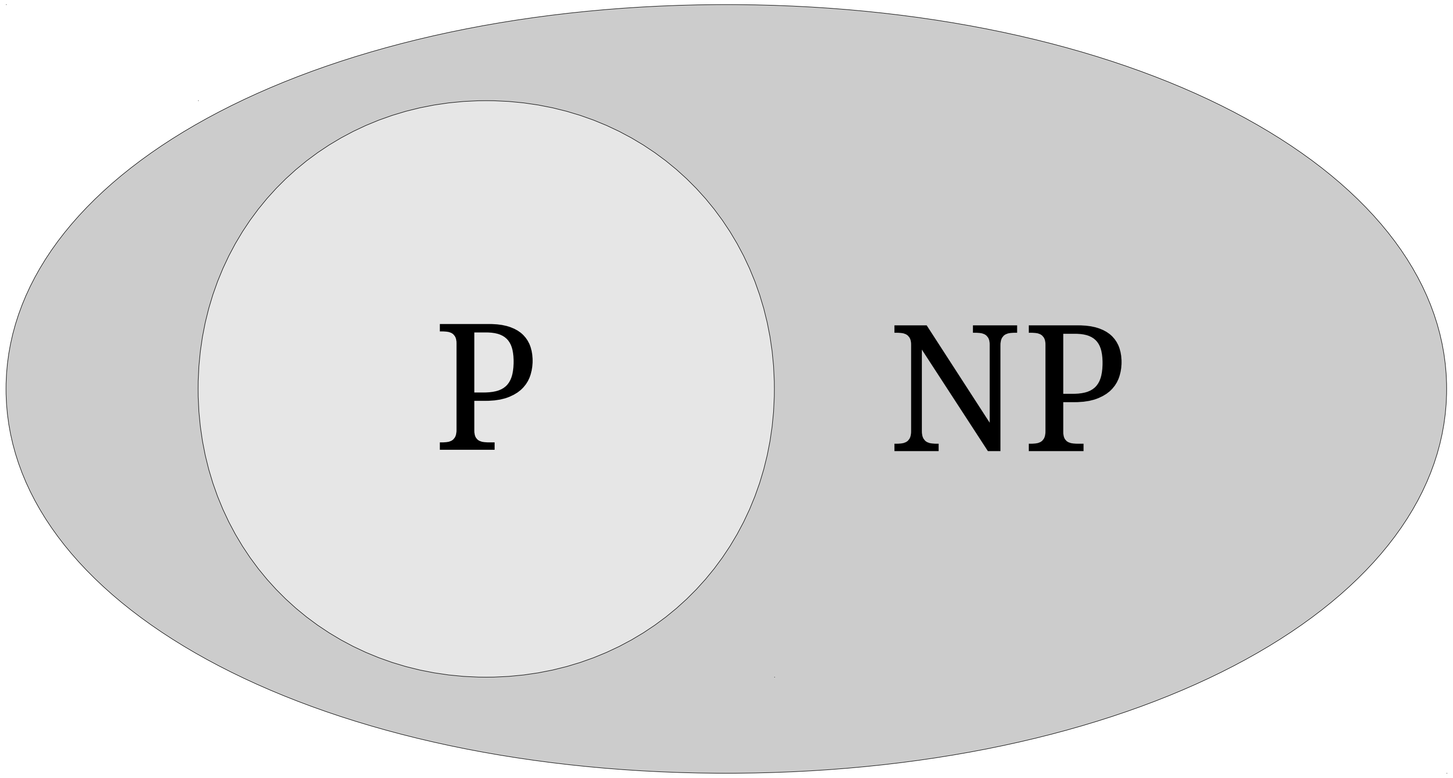
What is the connection between **P** and **NP**?

P = { *L* | There is a polynomial-time decider for *L* }

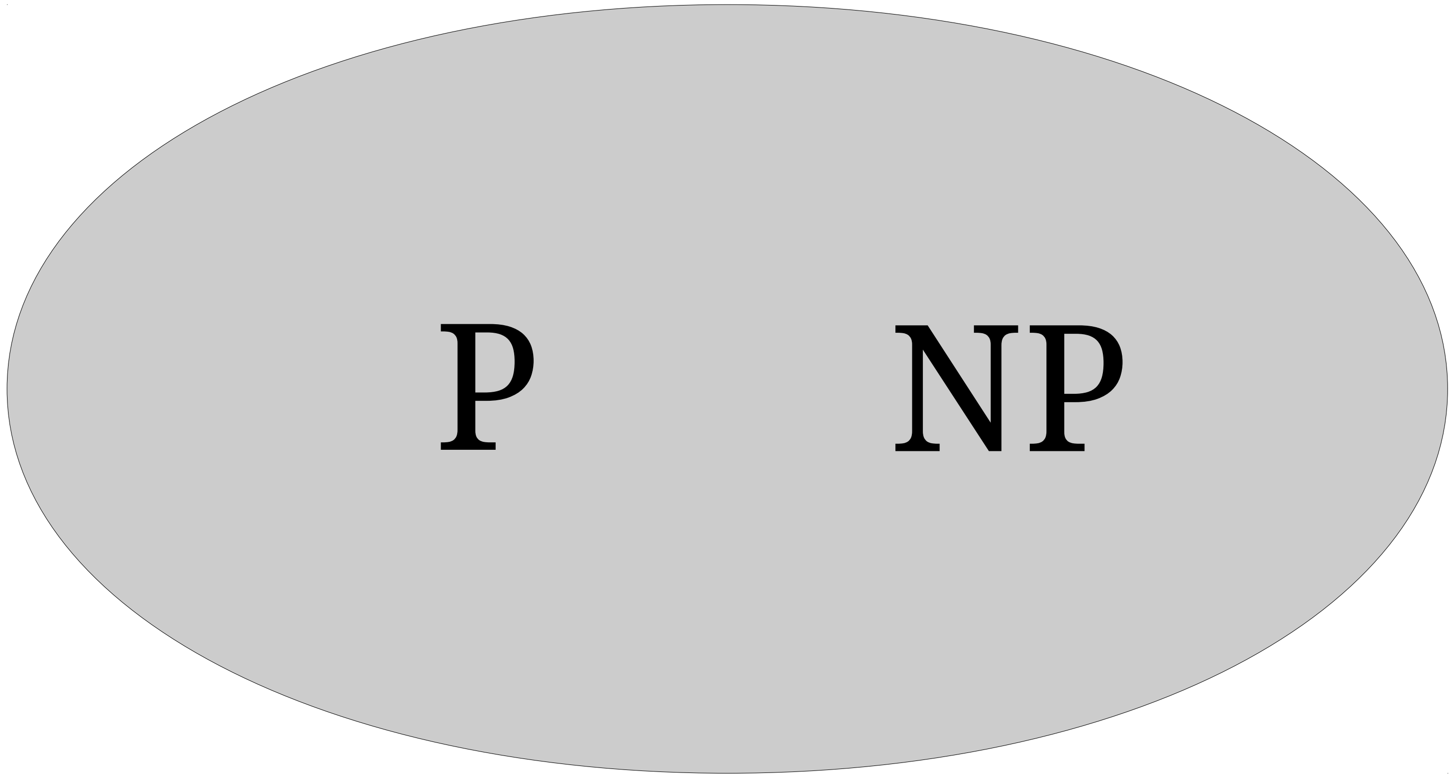
NP = { *L* | There is a polynomial-time verifier for *L* }

P ⊆ **NP**

Which Picture is Correct?



Which Picture is Correct?



Does $\mathbf{P} = \mathbf{NP}$?

$\mathbf{P} \overset{?}{=} \mathbf{NP}$

- The $\mathbf{P} \overset{?}{=} \mathbf{NP}$ question is the most important question in theoretical computer science.
- With the verifier definition of **NP**, one way of phrasing this question is

  If a solution to a problem can be ***checked*** *efficiently, can that problem be* ***solved*** *efficiently?*

- An answer either way will give fundamental insights into the nature of computation.

Why This Matters

- The following problems are known to be efficiently verifiable, but have no known efficient solutions:
    - Determining whether an electrical grid can be built to link up some number of houses for some price (Steiner tree problem).
    - Determining whether a simple DNA strand exists that multiple gene sequences could be a part of (shortest common supersequence).
    - Determining the best way to assign hardware resources in a compiler (optimal register allocation).
    - Determining the best way to distribute tasks to multiple workers to minimize completion time (job scheduling).
    - ***And many more***.
- If **P** = **NP**, ***all*** of these problems have efficient solutions.
- If **P** ≠ **NP**, ***none*** of these problems have efficient solutions.

Why This Matters

- If **P = NP**:
  - A huge number of seemingly difficult problems could be solved efficiently.
  - Our capacity to solve many problems will scale well with the size of the problems we want to solve.
- If **P ≠ NP**:
  - Enormous computational power would be required to solve many seemingly easy tasks.
  - Our capacity to solve problems will fail to keep up with our curiosity.

What We Know

- Resolving $\mathbf{P} \overset{?}{=} \mathbf{NP}$ has proven ***extremely difficult***.
- In the past 44 years:
  - Not a single correct proof either way has been found.
  - Many types of proofs have been shown to be insufficiently powerful to determine whether $\mathbf{P} \overset{?}{=} \mathbf{NP}$.
  - A majority of computer scientists believe $\mathbf{P} \neq \mathbf{NP}$, but this isn't a large majority.
- Interesting read: Interviews with leading thinkers about $\mathbf{P} \overset{?}{=} \mathbf{NP}$:
  - http://web.ing.puc.cl/~jabaier/iic2212/poll-1.pdf

The Million-Dollar Question

The Clay Mathematics Institute has offered a ***$1,000,000 prize*** to anyone who proves or disproves **P = NP**.

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Time-Out for Announcements!

Please evaluate this course on Axess.

Your feedback really does make a difference.

Problem Set Logistics

- Solutions to PS8 will go up later today.
- PS9 is due on Friday at 3:00PM.
  - ***This is a hard deadline!*** Don't tempt the network gremlins. Be sure to submit with plenty of breathing room!
  - As always, ask us questions on Piazza or in office hours!

Exam Logistics

- As a reminder, the final exam is on Monday from 3:30PM – 6:30PM. We'll release location information later this week.
- The exam is closed-book, closed-computer, and limited-note. You get one double-sided sheet of 8.5” × 11” notes decorated however you'd like.
- We'll release solutions to the practice final exams later this evening.

Practice Final Exam

- By popular demand, we'll be holding a practice final exam later this week.
  - Please fill out the form to let us know what works for you!
- We will be using Practice Final Exam 3 for the practice exam.
- Stop by, get some practice, get some feedback from the TAs, and see where to focus your study efforts!

Your Questions

"So if we prove $P \stackrel{?}{=} NP$, we get an automatic A+, regardless of other course performance?"

Yes, absolutely! That would be a really, really, really big deal. Like, you have no idea how big of a deal that would be. You'd totally change the course of CS theory. I'm fairly sure someone will give you a Ph.D for it, too!

"What are the rules for dead week for professors? It seems like "dead week" is just like any other week with lectures and psets for all my classes. ☹"

I never understood Dead Week as an undergrad here – it didn't seem like anything changed. You can take a look at the official policies that we get every quarter around this time if you'd like; they're publicly available at this link.

“You seem like a life-long learner. If so, what are you learning or reading about now? What do you think you'll learn or read about in the future?”

I'm reading up on landmark Supreme Court cases and listening to a bunch of podcasts on them. I'm also reading Mary Beard's “SPQR” about the history of the Roman Empire. I'm not sure what's next, actually! Any suggestions?

Back to CS103!

What do we know about $\mathbf{P} \overset{?}{=} \mathbf{NP}$?

Adapting our Techniques

A Problem

- The **R** and **RE** languages correspond to problems that can be decided and verified *period*, without any time bounds.
- To reason about what's in **R** and what's in **RE**, we used two key techniques:
  - ***Universality***: TMs can run other TMs as subroutines.
  - ***Self-Reference***: TMs can get their own source code.
- Why can't we just do that for **P** and **NP**?

Theorem (Baker-Gill-Solovay): Any proof that purely relies on universality and self-reference cannot resolve $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$.

Proof: Take CS154!

So how *are* we going to
reason about **P** and **NP**?

Reducibility

Maximum Matching

- Given an undirected graph *G*, a ***matching*** in *G* is a set of edges such that no two edges share an endpoint.
- A ***maximum matching*** is a matching with the largest number of edges.

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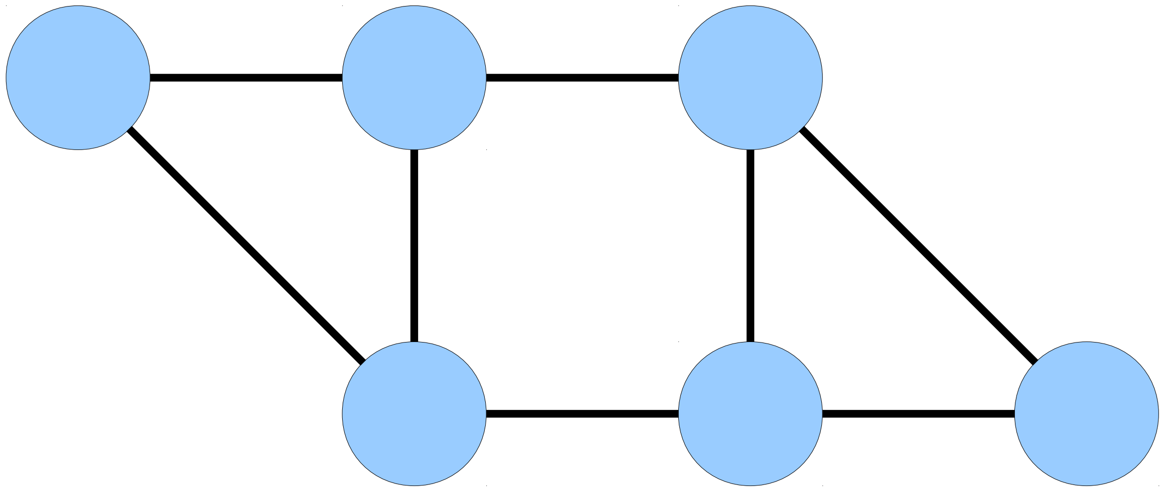

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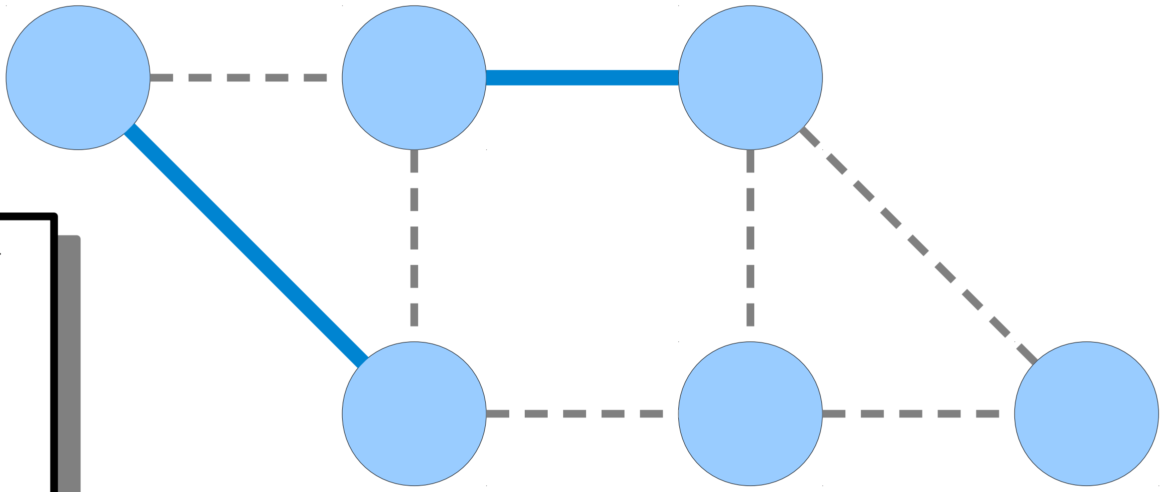

A matching, but not a maximum matching.

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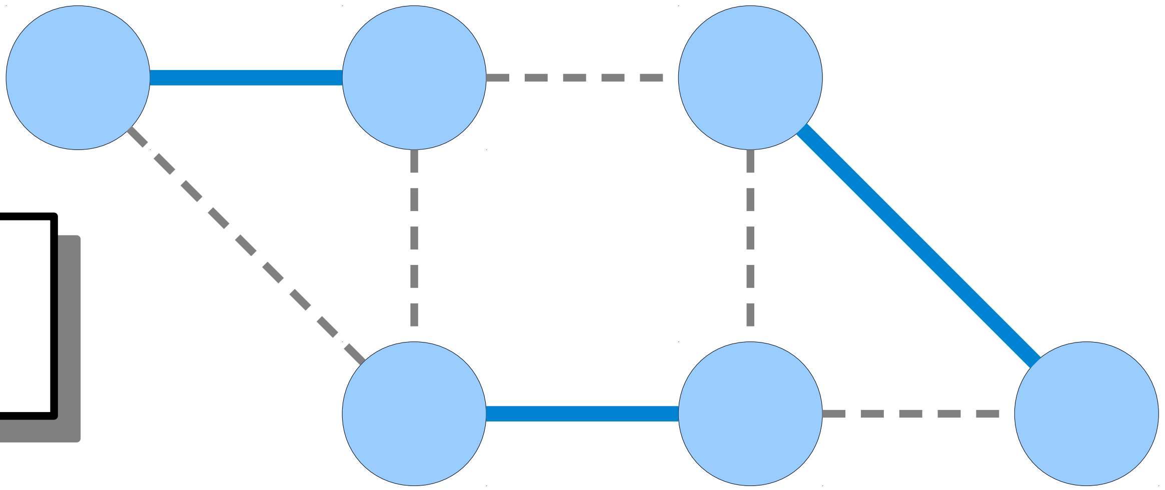

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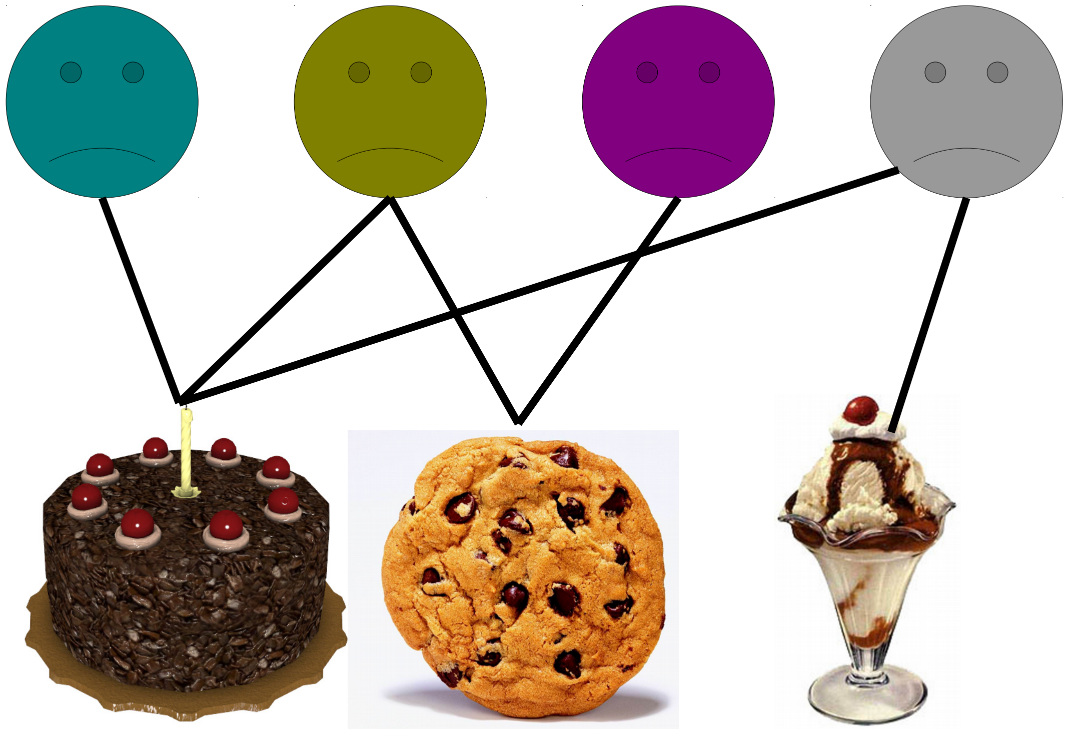

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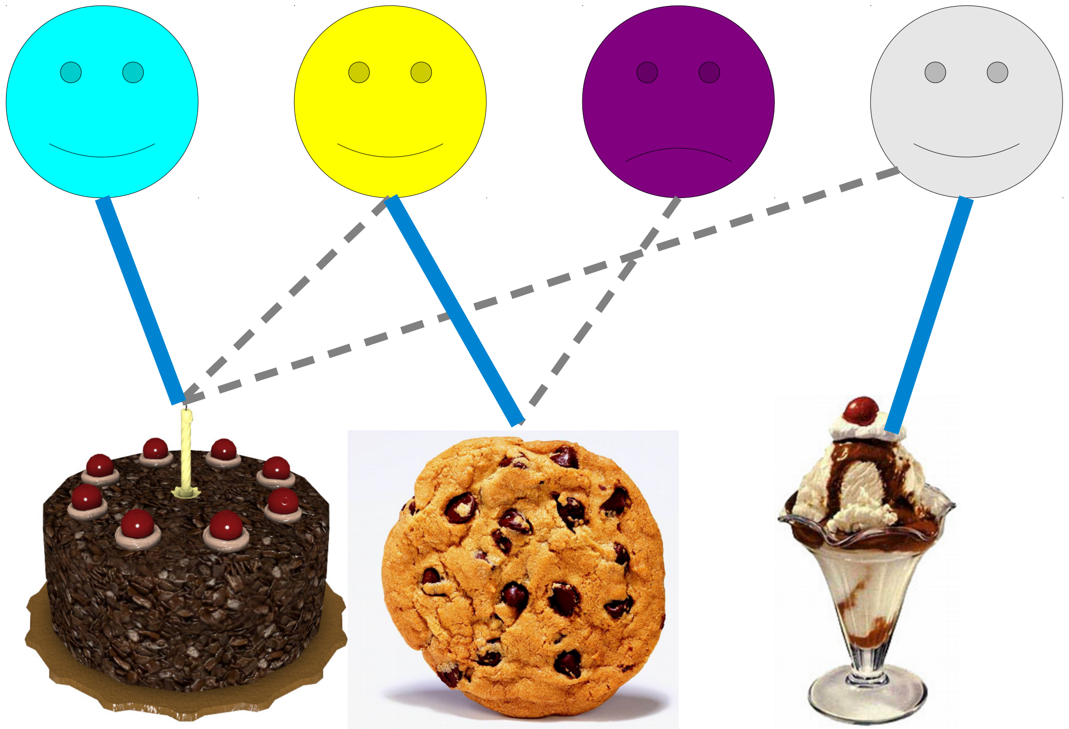
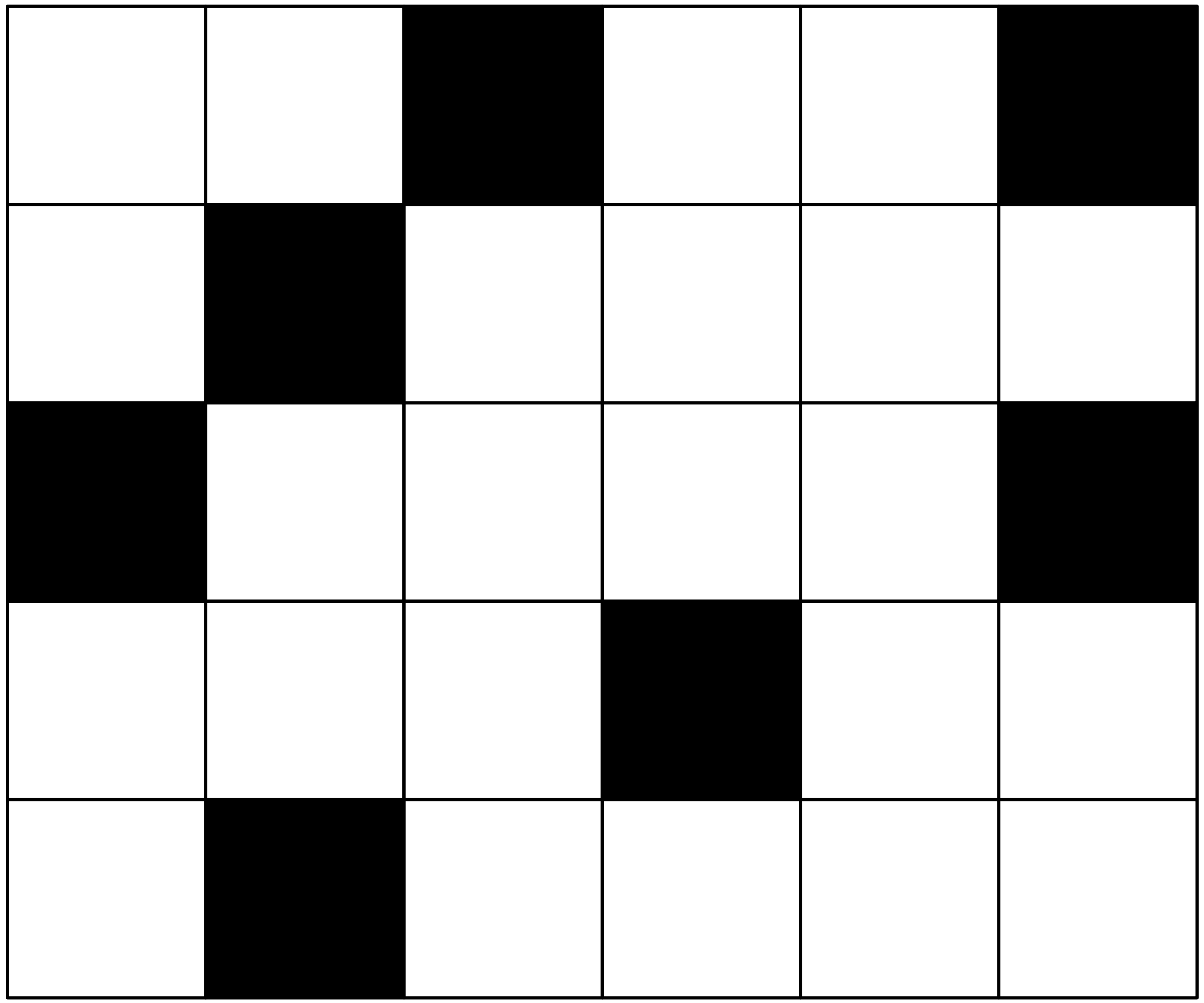
- Given an undirected graph *G*, a ***matching*** in *G* is a set of edges such that no two edges share an endpoint.
- A ***maximum matching*** is a matching with the largest number of edges.



Maximum Matching

- Jack Edmonds' paper “Paths, Trees, and Flowers” gives a polynomial-time algorithm for finding maximum matchings.
  - (This is the same Edmonds as in “Cobham-Edmonds Thesis.”)
- Using this fact, what other problems can we solve?

Domino Tiling



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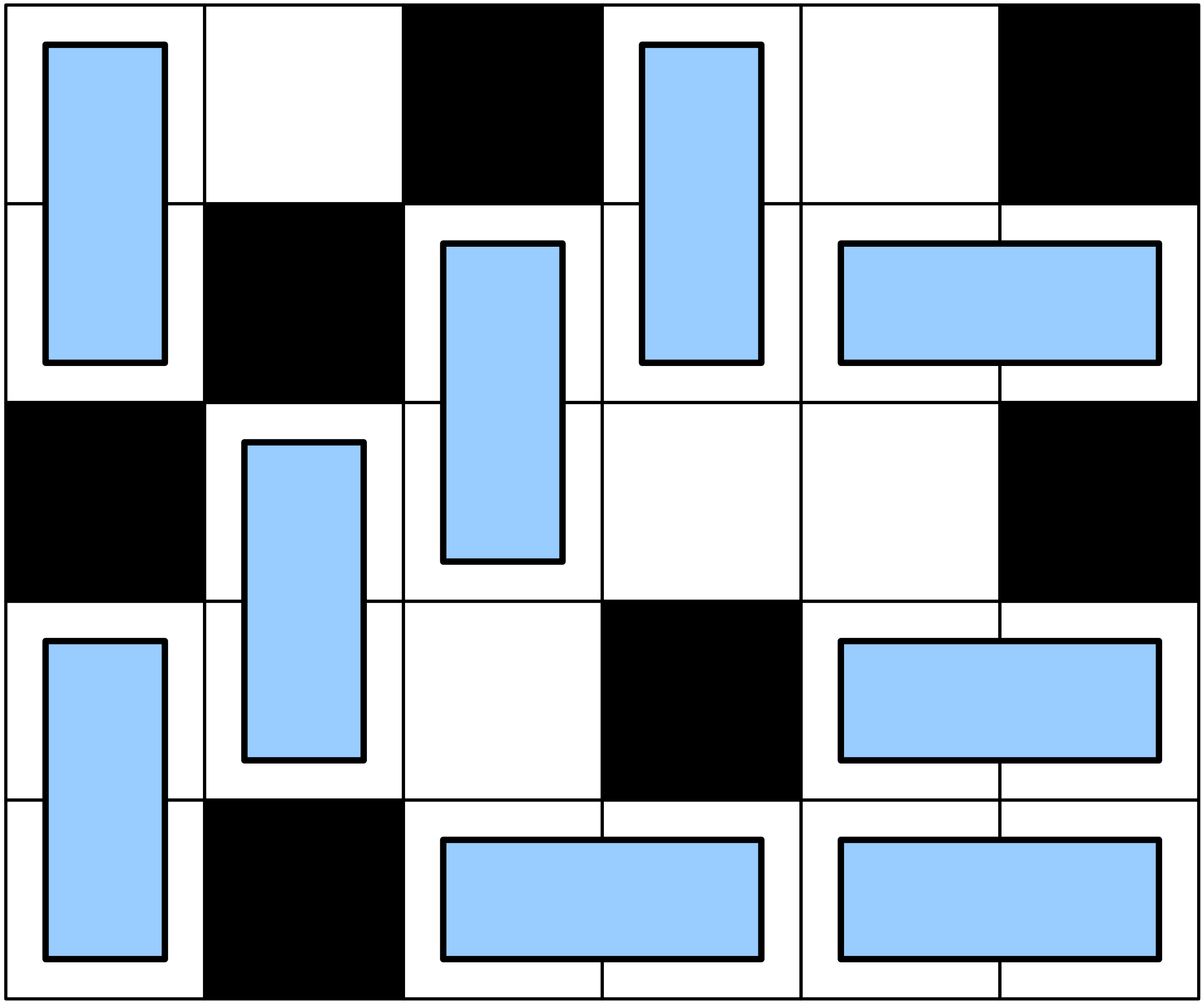

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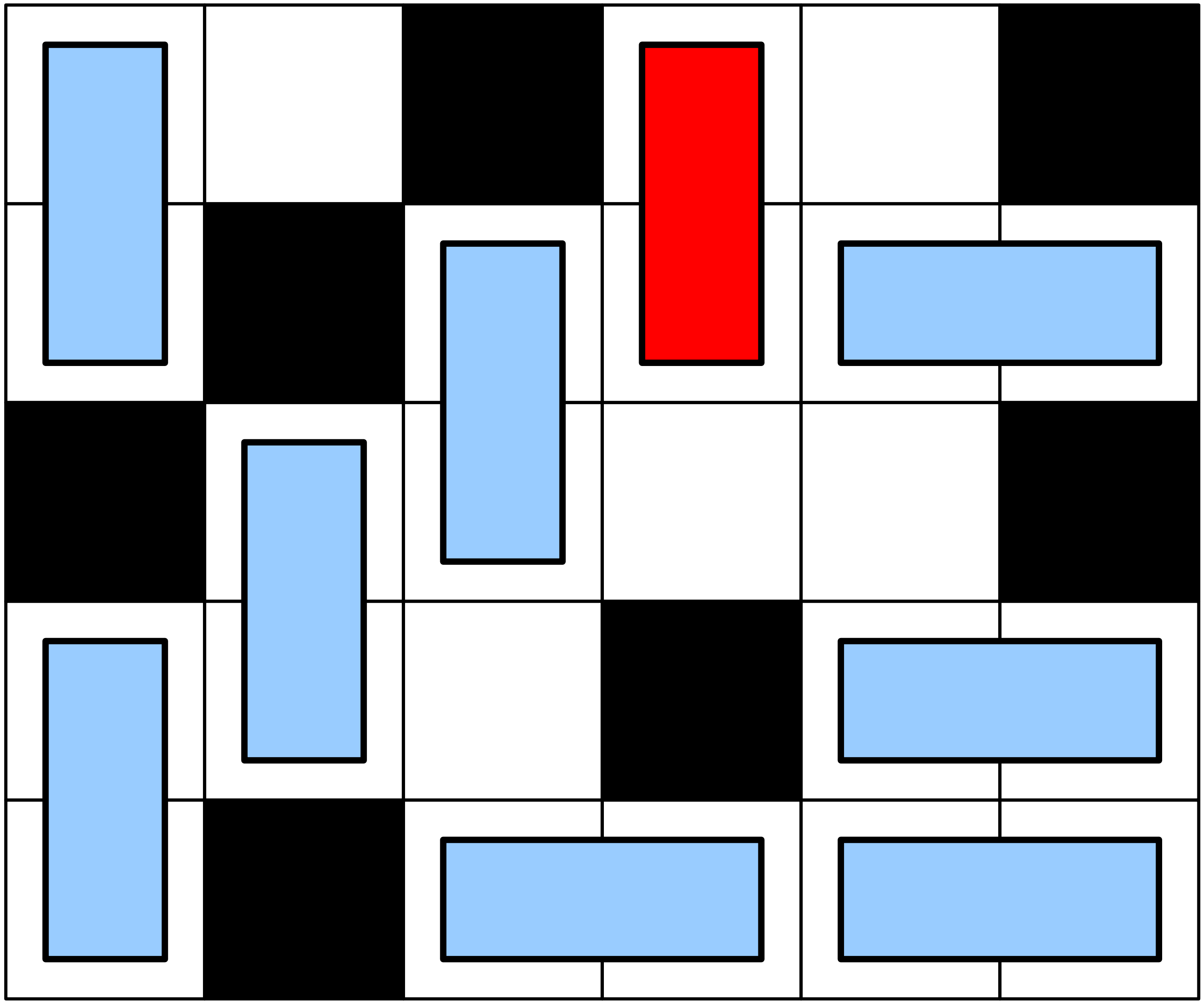

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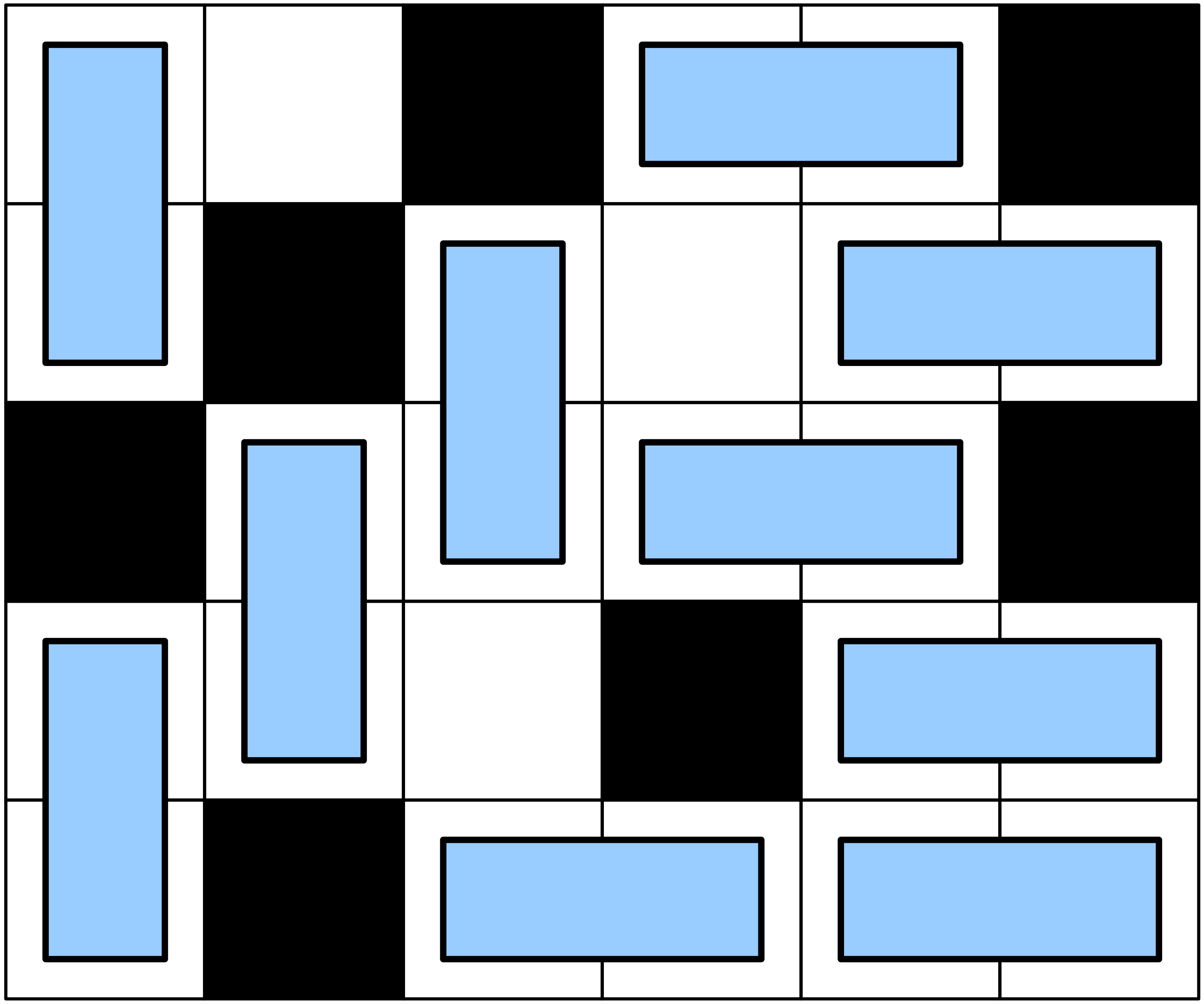

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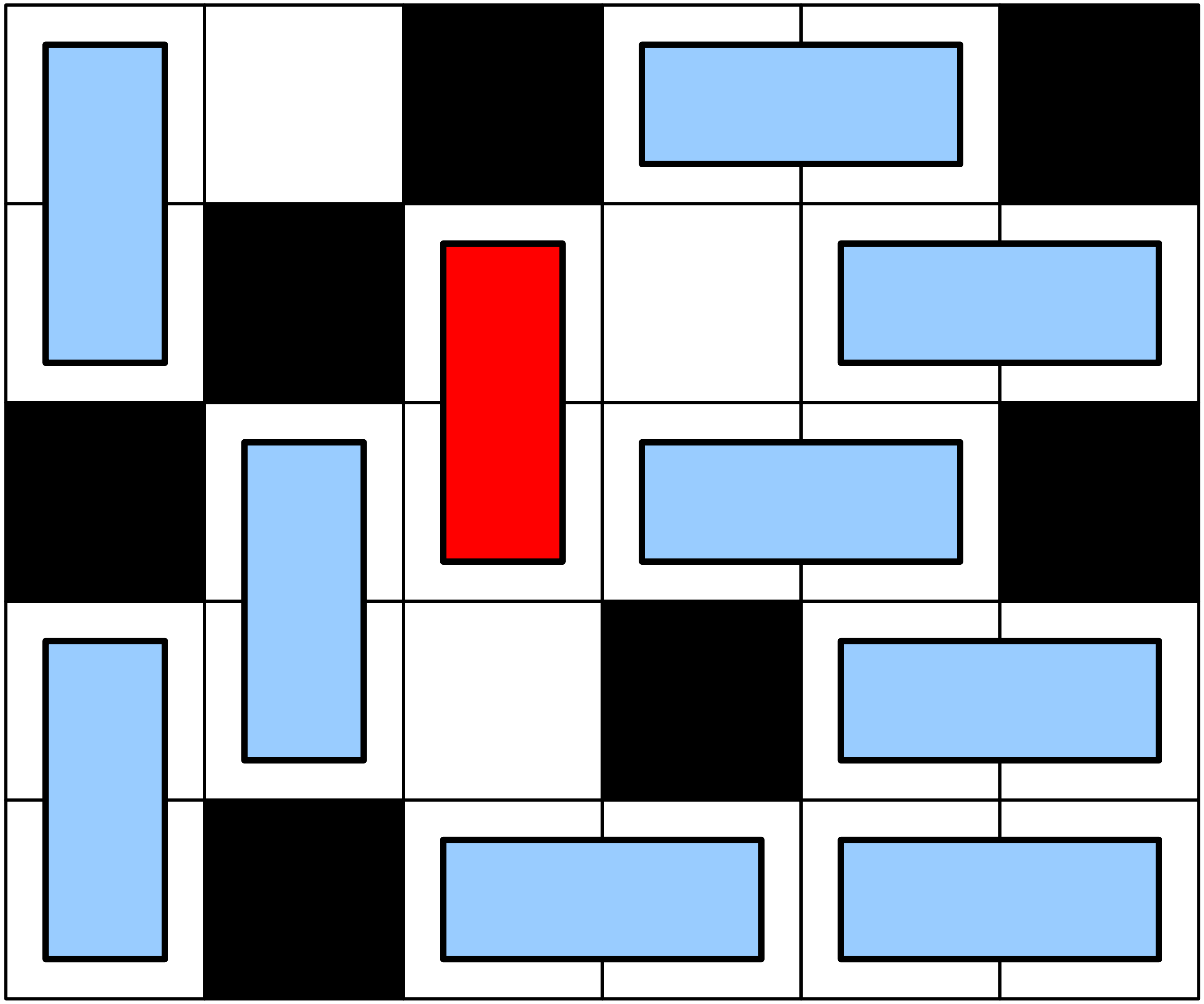

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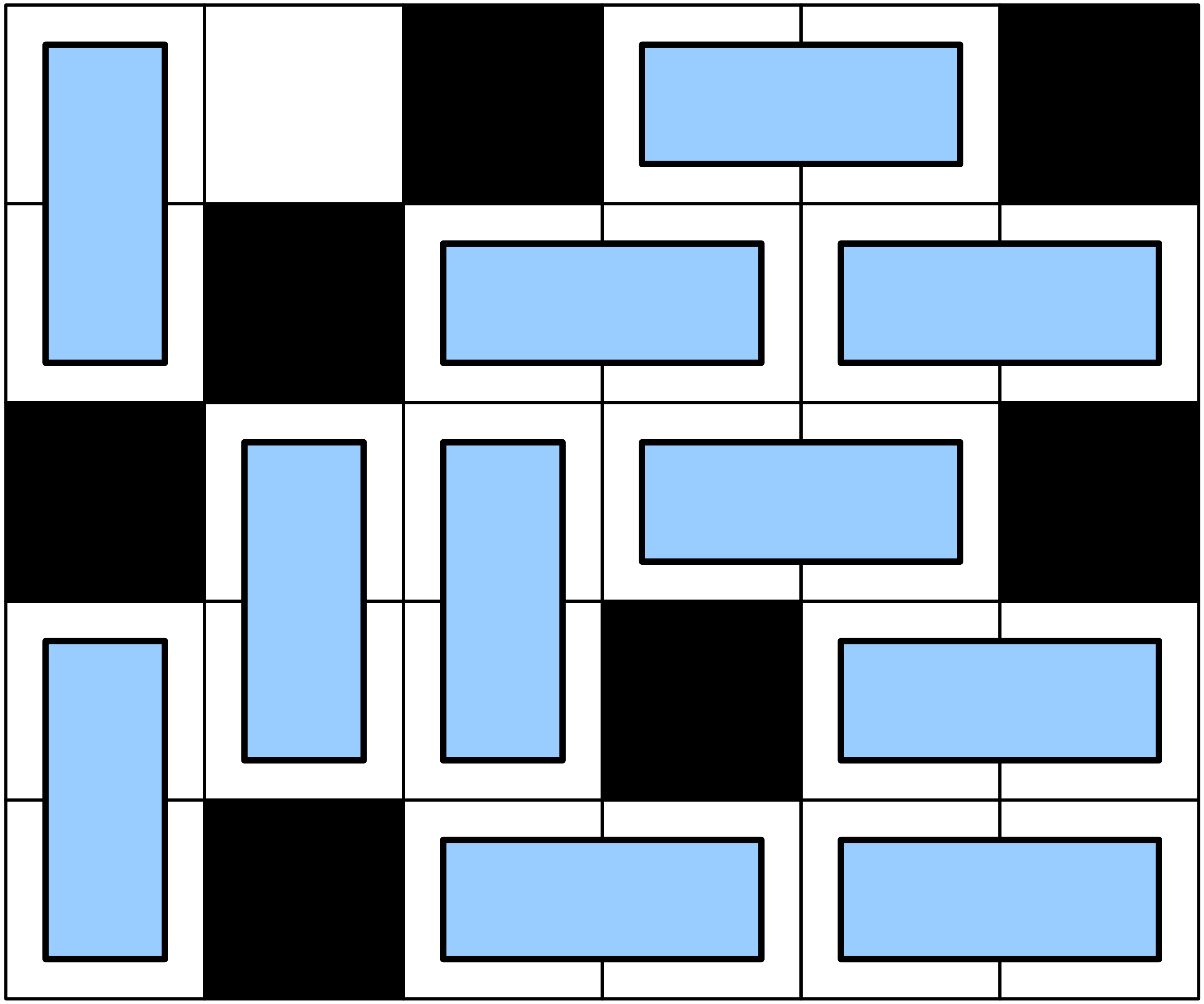


A Domino Tiling Reduction

- Let *MATCHING* be the language defined as follows:

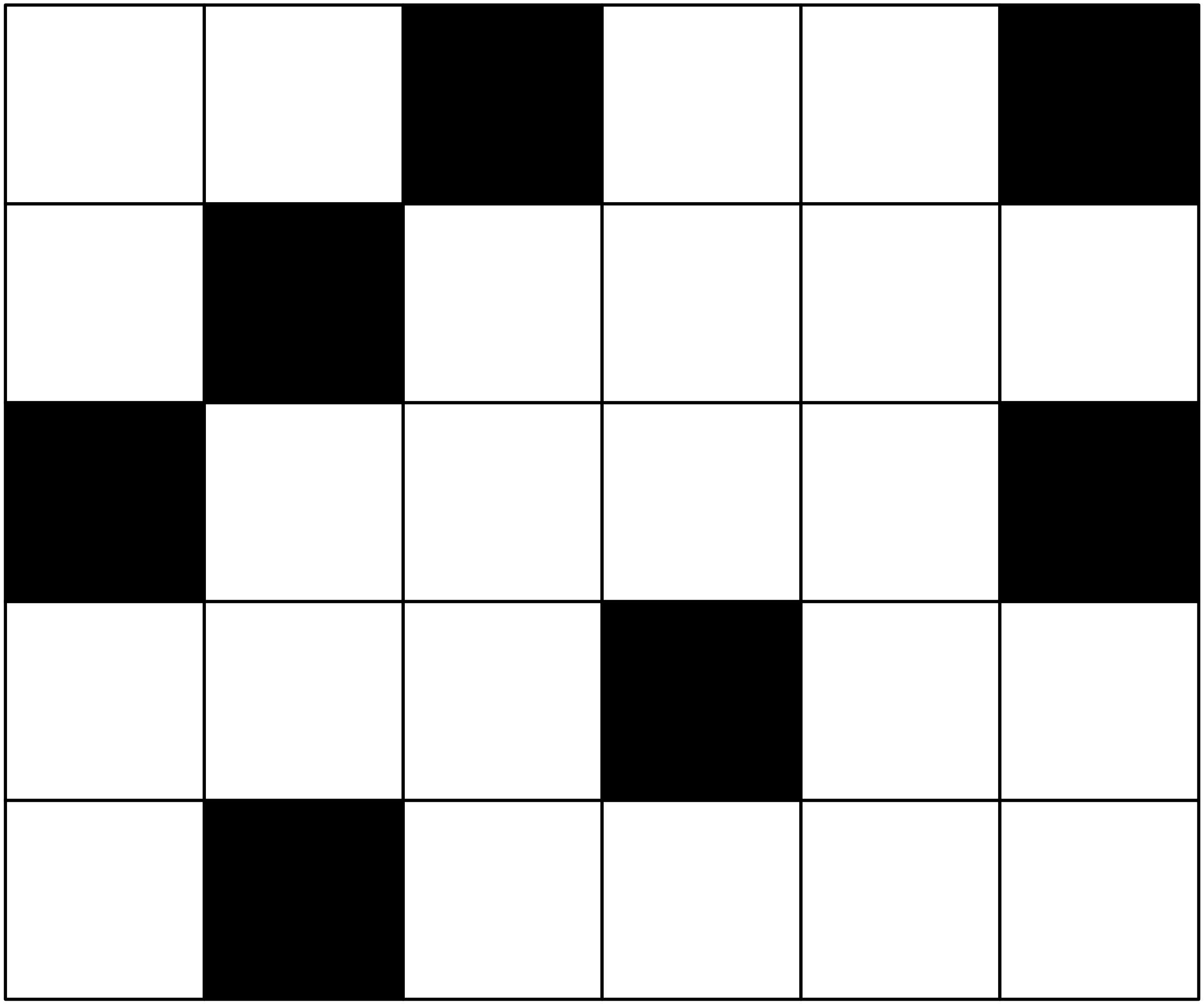
  MATCHING = { ⟨*G*, *k*⟩ | *G* is an undirected graph with a matching of size at least *k* }

- ***Theorem (Edmonds)***: *MATCHING* ∈ **P**.
- Let *DOMINO* be this language:

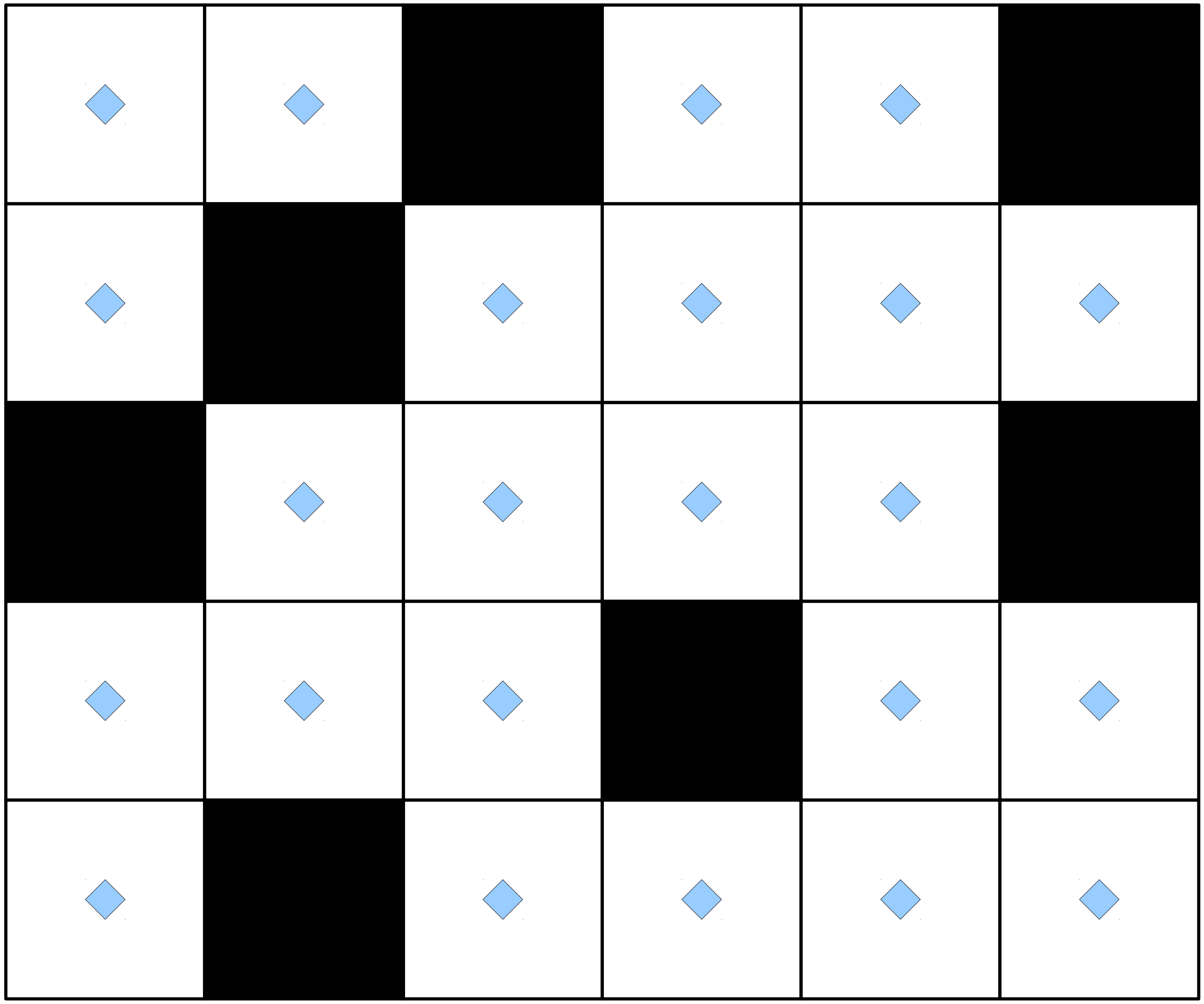
  DOMINO = { ⟨*D*, *k*⟩ | *D* is a grid and *k* nonoverlapping dominoes can be placed on *D*. }

- We'll use the fact that *MATCHING* ∈ **P** to prove that *DOMINO* ∈ **P**.

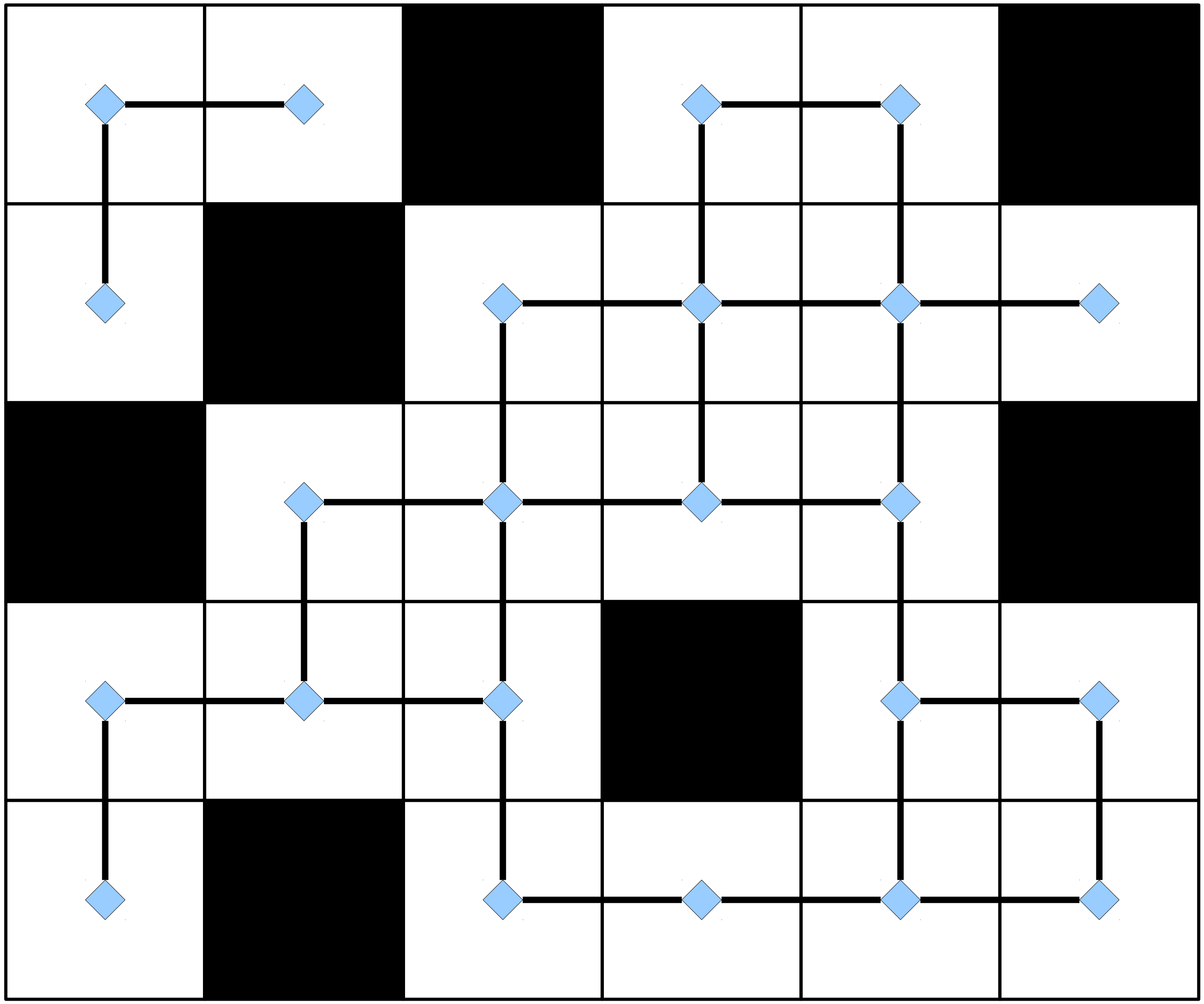
Solving Domino Tiling



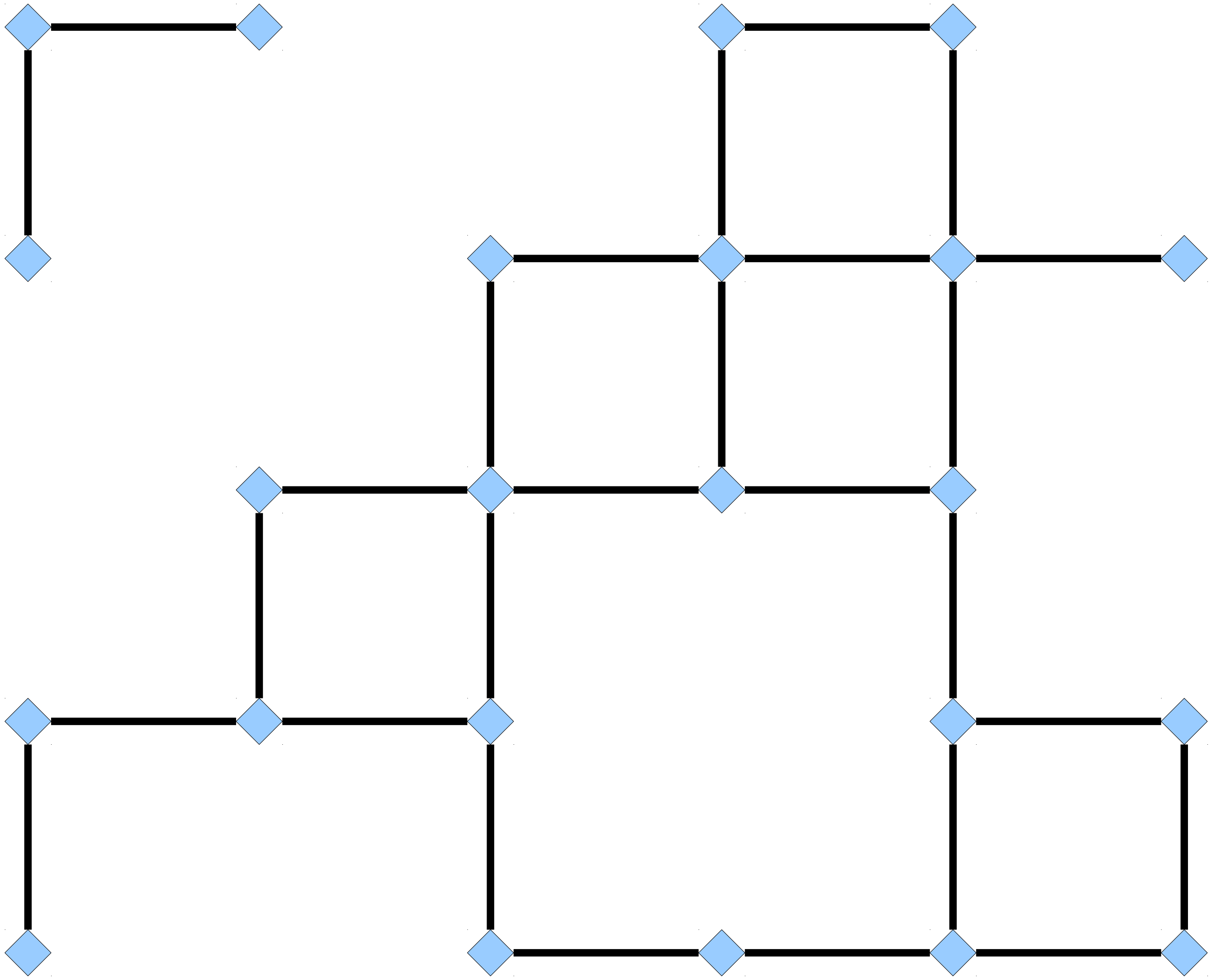
Solving Domino Tiling



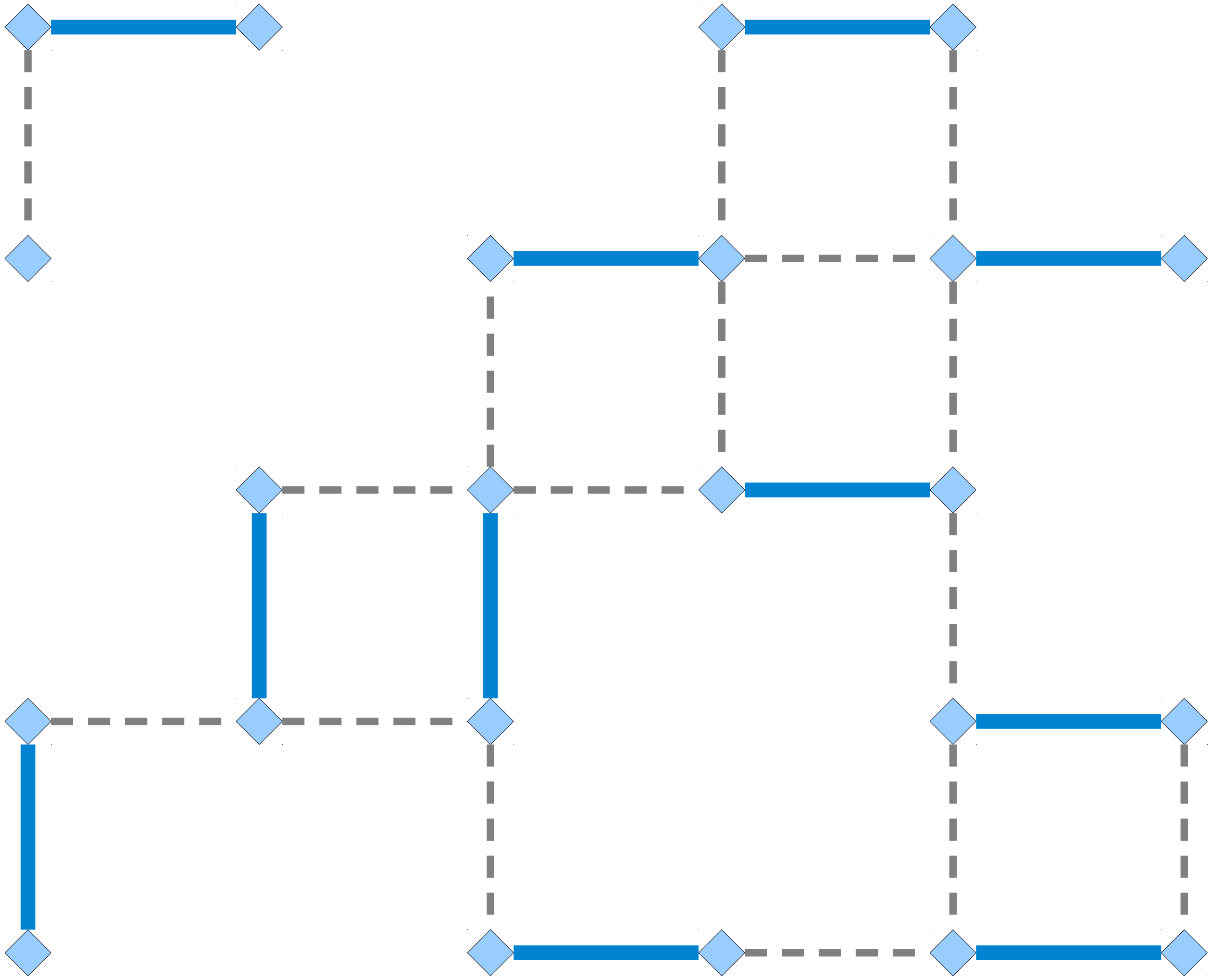
Solving Domino Tiling



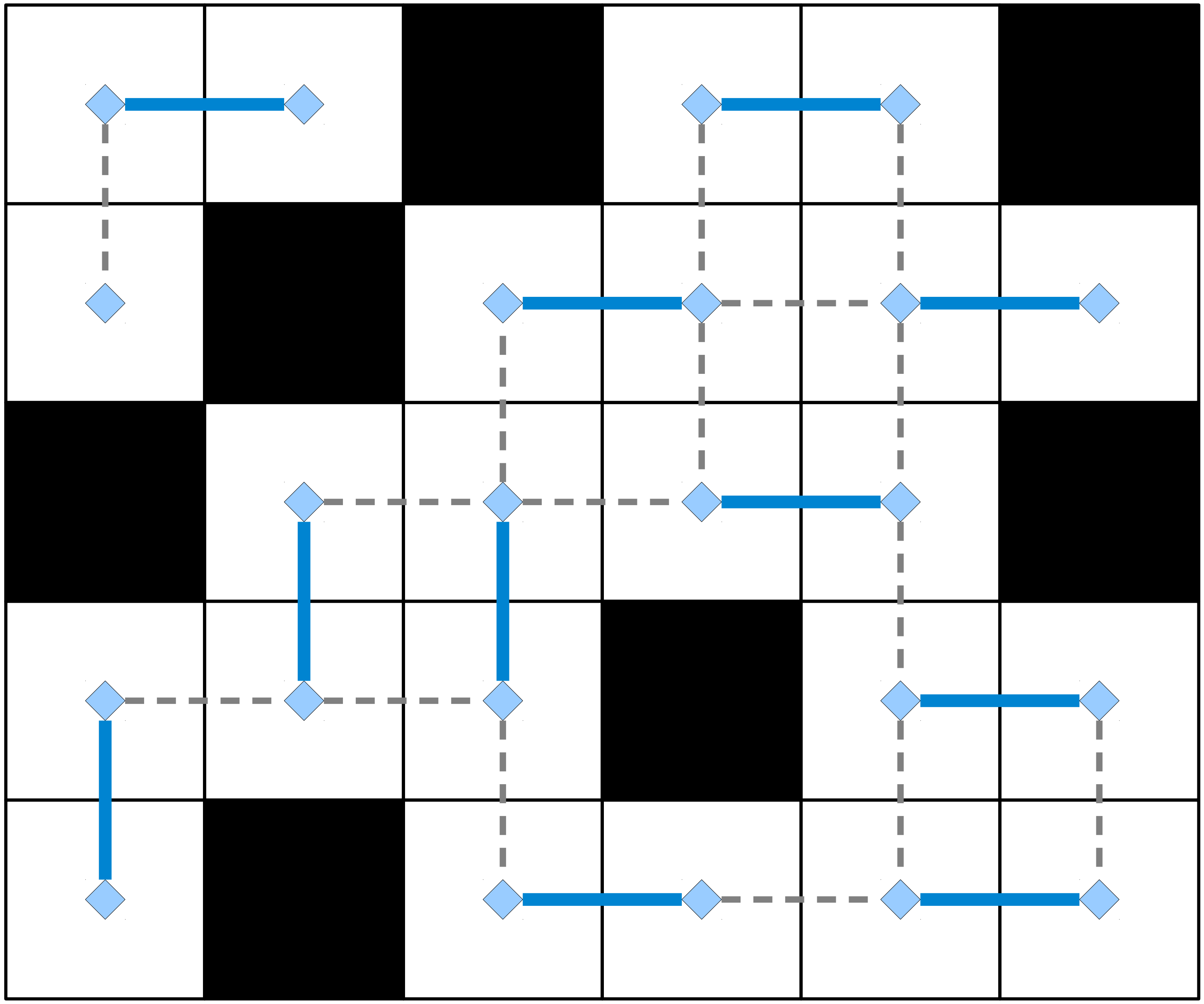
Solving Domino Tiling



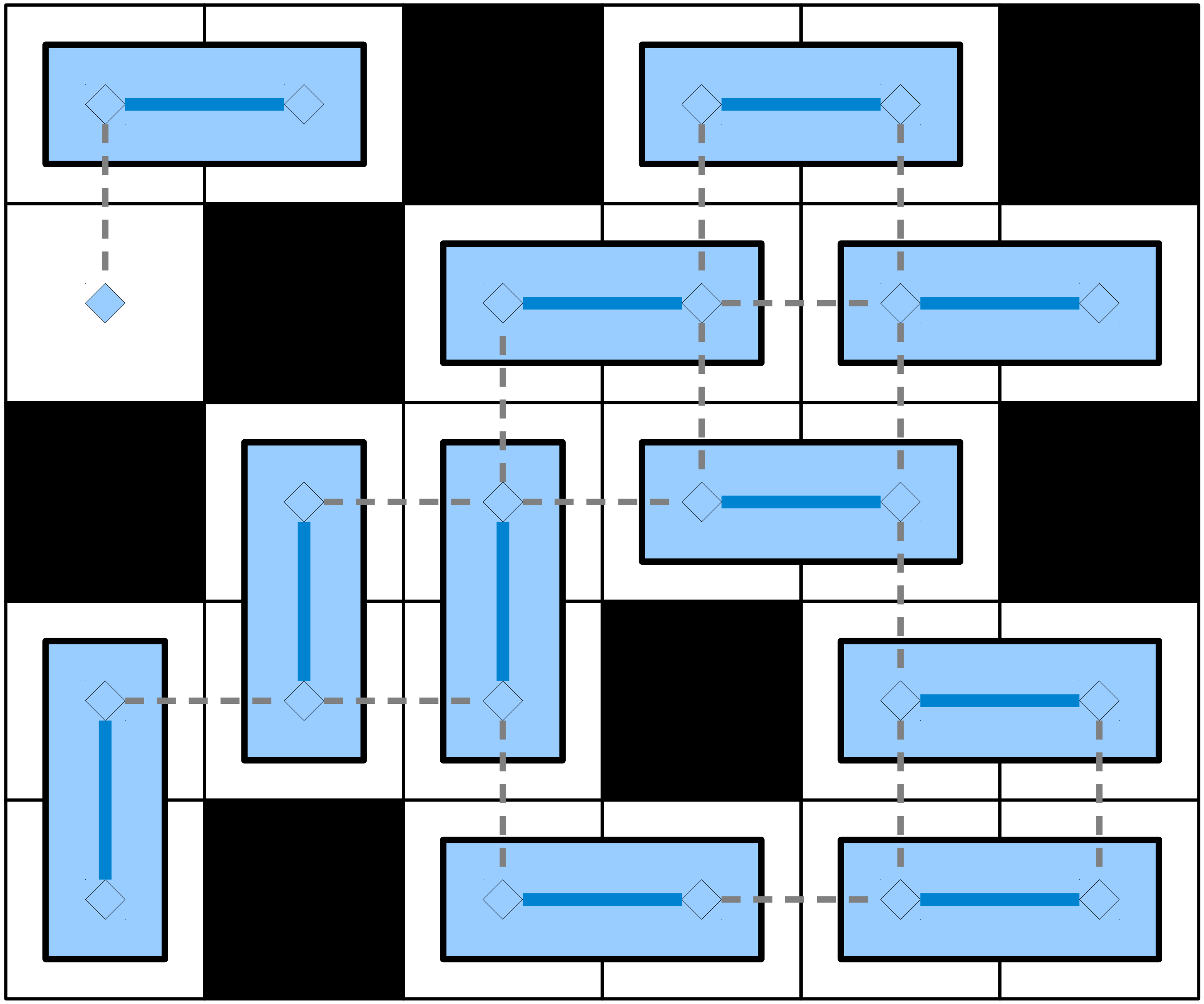
Solving Domino Tiling



Solving Domino Tiling



Solving Domino Tiling



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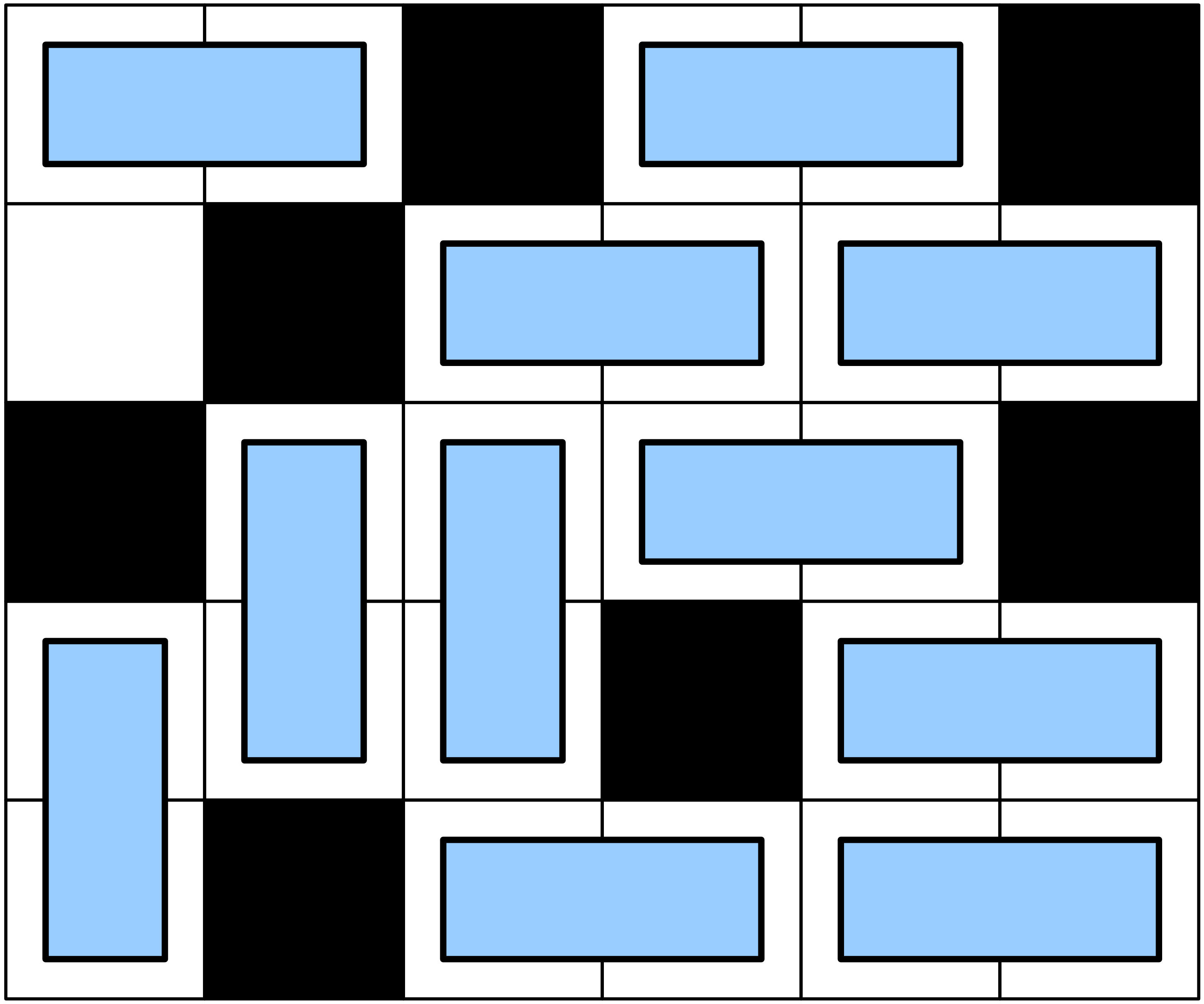

In Pseudocode

```
boolean canPlaceDominos(Grid $G$, int $k$) {
    return hasMatching(gridToGraph($G$), $k$);
}
```

Next Time

- ***More Reductions***
  - How do we link problems together?
- ***NP-Hardness***
  - A measure of difficulty.
- ***NP-Completeness***
  - The hardest problems in **NP**.