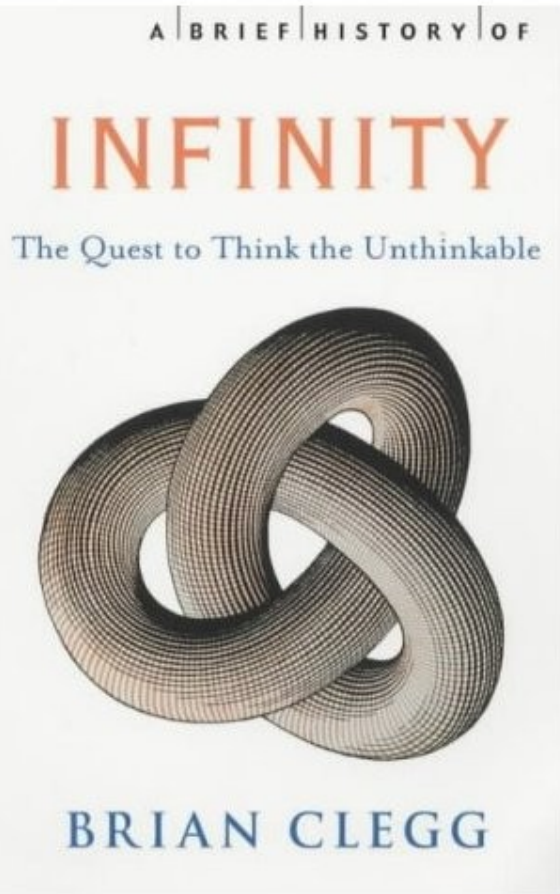
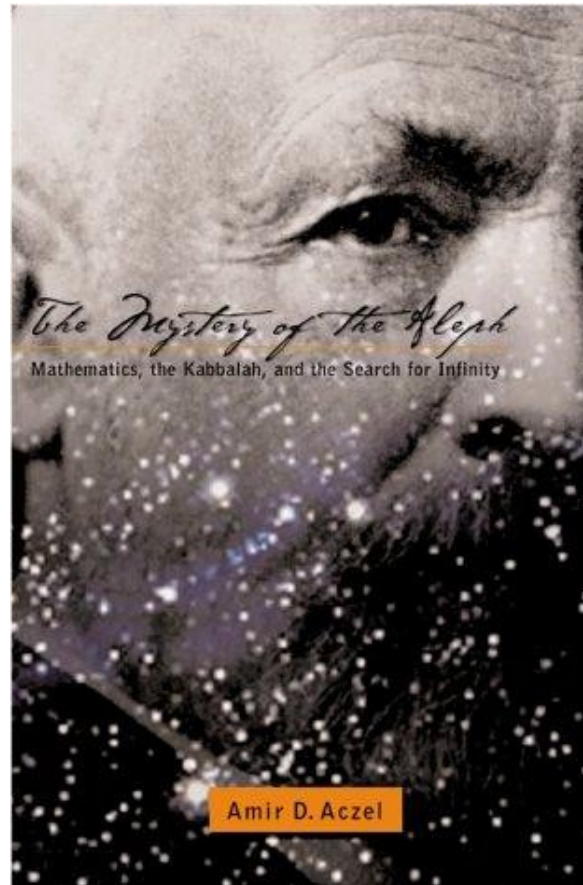


# Direct Proofs

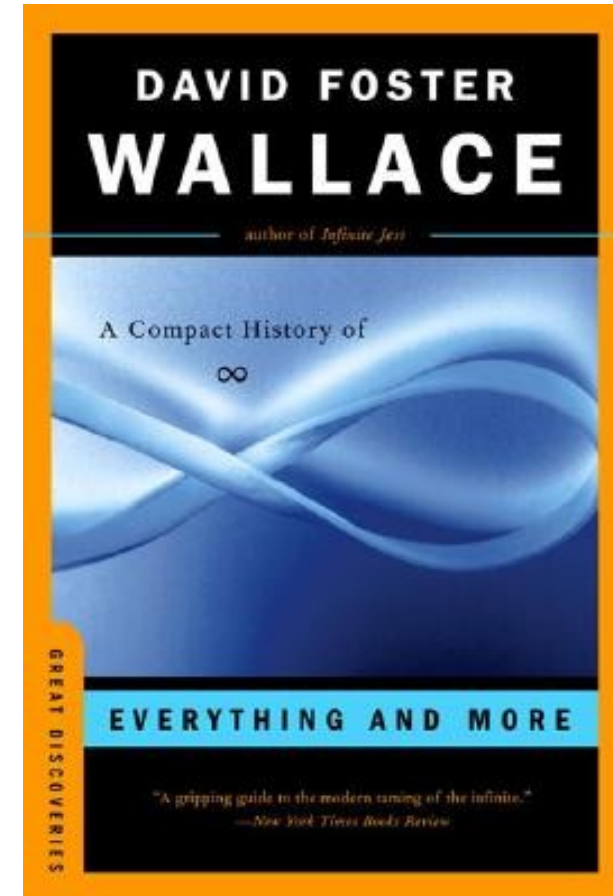
# Recommended Reading



*A Brief History of  
Infinity*



*The Mystery of the  
Aleph*



*Everything and More*

# Recommended Courses

Math 161: Set Theory

# Outline for Today

- ***Mathematical Proof***
  - What is a mathematical proof? What does a proof look like?
- ***Direct Proofs***
  - A versatile, powerful proof technique.
- ***Universal and Existential Statements***
  - What exactly are we trying to prove?
- ***Proofs on Set Theory***
  - Formalizing our reasoning.

What is a Proof?

A *proof* is an argument that demonstrates why a conclusion is true, subject to certain standards of truth.

A ***mathematical proof*** is an argument that demonstrates why a mathematical statement is true, following the rules of mathematics.

\*54·43.  $\vdash :: \alpha, \beta \in 2 = \Lambda . \equiv . \alpha \cup \beta \in 2$

*Dem.*

$\vdash . *54 \dots \vdash :: \alpha = \iota'x . \beta = \iota'y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cup \beta \in 2$   
[\*51·2]  $\dots \equiv . \iota'x \cup \iota'y \in 2 = \Lambda .$   
[\*13·]  $\dots \equiv . \alpha \cap \beta \in 2 \dots$  (1)

$\vdash . (1) \dots 11:35 . \supset$   
 $\vdash . (\alpha = \iota'x . \beta = \iota'y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta \in 2) \dots$  (2)

$\vdash . (2) \dots *52:1 . \supset \vdash . \text{Prop}$

From this proposition it will follow, when a certain relation has been defined, that 1 +



# Modern Proofs

# Two Quick Definitions

- An integer  $n$  is **even** if there is some integer  $k$  such that  $n = 2k$ .
  - This means that 0 is even.
- An integer  $n$  is **odd** if there is some integer  $k$  such that  $n = 2k + 1$ .
  - This means that 0 is not odd.
- We'll assume the following for now:
  - Every integer is either even or odd.
  - No integer is both even and odd.

# Our First Direct Proof

***Theorem:*** If  $n$  is an even integer, then  $n^2$  is even.

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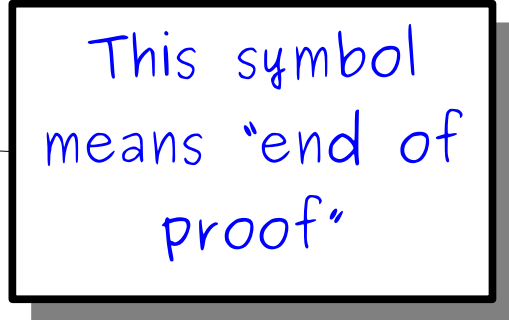
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Therefore,  $n^2$  is even. 



This symbol means "end of proof"

# Our First Direct Proof

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Therefore

To prove a statement of the form

**“If  $P$ , then  $Q$ ”**

Assume that  $P$  is true, then show that  $Q$  must be true as well.

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**Proof:** Let  $n$  be an even integer.

Since  $n$  is even, there is some integer  $k$  such that  $n = 2k$ .

This means that

From this, we can write  $n$  as  $2k$  (namely,  $2k$ ).

Therefore,  $n^2 = (2k)^2 = 4k^2$ .

This is the definition of an even integer. When writing a mathematical proof, it's common to call back to the definitions.



# Our First Direct Proof

***Theorem:*** If  $n$  is an even integer, then  $n^2$  is even.

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m  
Th

Notice how we use the value of  $k$  that we obtained above. Giving names to quantities, even if we aren't fully sure what they are, allows us to manipulate them. This is similar to variables in programs.

# Our First Direct Proof

**Theorem:** If  $n$  is an even integer, then  $n^2$  is even.

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Therefore,  $n^2$  is even. ■

Our ultimate goal is to prove that  $n^2$  is even. This means that we need to find some  $m$  such that  $n^2 = 2m$ . Here, we're explicitly showing how we can do that.

# Our First Direct Proof

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**Proof:** Let  $n$  be an even integer.

Since  $n$  is even, there is some integer  $k$  such that  $n = 2k$ .

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**Proof:** Let  $n$  be an even integer.

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This means

From this we get  $n^2 = 4k^2 = 2(2k^2)$ . Let  $m = 2k^2$  (name

Hey, that's what we were trying to show! We're done now.

Therefore,  $n^2$  is even. ■

# Our First Direct Proof

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Therefore,  $n^2$  is even. ■

That wasn't so bad! Let's do another one.



# Set Theory Review

- Recall from last time that we write  $x \in S$  if  $x$  is an element of set  $S$  and  $x \notin S$  if  $x$  is not an element of set  $S$ .
- If  $S$  and  $T$  are sets, we say that  $S$  is a subset of  $T$  (denoted  $S \subseteq T$ ) if the following statement is true:

**For every object  $x$ , if  $x \in S$ , then  $x \in T$ .**

- Let's explore some properties of the subset relation.

***Theorem:*** For any sets  $A$ ,  $B$ , and  $C$ , if  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

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How do we prove  
that this is true for  
*any* choice of sets?

# Proving Something Always Holds

- Many statements have the form

**For any  $x$ , [some-property] holds of  $x$ .**

- Examples:

For all integers  $n$ , if  $n$  is even,  $n^2$  is even.

For any sets  $A$ ,  $B$ , and  $C$ , if  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

For all sets  $S$ ,  $|S| < |\wp(S)|$ .

Everything that drowns me makes me wanna fly.

- How do we prove these statements when there are (potentially) infinitely many cases to check?

# Arbitrary Choices

- To prove that some property holds true for all possible  $x$ , show that no matter what choice of  $x$  you make, that property must be true.
- Start the proof by choosing  $x$  *arbitrarily*:
  - “Let  $x$  be an arbitrary even integer.”
  - “Let  $x$  be an arbitrary set containing 137.”
  - “Consider any  $x$ .”
  - “Consider an even integer  $x$ .”
- Demonstrate that the property holds true for this choice of  $x$ .

***Theorem:*** For any sets  $A$ ,  $B$ , and  $C$ , if  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

***Proof:***

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We're showing here that regardless of what  $A$ ,  $B$ , and  $C$  you pick, the result will still be true.

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**“If  $P$ , then  $Q$ ”**

Assume that  **$P$**  is true, then show that  **$Q$**  must be true as well.

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This is, by definition, what it means for  **$A \subseteq C$**  to be true. Our job will be to prove this statement.



**Theorem:** For any sets  $A$ ,  $B$ , and  $C$ , if  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

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Consider any  $x$  where  $x \in A$ . Since  $A \subseteq B$  and  $x \in A$ , we see that  $x \in B$ .

**Theorem:** For any sets  $A$ ,  $B$ , and  $C$ , if  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

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Consider any  $x$  where  $x \in A$ . Since  $A \subseteq B$  and  $x \in A$ , we see that  $x \in B$ . Similarly, since  $B \subseteq C$  and  $x \in B$ , we see that  $x \in C$ , which is what we needed to show.

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# Transitivity

- We just proved that if  $A$ ,  $B$  and  $C$  are sets where  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .
- This property of the  $\subseteq$  relation is called ***transitivity***. We'll revisit it later in the quarter.

***Theorem:*** For any sets  $A$ ,  $B$ ,  $C$ , and  $D$ , if  $A \subseteq B$ ,  
 $B \subseteq C$ , and  $C \subseteq D$ , then  $A \subseteq D$ .

***Theorem:*** For any sets  $A$ ,  $B$ ,  $C$ , and  $D$ , if  $A \subseteq B$ ,  
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***Proof:***

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Didn't we just prove something about this?

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We're heavily leveraging our previous proof here. *This is extremely common!* Most of mathematics is about building on earlier results.

# The Story So Far

- If you need to prove an implication (a statement of the form “if  $P$ , then  $Q$ ”), you should assume  $P$  is true, then prove  $Q$  is true.
- To prove a statement of the form “for all  $x$ , some property  $P(x)$  is true,” state that you're choosing an arbitrary  $x$ , then prove that  $P(x)$  must be true.
- Proofs usually call back to some key terms or definitions (here, even numbers, and subsets).
- Proofs build on top of one another. Once you do enough math, you start to use older proofs as building blocks in larger proofs.

# How Not To Write Proofs



# An Incorrect Proof

**Theorem:** For all sets  $A$  and  $B$ , we have  $A \subseteq A \cap B$ .

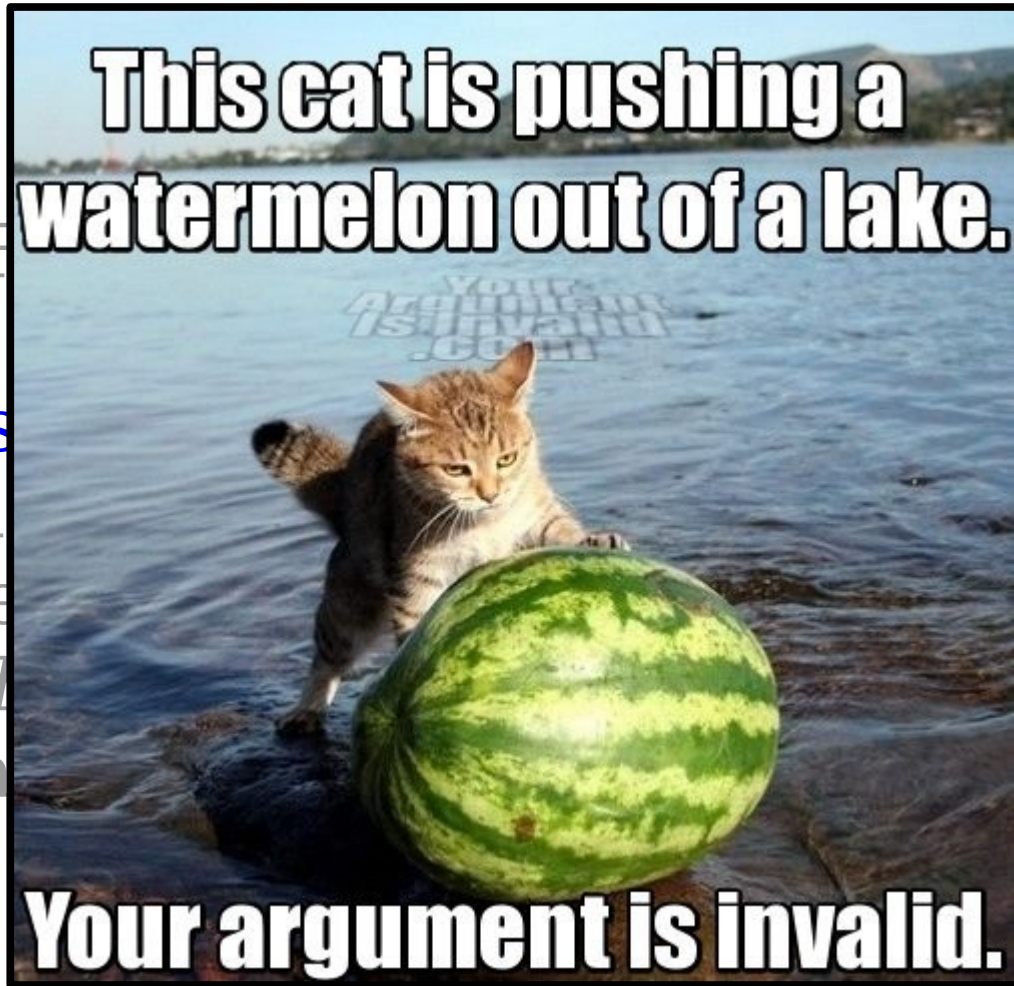
**Proof:** Consider two arbitrary sets, say,  $A = \emptyset$  and  $B = \mathbb{N}$ . Since  $\emptyset$  is a subset of every set and  $A = \emptyset$ , we see that  $A \subseteq A \cap B$ . Since our choices of  $A$  and  $B$  were arbitrary, we conclude that if  $A$  and  $B$  are any sets, then  $A \subseteq A \cap B$ . ■

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adjective /'ärbi,trerē/

1. Based on random choice or personal whim, rather than any reason or system - *“his mealtimes were entirely arbitrary”*
2. *(of power or a ruling body)* Unrestrained and autocratic in the use of authority - *“arbitrary rule by King and bishops has been made impossible”*
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Use this  
definition...

To prove something is true for all  $x$ ,  
don't choose an  $x$  and base the proof  
off of your choice.

Instead, leave  $x$  unspecified  
and show that no matter what  $x$  is,  
the specified property must hold.

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If you want to prove that  $P$  implies  $Q$ ,  
assume  $P$  and prove  $Q$ .

***Don't*** assume  $Q$  and then prove  $P$ !

# Special Classes of Statements

# Universal and Existential Statements

# An Entirely Different Proof

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*Theorem:* **There exists** a natural number  $n > 0$  **such that** the sum of all natural numbers less than  $n$  is equal to  $n$ .

This is a fundamentally different type of proof that what we've done before. Instead of showing that every object has some property, we want to show that some object has a given property.

# Universal vs. Existential Statements

- A ***universal statement*** is a statement of the form  
**For all  $x$ , [some-property] holds for  $x$ .**
- We've seen how to prove these statements.
- An ***existential statement*** is a statement of the form  
**There is some  $x$  where [some-property] holds for  $x$ .**
- How do you prove an existential statement?



# Proving an Existential Statement

- Over the course of the quarter, we will see several different ways to prove an existential statement.
- ***Simplest approach:*** Just go and find some  $x$  where the property  $P(x)$  is true.
  - In our case, we need to find a positive natural number  $n$  such that that sum of all smaller natural numbers is equal to  $n$ .
  - Can we find one?

# An Entirely Different Proof

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# An Entirely Different Proof

**Theorem:** There exists a natural number  $n > 0$  such that the sum of all natural numbers less than  $n$  is equal to  $n$ .

**Proof:** Take  $n = 3$ .

The three natural numbers smaller than three are 0, 1, and 2.

Notice that  $0 + 1 + 2 = 3$ .

Therefore, there is a natural number greater than zero equal to the sum of all smaller natural numbers. ■

# Biconditional Statements

# Biconditionals

- You often see statements like these in mathematics:
  - The natural number  $n$  is even *if and only if*  $n^2$  is even.
  - $G$  is bipartite *if and only if*  $G$  has no odd cycles.
  - $L \in \mathbf{RE}$  *if and only if* there is a verifier for  $L$ .
- All of these statements involve the phrase “if and only if.”
- The statement “ $P$  if and only if  $Q$ ” means the following:

**If  $P$ , then  $Q$ , and if  $Q$ , then  $P$ .**
- In other words, both  $P$  and  $Q$  imply each other.
- These statements are called ***biconditional statements***. To prove a biconditional statement, you usually prove both implications separately.

# Set Equality

- If  $A$  and  $B$  are sets, we say that  $A = B$  precisely when the following statement is true:

**For any object  $x$ ,  $x \in A$  if and only if  $x \in B$ .**

- (This is called the *axiom of extensionality*.)
- In practice, this definition is tricky to work with.
- It's often easier to use the following result to show that two sets are equal:

**For any sets  $A$  and  $B$ ,  
if  $A \subseteq B$  and  $B \subseteq A$ , then  $A = B$ .**

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***Proof:***



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**Time-Out for Announcements!**

# A Quick Introduction

# APPLY TO JOIN CS+SOCIAL GOOD

Empower 150+ students to build impactful projects and gain exposure to the tech + social impact space by leading classes such as CS50, Studio, CS1065, and CS53

Start conversations about the role tech can play in issues like poverty, mental health, and discrimination by organizing mixers and discussions

Work directly with Silicon Valley companies, nonprofits, and community organizers to change tech culture for the better

Meet and collaborate with likeminded students who are passionate and curious about using CS to make a positive impact

To learn more about our work, visit [cs4good.com](http://cs4good.com)



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**JOIN THE TEAM AT: [bit.ly/cssg1718](http://bit.ly/cssg1718)**  
**FORM CLOSING SATURDAY, OCT 7 AT MIDNIGHT**

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# CS Research Opportunities

- Interested in trying out CS research? Here's some ways to do so:
  - Want to see what undergrad research is like? There's a CURIS poster session **tomorrow, 4:30PM - 6:00PM** on the **Packard lawn**.
  - Interested in seeing what our professors are up to? Check out **CS300**, a seminar course on research opportunities.
  - Want to join in the fun? **CURIS** is now open year-round and is a great way to try your hand at research.
- If you have any questions, feel free to contact [Griffin Dietz](#) or [Kexin Rong](#).

# Reading Recommendations

- We've released two handouts you should read over:
  - Handout 06: How to Succeed in CS103
  - Handout 07: Set Theory Definitions.
- Additionally, if you haven't yet read over the Guide to Elements and Subsets, we'd recommend doing so.
- Finally, we strongly recommend reading over Chapter 1 and Chapter 2 of the online course reader to get some more background with proofs and set theory.



# Piazza

- We have a Piazza site for CS103.
- Sign in to [www.piazza.com](http://www.piazza.com) and search for the course CS103 to sign in.
- Feel free to ask us questions!
- ***Use the site to find a partner for the problem sets!***

# Problem Set 0

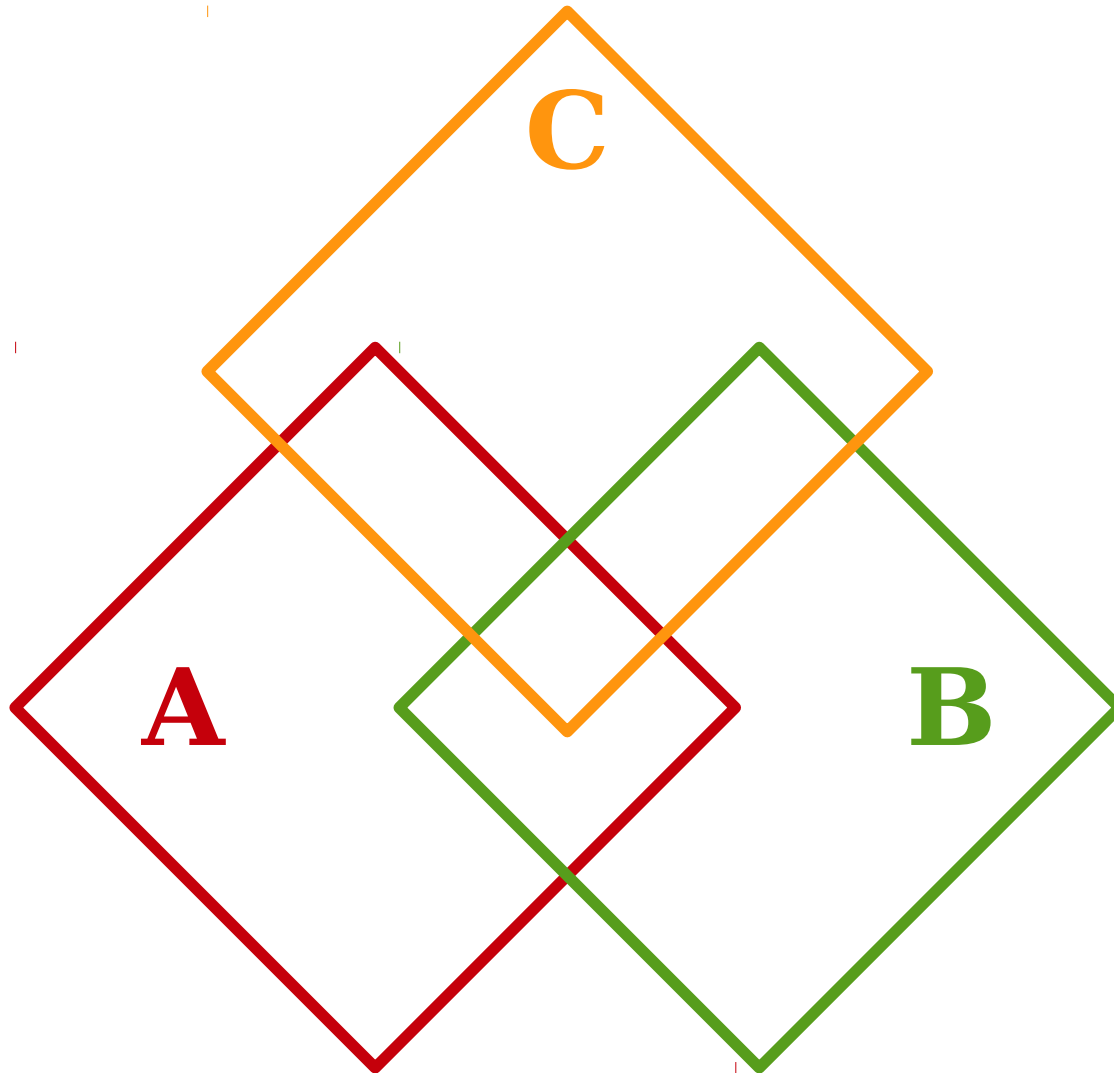
- Problem Set 0 went out on Monday. It's due this Friday at 2:30PM.
  - We have some office hours scheduled over the week to help with Qt Creator issues. Feel free to stop on by! A schedule is on the course website.
  - Even though this just involves setting up your compiler and submitting things, please start this one early. If you start things on Friday morning, we can't help you troubleshoot Qt Creator issues!

Back to CS103!

# Proofs on Set Combinations

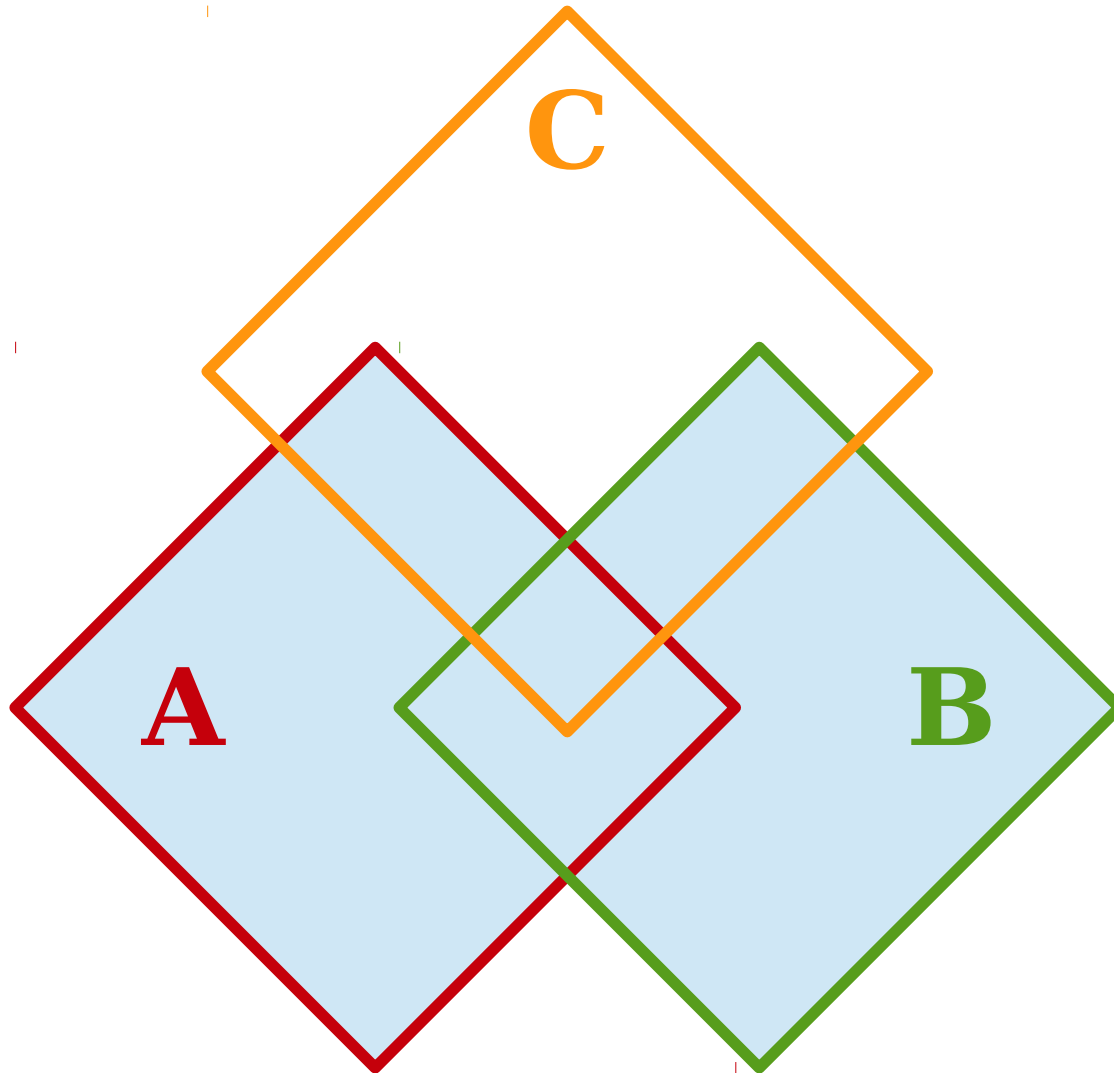
**Theorem:** Let  $A$ ,  $B$ , and  $C$  be any sets. Then

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$



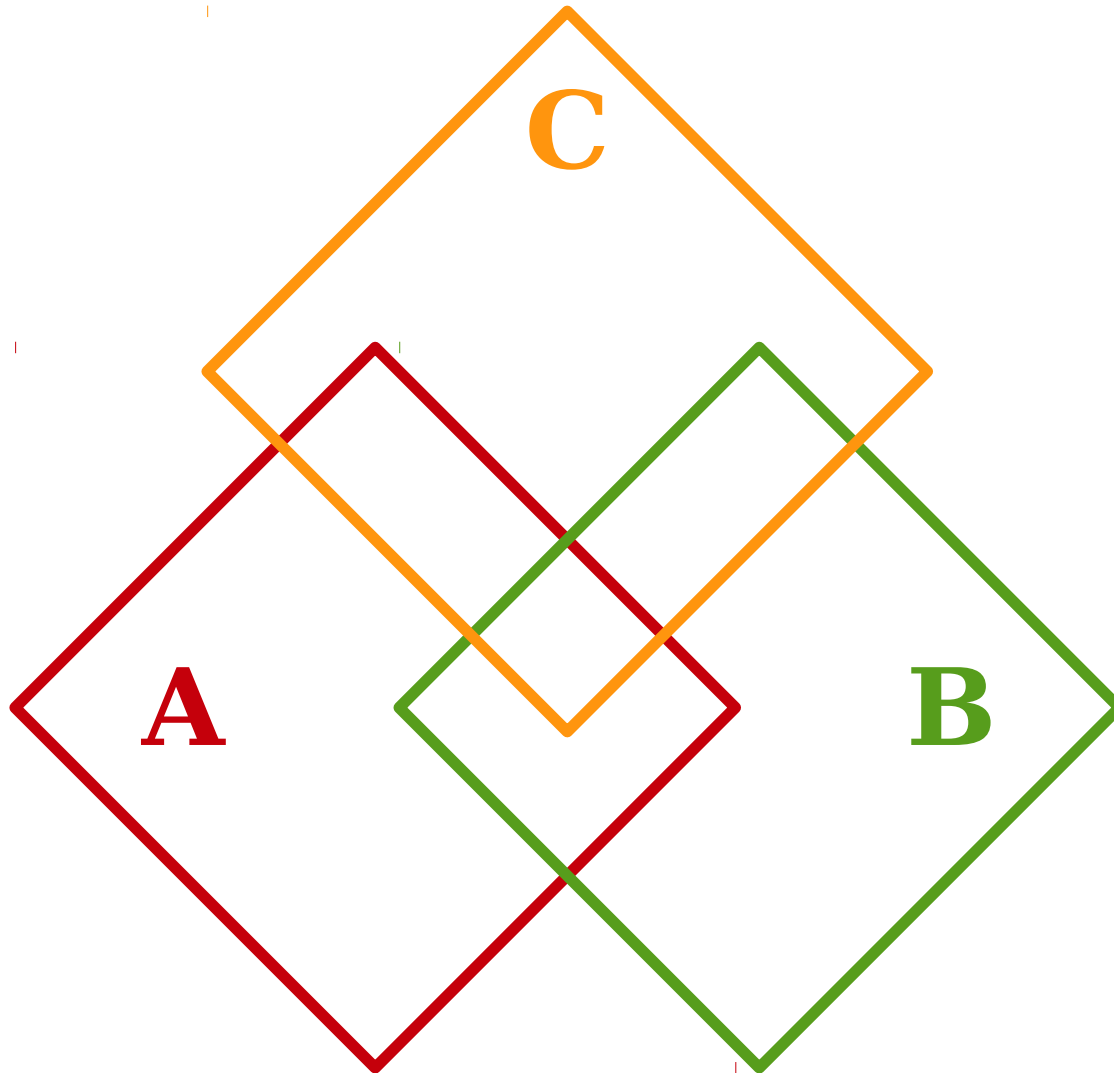
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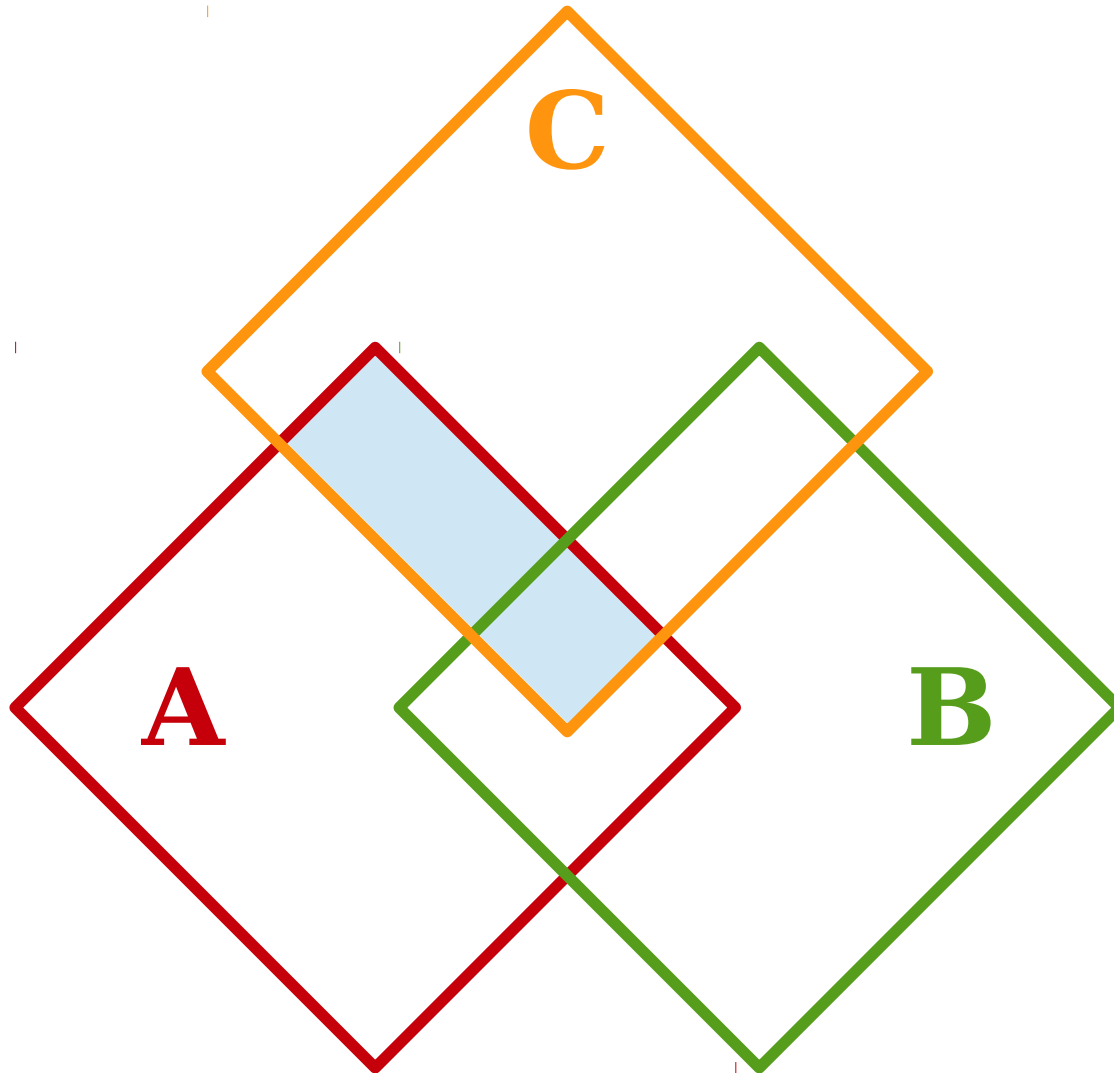
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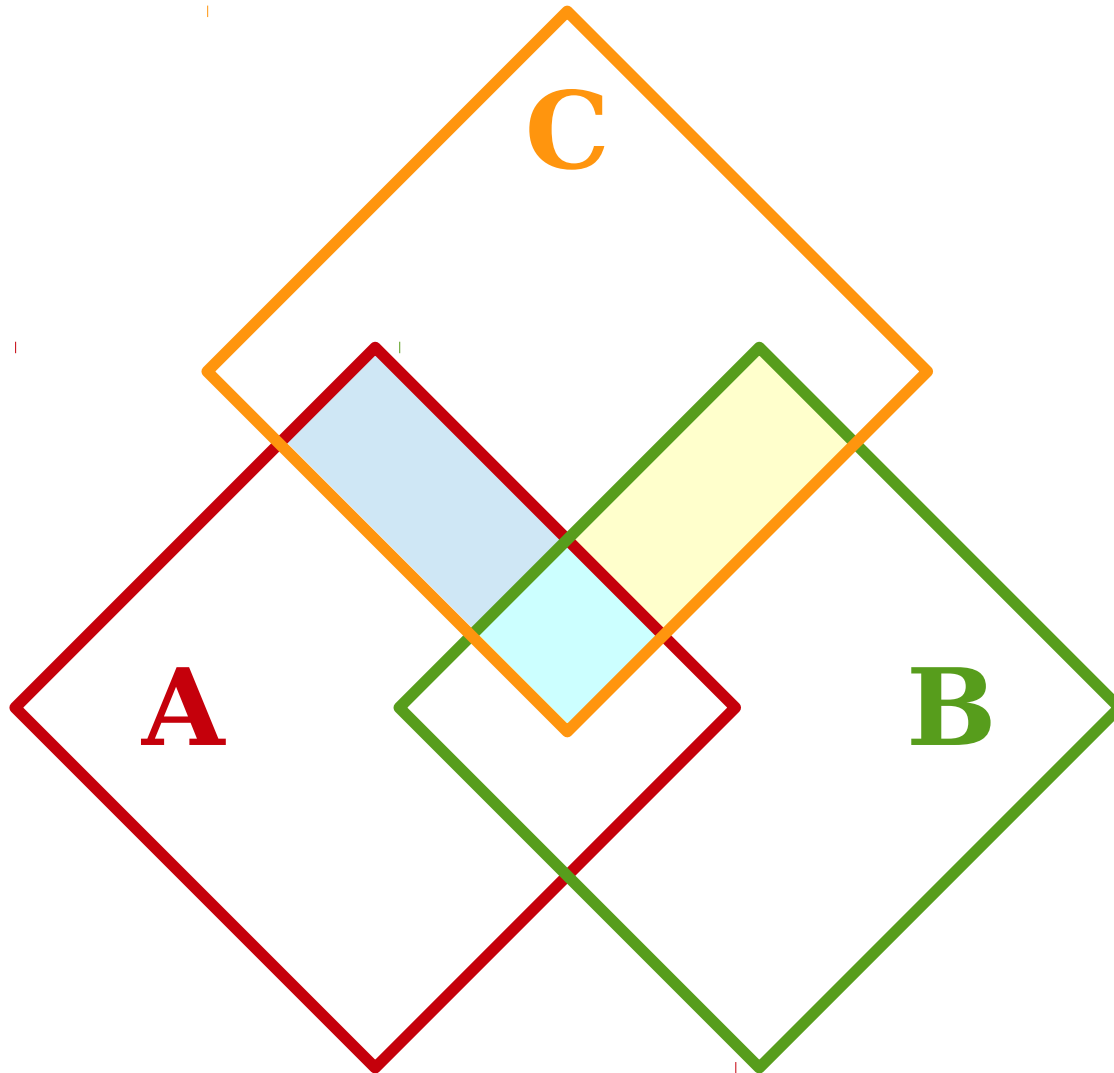
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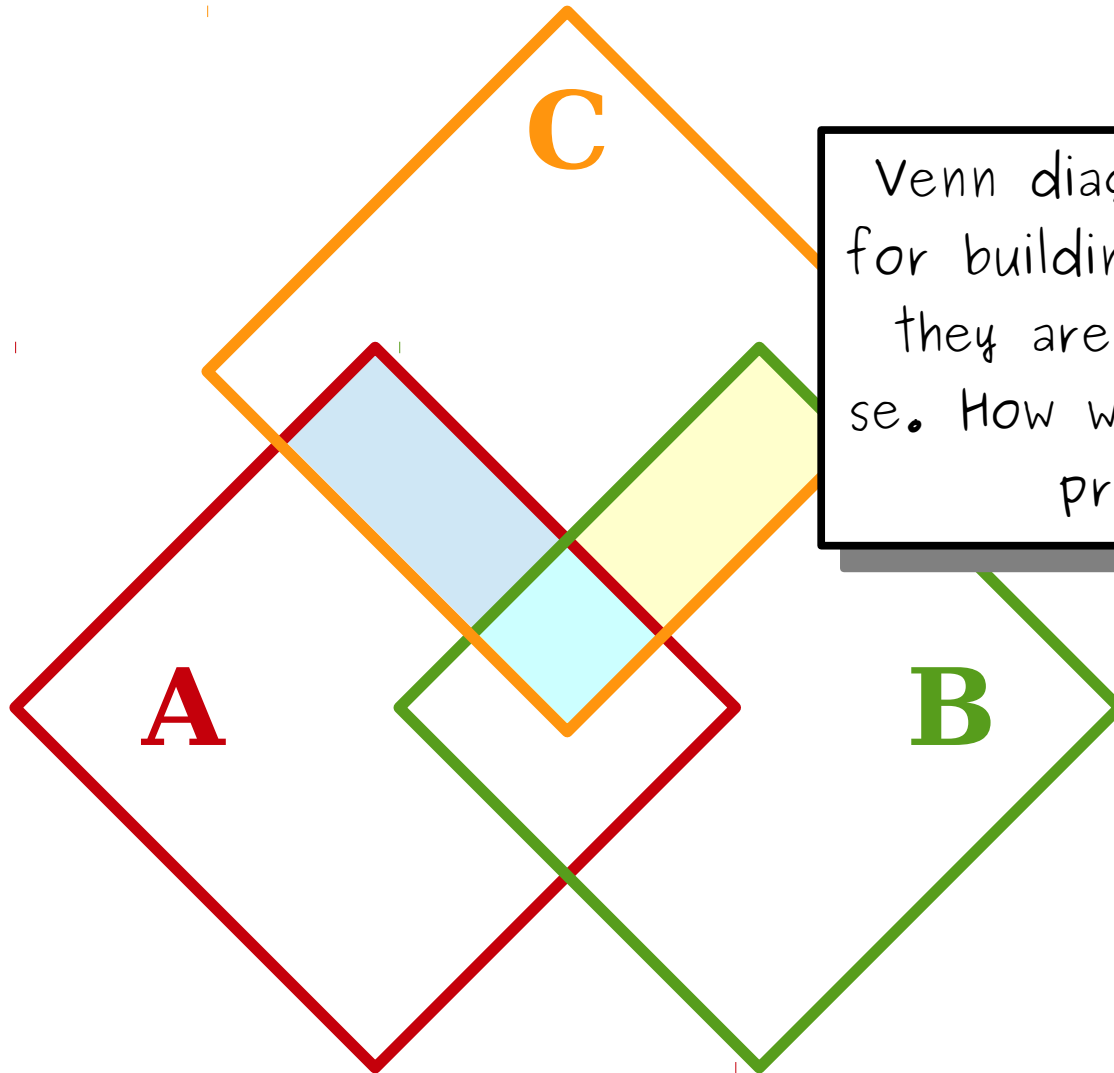
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Venn diagrams are great for building intuitions, but they aren't proofs per se. How would we formally prove this?

**Theorem:** Let  $A$ ,  $B$ , and  $C$  be any sets. Then

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

To prove this, we'll prove each is a subset of the other:

$$(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C)$$

$$(A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap C$$

# Relations Between Proofs

- Proofs often build off of one another: large results are almost often accomplished by building off of previous work.
  - Like writing a large program - split the work into smaller methods, across different classes, etc. instead of putting the whole thing into `main`.
- A result that is proven specifically as a stepping stone toward a larger result is called a *lemma*.
- We'll prove each of these smaller statements as a lemma for part of the larger proof. It's decomposition, proof style!

***Lemma 1:*** For any sets  $A$ ,  $B$ , and  $C$ , the following is true:

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... what, exactly?

# The Need for Formalisms

- In all of our previous proofs, we've used formal definitions for terms that we have a good intuitive feel for.
  - Even numbers, subsets, set equality, etc.
- Mathematics revolves around formal definitions. Definitions give us a clear sense of what something means and guides the structure of our proofs.

# Set Operations

- Last time, we introduced four operations for combining sets together:
  - Union:  $S \cup T$
  - Intersection:  $S \cap T$
  - Difference:  $S - T$
  - Symmetric Difference:  $S \Delta T$
- While we may have an intuitive feel for these terms, we haven't actually defined them anywhere. Therefore, we can't rigorously reason about them in proofs.
- Let's go fix that!

# Set Operations

- The union of two sets  $S$  and  $T$  is defined as follows:

$$\mathbf{S \cup T = \{ x \mid x \in S \text{ or } x \in T \text{ (or both) } \}}$$

- As a result, if  $x$  is any object, then  $x \in S \cup T$  if and only if  $x \in S$  or  $x \in T$  (or both).
- The intersection of two sets  $S$  and  $T$  is defined as follows:

$$\mathbf{S \cap T = \{ x \mid x \in S \text{ and } x \in T \}}$$

- As a result, if  $x$  is any object, then  $x \in S \cap T$  if and only if  $x \in S$  and  $x \in T$ .
- We can use these definitions and these properties to write formal proofs about unions and intersections.

**Lemma 1:** For any sets  $A$ ,  $B$ , and  $C$ , the following is true:

$$(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C).$$

**Proof:** Consider any arbitrary sets  $A$ ,  $B$ , and  $C$ . We need to prove that  $(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C)$ . To do so, consider an arbitrary  $x \in (A \cup B) \cap C$ . We will prove that  $x \in (A \cap C) \cup (B \cap C)$ .

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This might be a problem – we can't say for certain which of  $A$  or  $B$  will contain  $x$ !  
Fortunately, there's a way around this.

**Lemma 1:** For any sets  $A$ ,  $B$ , and  $C$ , the following is true:

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This is called a **proof by cases** (alternatively, a **proof by exhaustion**) and works by showing that the theorem is true regardless of what specific outcome arises.

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**Case 1:  $x \in A$ .** Then since  $x \in A$  and  $x \in C$ , we know that  $x \in A \cap C$ . Therefore,  $x \in (A \cap C) \cup (B \cap C)$ .

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**Case 1:  $x \in A$ .** Then since  $x \in A$  and  $x \in C$ , we know that  $x \in A \cap C$ . Therefore,  $x \in (A \cap C) \cup (B \cap C)$ .

**Case 2:  $x \in B$ .** Then since  $x \in B$  and  $x \in C$ , we know that  $x \in B \cap C$ . Therefore,  $x \in (A \cap C) \cup (B \cap C)$ .

In both cases, we see that  $x \in (A \cap C) \cup (B \cap C)$ , which is what we needed to show.

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Since  $x \in (A \cup B) \cap C$ , we know that  $x \in A \cup B$  and that  $x \in C$ . Because  $x \in A \cup B$ , we know that  $x \in A$  or  $x \in B$  (or both). We consider two cases:

**Case 1:  $x \in A$ .** Then since  $x \in C$ , we know that  $x \in A \cap C$ . Therefore,  $x \in (A \cap C) \cup (B \cap C)$ .

**Case 2:  $x \in B$ .** Then since  $x \in C$ , we know that  $x \in B \cap C$ . Therefore,  $x \in (A \cap C) \cup (B \cap C)$ .

After splitting into cases, it's a good idea to summarize what you just did so that the reader knows what to take away from it.

In both cases, we see that  $x \in (A \cap C) \cup (B \cap C)$ , which is what we needed to show.

**Lemma 1:** For any sets  $A$ ,  $B$ , and  $C$ , the following is true:

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Since  $x \in (A \cup B) \cap C$ , we know that  $x \in A \cup B$  and that  $x \in C$ . Because  $x \in A \cup B$ , we know that  $x \in A$  or  $x \in B$  (or both). We consider two cases:

**Case 1:  $x \in A$ .** Then since  $x \in A$  and  $x \in C$ , we know that  $x \in A \cap C$ . Therefore,  $x \in (A \cap C) \cup (B \cap C)$ .

**Case 2:  $x \in B$ .** Then since  $x \in B$  and  $x \in C$ , we know that  $x \in B \cap C$ . Therefore,  $x \in (A \cap C) \cup (B \cap C)$ .

In both cases, we see that  $x \in (A \cap C) \cup (B \cap C)$ , which is what we needed to show. ■



**Lemma 2:** For any sets  $A$ ,  $B$ , and  $C$ , the following is true:

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Since  $x \in (A \cap C) \cup (B \cap C)$ , we know that either  $x \in A \cap C$  or that  $x \in B \cap C$  (or both). We consider two cases:

**Case 1:  $x \in A \cap C$ .** This means that  $x \in A$  and  $x \in C$ . Since  $x \in A$ , we see that  $x \in A \cup B$ . Therefore, we see that  $x \in (A \cup B) \cap C$ .

**Case 2:  $x \in B \cap C$ .** This means that  $x \in B$  and  $x \in C$ . Since  $x \in B$ , we see that  $x \in A \cup B$ . Therefore, we see that  $x \in (A \cup B) \cap C$ .

In both cases, we see that  $x \in (A \cup B) \cap C$ , which is what we needed to show. ■

**Theorem:** For any sets  $A$ ,  $B$ , and  $C$ , the following is true:

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# What We've Covered

- ***What is a mathematical proof?***
  - An argument – mostly written in English – outlining a mathematical argument.
- ***What is a direct proof?***
  - It's a proof where you begin from some initial assumptions and reason your way to the conclusion.
- ***What are universal and existential statements?***
  - Universal statements make a claim about all objects of one type. Existential statements make claims about at least one object of some type.
- ***How do we write proofs about set theory?***
  - By calling back to definitions! Definitions are key.

# Next Time

- ***Indirect Proofs***
  - How do you prove something without actually proving it?
- ***Mathematical Implications***
  - What exactly does “if  $P$ , then  $Q$ ” mean?
- ***Proof by Contrapositive***
  - A helpful technique for proving implications.
- ***Proof by Contradiction***
  - Proving something is true by showing it can't be false.