

Unsolvable Problems

Part Two

Outline for Today

- ***Recap from Last Time***
 - Where are we, again?
- ***A Different Perspective on RE***
 - What exactly does “recognizability” mean?
- ***Verifiers***
 - A new approach to problem-solving.
- ***Beyond RE***
 - Monstrously hard problems!

Recap from Last Time

Self-Referential Programs

- ***Claim:*** Any program can be augmented to include a method called `mySource()` that returns a string representation of its source code.
- ***Theorem:*** It is possible to build Turing machines that get their own encodings and perform arbitrary computations on them.

What does this program do?

```
bool willAccept(string program, string input) {  
    /* ... some implementation ... */  
}  
  
int main() {  
    string me = mySource();  
    string input = getInput();  
  
    if (willAccept(me, input)) {  
        reject();  
    } else {  
        accept();  
    }  
}
```

What happens if...

... this program accepts its input?
It rejects the input!

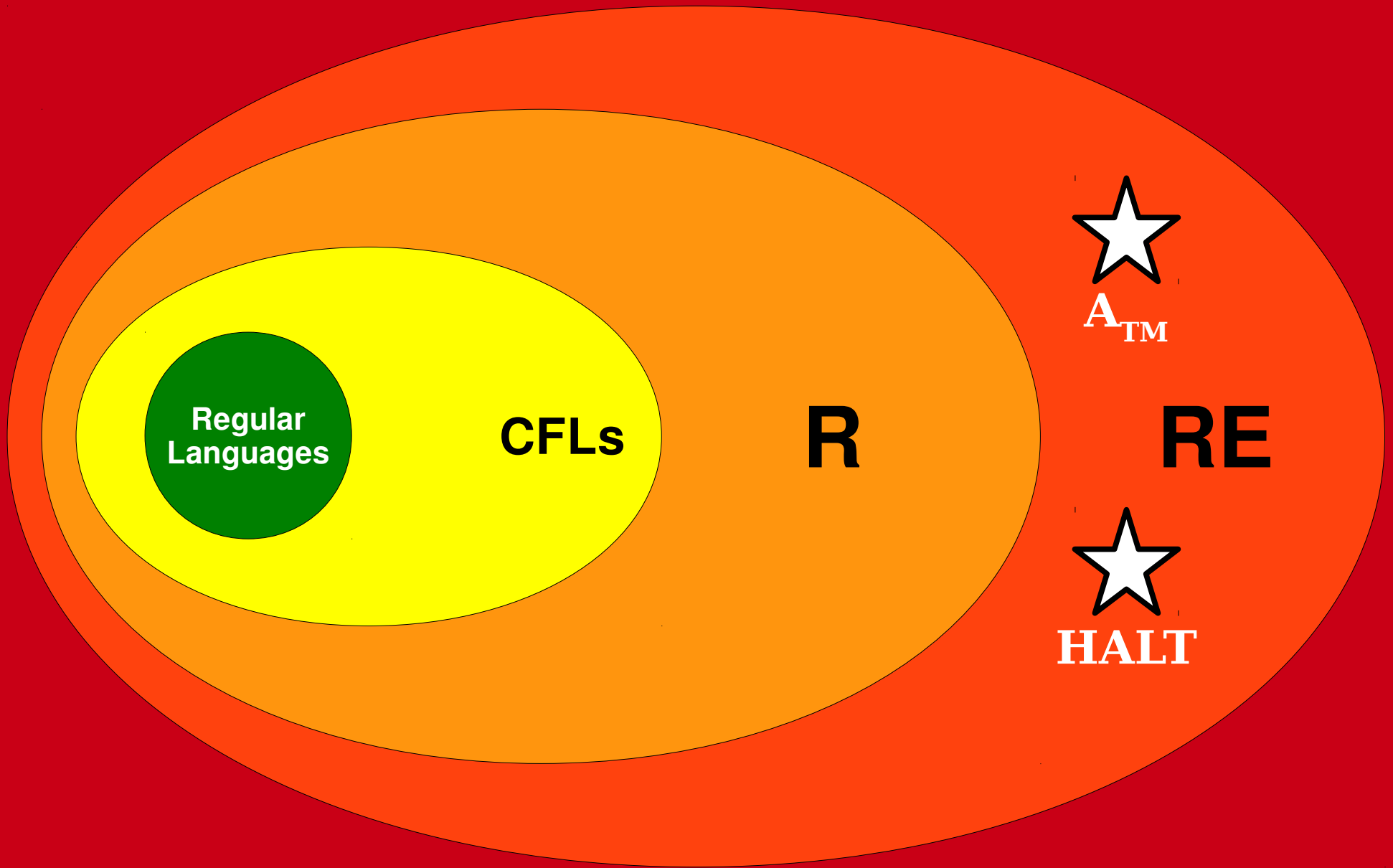
... this program doesn't accept its input?
It accepts the input!

What does this program do?

```
bool willHalt(string program, string input) {  
    /* ... some implementation ... */  
}  
  
int main() {  
    string me = mySource();  
    string input = getInput();  
  
    if (willHalt(me, input)) {  
        while (true) {  
            // loop infinitely  
        }  
    } else {  
        accept();  
    }  
}
```

What happens if...

- ... this program halts on this input?
It loops on the input!
- ... this program loops on this input?
It halts on the input!



All Languages

New Stuff!

Beyond **R** and **RE**

Beyond **R** and **RE**

- We've now seen how to use self-reference as a tool for showing undecidability (finding languages not in **R**).
- We still have not broken out of **RE** yet, though.
- To do so, we will need to build up a better intuition for the class **RE**.

What exactly is the class **RE**?

RE, Formally

- Recall that the class **RE** is the class of all recognizable languages:

$$\mathbf{RE} = \{ L \mid \text{there is a TM } M \text{ where } \mathcal{L}(M) = L \}$$

- Since $\mathbf{R} \neq \mathbf{RE}$, there is no general way to “solve” problems in the class **RE**, if by “solve” you mean “make a computer program that can always tell you the correct answer.”
- So what exactly *are* the sorts of languages in **RE**?

Key Intuition:

A language L is in **RE** if, for any string w , if you are *convinced* that $w \in L$, there is some way you could prove that to someone else.

Verification

		7		6		1		
					3		5	2
3			1		5	9		7
6		5		3		8		9
	1						2	
8		2		1		5		4
1		3	2		7			8
5	7		4					
		4		8		7		

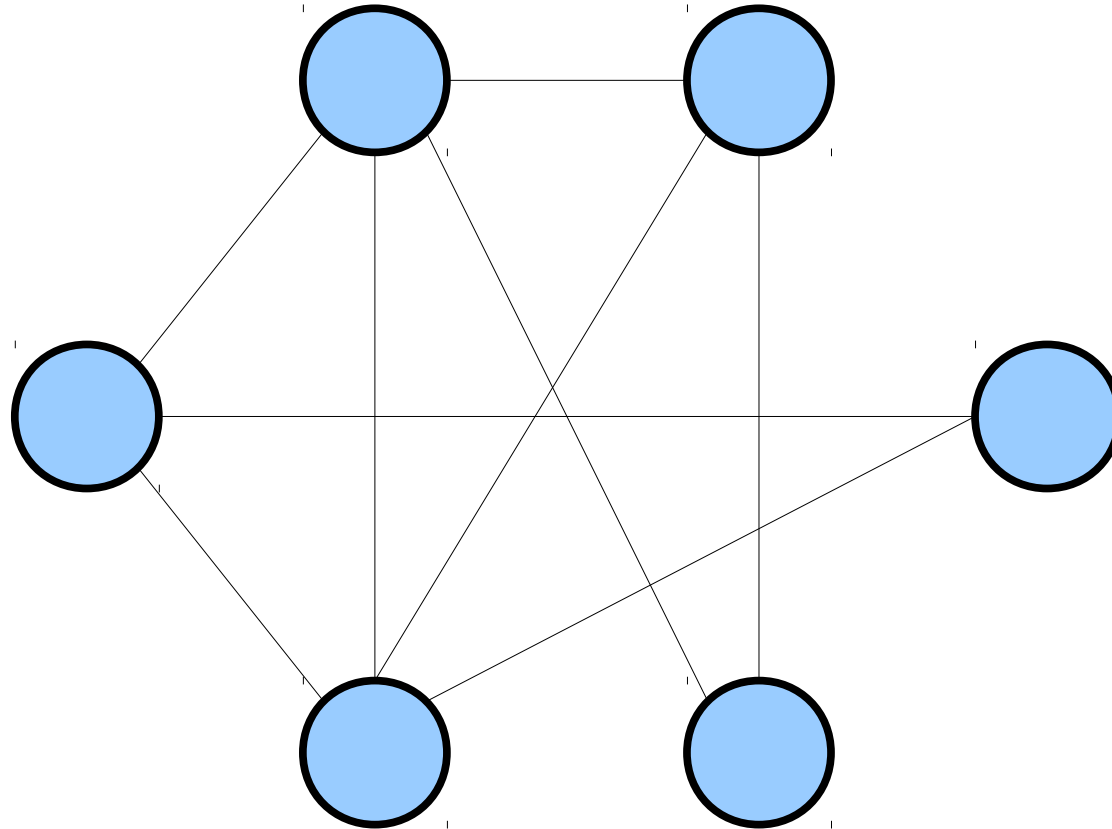
Does this Sudoku puzzle
have a solution?

Verification

2	5	7	9	6	4	1	8	3
4	9	1	8	7	3	6	5	2
3	8	6	1	2	5	9	4	7
6	4	5	7	3	2	8	1	9
7	1	9	5	4	8	3	2	6
8	3	2	6	1	9	5	7	4
1	6	3	2	5	7	4	9	8
5	7	8	4	9	6	2	3	1
9	2	4	3	8	1	7	6	5

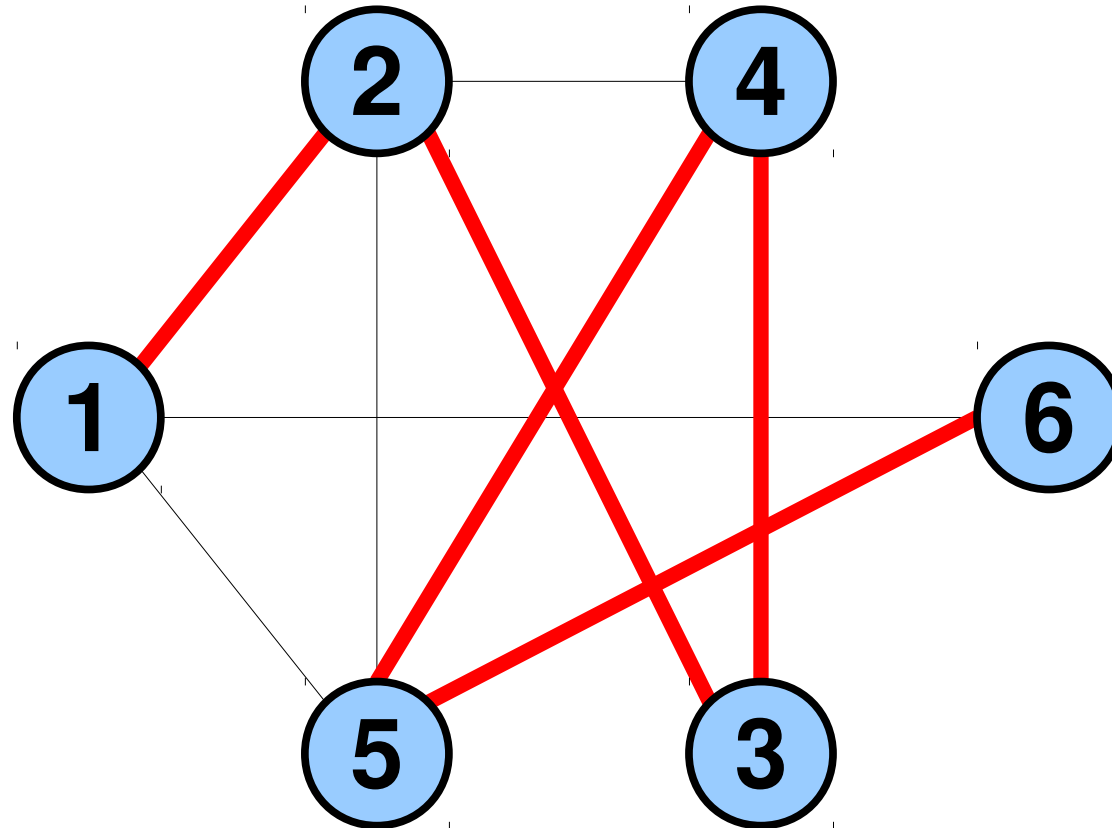
Does this Sudoku puzzle
have a solution?

Verification



Does this graph have a ***Hamiltonian path*** (a simple path that passes through every node exactly once?)

Verification



Does this graph have a ***Hamiltonian path*** (a simple path that passes through every node exactly once?)

Verification

11

Does the hailstone sequence
terminate for this number?

Verification

11

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

34

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

17

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

52

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

26

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

13

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?

Verification

40

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

20

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?

Verification

10

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?

Verification

5

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

16

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

8

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

4

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

2

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

1

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

		7		6		1		
					3		5	2
3			1		5	9		7
6		5		3		8		9
	1						2	
8		2		1		5		4
1		3	2		7			8
5	7		4					
		4		8		7		

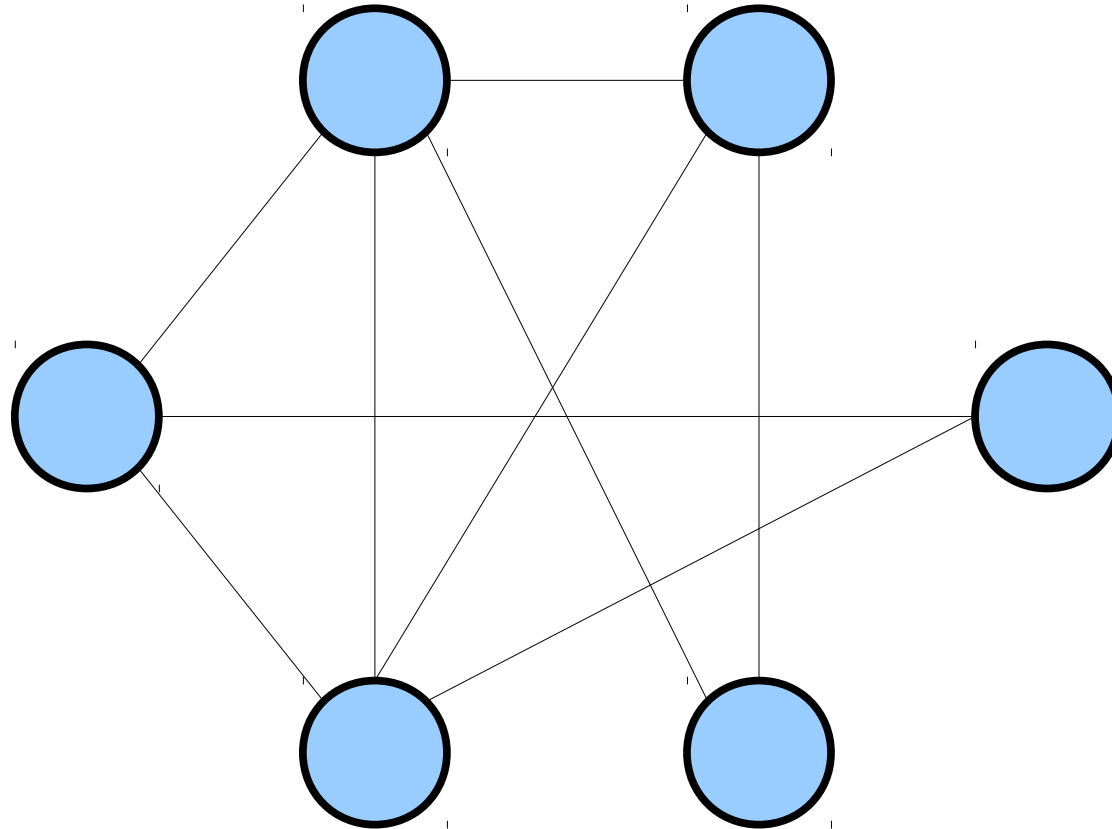
Does this Sudoku puzzle
have a solution?

Verification

1	1	7	1	6	1	1	1	1
1	1	1	1	1	3	1	5	2
3	1	1	1	1	5	9	1	7
6	1	5	1	3	1	8	1	9
1	1	1	1	4	1	1	2	1
8	1	2	1	1	1	5	1	4
1	1	3	2	1	7	1	1	8
5	7	1	4	1	1	1	1	1
1	1	4	1	8	1	7	1	1

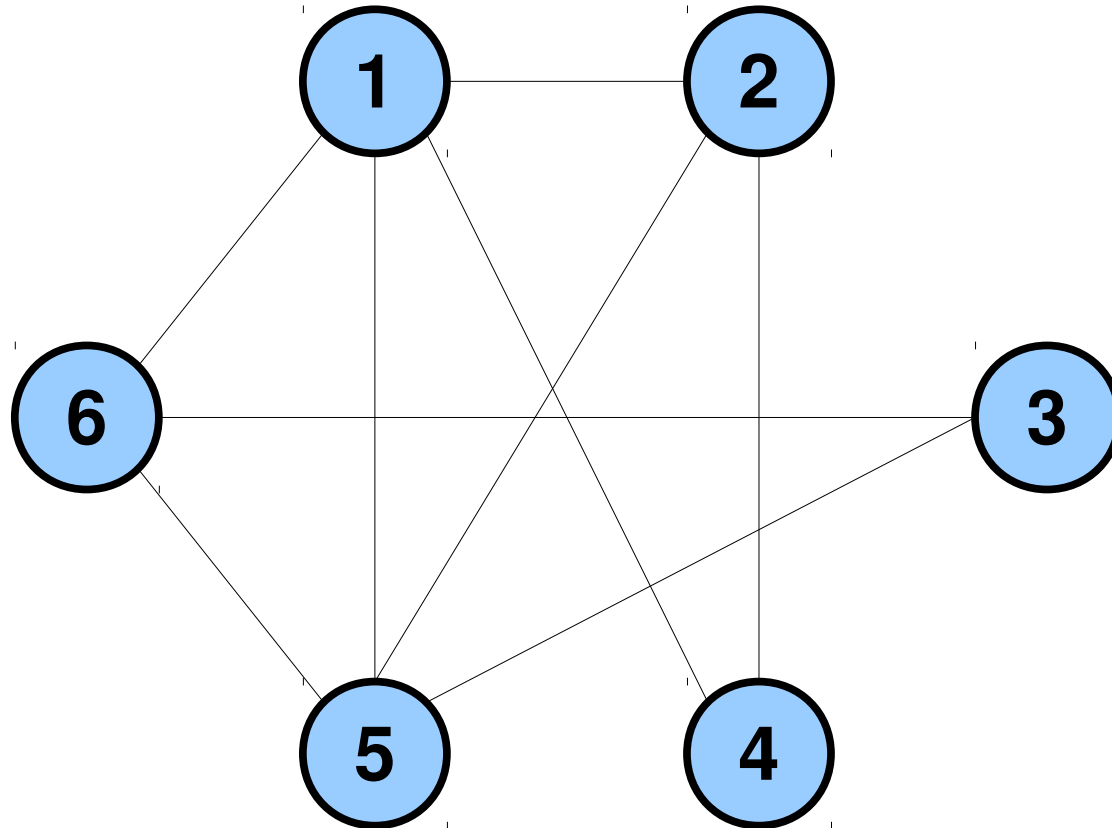
Does this Sudoku puzzle
have a solution?

Verification



Does this graph have a ***Hamiltonian path*** (a simple path that passes through every node exactly once?)

Verification



Does this graph have a ***Hamiltonian path*** (a simple path that passes through every node exactly once?)

Verification

11

Does the hailstone sequence
terminate for this number?

Verification

11

Try running five steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

34

Try running five steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

17

Try running five steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

52

Try running five steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

26

Try running five steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

13

Try running five steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

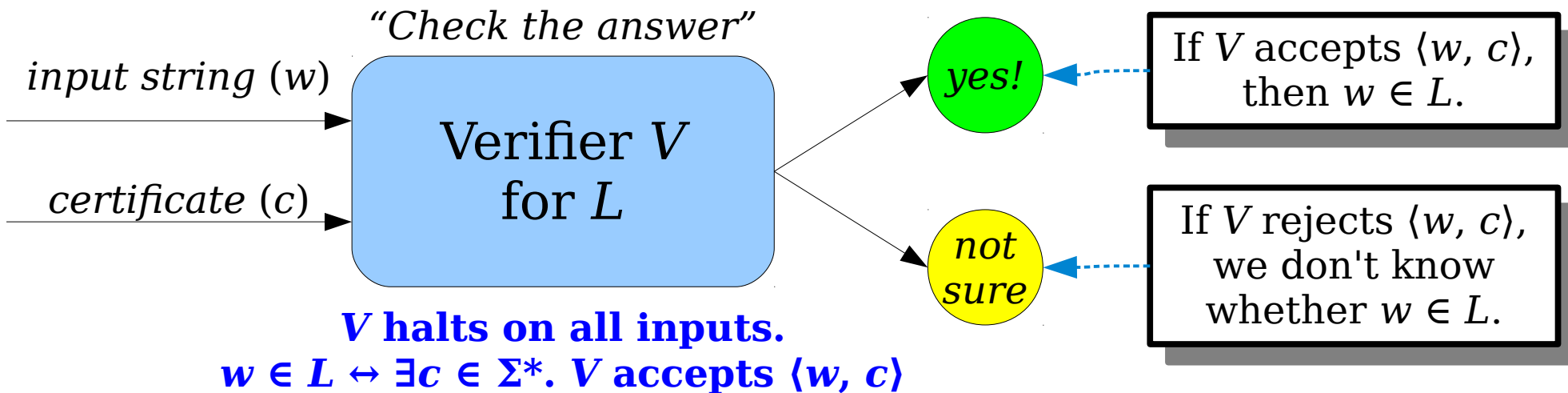
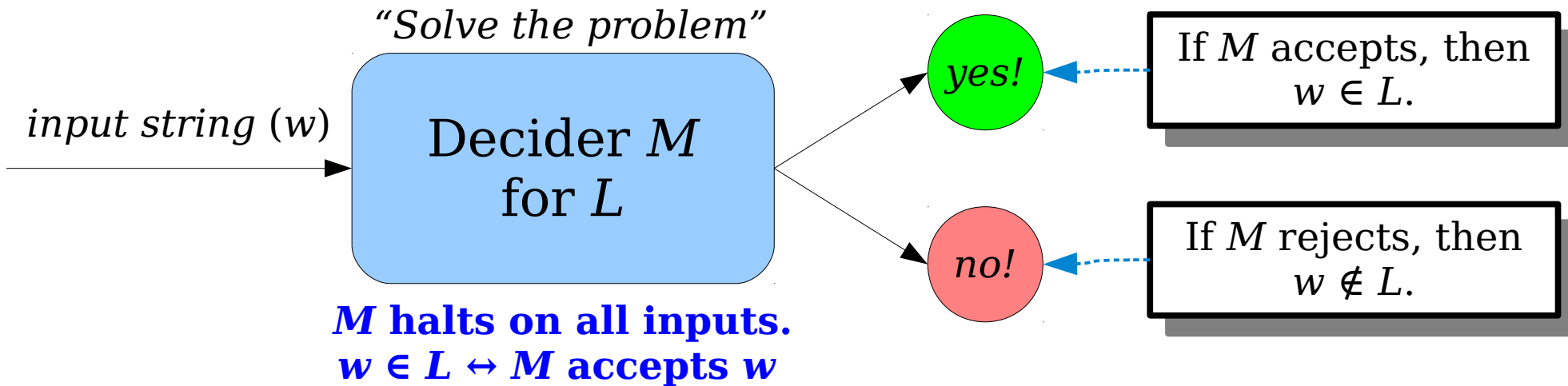
Verification

- In each of the preceding cases, we were given some problem and some evidence supporting the claim that the answer is “yes.”
- Given the correct evidence, we can be certain that the answer is indeed “yes.”
- Given incorrect evidence, we aren't sure whether the answer is “yes.”
 - Maybe there's *no* evidence saying that the answer is “yes,” or maybe there is some evidence, but just not the evidence we were given.
- Let's formalize this idea.

Verifiers

- A **verifier** for a language L is a TM V with the following properties:
 - V halts on all inputs.
 - For any string $w \in \Sigma^*$, the following is true:
$$w \in L \leftrightarrow \exists c \in \Sigma^*. V \text{ accepts } \langle w, c \rangle$$
- A string c where V accepts $\langle w, c \rangle$ is called a **certificate** for w .
- Intuitively, what does this mean?

Deciders and Verifiers



Verifiers

- A **verifier** for a language L is a TM V with the following properties:
 - V halts on all inputs.
 - For any string $w \in \Sigma^*$, the following is true:
$$w \in L \iff \exists c \in \Sigma^*. V \text{ accepts } \langle w, c \rangle$$
- Some notes about V :
 - If V accepts $\langle w, c \rangle$, then we're guaranteed $w \in L$.
 - If V does not accept $\langle w, c \rangle$, then either
 - $w \in L$, but you gave the wrong c , or
 - $w \notin L$, so no possible c will work.

Verifiers

- A **verifier** for a language L is a TM V with the following properties:
 - V halts on all inputs.
 - For any string $w \in \Sigma^*$, the following is true:

$$w \in L \leftrightarrow \exists c \in \Sigma^*. V \text{ accepts } \langle w, c \rangle$$

- Some notes about V :
 - Notice that c is existentially quantified. Any string $w \in L$ must have at least one c that causes V to accept, and possibly more.
 - V is required to halt, so given any potential certificate c for w , you can check whether the certificate is correct.

Verifiers

- A **verifier** for a language L is a TM V with the following properties:
 - V halts on all inputs.
 - For any string $w \in \Sigma^*$, the following is true:
$$w \in L \iff \exists c \in \Sigma^*. V \text{ accepts } \langle w, c \rangle$$
- Some notes about V :
 - Notice that $\mathcal{L}(V) \neq L$. (*Good question: what is $\mathcal{L}(V)$?*)
 - The job of V is just to check certificates, not to decide membership in L .

Verifiers

- A **verifier** for a language L is a TM V with the following properties:
 - V halts on all inputs.
 - For any string $w \in \Sigma^*$, the following is true:
$$w \in L \iff \exists c \in \Sigma^*. V \text{ accepts } \langle w, c \rangle$$
- Some notes about V :
 - Although this formal definition works with a string c , remember that c can be an encoding of some other object.
 - In practice, c will likely just be “some other auxiliary data that helps you out.”

Some Verifiers

- Let L be the following language:

$$L = \{ \langle n \rangle \mid n \in \mathbb{N} \text{ and the hailstone sequence terminates for } n \}$$

- Let's see how to build a verifier for L .

Verification

11

Does the hailstone sequence
terminate for this number?

Verification

11

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

34

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?

Verification

17

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

52

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

26

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

13

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?

Verification

40

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

20

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

10

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

5

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

16

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?

Verification

8

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

4

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

2

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

1

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Some Verifiers

- Let L be the following language:

$L = \{ \langle n \rangle \mid n \in \mathbb{N} \text{ and the hailstone sequence terminates for } n \}$

```
bool checkHailstone(int n, int k) {  
    for (int i = 0; i < k; i++) {  
        if (n % 2 == 0) n /= 2;  
        else n = 3*n + 1;  
    }  
    return n == 1;  
}
```

- Do you see why $\langle n \rangle \in L$ iff there is some k such that `checkHailstone(n, k)` returns true?
- Do you see why `checkHailstone` always halts?

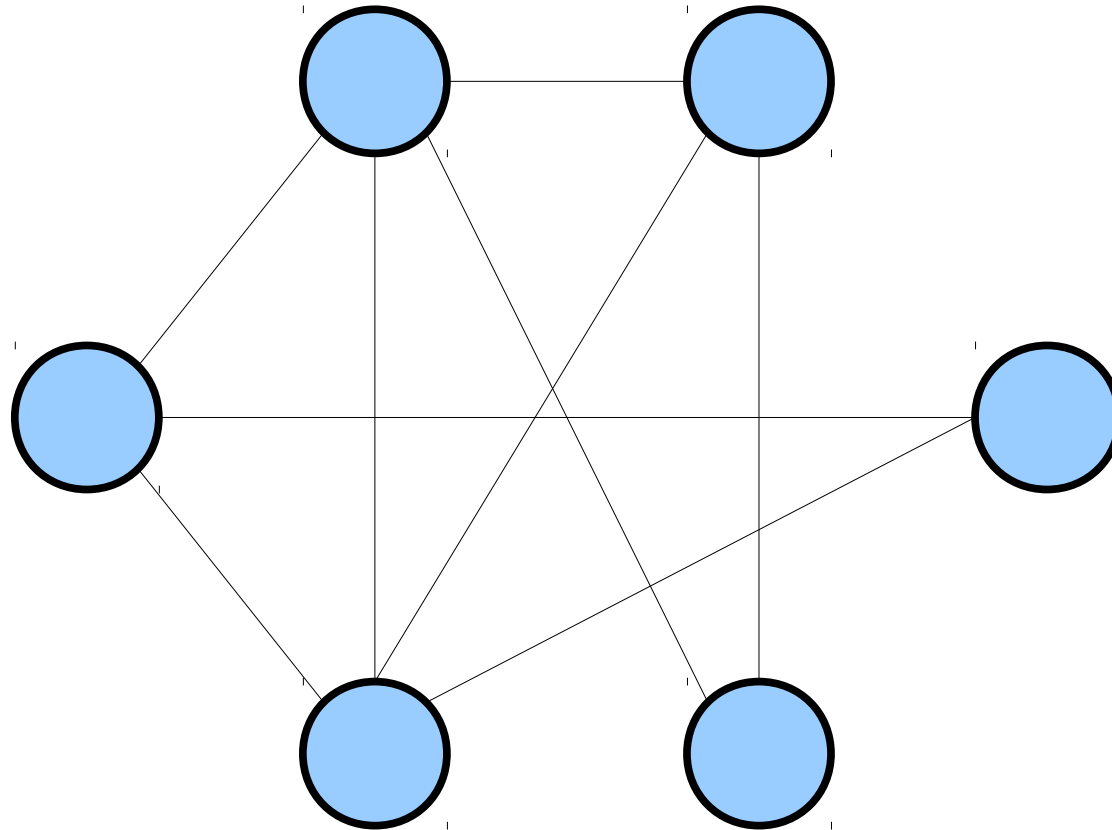
Some Verifiers

- Let L be the following language:

$$L = \{ \langle G \rangle \mid G \text{ is a graph and } G \text{ has a Hamiltonian path } \}$$

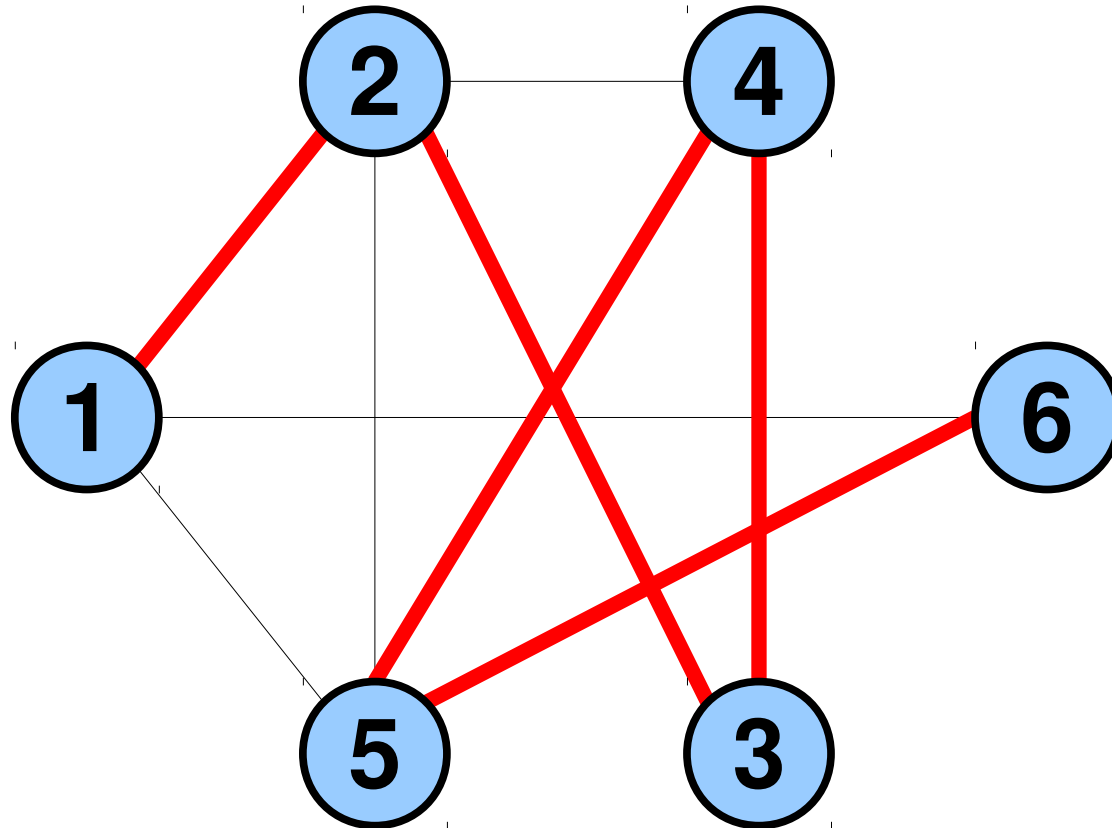
- (A Hamiltonian path is a simple path that visits every node in the graph.)
- Let's see how to build a verifier for L .

Verification



Is there a simple path that goes through every node exactly once?

Verification



Is there a simple path that goes through every node exactly once?

Some Verifiers

- Let L be the following language:

$L = \{ \langle G \rangle \mid G \text{ is a graph with a Hamiltonian path} \}$

```
bool checkHamiltonian(Graph G, vector<Node> p) {  
    if (p.size() != G.numNodes()) return false;  
    if (containsDuplicate(p)) return false;  
  
    for (size_t i = 0; i < p.size() - 1; i++) {  
        if (!G.hasEdge(p[i], p[i+1])) return false;  
    }  
    return true;  
}
```

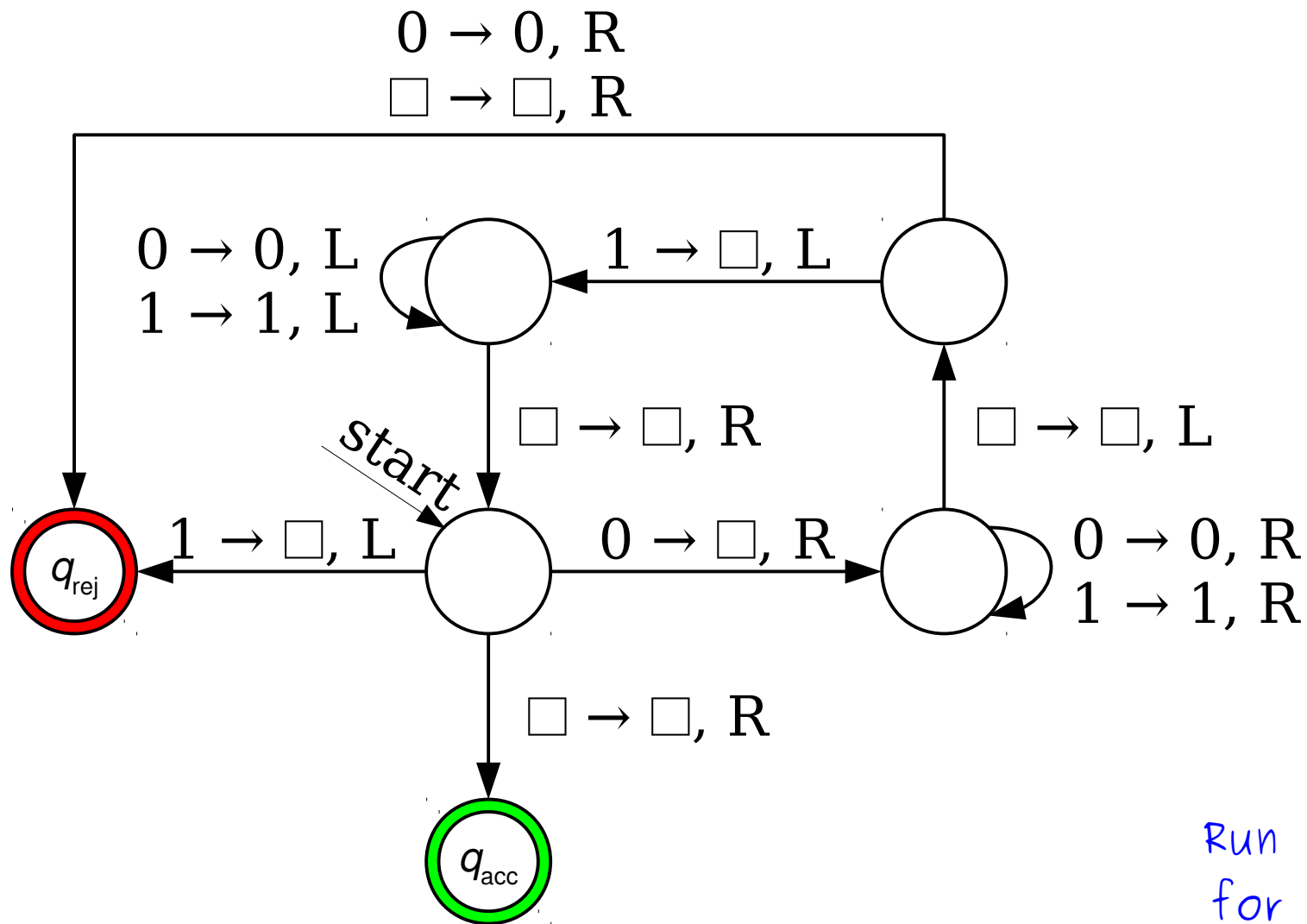
- Do you see why $\langle G \rangle \in L$ iff there is a p where `checkHamiltonian(G, p)` returns true?
- Do you see why `checkHamiltonian` always halts?

Some Verifiers

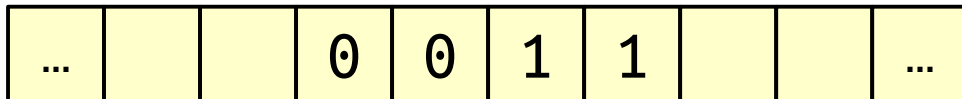
- Consider A_{TM} :

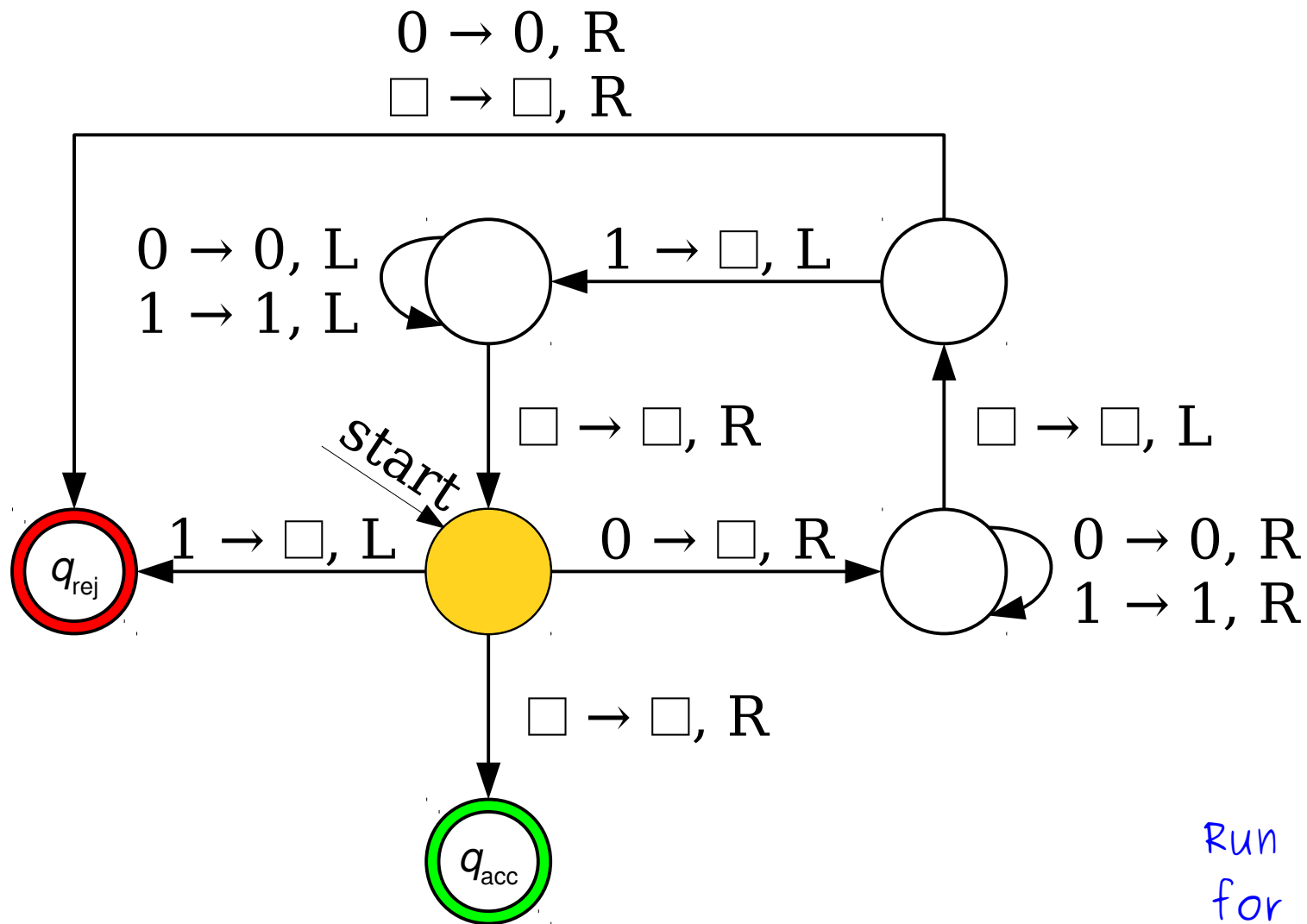
$$A_{\text{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}.$$

- This is a *canonical* example of an undecidable language. There's no way, in general, to tell whether a TM M will accept a string w .
- Although this language is undecidable, it's an **RE** language, and it's possible to build a verifier for it!

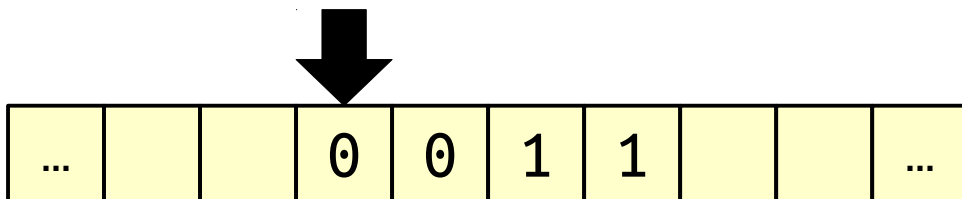


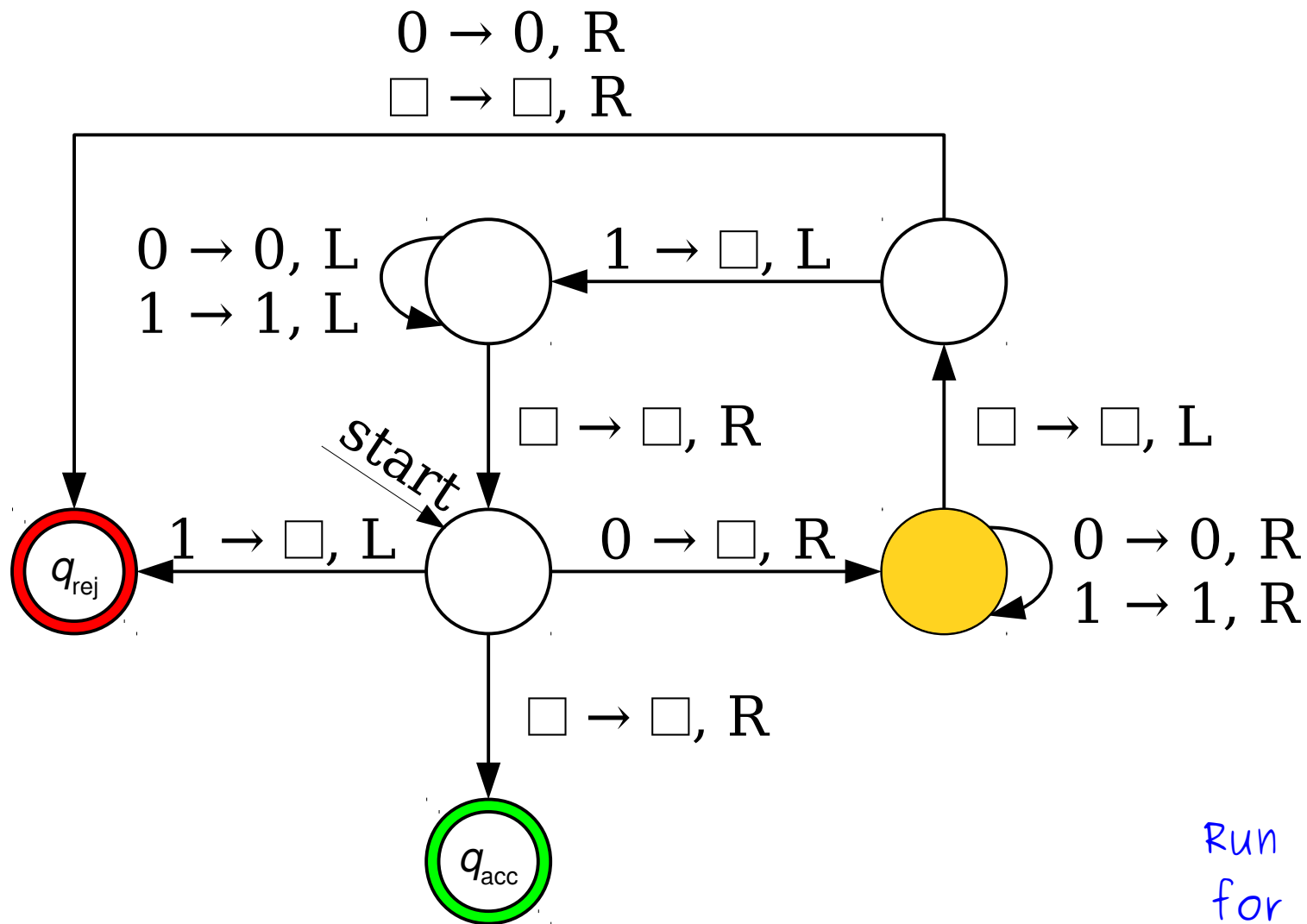
Run this TM
for fifteen
steps.



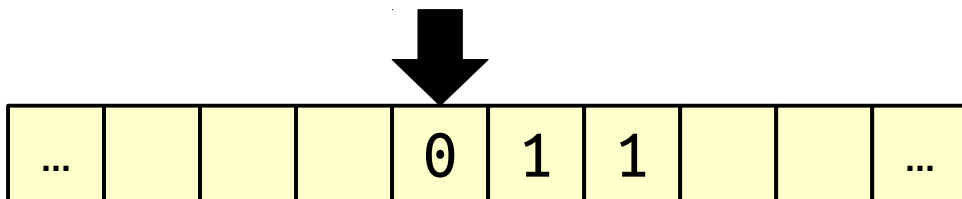


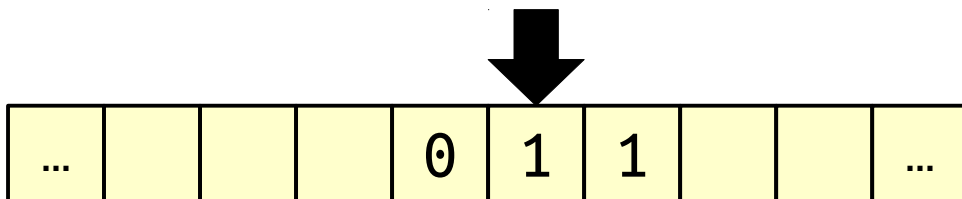
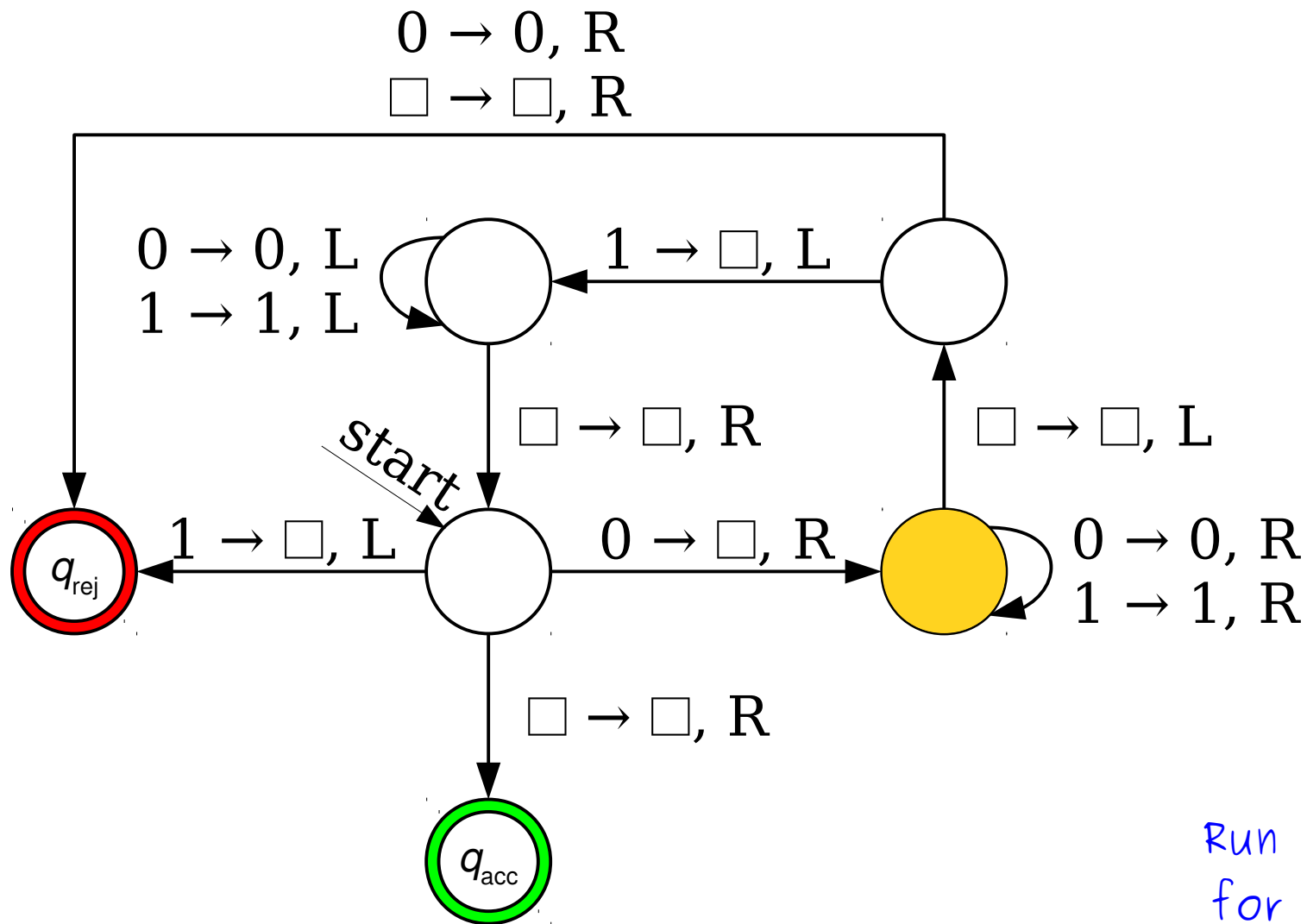
Run this TM
for fifteen
steps.

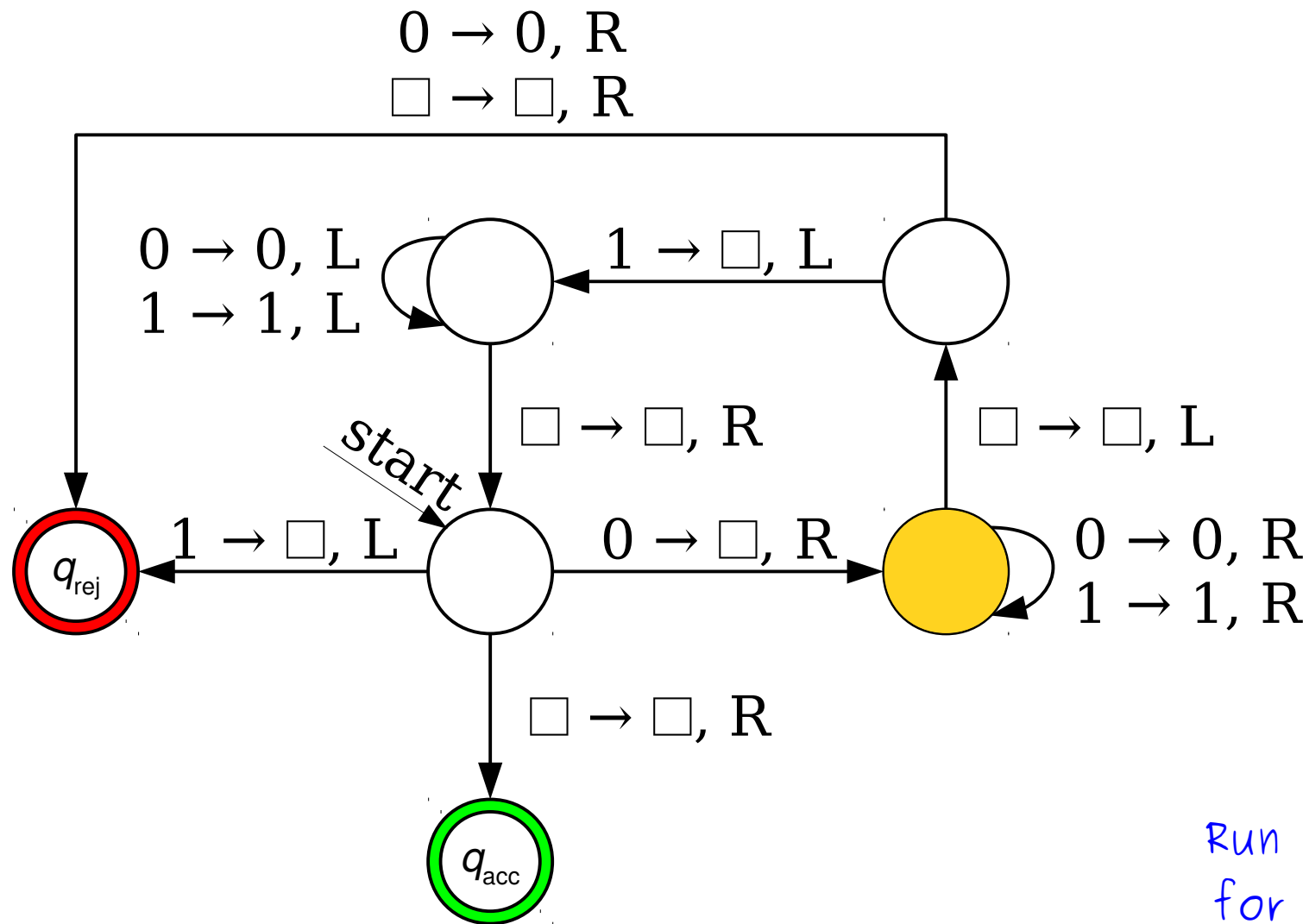




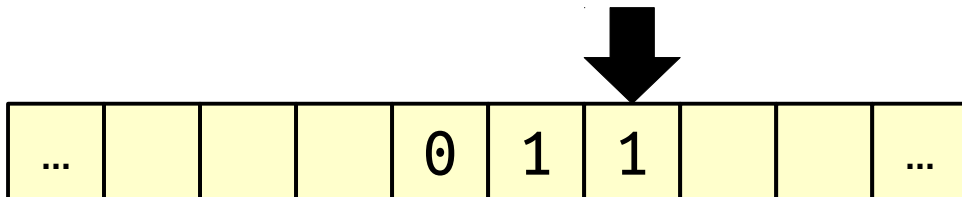
Run this TM
for fifteen
steps.

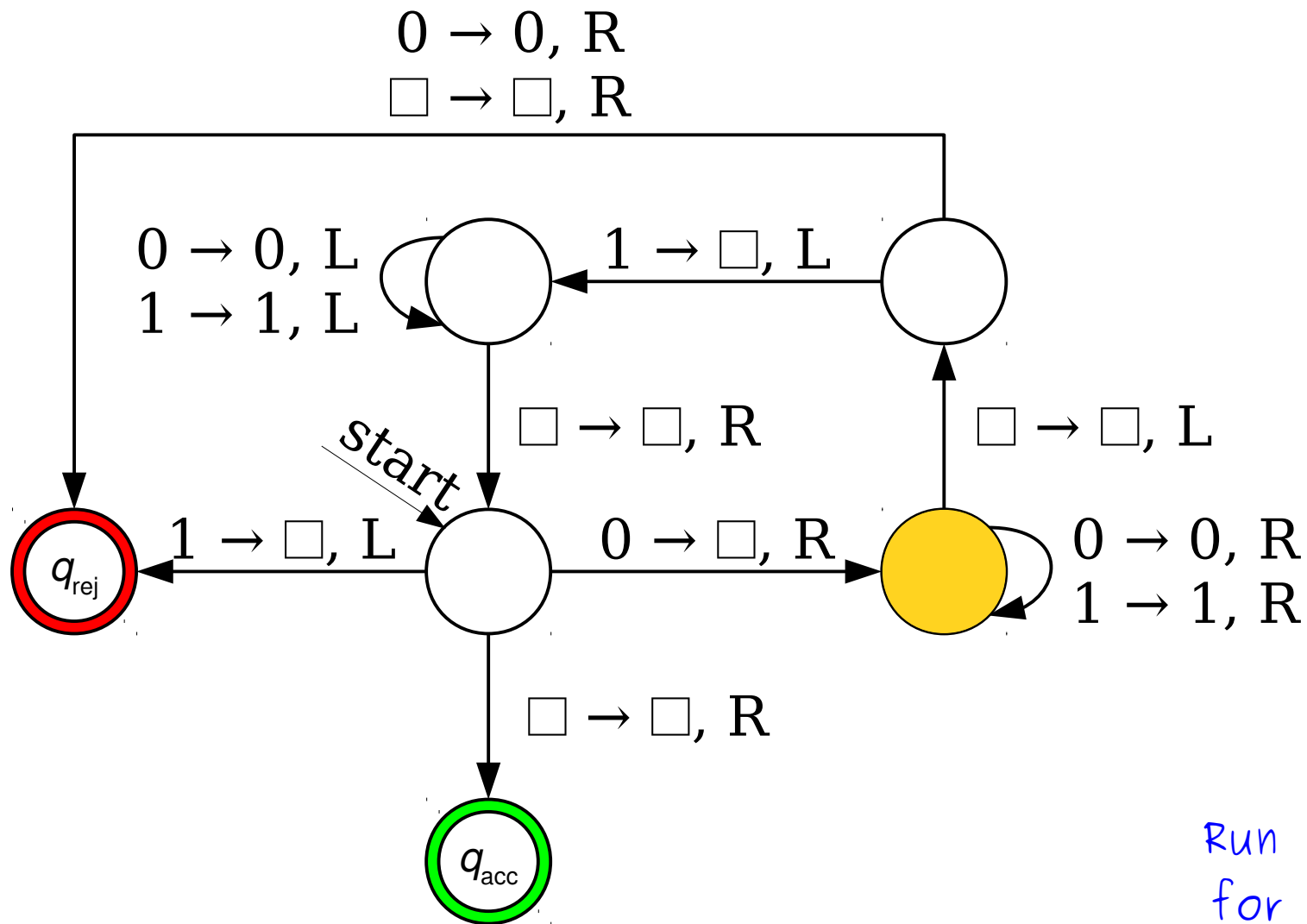




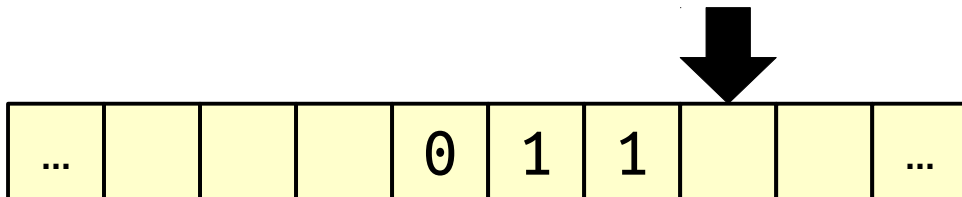


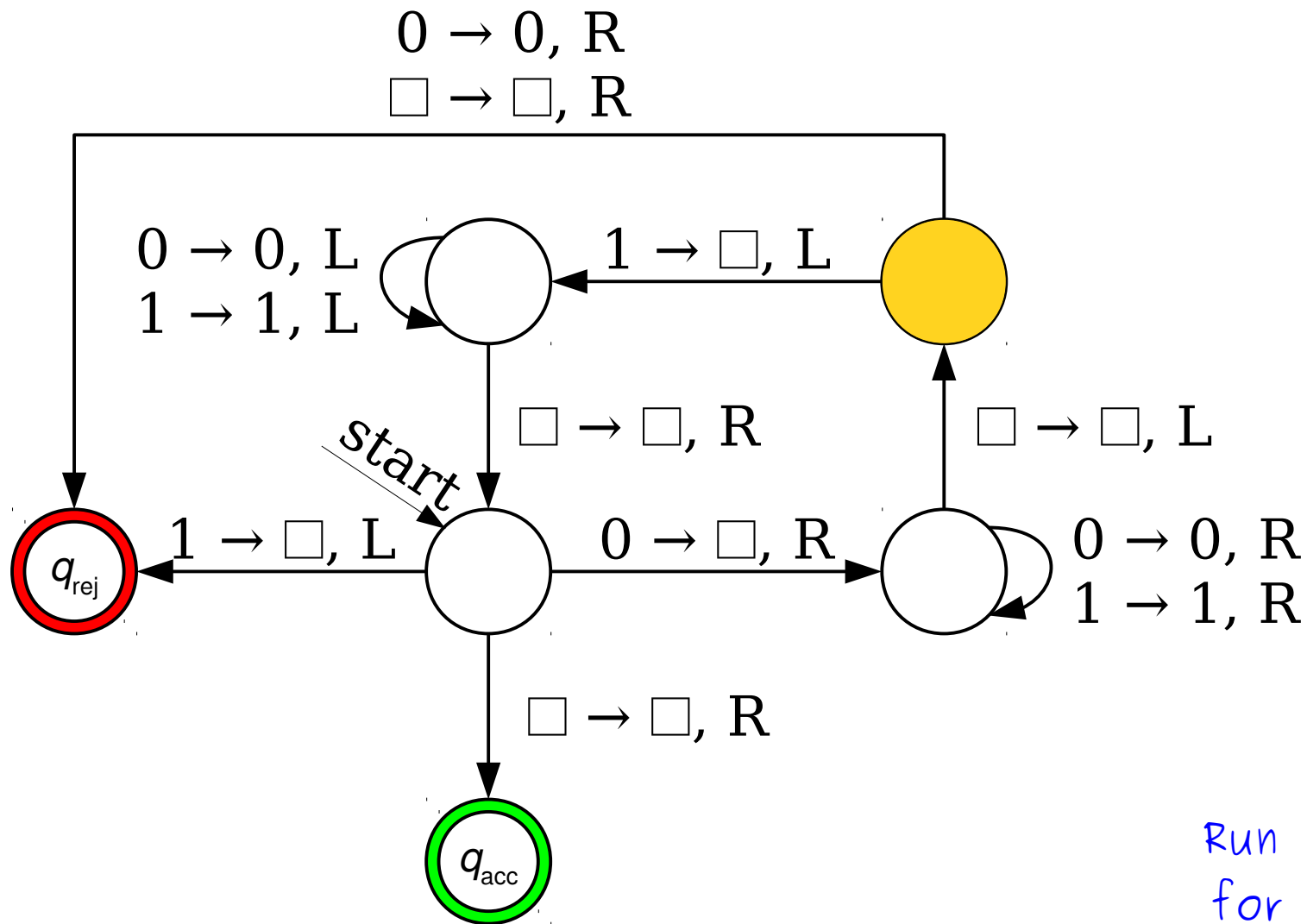
Run this TM
for fifteen
steps.



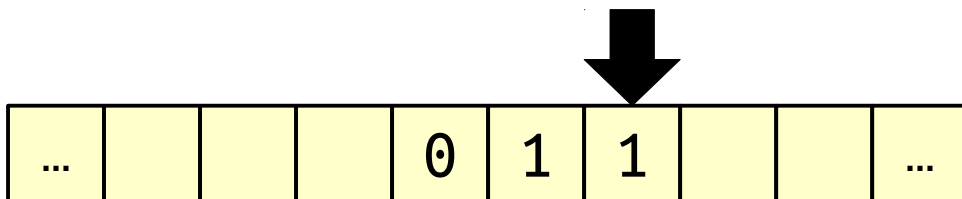


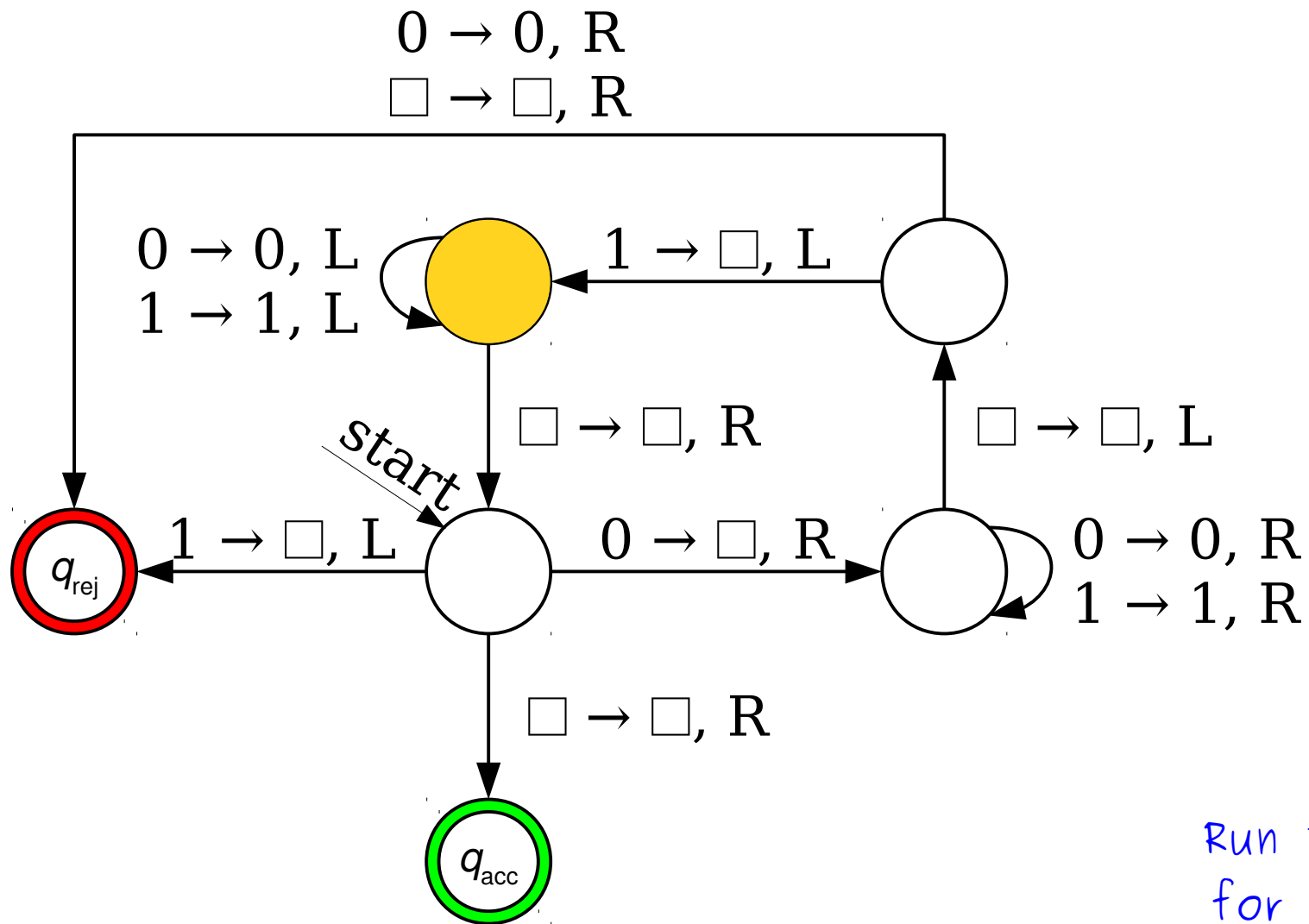
Run this TM
for fifteen
steps.



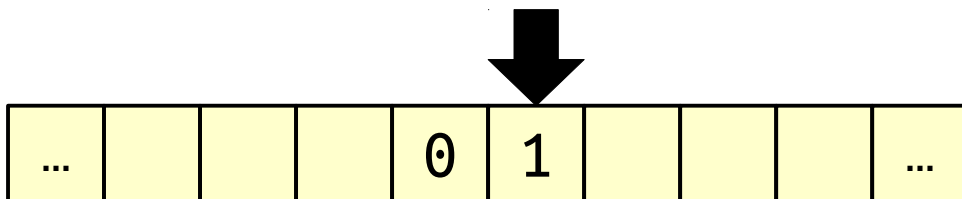


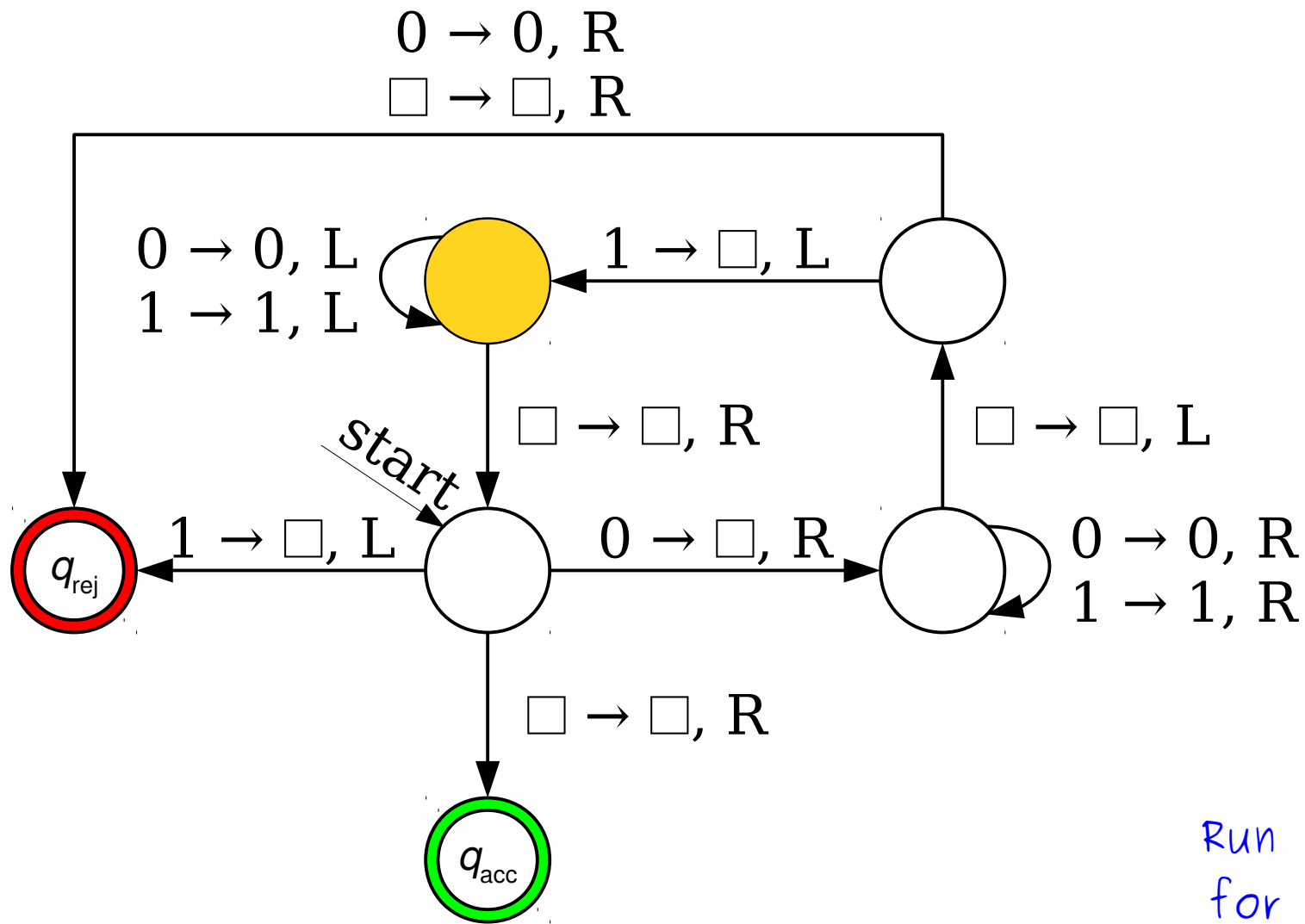
Run this TM
for fifteen
steps.



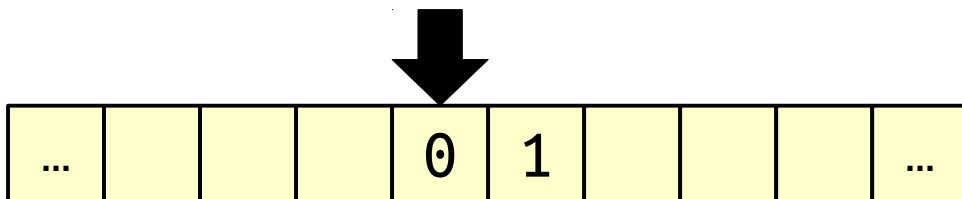


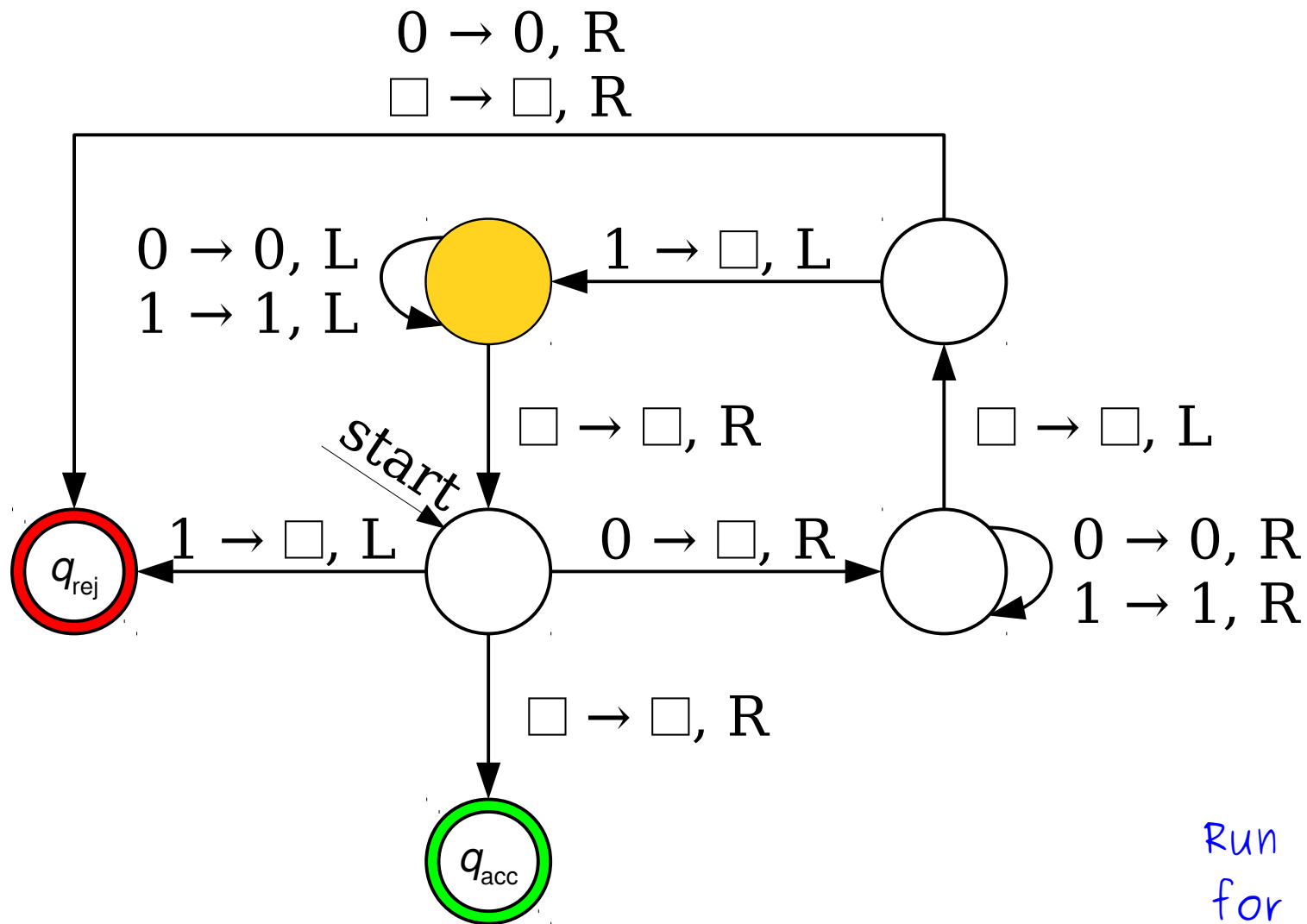
Run this TM
for fifteen
steps.



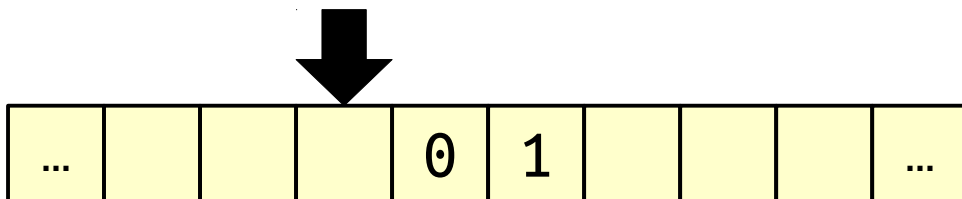


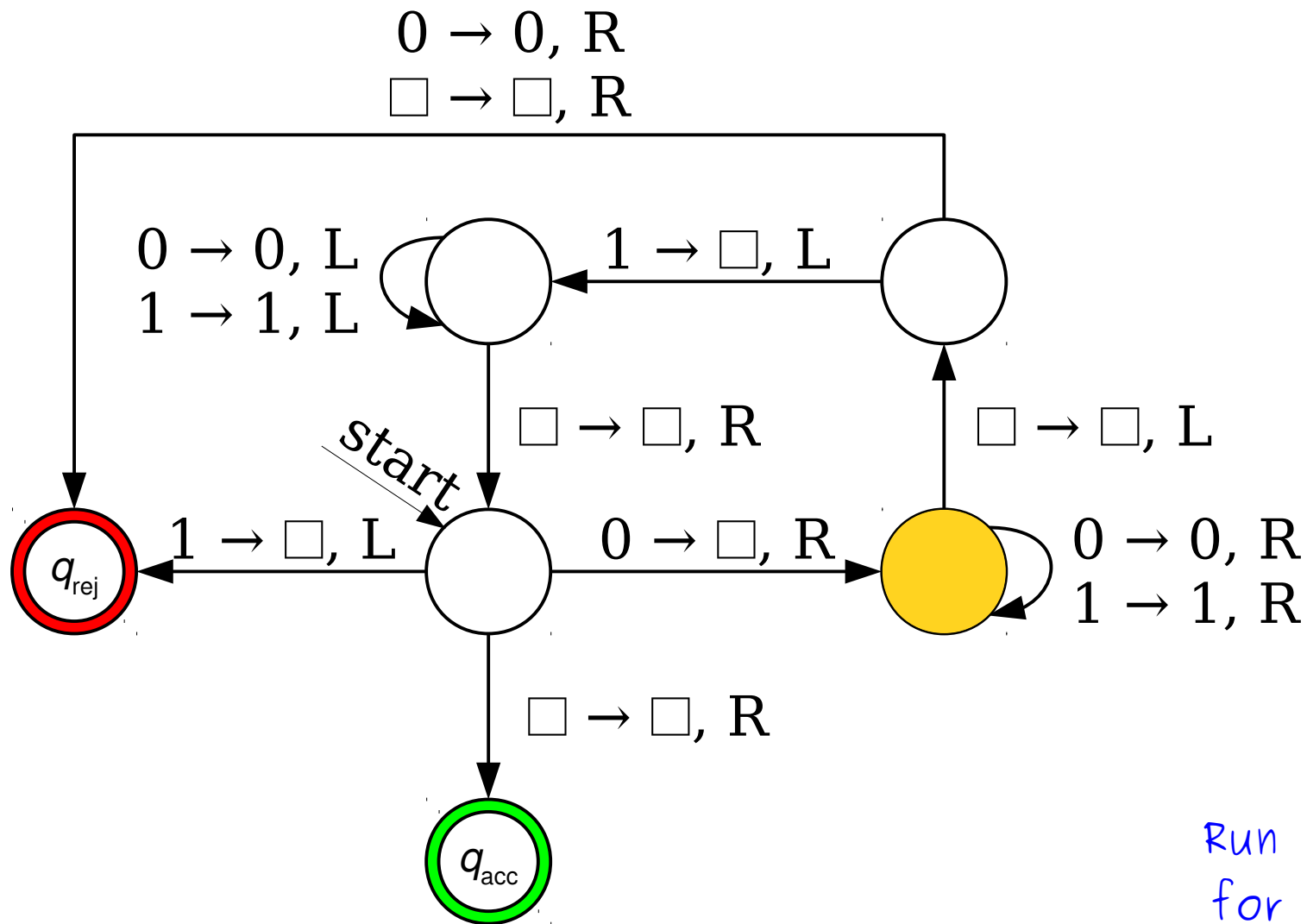
Run this TM
for fifteen
steps.



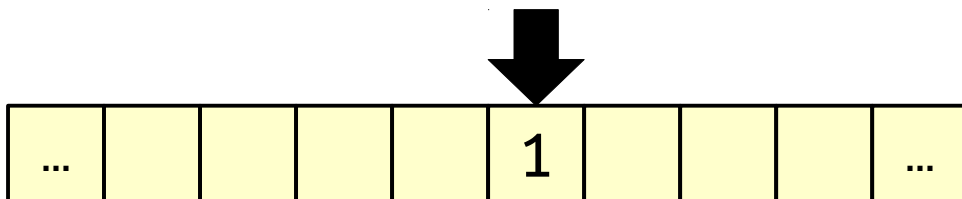


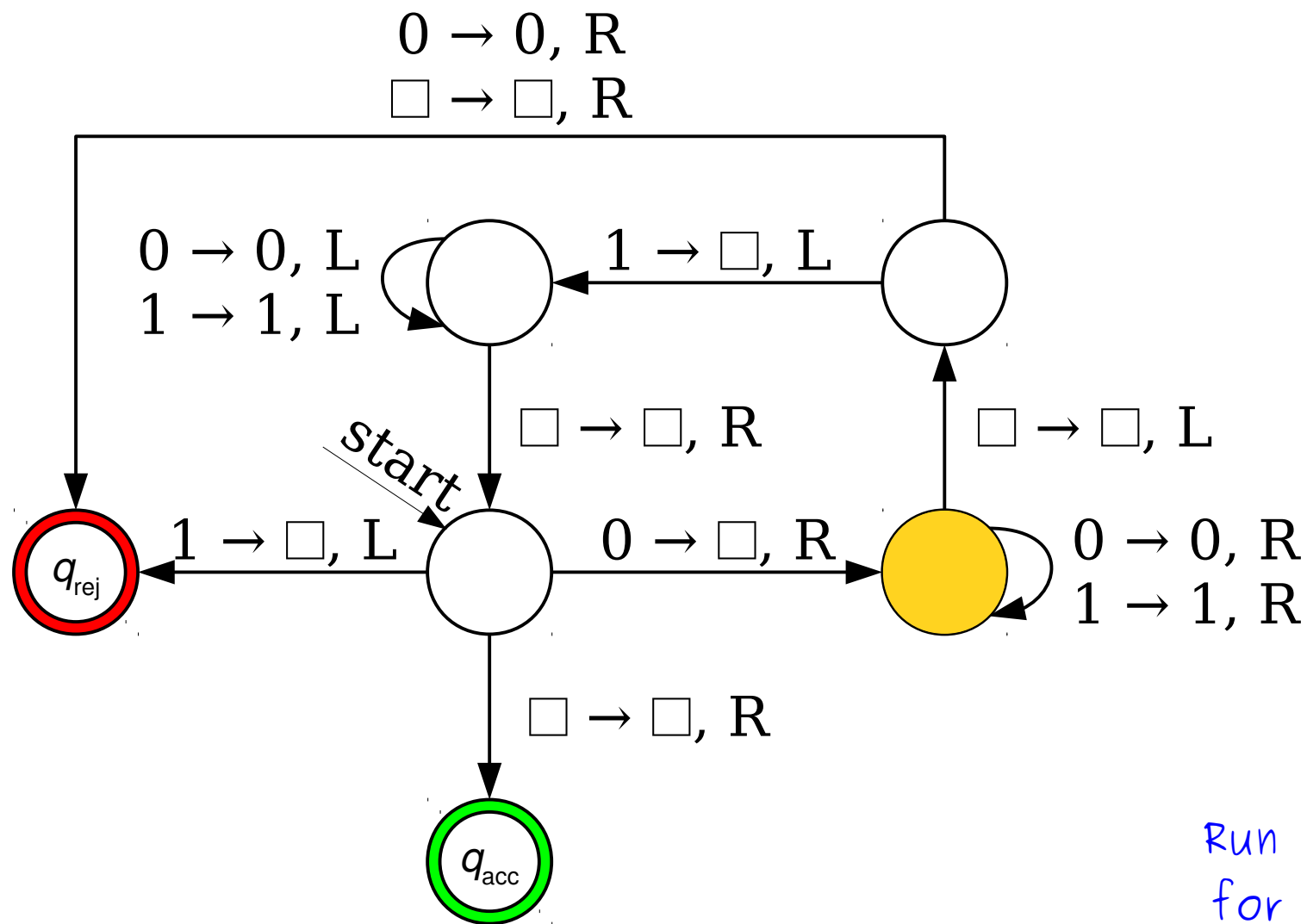
Run this TM
for fifteen
steps.



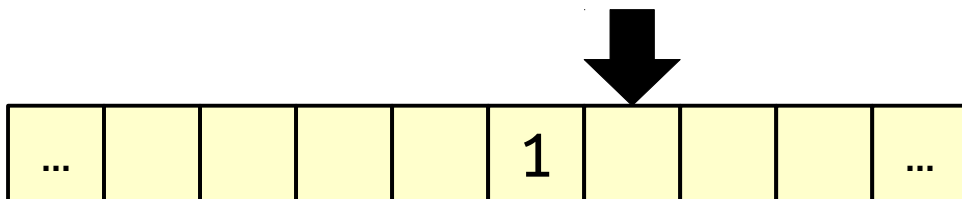


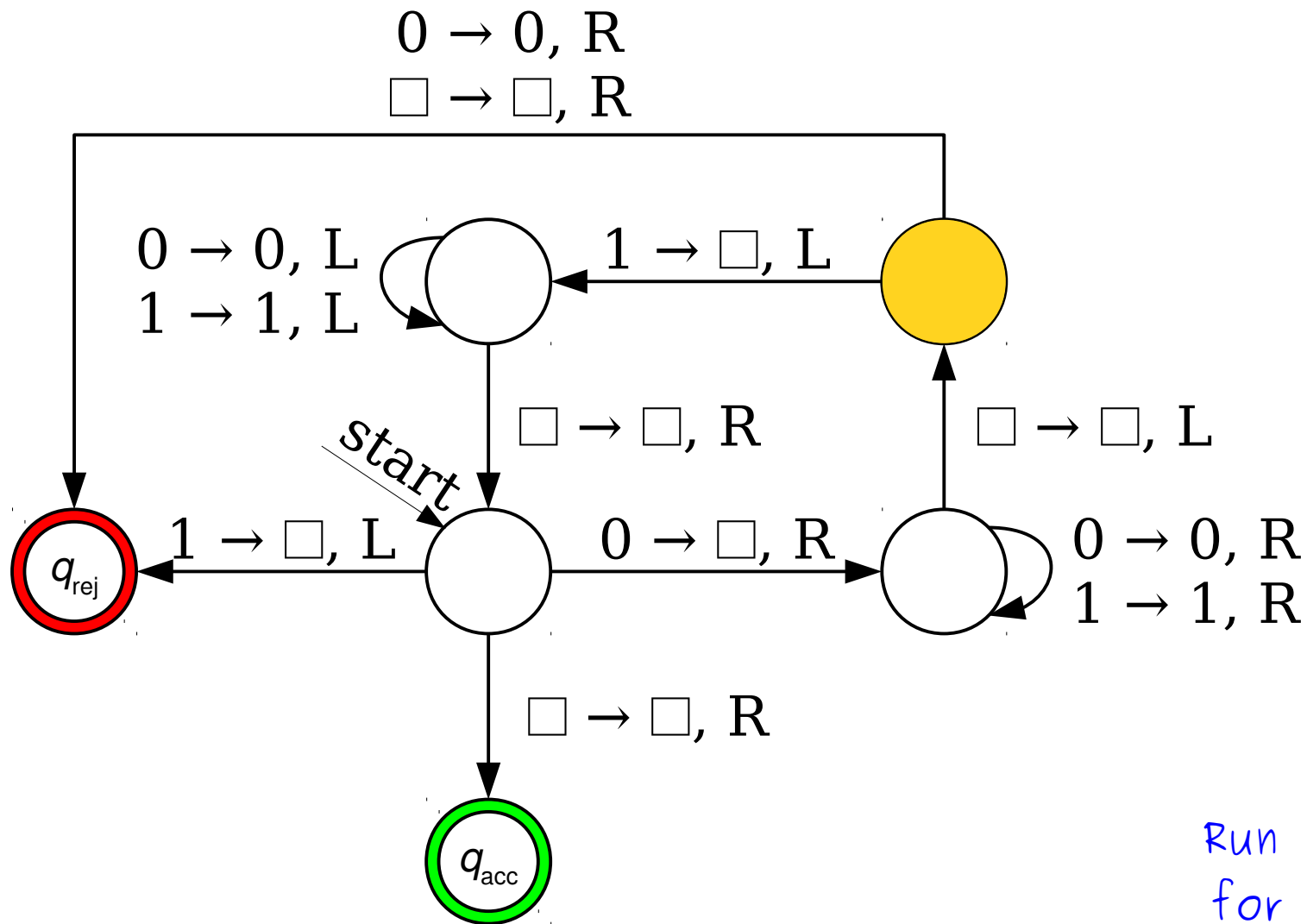
Run this TM
for fifteen
steps.



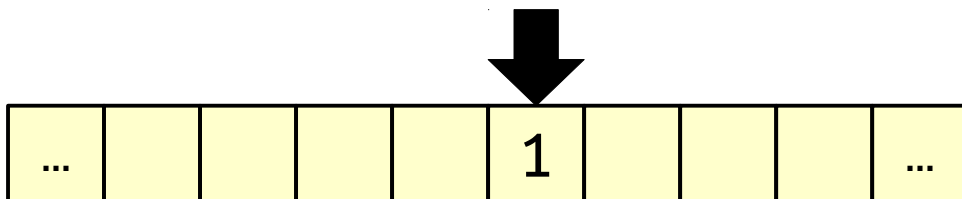


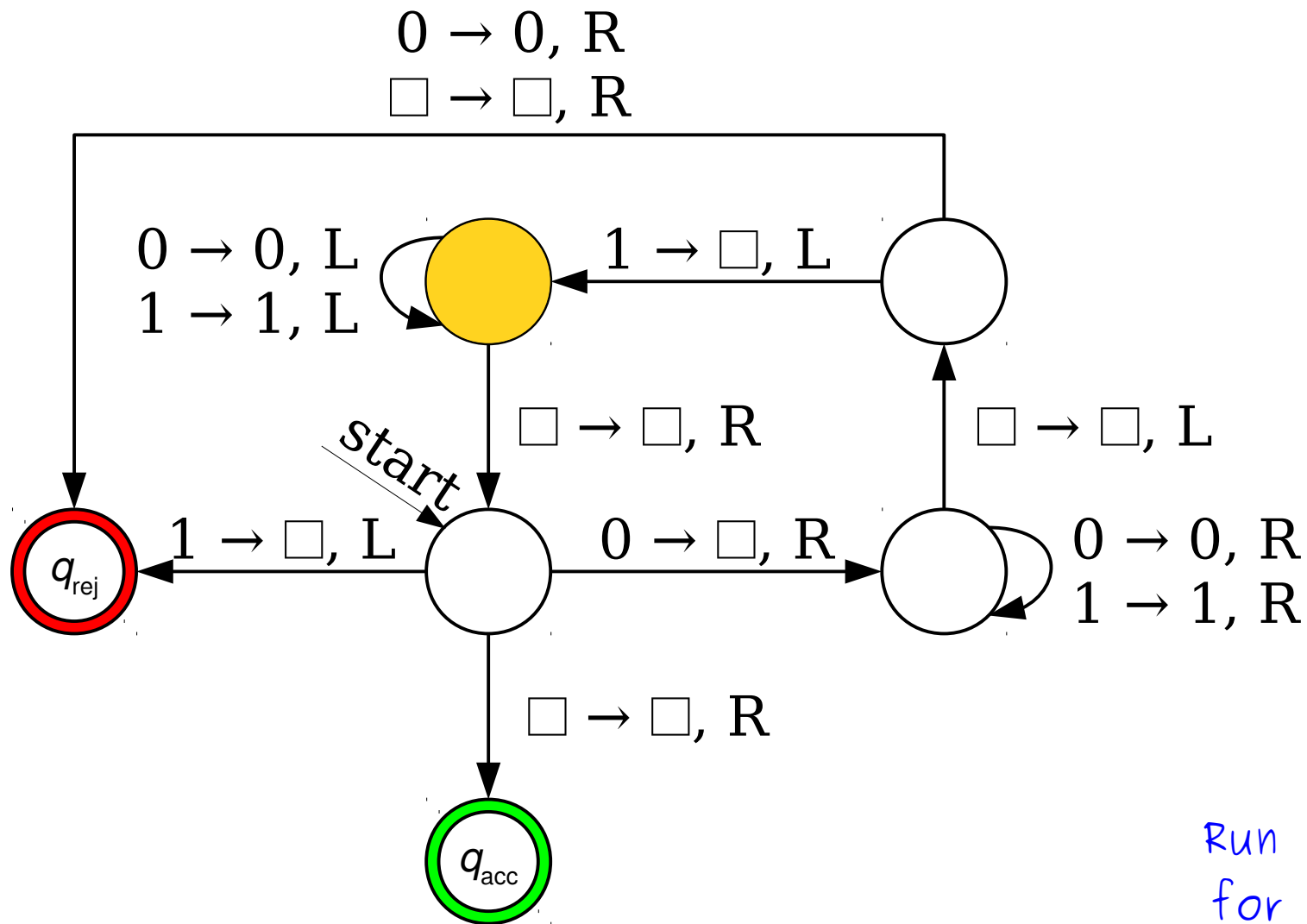
Run this TM
for fifteen
steps.



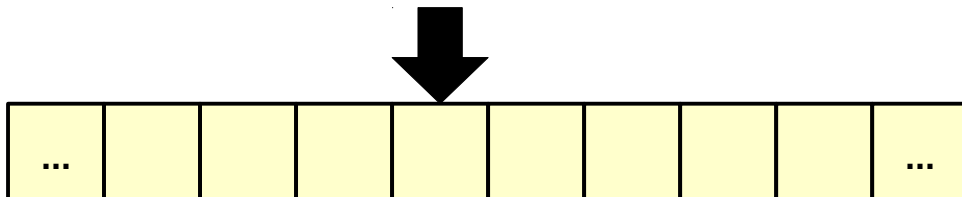


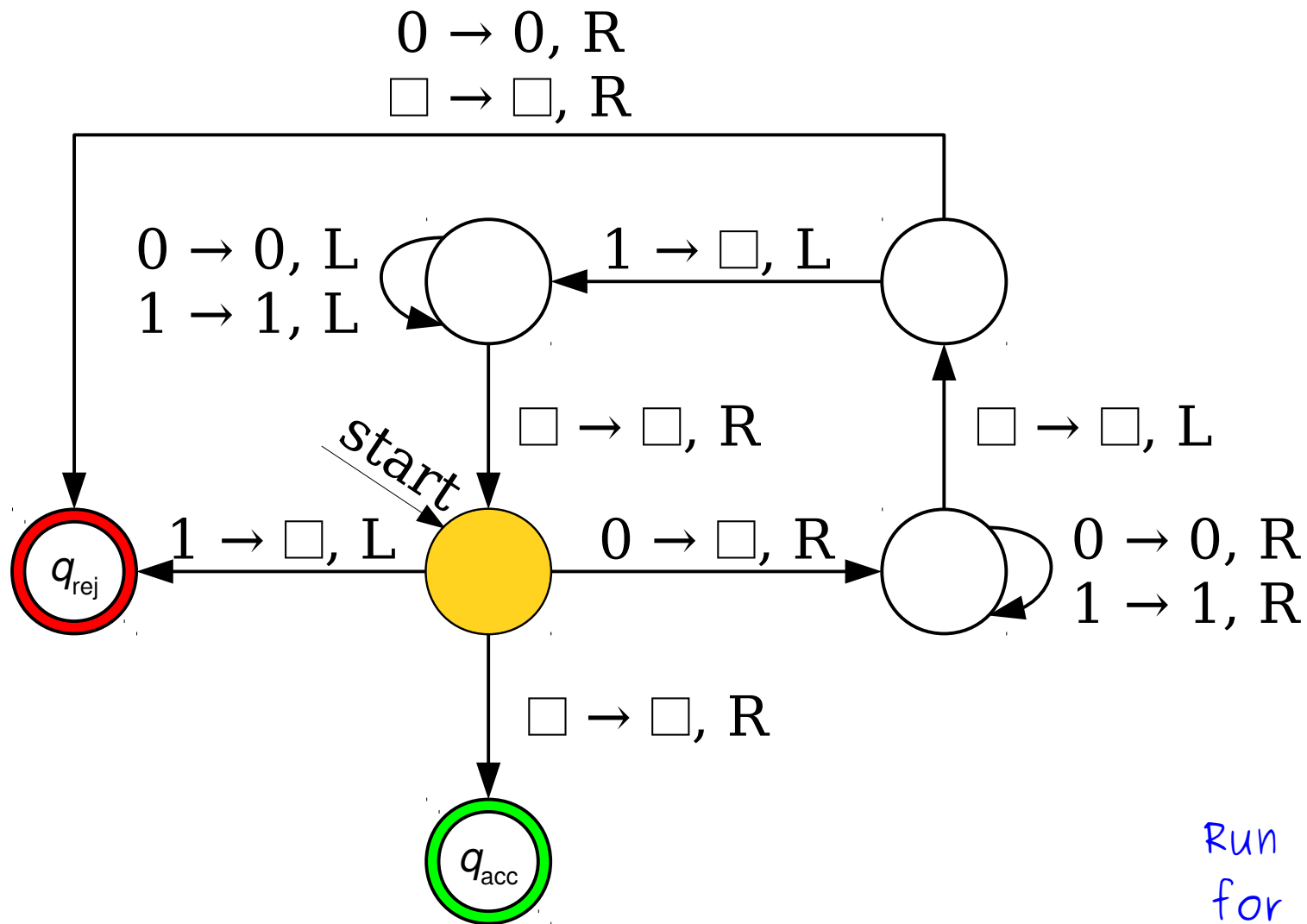
Run this TM
for fifteen
steps.



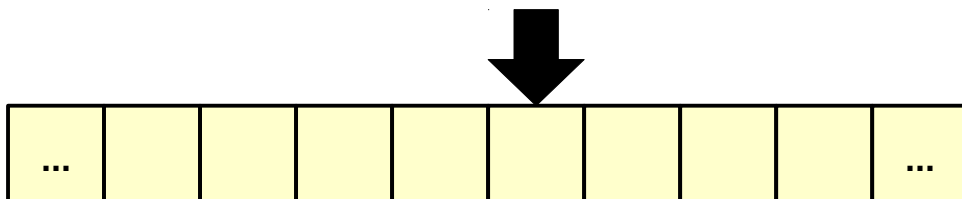


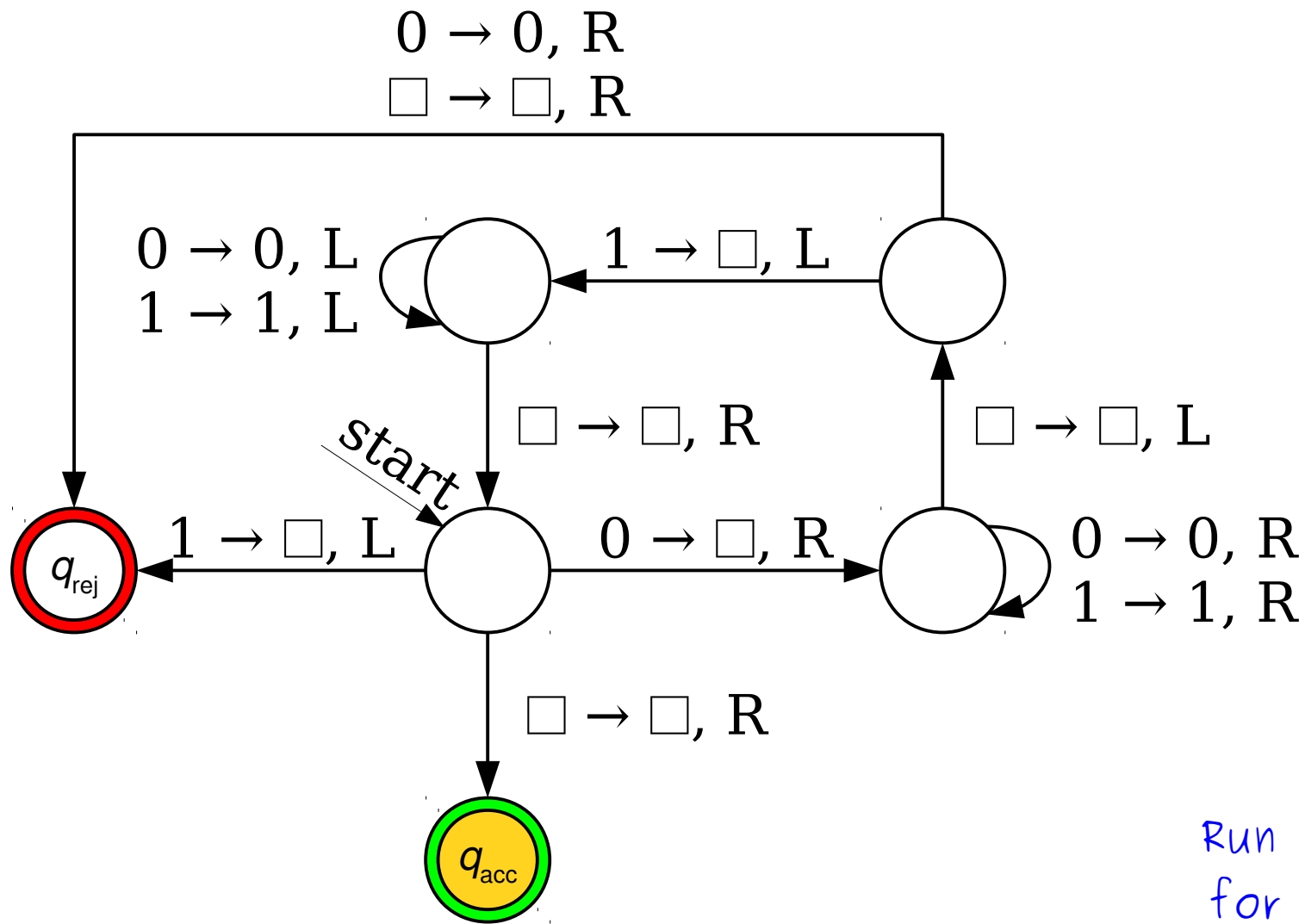
Run this TM
for fifteen
steps.



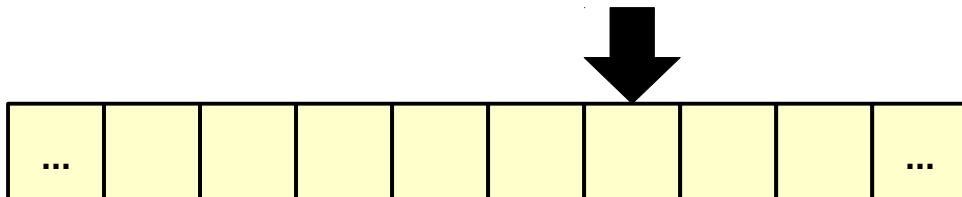


Run this TM
for fifteen
steps.





Run this TM
for fifteen
steps.



Some Verifiers

- Consider A_{TM} :

$$A_{\text{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$$

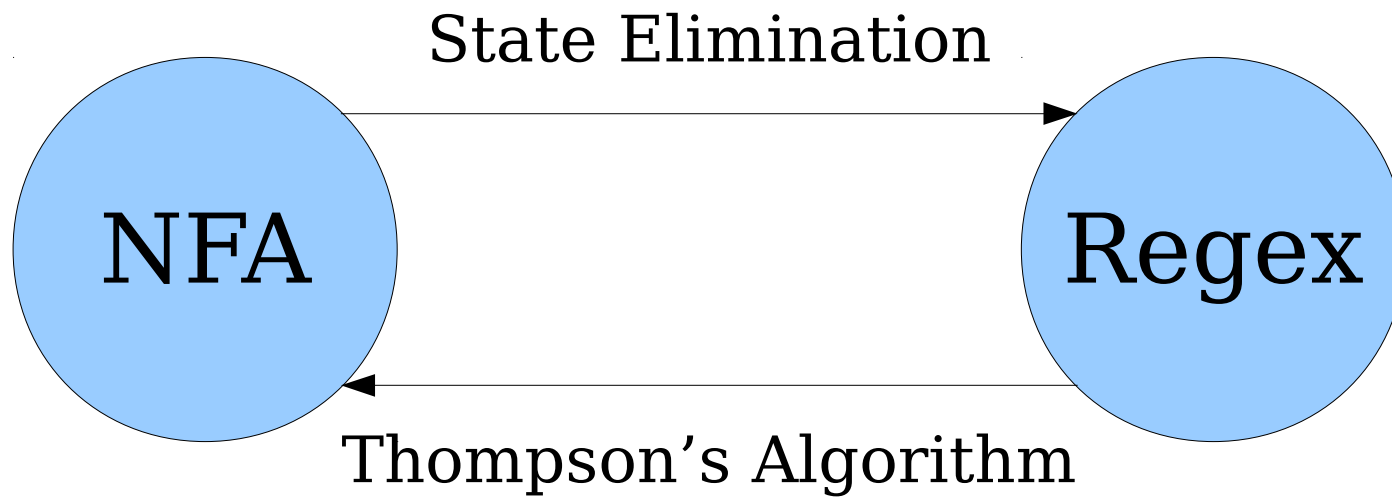
```
bool checkWillAccept(TM M, string w, int k) {  
    set up a simulation of M running on w;  
    for (int i = 0; i < k; i++) {  
        simulate the next step of M running on w;  
    }  
    return whether M is in an accepting state;  
}
```

- Do you see why M accepts w iff there is some k such that `checkWillAccept(M, w, k)` returns true?
- Do you see why `checkWillAccept` always halts?

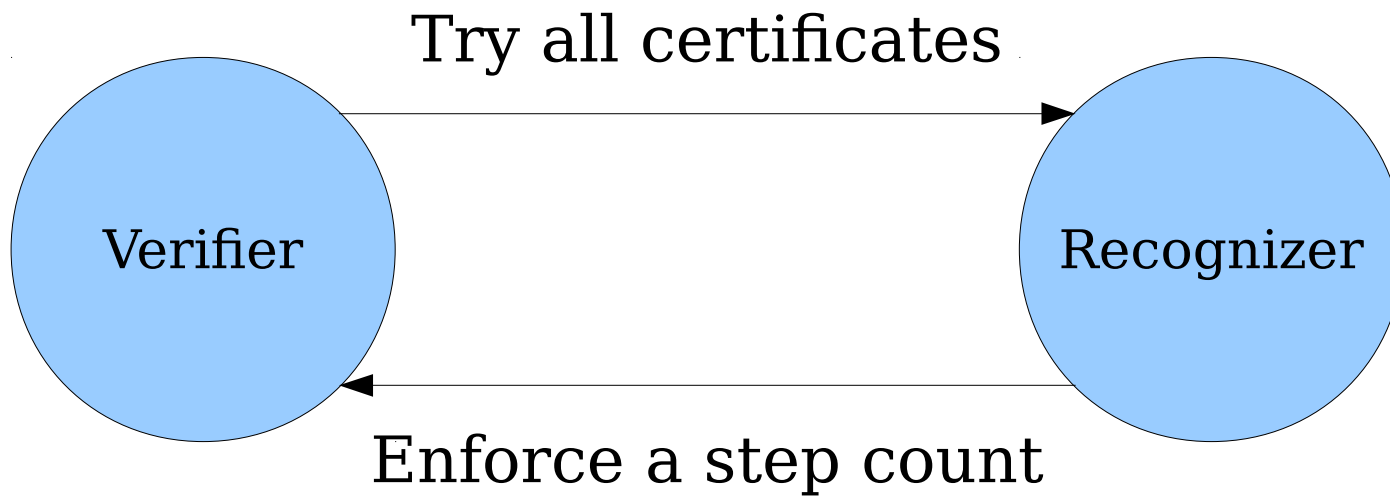
What languages are verifiable?

Theorem: If L is a language, then there is a verifier for L if and only if $L \in \mathbf{RE}$.

Where We've Been



Where We're Going

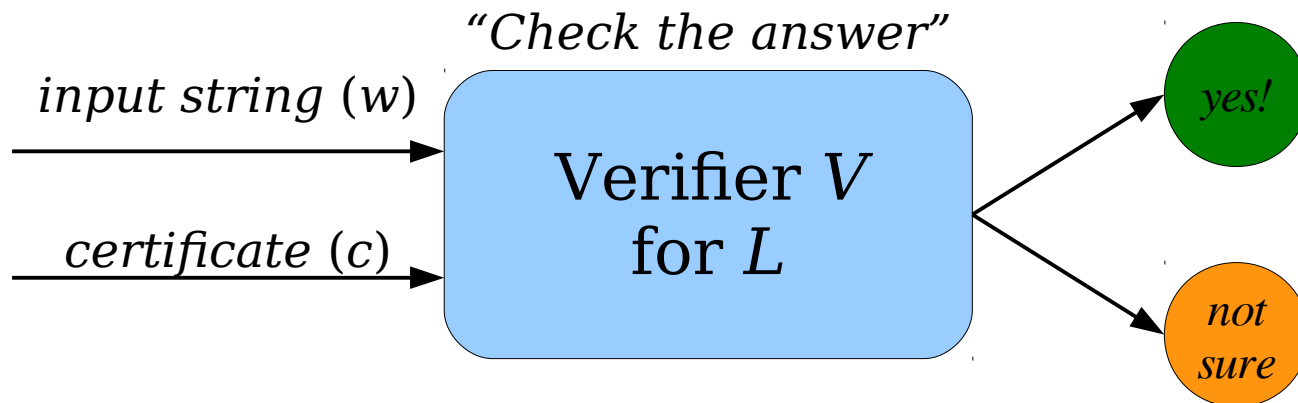


Verifiers and **RE**

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .

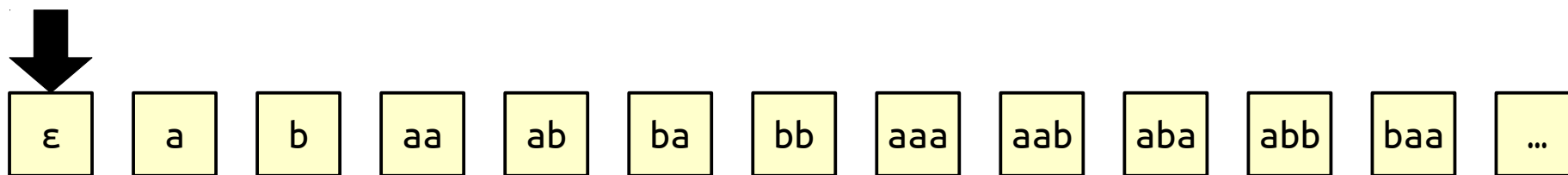
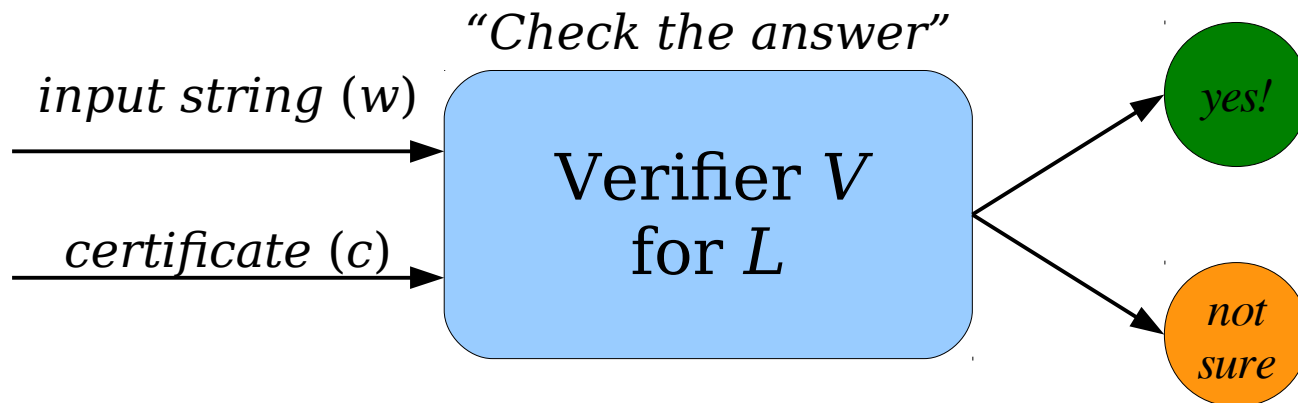
Verifiers and **RE**

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



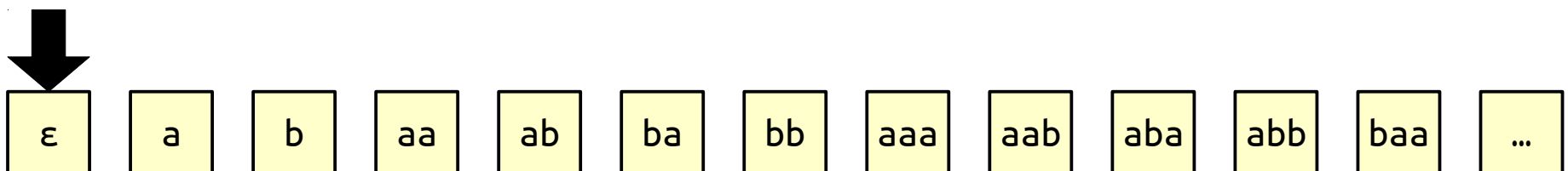
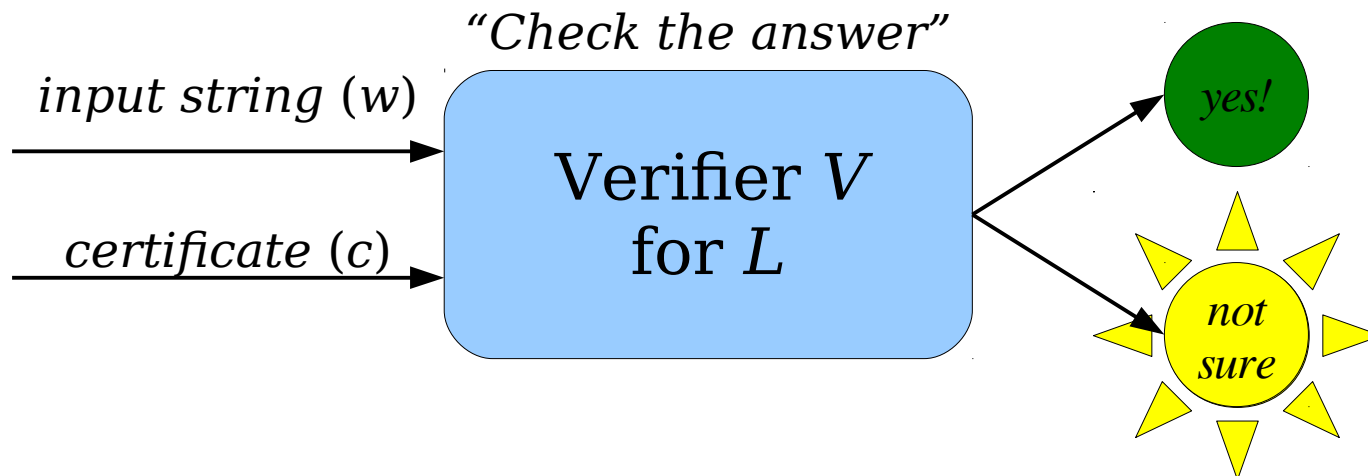
Verifiers and RE

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



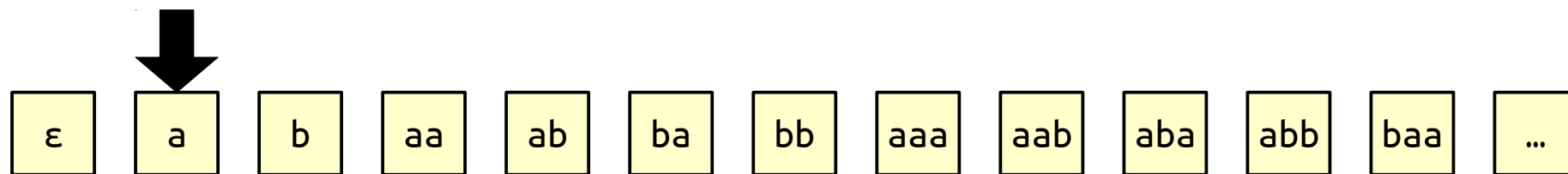
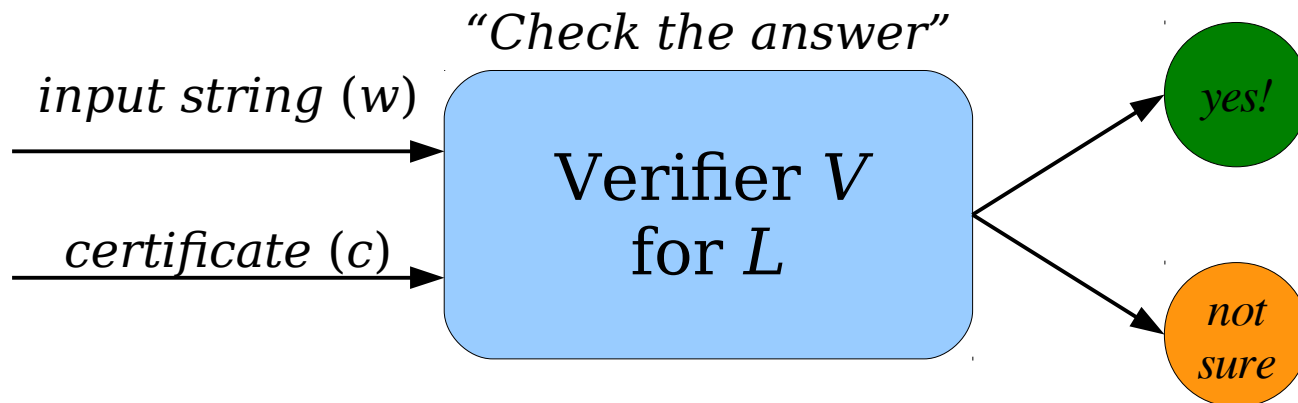
Verifiers and RE

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



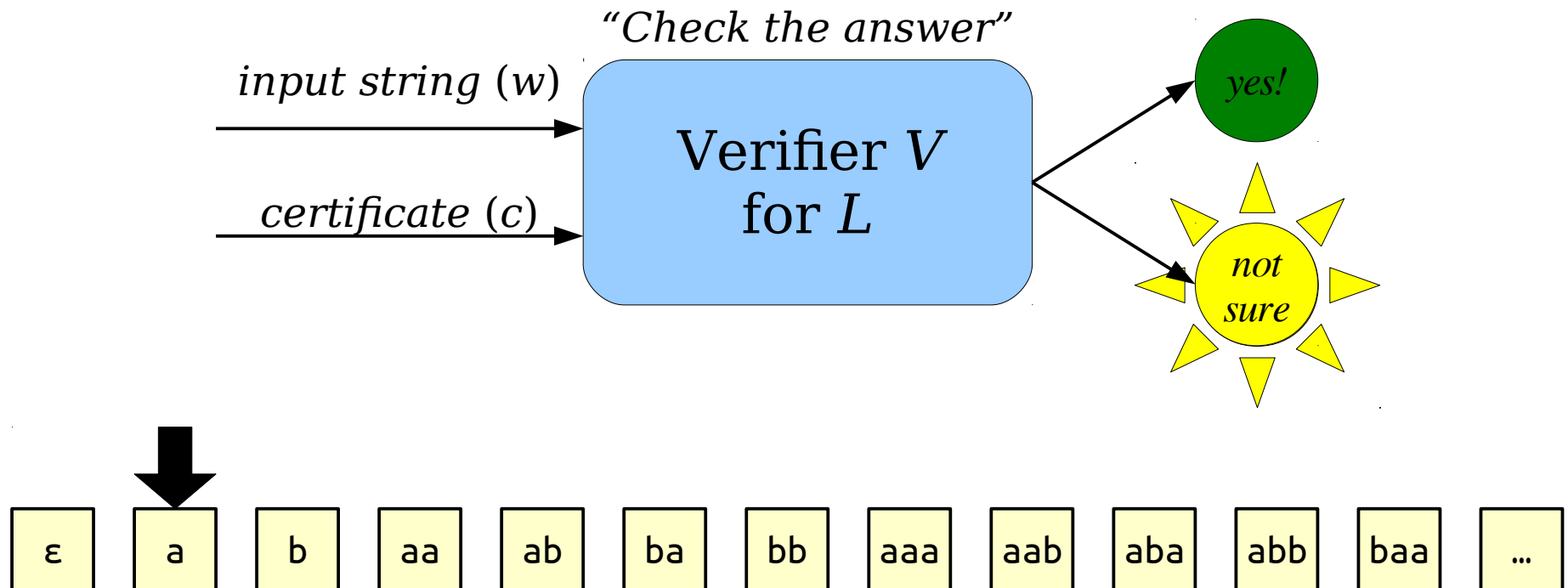
Verifiers and RE

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



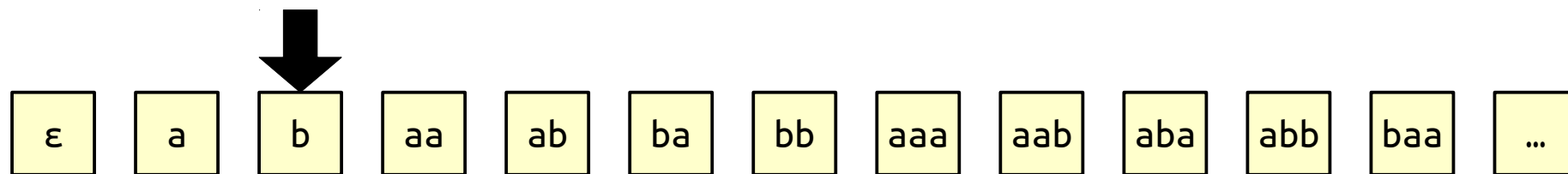
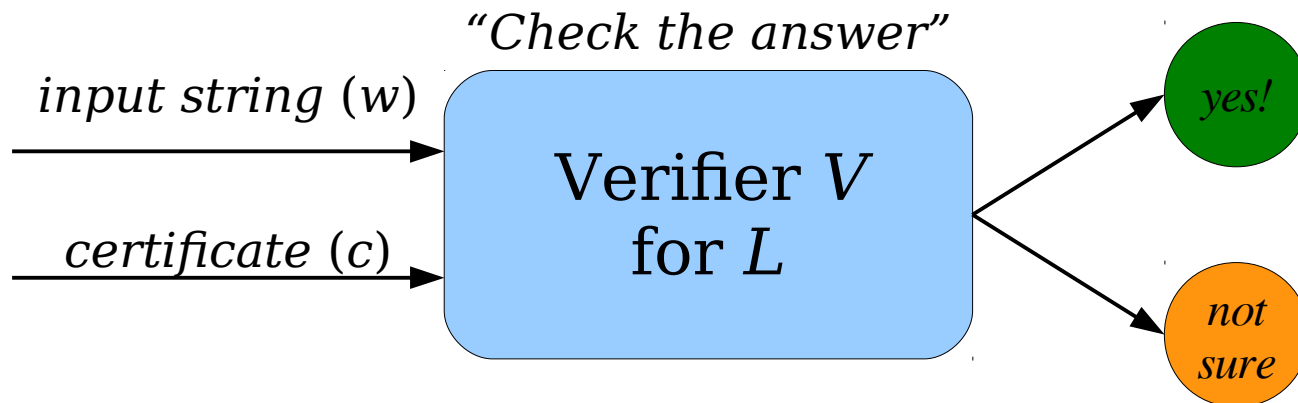
Verifiers and RE

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



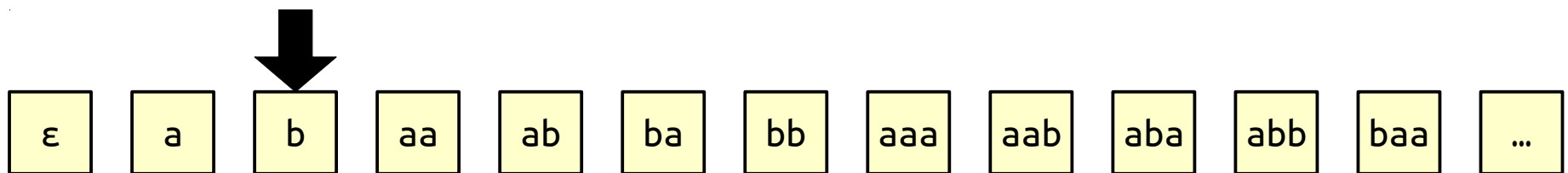
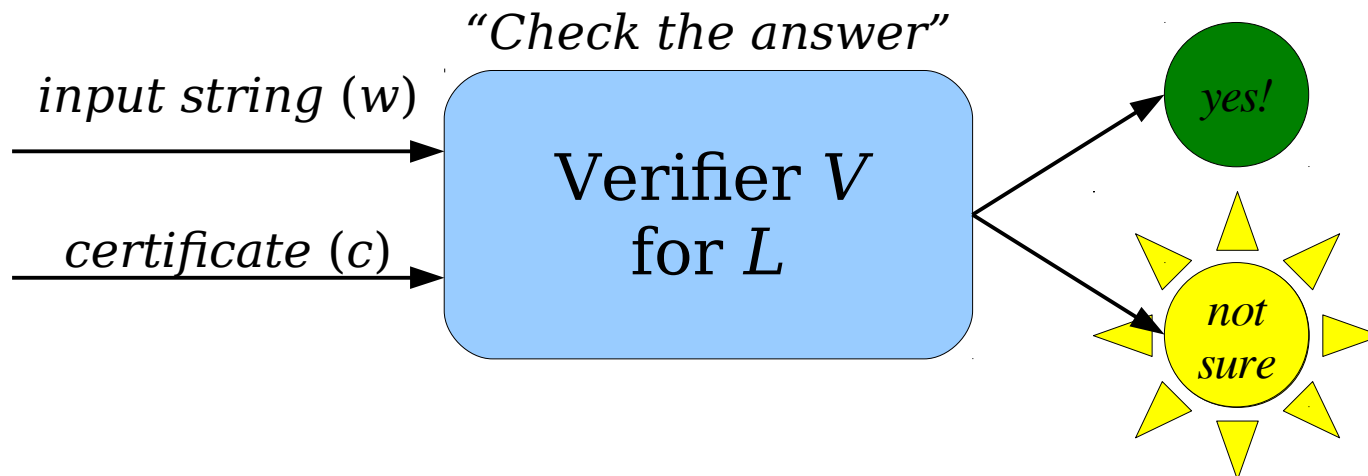
Verifiers and RE

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



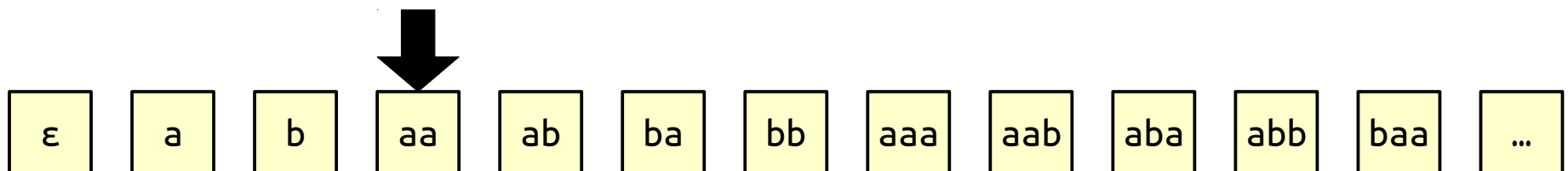
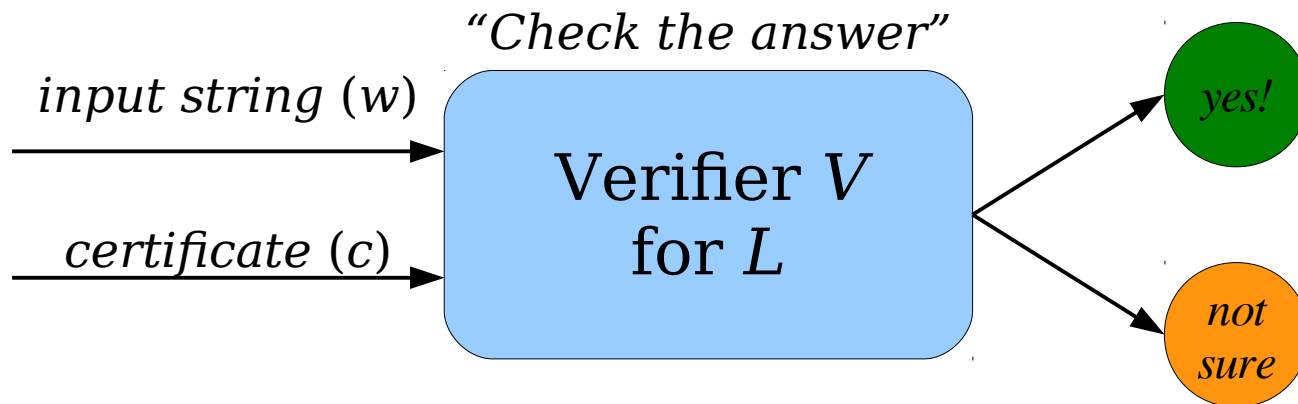
Verifiers and RE

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



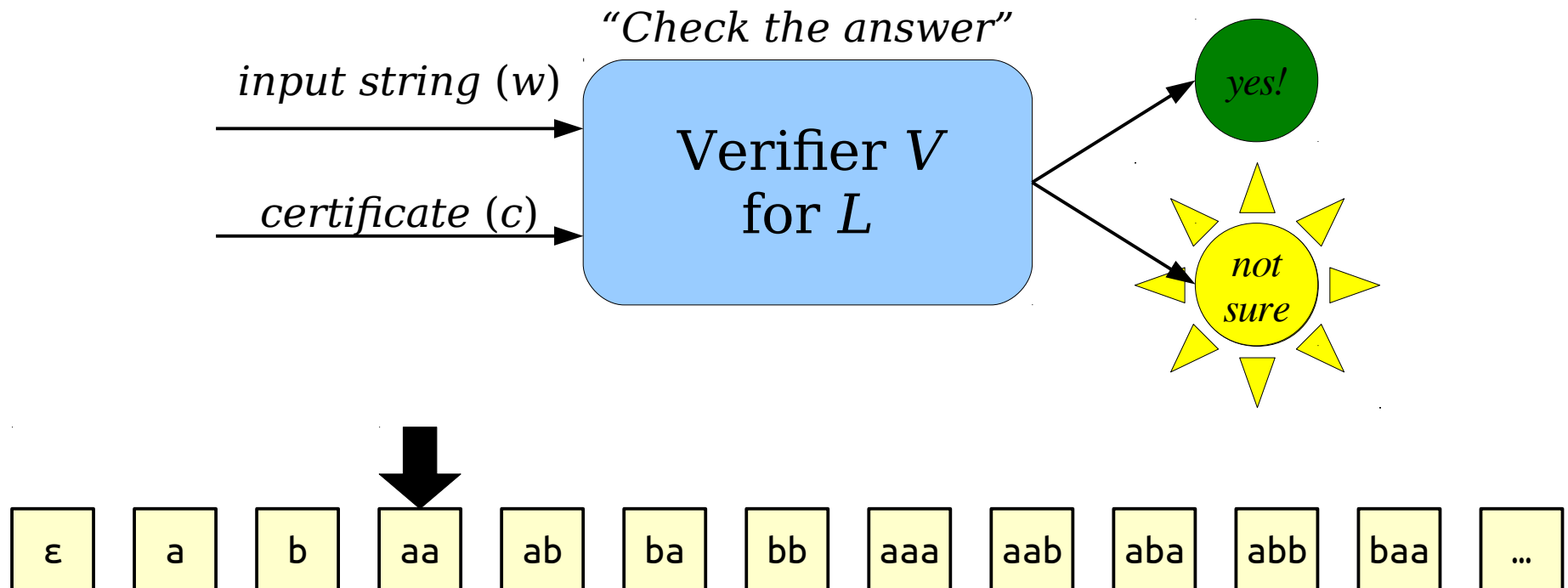
Verifiers and **RE**

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



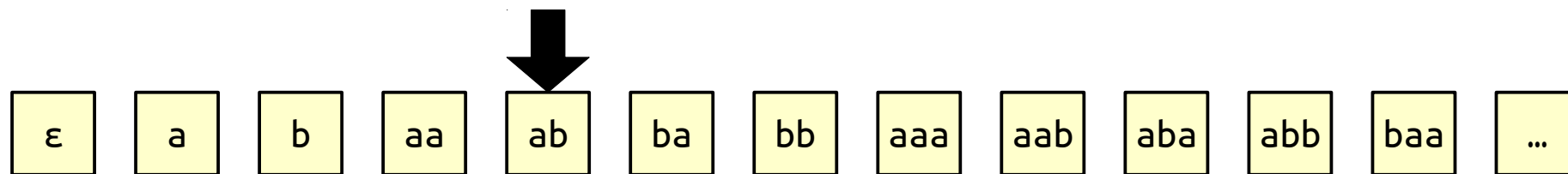
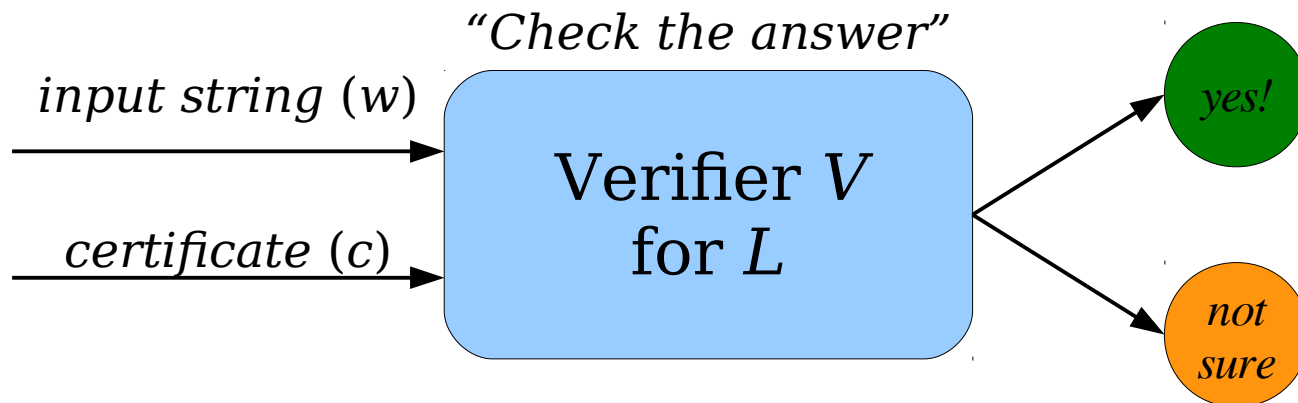
Verifiers and **RE**

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



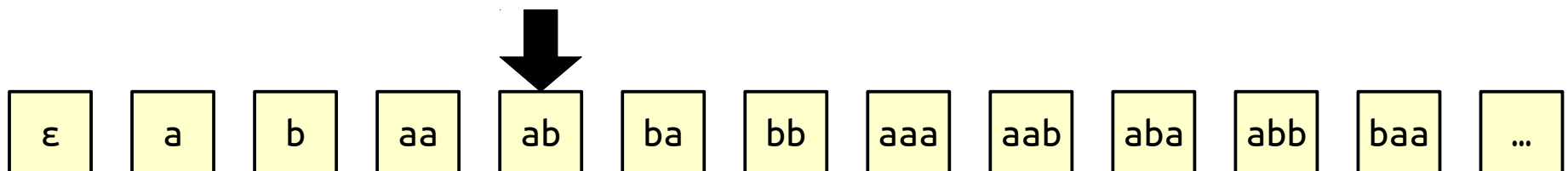
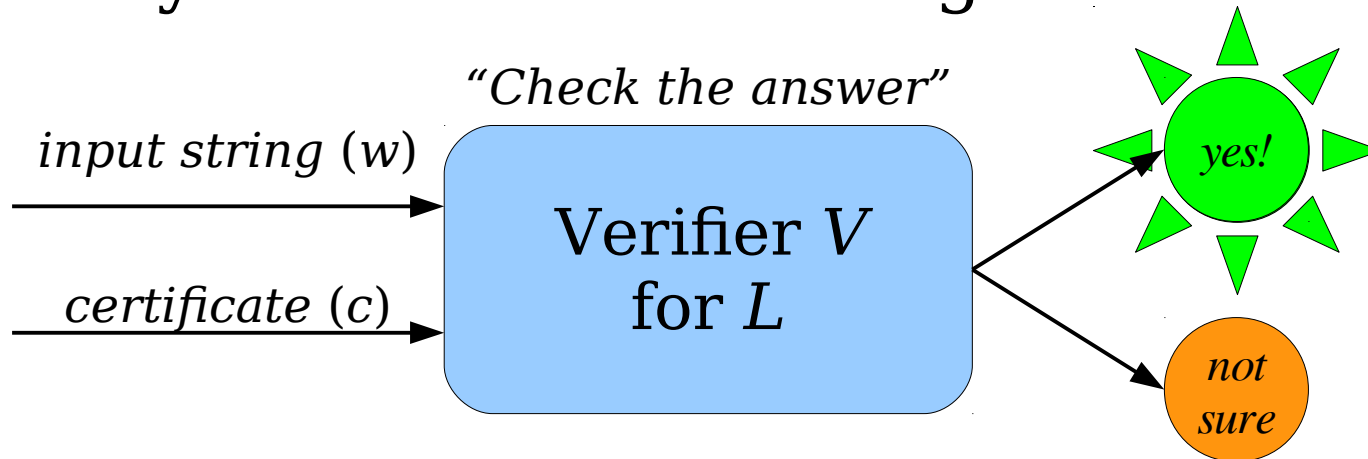
Verifiers and **RE**

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



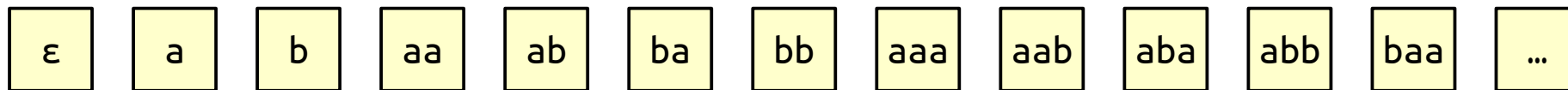
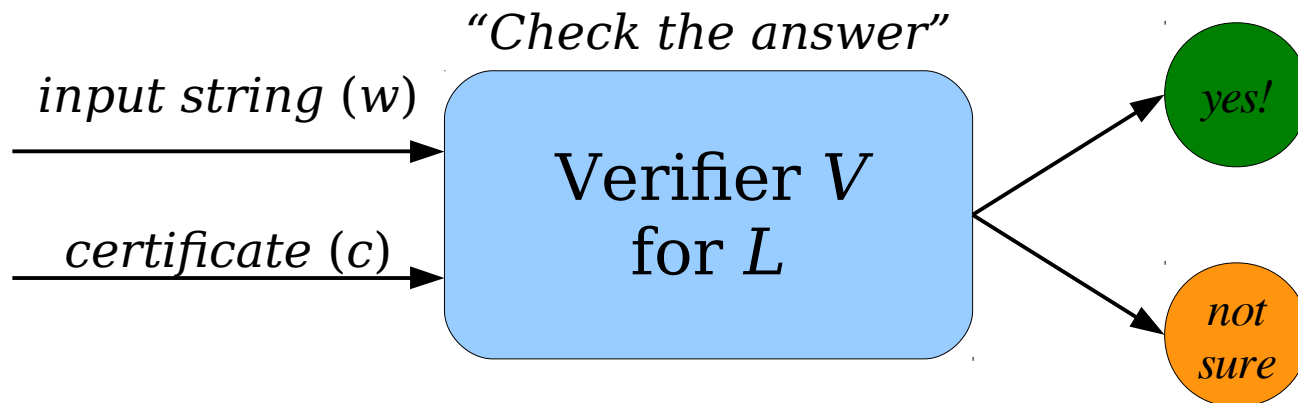
Verifiers and RE

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



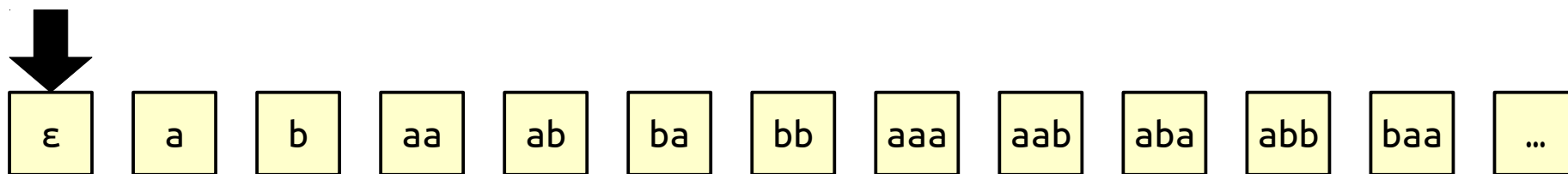
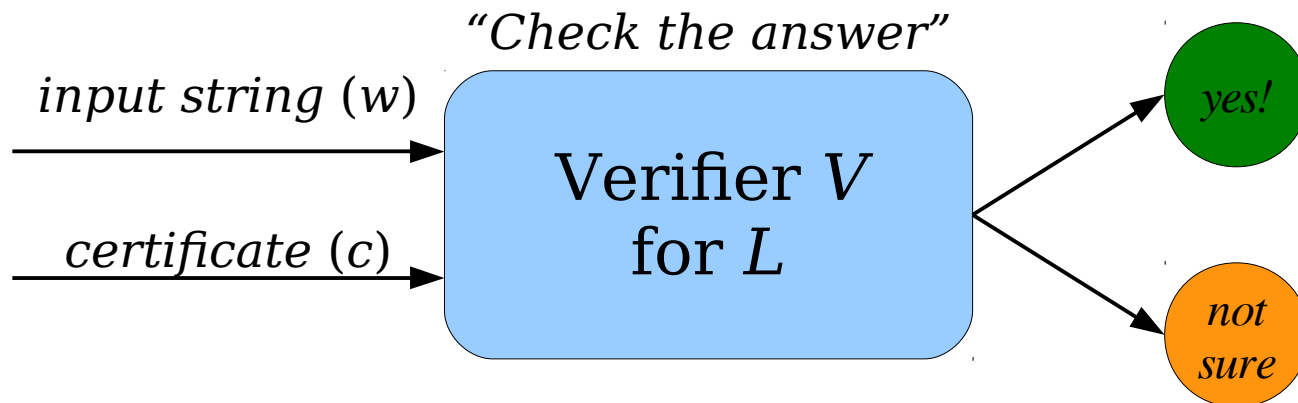
Verifiers and **RE**

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



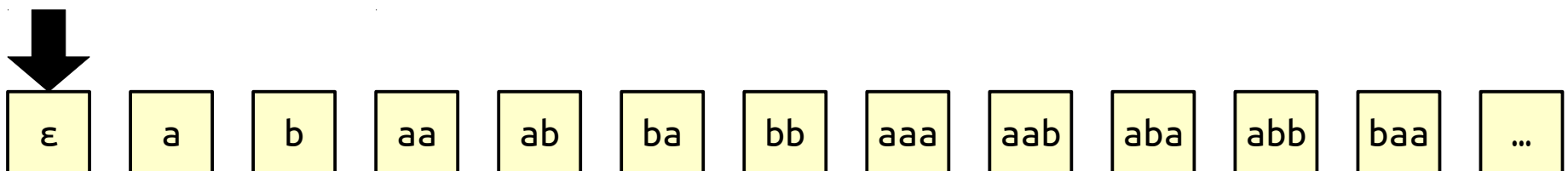
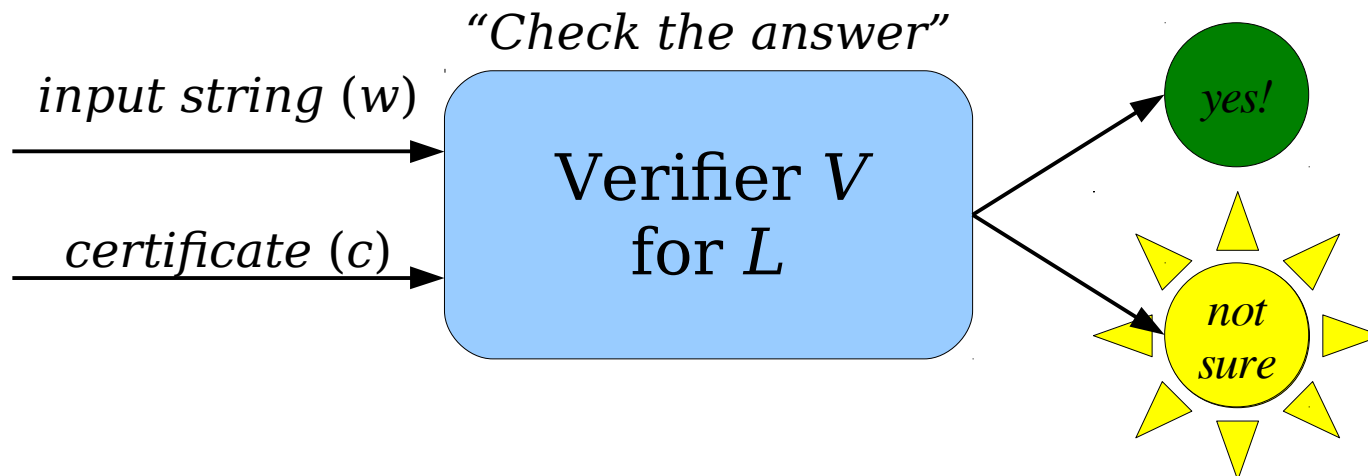
Verifiers and RE

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



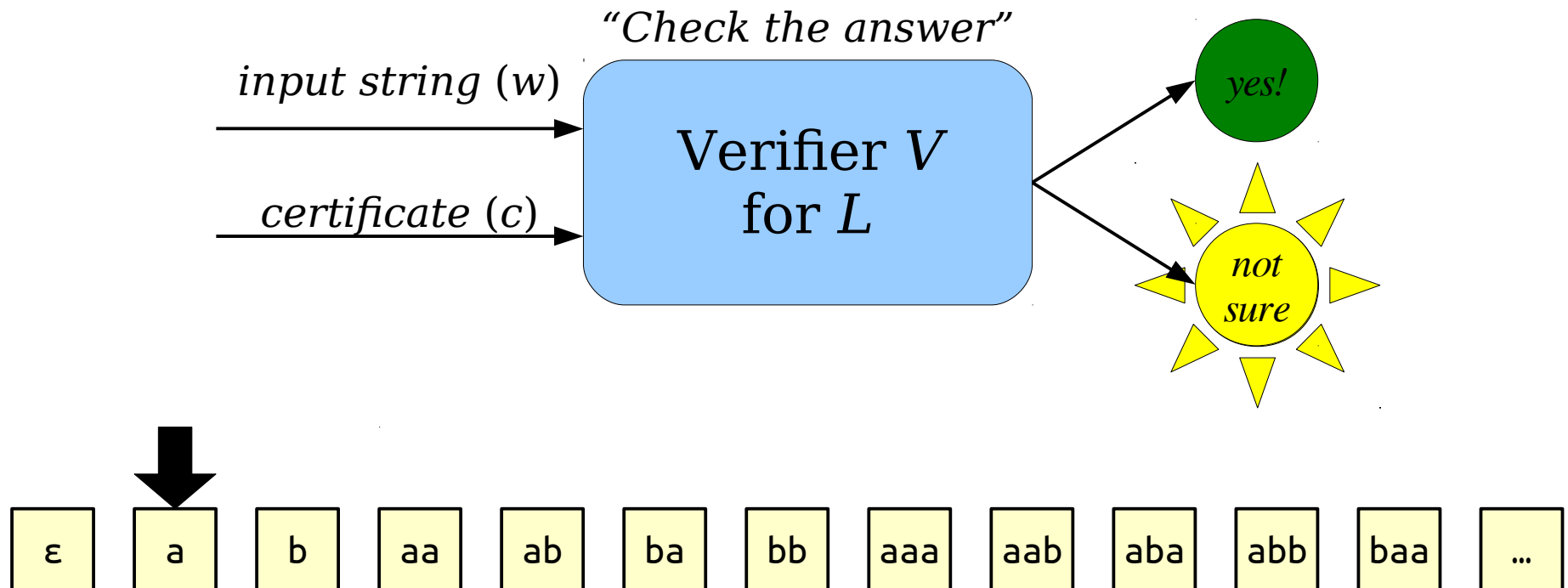
Verifiers and RE

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



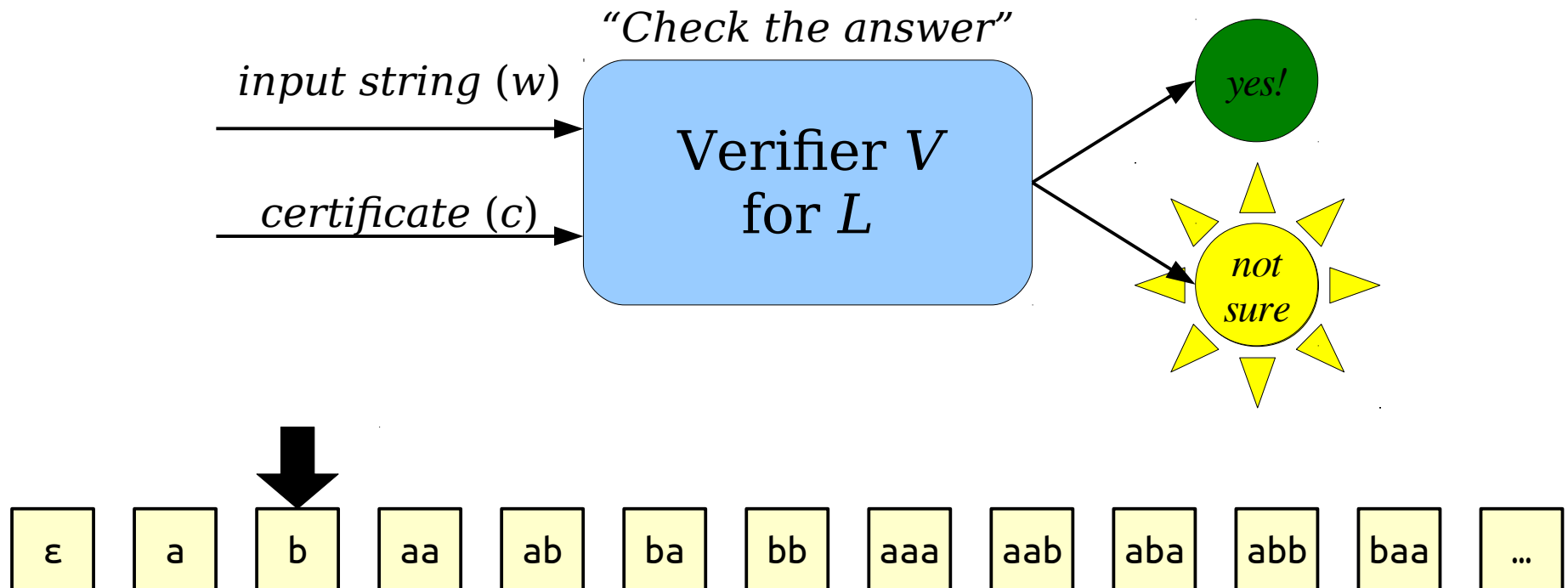
Verifiers and **RE**

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



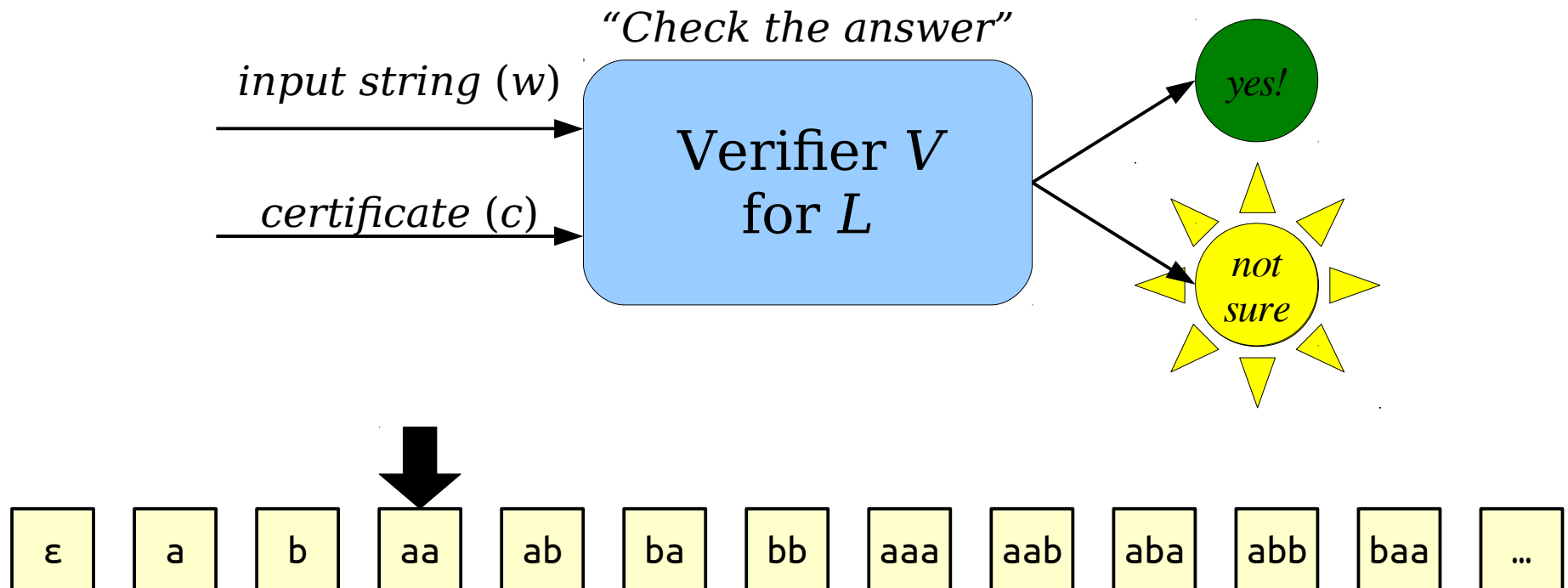
Verifiers and RE

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



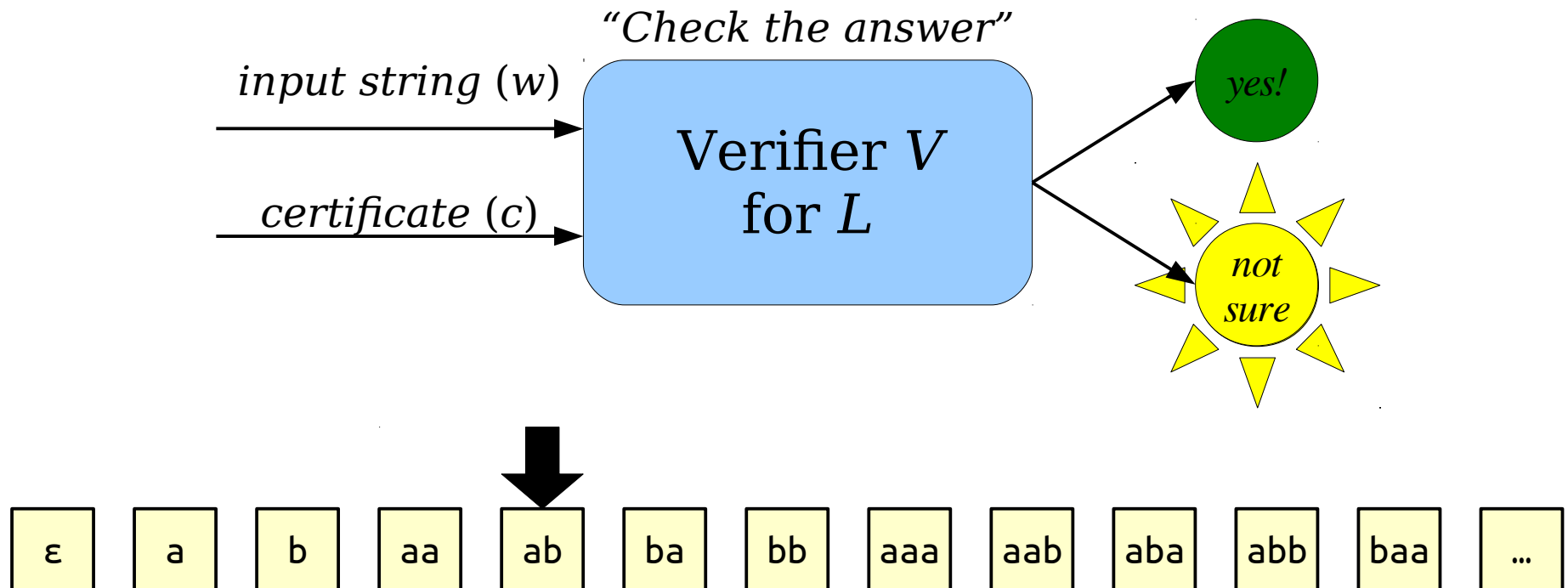
Verifiers and **RE**

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



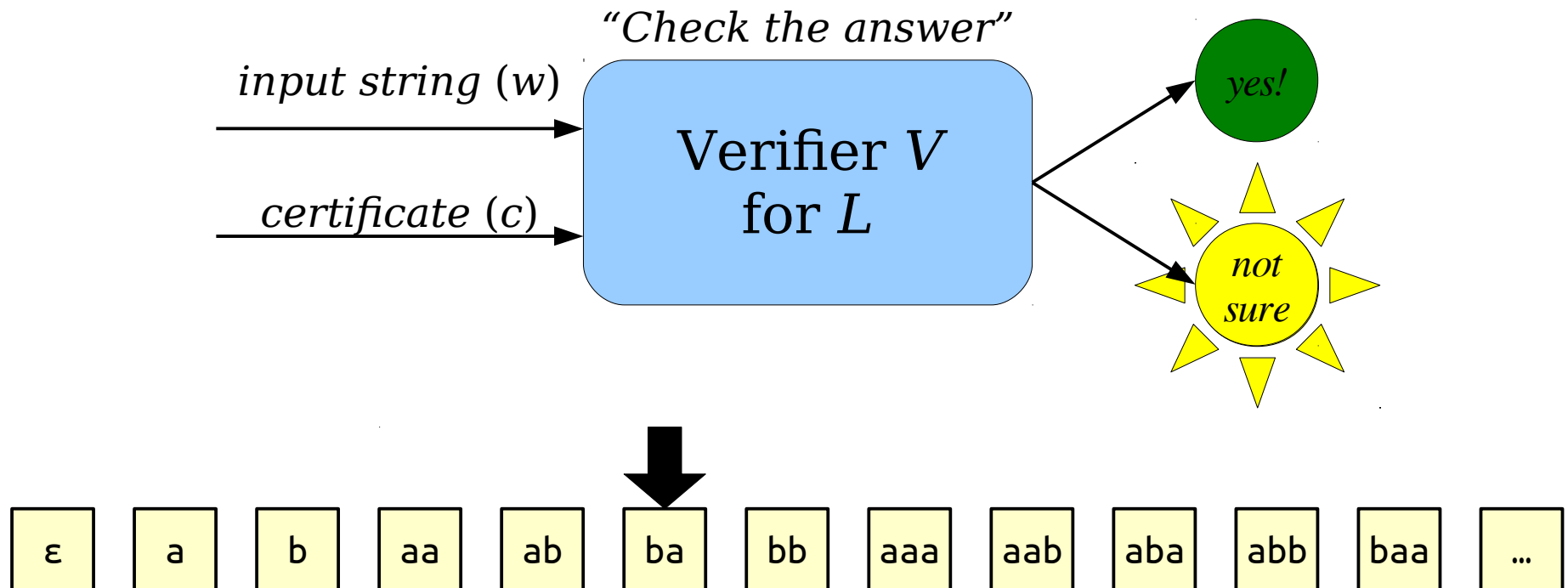
Verifiers and **RE**

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



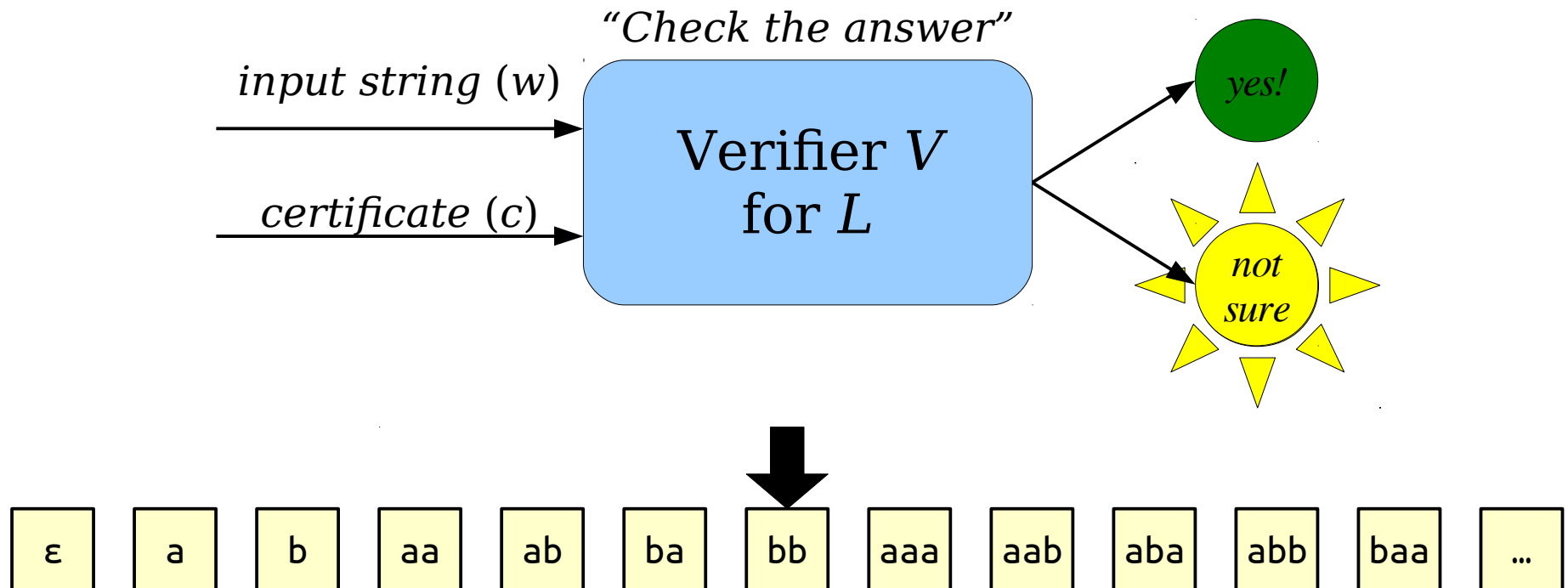
Verifiers and RE

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



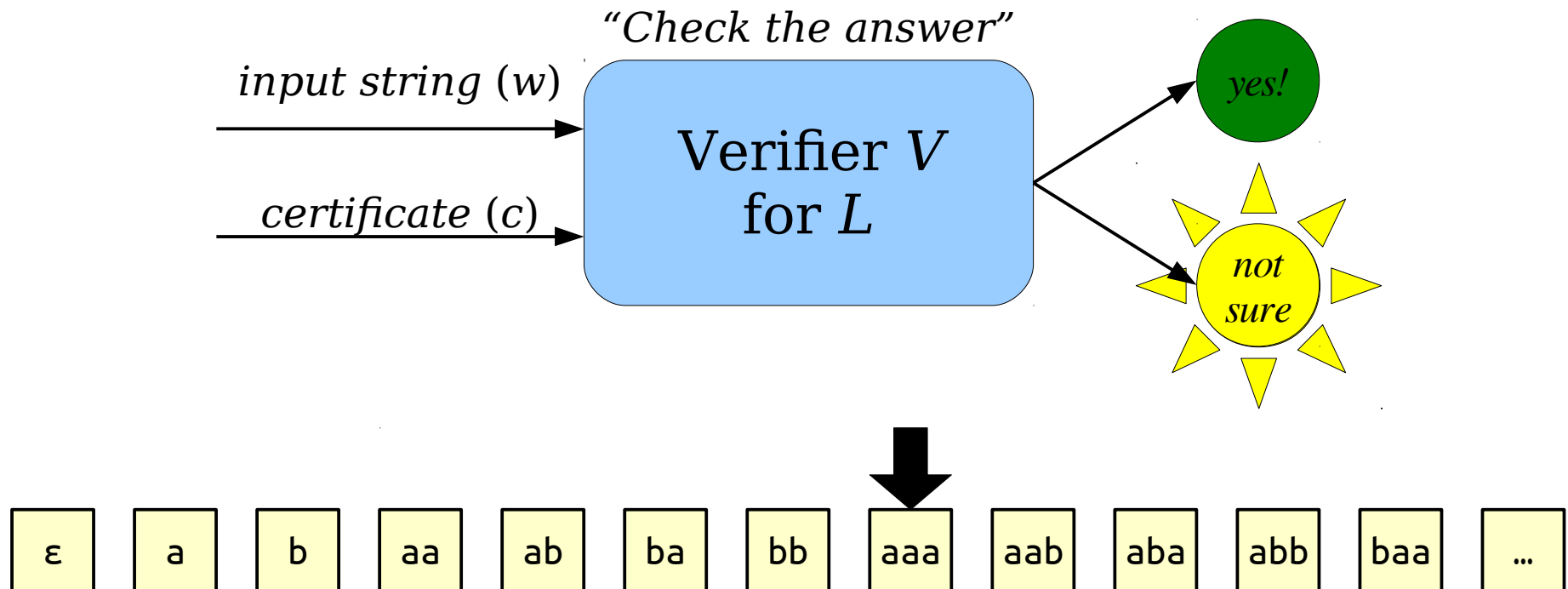
Verifiers and RE

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



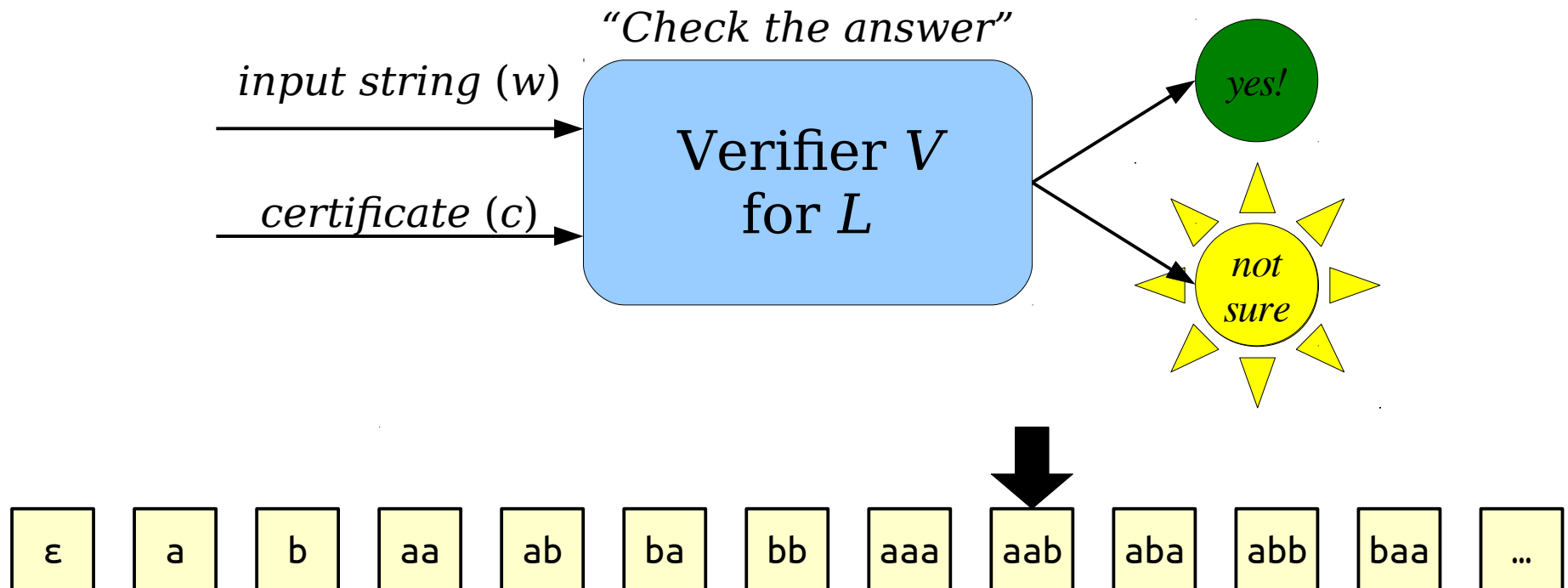
Verifiers and RE

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



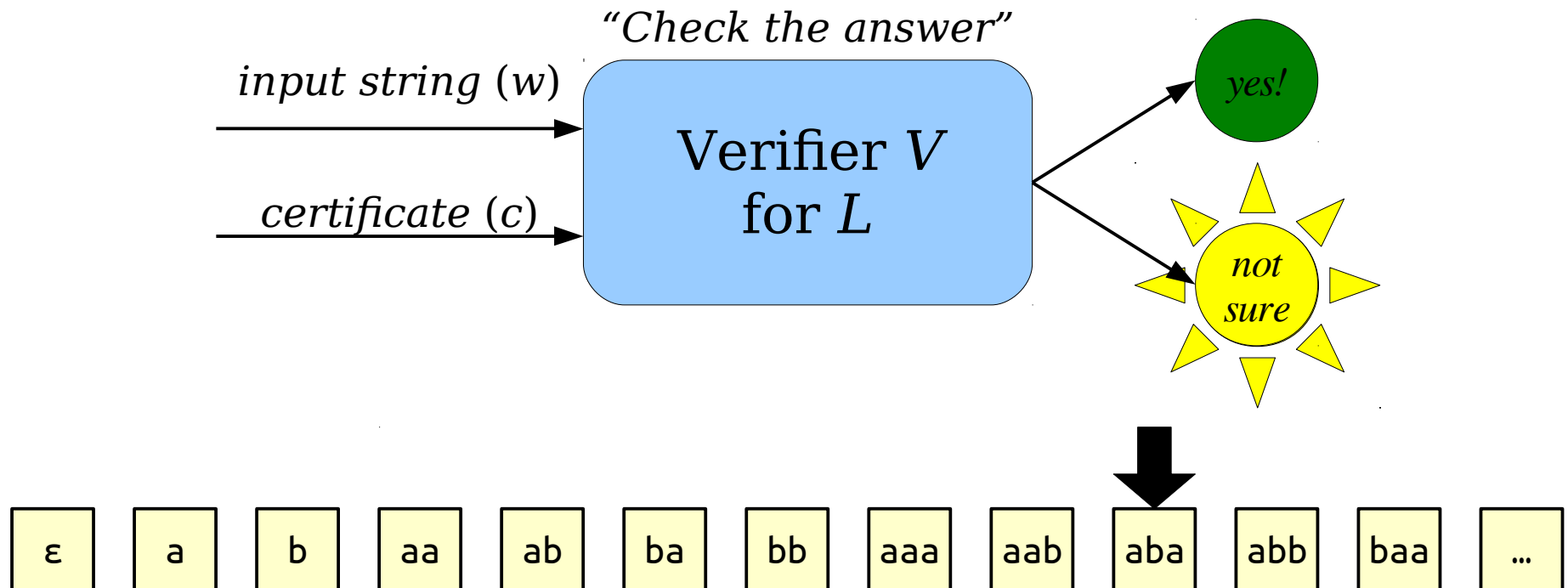
Verifiers and RE

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



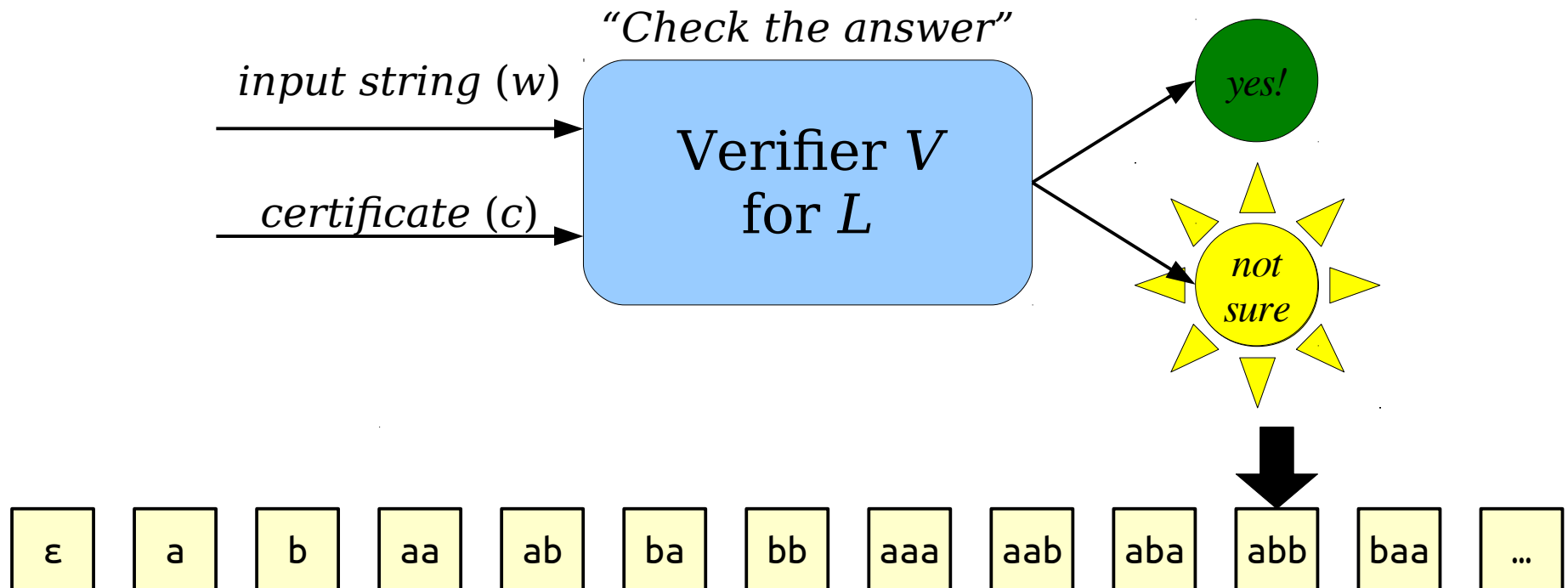
Verifiers and RE

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



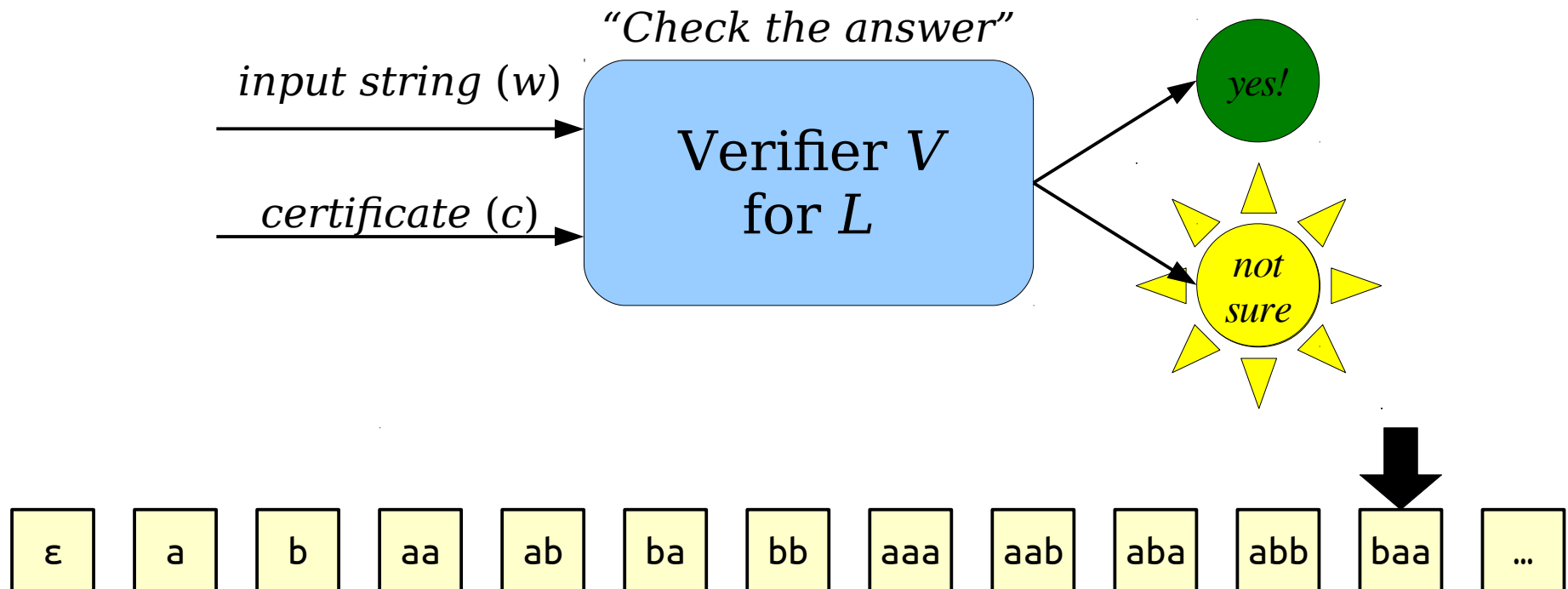
Verifiers and RE

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



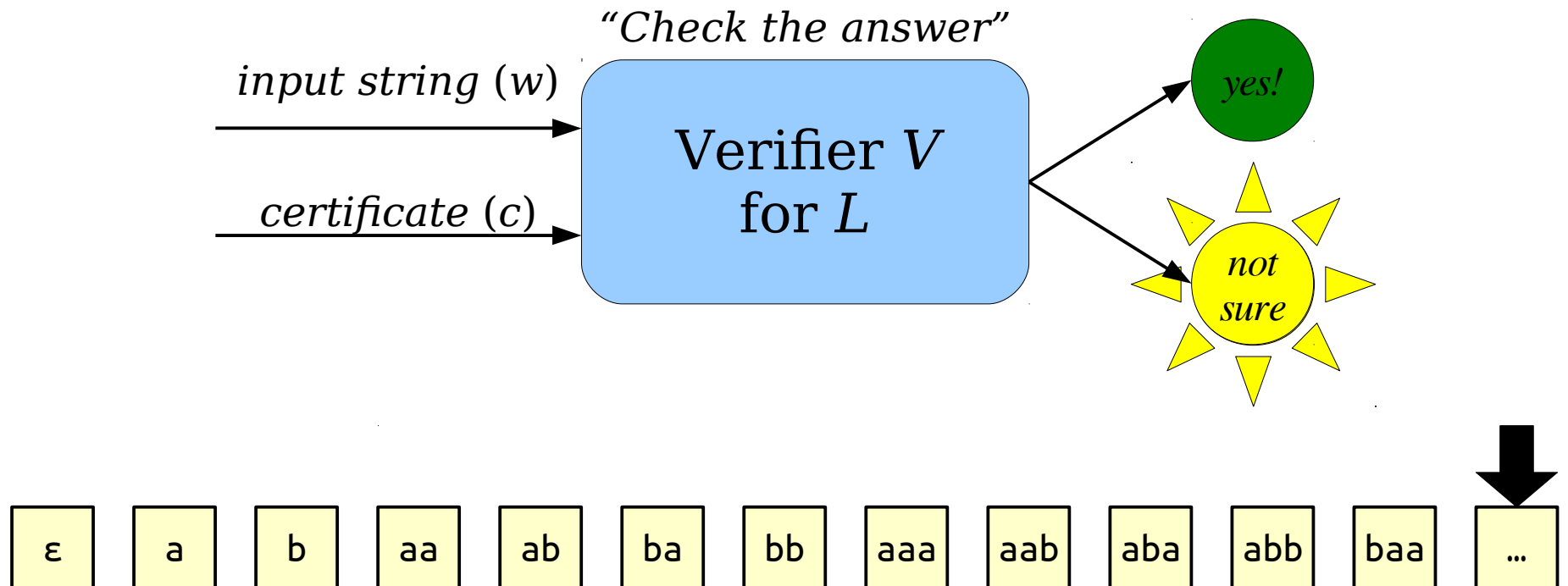
Verifiers and RE

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



Verifiers and **RE**

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof goal:** Given a verifier V for a language L , find a way to construct a recognizer M for L .



Verifiers and RE

- **Theorem:** If there is a verifier V for a language L , then $L \in \mathbf{RE}$.
- **Proof idea:** Build a recognizer that tries every possible certificate to see if $w \in L$.
- **Proof sketch:** Consider this program:

```
bool isInL(string w) {  
    for (i from 0 to  $\infty$ ) {  
        for (each string c of length i) {  
            if (V accepts  $\langle w, c \rangle$ ) return true;  
        }  
    }  
}
```

If $w \in L$, then there is a $c \in \Sigma^*$ such that V accepts $\langle w, c \rangle$. This program tries all possible strings as certificates, so it eventually finds c , watches V accept $\langle w, c \rangle$, then and accepts w . If $w \notin L$, there is no $c \in \Sigma^*$ where V accepts $\langle w, c \rangle$, so this program loops on w . ■

Verifiers and **RE**

- **Theorem:** If $L \in \mathbf{RE}$, then there is a verifier for L .
- **Proof goal:** Beginning with a recognizer M for the language L , show how to construct a verifier V for L .
- The challenges:
 - A recognizer M is not required to halt on all inputs. A verifier V must always halt.
 - A recognizer M takes in one single input. A verifier V takes in two inputs.
- We'll need to find a way of reconciling these requirements.

Recall: If M is a recognizer for a language L , then M accepts w iff $w \in L$.

Key insight: If M accepts a string w , it always does so in a finite number of steps.

Idea: Adapt the verifier for A_{TM} into a more general construction that turns any recognizer into a verifier by running it for a fixed number of steps.

Verifiers and RE

- **Theorem:** If $L \in \mathbf{RE}$, then there is a verifier for L .
- **Proof sketch:** Let L be an \mathbf{RE} language and let M be a recognizer for it. The following piece of code is a verifier for L :

```
bool checkInL(string w, int k) {  
    run M on w for k steps;  
    if (M is in an accepting state) return true;  
    else return false;  
}
```

If $w \in L$, then M accepts w in some number of steps (call it c). Calling `checkInL(w, c)` then returns true. Conversely, if there is a k where `checkInL(w, k)` returns true, then M accepts w , so $w \in L$. ■

RE and Proofs

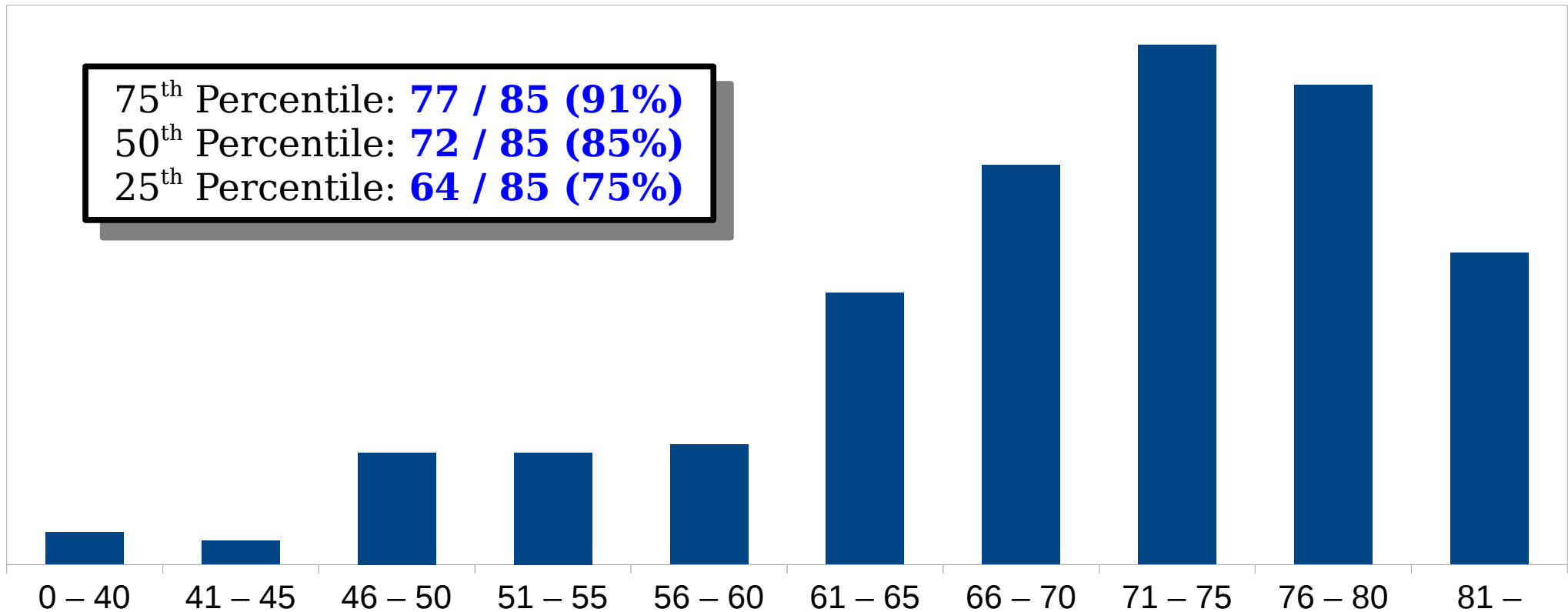
- Verifiers and recognizers give two different perspectives on the “proof” intuition for **RE**.
- Verifiers are explicitly built to check proofs that strings are in the language.
 - If you know that some string w belongs to the language and you have the proof of it, you can convince someone else that $w \in L$.
- You can think of a recognizer as a device that “searches” for a proof that $w \in L$.
 - If it finds it, great!
 - If not, it might loop forever.

RE and Proofs

- If the **RE** languages represent languages where membership can be proven, what does a non-**RE** language look like?
- Intuitively, a language is *not* in **RE** if there is no general way to prove that a given string $w \in L$ actually belongs to L .
- In other words, even if you knew that a string was in the language, you may never be able to convince anyone of it!

Time-Out for Announcements!

Problem Set Seven Graded



Problem Set Nine

- Problem Set Eight was due today at 2:30PM.
 - You *can* use late days here to extend the deadline as far as Monday at 2:30PM, but don't recommend this.
- Problem Set Nine goes out today. It's due next Friday at 2:30PM.
 - Play around with the limits of **R** and **RE** languages – the upper extent of computation!
 - See how everything fits together!
- Due to university policies, ***no late submissions will be accepted for PS9***. Please budget at least two hours before the deadline to submit the assignment.

The Last Two Guides

- We've posted two final guides to the course website:
 - The ***Guide to Self-Reference***, which talks about proofs of undecidability via self-reference.
 - The ***Guide to the Lava Diagram***, which provides an intuition for how different classes of languages relate to one another.
- Give these a read – there's a ton of useful information in there!

Final Exam Logistics

- Our final exam is Monday, December 11th from 3:30PM – 6:30PM, location TBA.
 - Sorry about how soon that is – the registrar picked this time, not us. If we had a choice, it would be on the last day of finals week.
- The exam is cumulative. You're responsible for topics from PS1 – PS9 and all of the lectures.
- As with the midterms, the exam is closed-book, closed-computer, and limited-note. You can bring one double-sided sheet of 8.5" × 11" notes with you to the exam, decorated any way you'd like.
- Students with OAE accommodations: if we don't yet have your OAE letter, please send it to us ASAP.

Preparing for the Exam

- We've posted *five* practice final exams, with solutions, to the course website.
- These exams are essentially the final exams we've given out in the last five quarters, with a few tweaks and modifications.
- Practice Final 1 and Practice Final 2 are the two most recent exams and should give you the best indicator of the expected topic coverage.
- And don't forget that Extra Practice Problems 3 is available online. After today's lecture, you know enough to take on any of those questions, including the starred ones.

Your Questions

“carta says 30+% of credit/no credit students get no credit but the sample size is pretty low. is that accurate?”

I actually don't know - I compute raw grades as letter grades and use a script that Marty Stepp put together to enter grades into Axxess, so I never actually learn who's taking the class for a letter grade and who's taking it on a CR/NC scale.

It wouldn't surprise me if this was true, though, since I'd imagine that students who opt to take the class CR/NC are not a uniformly-random sample of all students.

“Is it possible to create a DFA/NFA to model the human brain?”

I honestly have no idea, and I don't think anyone has a definitive answer to this question. There are tons of crazy psychological phenomena (e.g. transient global amnesia) that suggest that our cognitive processing might be more like a DFA than we may want to believe!

But then again, the human experience is so much more than just being Brains in Jars, and the best parts of life are often the ones that fall outside the realm of pure cognition.

Back to CS103!

Finding Non-**RE** Languages

Finding Non-**RE** Languages

- Right now, we know that non-**RE** languages exist, but we have no idea what they look like.
- How might we find one?

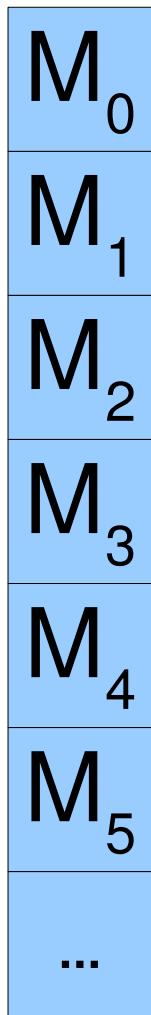
Languages, TMs, and TM Encodings

- Recall: The language of a TM M is the set

$$\mathcal{L}(M) = \{ w \in \Sigma^* \mid M \text{ accepts } w \}$$

- Some of the strings in this set might be descriptions of TMs.
- What happens if we list off all Turing machines, looking at how those TMs behave given other TMs as input?

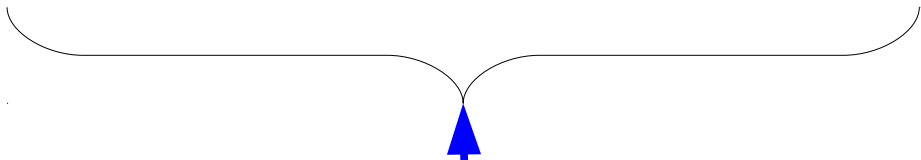
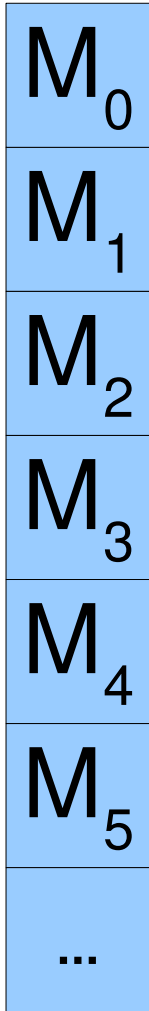
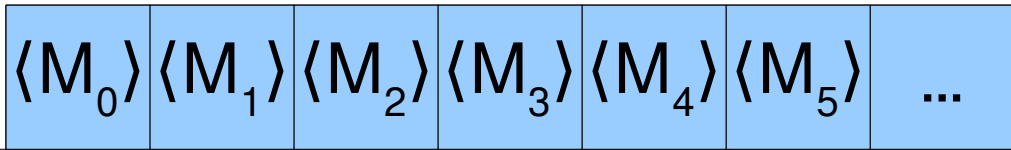
M_0
M_1
M_2
M_3
M_4
M_5
...



All Turing machines,
listed in some order.

$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
-----------------------	-----------------------	-----------------------	-----------------------	-----------------------	-----------------------	-----

M_0
M_1
M_2
M_3
M_4
M_5
...



All descriptions of TMs, listed in the same order.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1							
M_2							
M_3							
M_4							
M_5							
...							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2							
M_3							
M_4							
M_5							
...							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3							
M_4							
M_5							
...							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4							
M_5							
...							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5							
...							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

Acc	Acc	Acc	No	Acc	No	...
-----	-----	-----	----	-----	----	-----

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

Flip all "accept"
to "no" and
vice-versa

No	No	No	Acc	No	Acc	...
----	----	----	-----	----	-----	-----

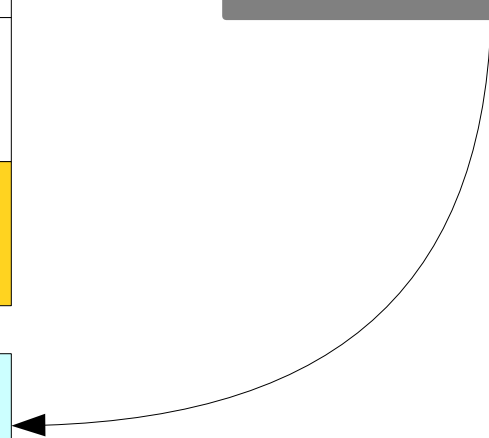
	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

No	No	No	Acc	No	Acc	...
----	----	----	-----	----	-----	-----

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

No	No	No	Acc	No	Acc	...
----	----	----	-----	----	-----	-----

What TM has this behavior?



	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

No	No	No	Acc	No	Acc	...
----	----	----	-----	----	-----	-----

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

No	No	No	Acc	No	Acc	...
----	----	----	-----	----	-----	-----

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

No	No	No	Acc	No	Acc	...
----	----	----	-----	----	-----	-----

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

No	No	No	Acc	No	Acc	...
----	----	----	-----	----	-----	-----

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

No	No	No	Acc	No	Acc	...
----	----	----	-----	----	-----	-----

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

No	No	No	Acc	No	Acc	...
----	----	----	-----	----	-----	-----

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

No	No	No	Acc	No	Acc	...
----	----	----	-----	----	-----	-----

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

No	No	No	Acc	No	Acc	...
----	----	----	-----	----	-----	-----

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

No	No	No	Acc	No	Acc	...
----	----	----	-----	----	-----	-----

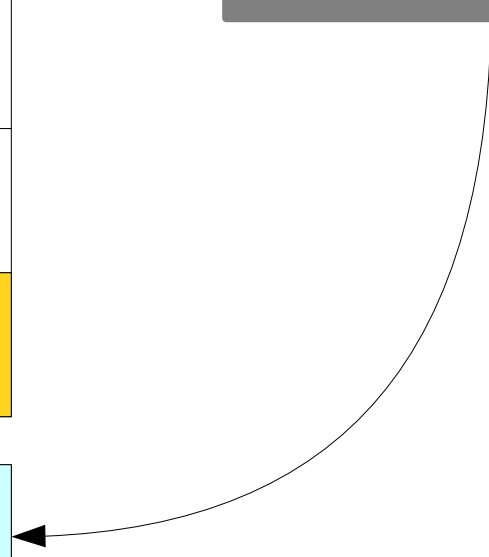
	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

No	No	No	Acc	No	Acc	...
----	----	----	-----	----	-----	-----

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

No No No Acc No Acc ...

No TM has this behavior!



	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

No	No	No	Acc	No	Acc	...
----	----	----	-----	----	-----	-----

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

No	No	No	Acc	No	Acc	...
----	----	----	-----	----	-----	-----

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

No	No	No	Acc	No	Acc	...
----	----	----	-----	----	-----	-----

“The language of all TMs that do not accept their own description.”

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

$\{ \langle M \rangle \mid M \text{ is a TM that does not accept } \langle M \rangle \}$

No	No	No	Acc	No	Acc	...
----	----	----	-----	----	-----	-----

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$...
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
...

$\{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathcal{L}(M) \}$

No	No	No	Acc	No	Acc	...
----	----	----	-----	----	-----	-----

Diagonalization Revisited

- The ***diagonalization language***, which we denote L_D , is defined as

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathcal{L}(M) \}$$

- That is, L_D is the set of descriptions of Turing machines that do not accept themselves.

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathcal{L}(M) \}$$

Theorem: $L_D \notin \mathbf{RE}$.

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathcal{L}(M) \}$$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$.

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathcal{L}(M) \}$$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathcal{L}(R) = L_D$.

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathcal{L}(M) \}$$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathcal{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathcal{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (1)$$

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathcal{L}(M) \}$$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathcal{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathcal{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (1)$$

Because $\mathcal{L}(R) = L_D$, we know that a string belongs to one set if and only if it belongs to the other.

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathcal{L}(M) \}$$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathcal{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathcal{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (1)$$

From the definition of L_D , we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathcal{L}(M)$.

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathcal{L}(M) \}$$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathcal{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathcal{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (1)$$

From the definition of L_D , we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathcal{L}(M)$. Combining this with statement (1) tells us that

$$\langle M \rangle \notin \mathcal{L}(M) \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (2)$$

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathcal{L}(M) \}$$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathcal{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathcal{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (1)$$

From the definition of L_D , we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathcal{L}(M)$. Combining this with statement (1) tells us that

$$\langle M \rangle \notin \mathcal{L}(M) \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (2)$$

We've replaced the left-hand side of this biconditional with an equivalent statement.

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathcal{L}(M) \}$$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathcal{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathcal{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (1)$$

From the definition of L_D , we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathcal{L}(M)$. Combining this with statement (1) tells us that

$$\langle M \rangle \notin \mathcal{L}(M) \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (2)$$

Since our choice of M was arbitrary, we see that statement (2) holds for any TM M .

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathcal{L}(M) \}$$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathcal{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathcal{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (1)$$

From the definition of L_D , we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathcal{L}(M)$. Combining this with statement (1) tells us that

$$\langle M \rangle \notin \mathcal{L}(M) \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (2)$$

Since our choice of M was arbitrary, we see that statement (2) holds for any TM M .

A nice consequence of a universally-quantified statement is that it should work in all cases.

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathcal{L}(M) \}$$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathcal{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathcal{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (1)$$

From the definition of L_D , we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathcal{L}(M)$. Combining this with statement (1) tells us that

$$\langle M \rangle \notin \mathcal{L}(M) \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (2)$$

Since our choice of M was arbitrary, we see that statement (2) holds for any TM M . In particular, this means that statement (2) holds for the TM R , which tells us that

$$\langle R \rangle \notin \mathcal{L}(R) \text{ iff } \langle R \rangle \in \mathcal{L}(R). \quad (3)$$

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathcal{L}(M) \}$$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathcal{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathcal{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (1)$$

From the definition of L_D , we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathcal{L}(M)$. Combining this with statement (1) tells us that

$$\langle M \rangle \notin \mathcal{L}(M) \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (2)$$

Since our choice of M was arbitrary, we see that statement (2) holds for any TM M . In particular, this means that statement (2) holds for the TM R , which tells us that

$$\langle R \rangle \notin \mathcal{L}(R) \text{ iff } \langle R \rangle \in \mathcal{L}(R). \quad (3)$$

This is clearly impossible.

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathcal{L}(M) \}$$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathcal{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathcal{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (1)$$

From the definition of L_D , we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathcal{L}(M)$. Combining this with statement (1) tells us that

$$\langle M \rangle \notin \mathcal{L}(M) \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (2)$$

Since our choice of M was arbitrary, we see that statement (2) holds for any TM M . In particular, this means that statement (2) holds for the TM R , which tells us that

$$\langle R \rangle \notin \mathcal{L}(R) \text{ iff } \langle R \rangle \in \mathcal{L}(R). \quad (3)$$

This is clearly impossible. We have reached a contradiction, so our assumption must have been wrong.

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathcal{L}(M) \}$$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathcal{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathcal{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (1)$$

From the definition of L_D , we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathcal{L}(M)$. Combining this with statement (1) tells us that

$$\langle M \rangle \notin \mathcal{L}(M) \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (2)$$

Since our choice of M was arbitrary, we see that statement (2) holds for any TM M . In particular, this means that statement (2) holds for the TM R , which tells us that

$$\langle R \rangle \notin \mathcal{L}(R) \text{ iff } \langle R \rangle \in \mathcal{L}(R). \quad (3)$$

This is clearly impossible. We have reached a contradiction, so our assumption must have been wrong. Thus $L_D \notin \mathbf{RE}$.

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathcal{L}(M) \}$$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathcal{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathcal{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (1)$$

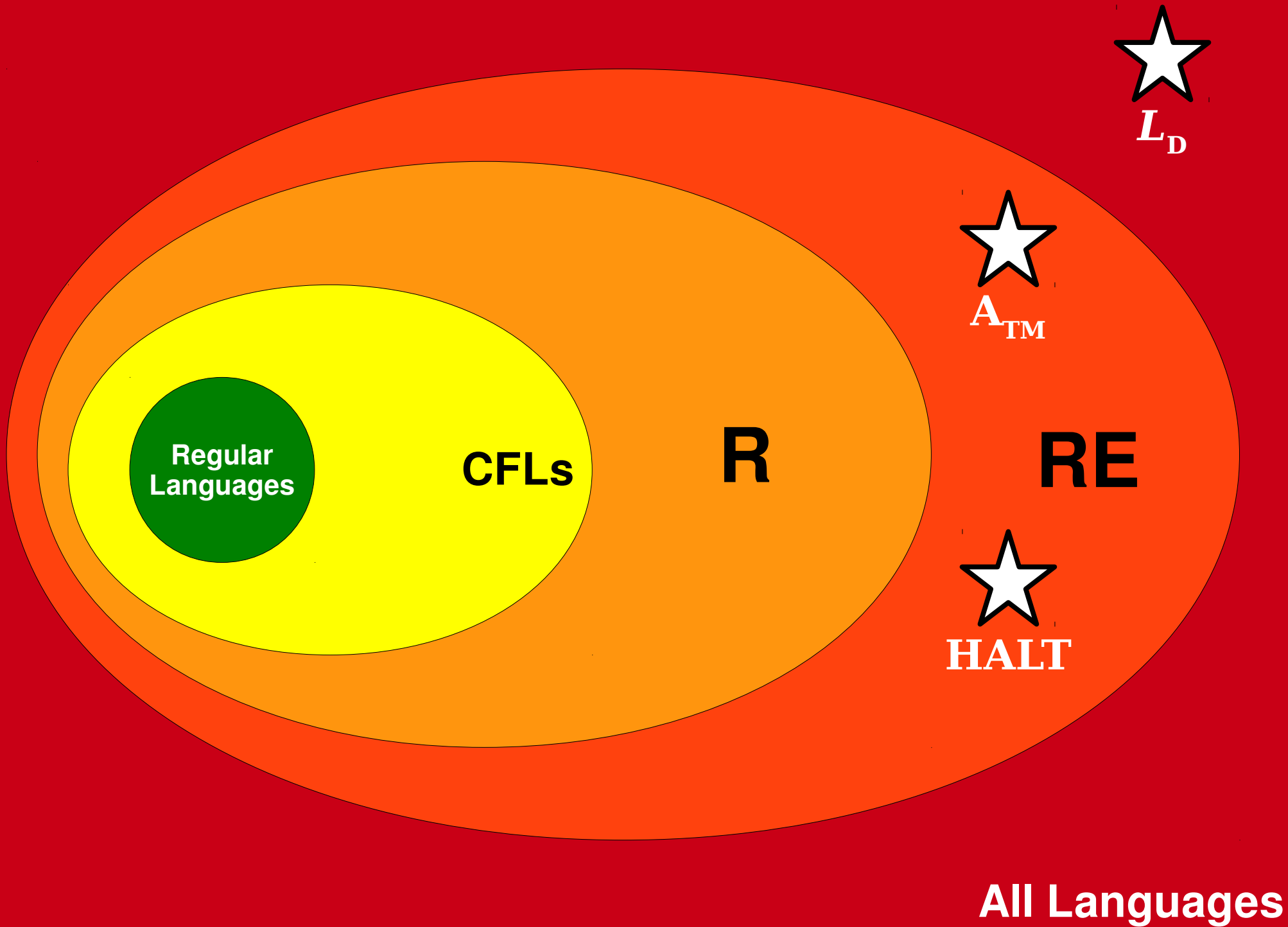
From the definition of L_D , we see that $\langle M \rangle \in L_D$ iff $\langle M \rangle \notin \mathcal{L}(M)$. Combining this with statement (1) tells us that

$$\langle M \rangle \notin \mathcal{L}(M) \text{ iff } \langle M \rangle \in \mathcal{L}(R). \quad (2)$$

Since our choice of M was arbitrary, we see that statement (2) holds for any TM M . In particular, this means that statement (2) holds for the TM R , which tells us that

$$\langle R \rangle \notin \mathcal{L}(R) \text{ iff } \langle R \rangle \in \mathcal{L}(R). \quad (3)$$

This is clearly impossible. We have reached a contradiction, so our assumption must have been wrong. Thus $L_D \notin \mathbf{RE}$. ■



What This Means

- On a deeper philosophical level, the fact that non-**RE** languages exist supports the following claim:

There are statements that are true but not provable.

- Intuitively, given any non-**RE** language, there will be some string in the language that *cannot* be proven to be in the language.
- This result can be formalized as a result called ***Gödel's incompleteness theorem***, one of the most important mathematical results of all time.
- Want to learn more? Take Phil 152 or CS154!

What This Means

- On a more philosophical note, you could interpret the previous result in the following way:

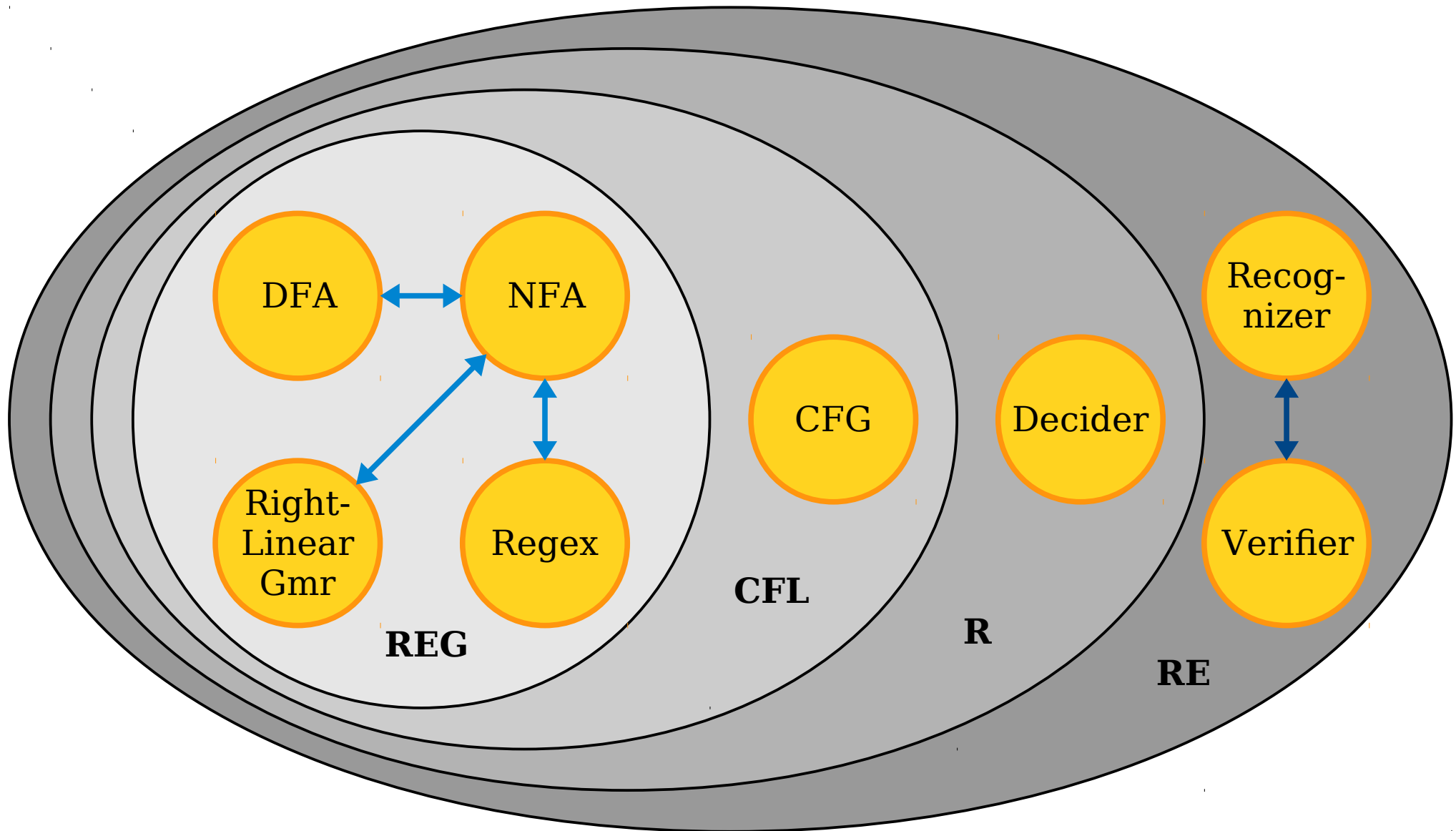
There are inherent limits about what mathematics can teach us.

- There's no automatic way to do math. There are true statements that we can't prove.
- That doesn't mean that mathematics is worthless. It just means that we need to temper our expectations about it.

Where We Stand

- We've just done a crazy, whirlwind tour of computability theory:
 - ***The Church-Turing thesis*** tells us that TMs give us a mechanism for studying computation in the abstract.
 - ***Universal computers*** – computers as we know them – are not just a stroke of luck. The existence of the universal TM ensures that such computers must exist.
 - ***Self-reference*** is an inherent consequence of computational power.
 - ***Undecidable problems*** exist partially as a consequence of the above and indicate that there are statements whose truth can't be determined by computational processes.
 - ***Unrecognizable problems*** are out there and can be discovered via diagonalization. They imply there are limits to mathematical proof.

The Big Picture



Where We've Been

- The class **R** represents problems that can be solved by a computer.
- The class **RE** represents problems where “yes” answers can be verified by a computer.

Where We're Going

- The class **P** represents problems that can be solved *efficiently* by a computer.
- The class **NP** represents problems where “yes” answers can be verified *efficiently* by a computer.

Next Time

- ***Introduction to Complexity Theory***
 - Not all decidable problems are created equal!
- ***The Classes P and NP***
 - Two fundamental and important complexity classes.
- ***The $P \stackrel{?}{=} NP$ Question***
 - A literal million-dollar question!