Complexity Theory

Part Two

Recap from Last Time

The Cobham-Edmonds Thesis

A language L can be ***decided efficiently*** if there is a TM that decides it in polynomial time.

Equivalently, L can be decided efficiently if it can be decided in time $O(n^k)$ for some $k \in \mathbb{N}$.

Like the Church-Turing thesis, this is ***not*** a theorem!

It's an assumption about the nature of efficient computation, and it is somewhat controversial.

The Complexity Class **P**

- The ***complexity class* P** (for ***p***olynomial time) contains all problems that can be solved in polynomial time.
- Formally:

$$\mathbf{P} = \{ L \mid \text{There is a polynomial-time decider for } L \}$$

- Assuming the Cobham-Edmonds thesis, a language is in **P** if it can be decided efficiently.

Polynomial-Time Verifiers

- A ***polynomial-time verifier*** for L is a TM V such that
  - V halts on all inputs.
  - $w \in L$ iff $\exists c \in \Sigma^*$. V accepts $\langle w, c \rangle$.
  - V's runtime is a polynomial in $|w|$ (that is, V's runtime is $O(|w|^k)$ for some integer k)

The Complexity Class **NP**

- The complexity class **NP** (***nondeterministic polynomial time***) contains all problems that can be verified in polynomial time.
- Formally:

$$\textbf{NP} = \{ L \mid \text{There is a polynomial-time verifier for } L \}$$

- The name **NP** comes from another way of characterizing **NP**. If you introduce *nondeterministic Turing machines* and appropriately define "polynomial time," then **NP** is the set of problems that an NTM can solve in polynomial time.

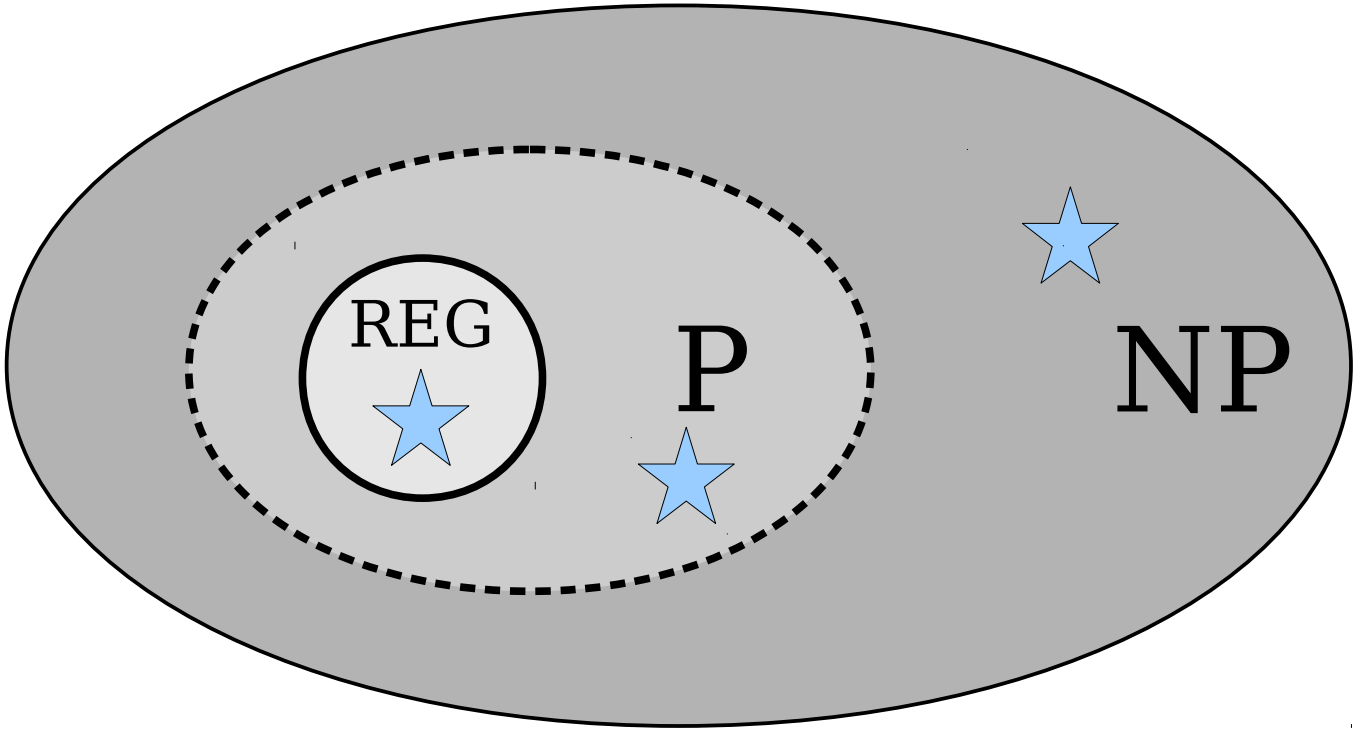
Theorem (Baker-Gill-Solovay): Any proof that purely relies on universality and self-reference cannot resolve $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$.

Proof: Take CS154!

So how *are* we going to reason about **P** and **NP**?

New Stuff!

A Challenge



Problems in **NP** vary widely in their difficulty, even if **P** = **NP**.

How can we rank the relative difficulties of problems?

Reducibility

Maximum Matching

- Given an undirected graph G, a ***matching*** in G is a set of edges such that no two edges share an endpoint.
- A ***maximum matching*** is a matching with the largest number of edges.

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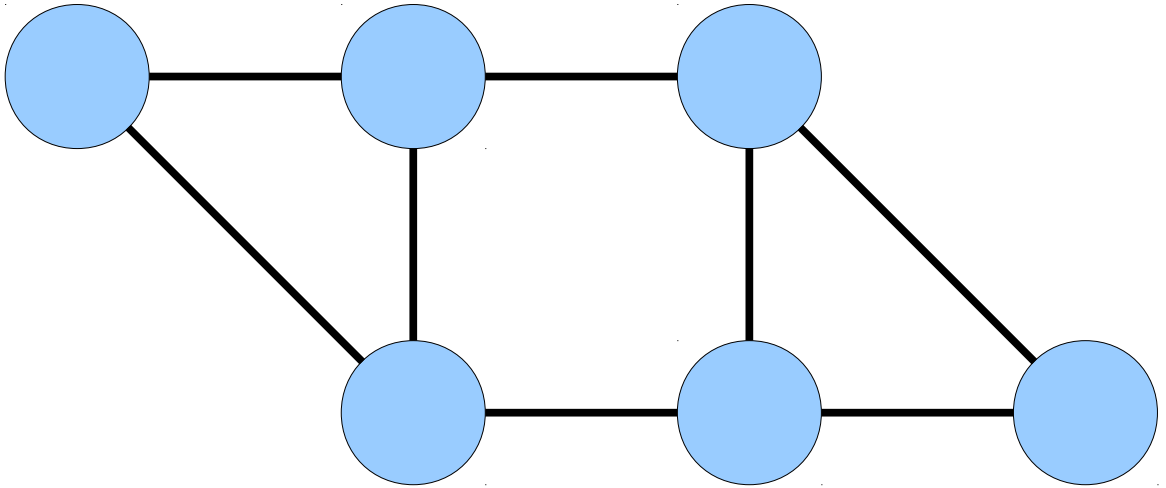

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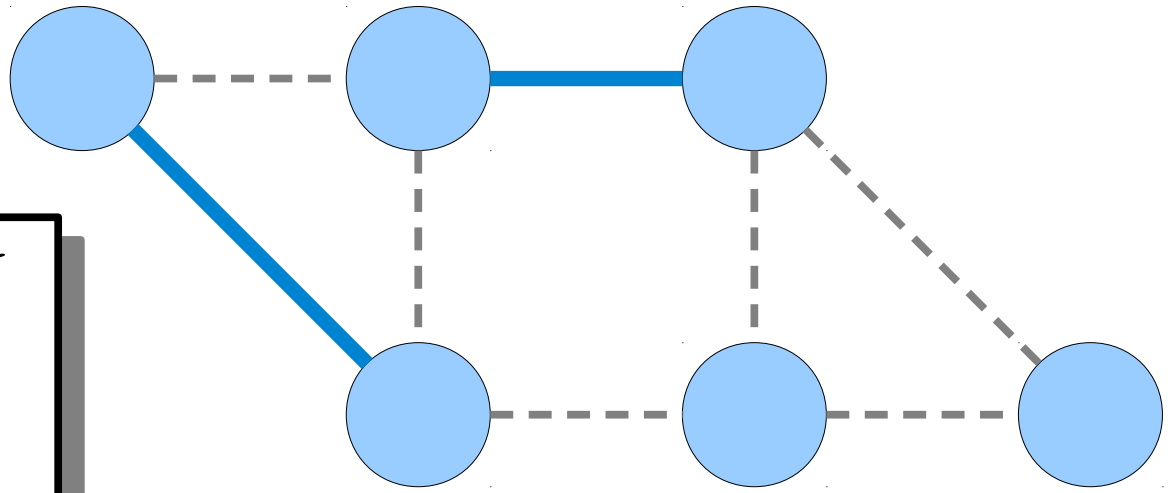

A matching, but not a maximum matching.

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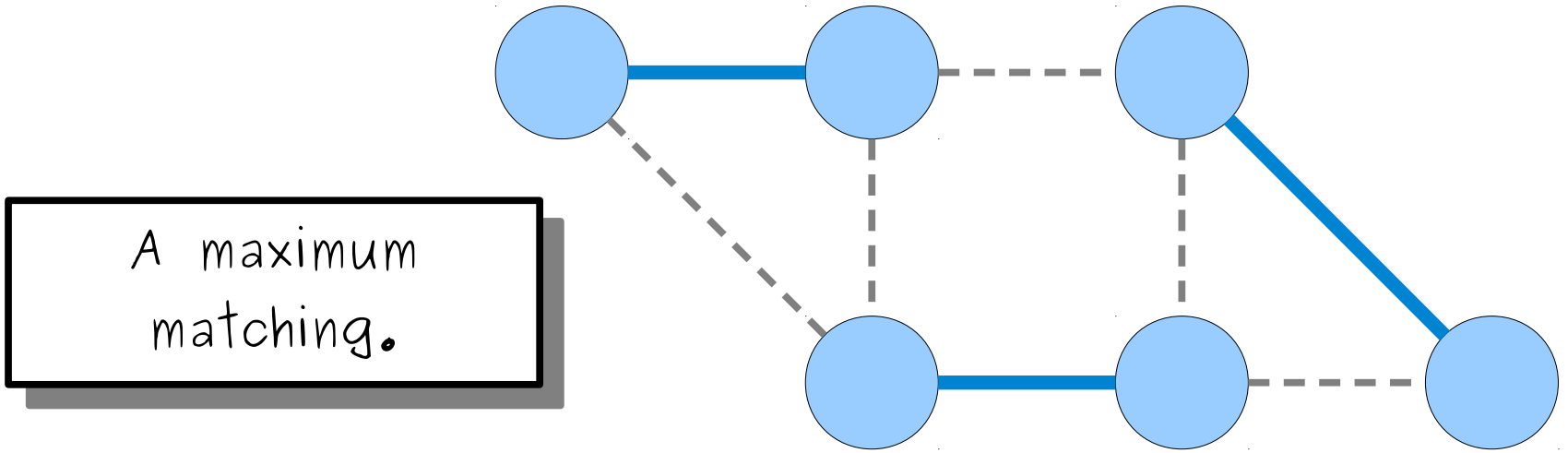

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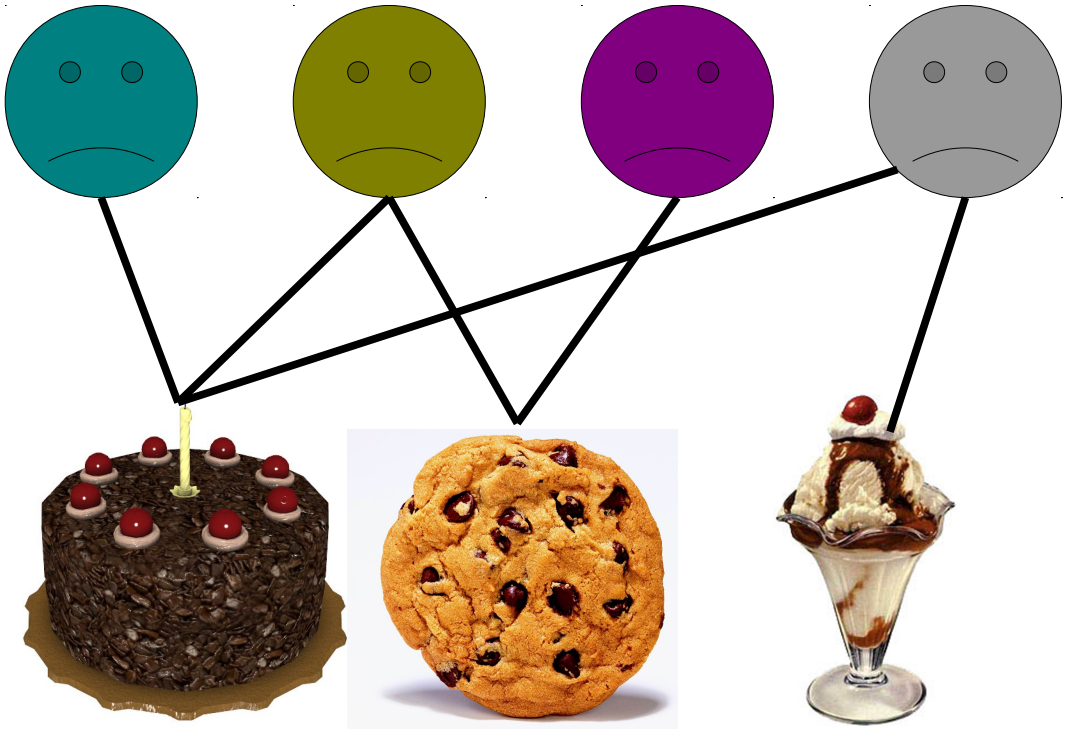

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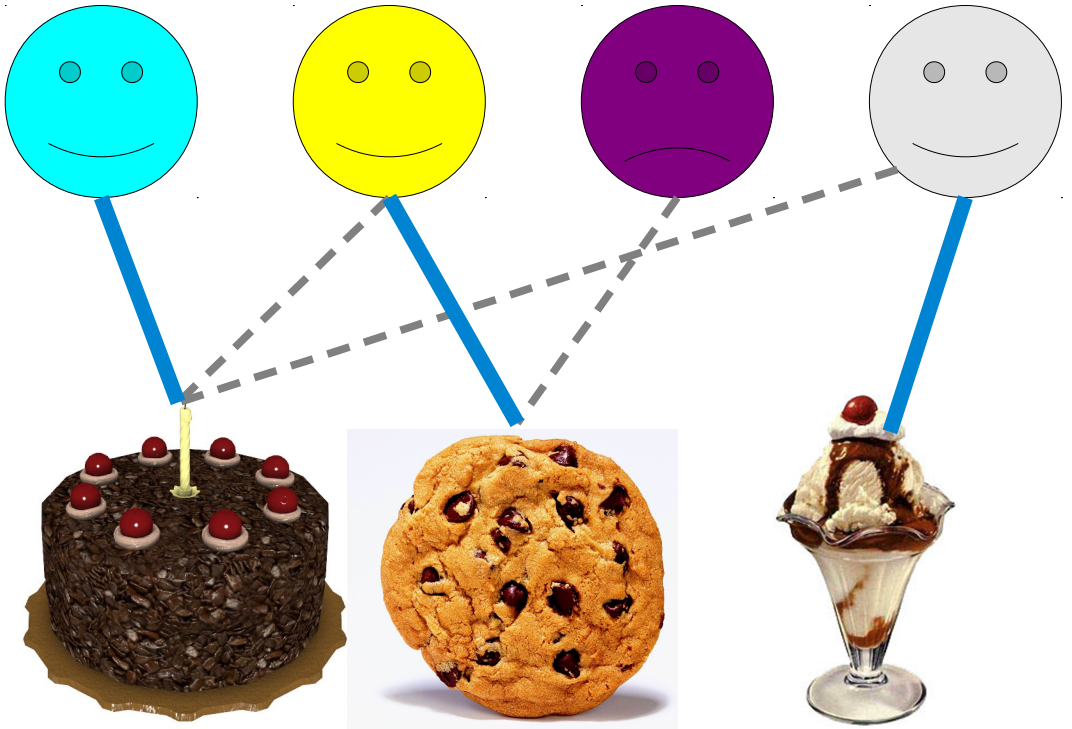
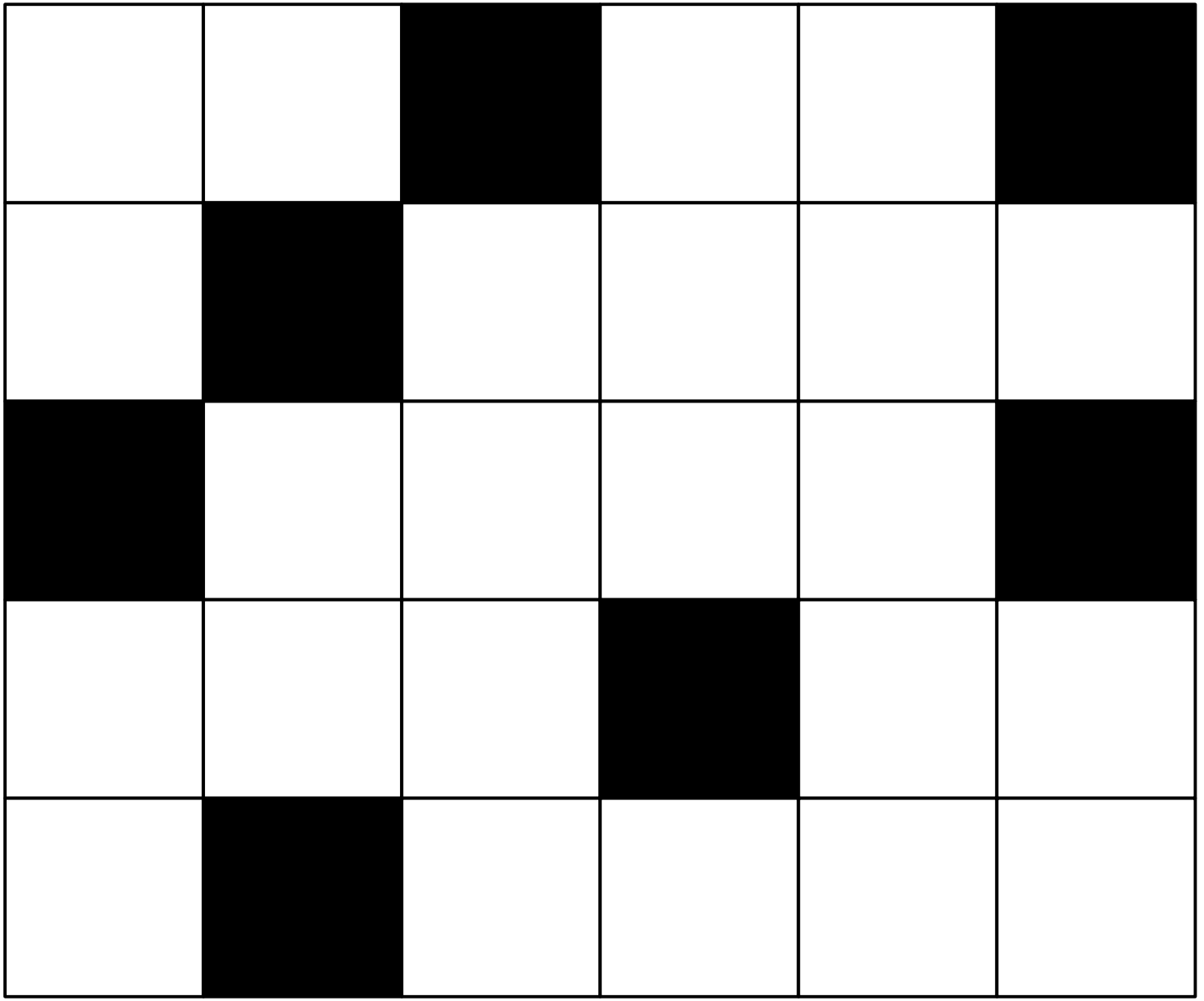


Maximum Matching

- Jack Edmonds' paper “Paths, Trees, and Flowers” gives a polynomial-time algorithm for finding maximum matchings.
  - (This is the same Edmonds as in “Cobham-Edmonds Thesis.”)
- Using this fact, what other problems can we solve?

Domino Tiling



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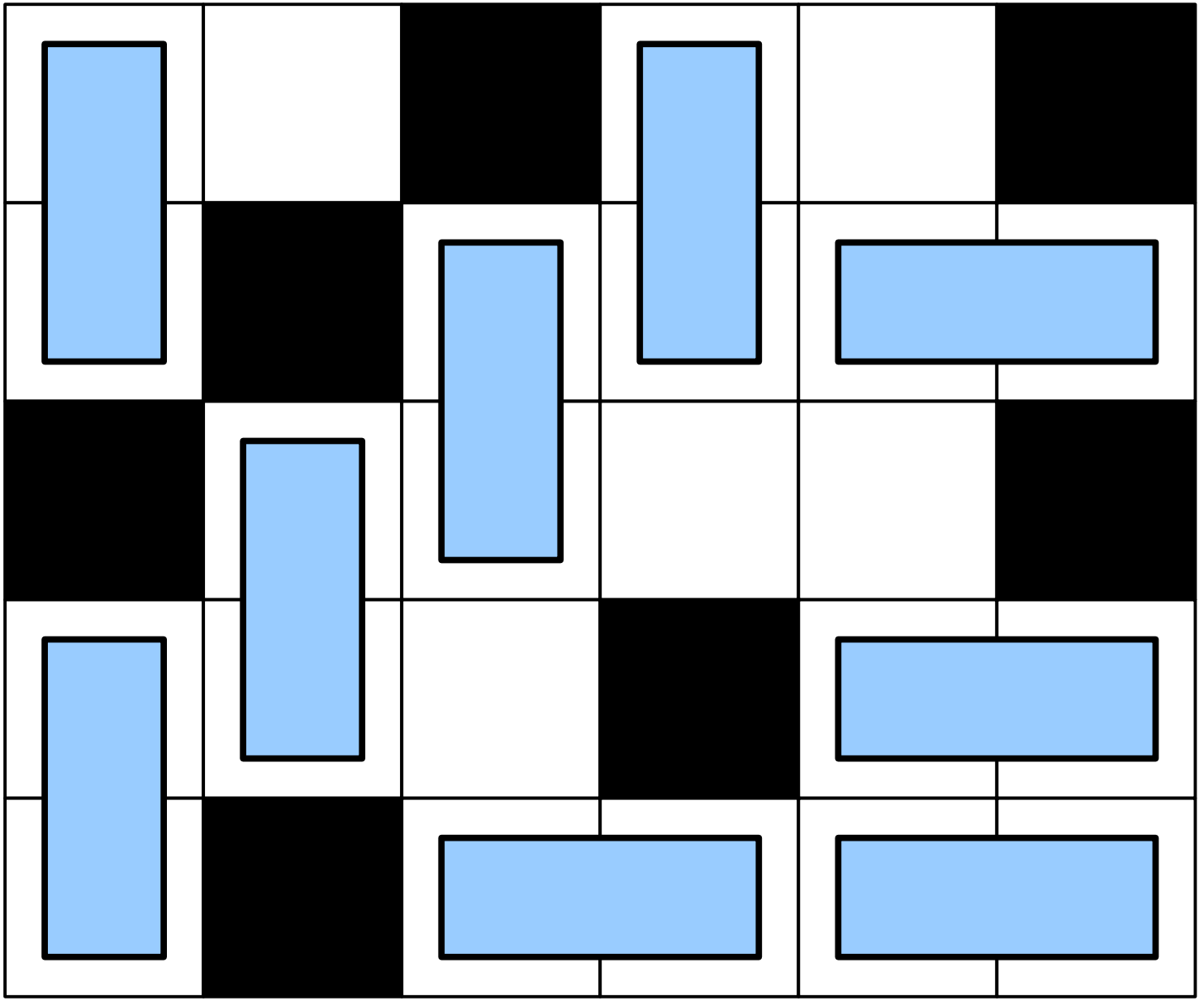

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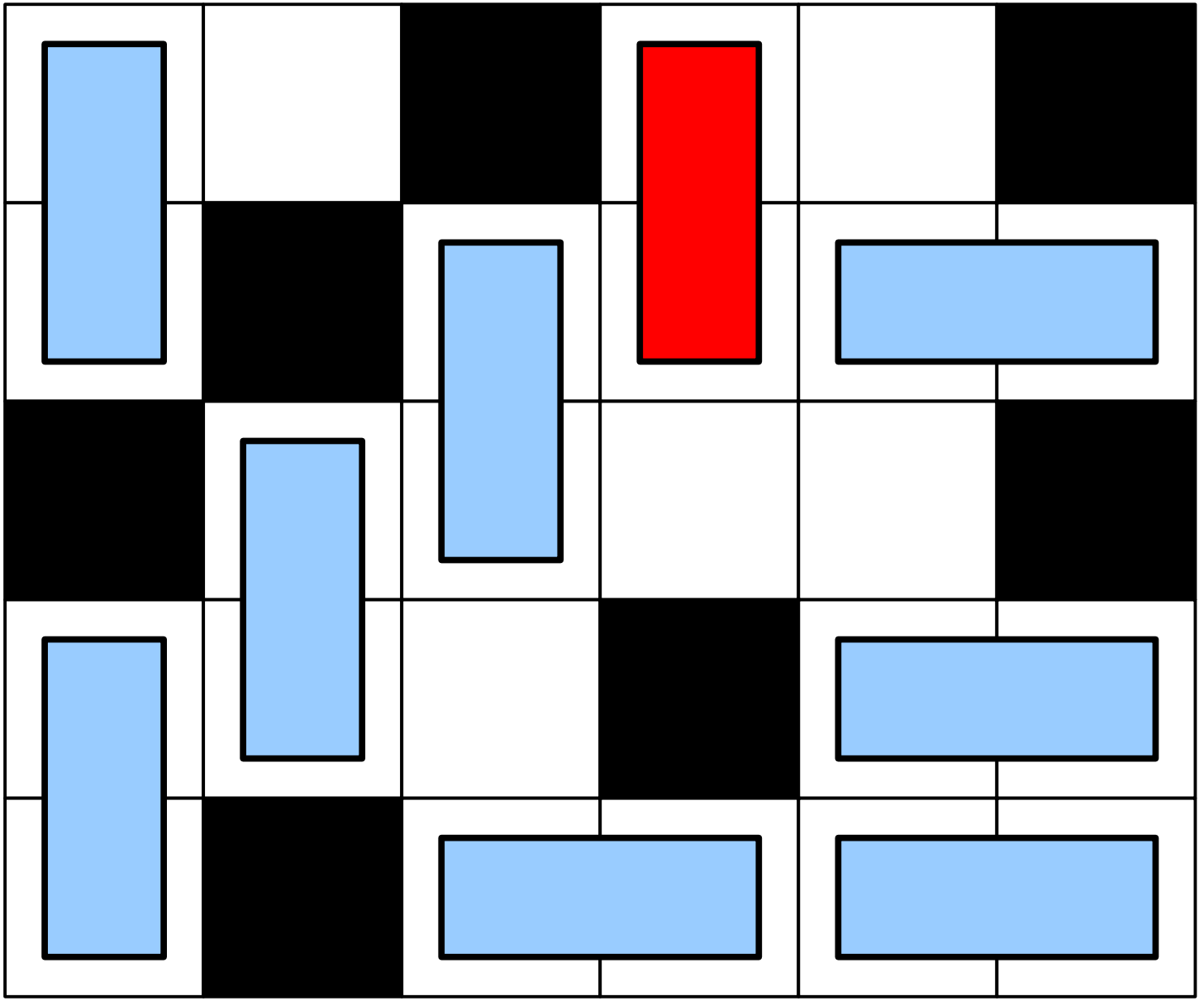

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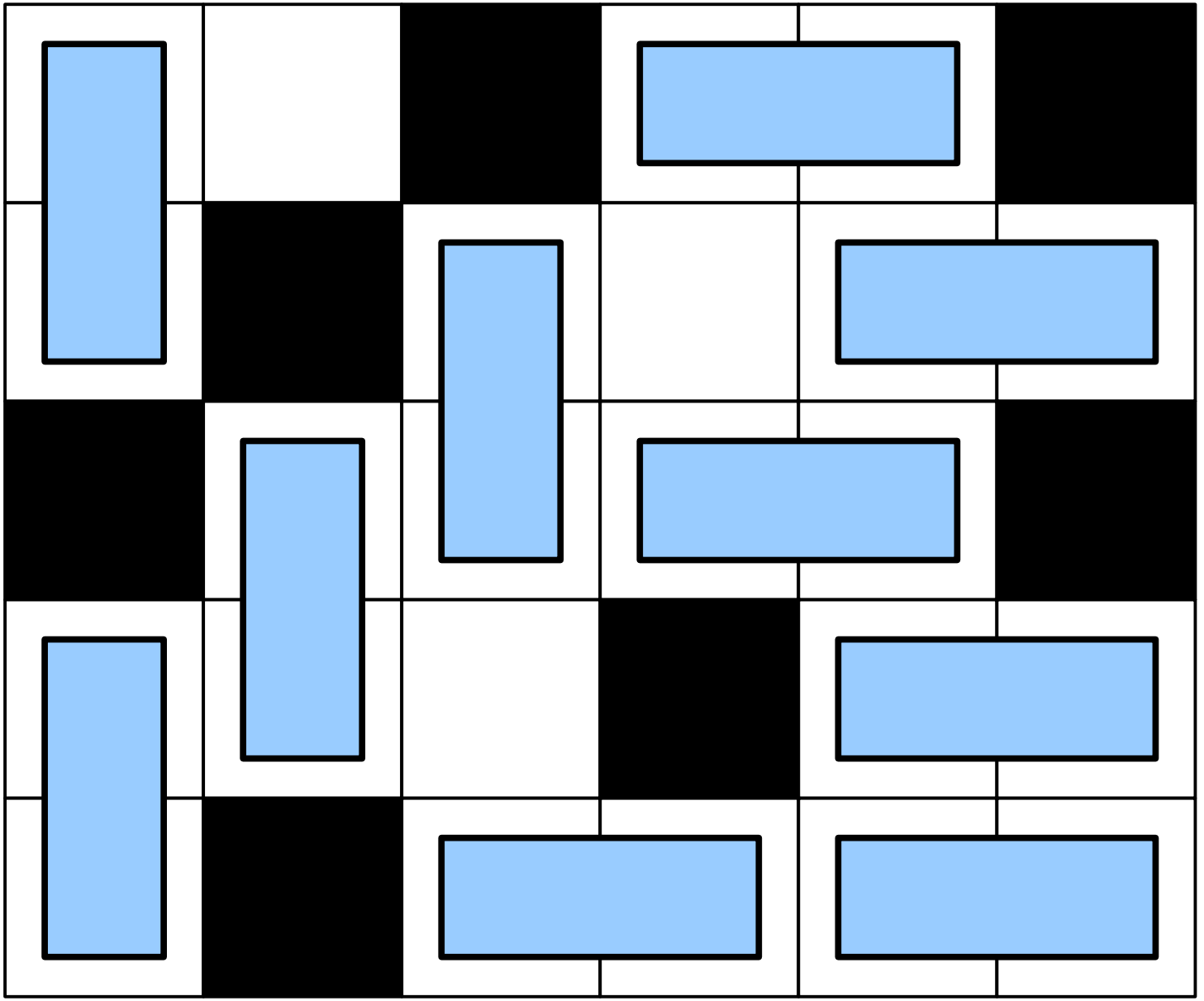

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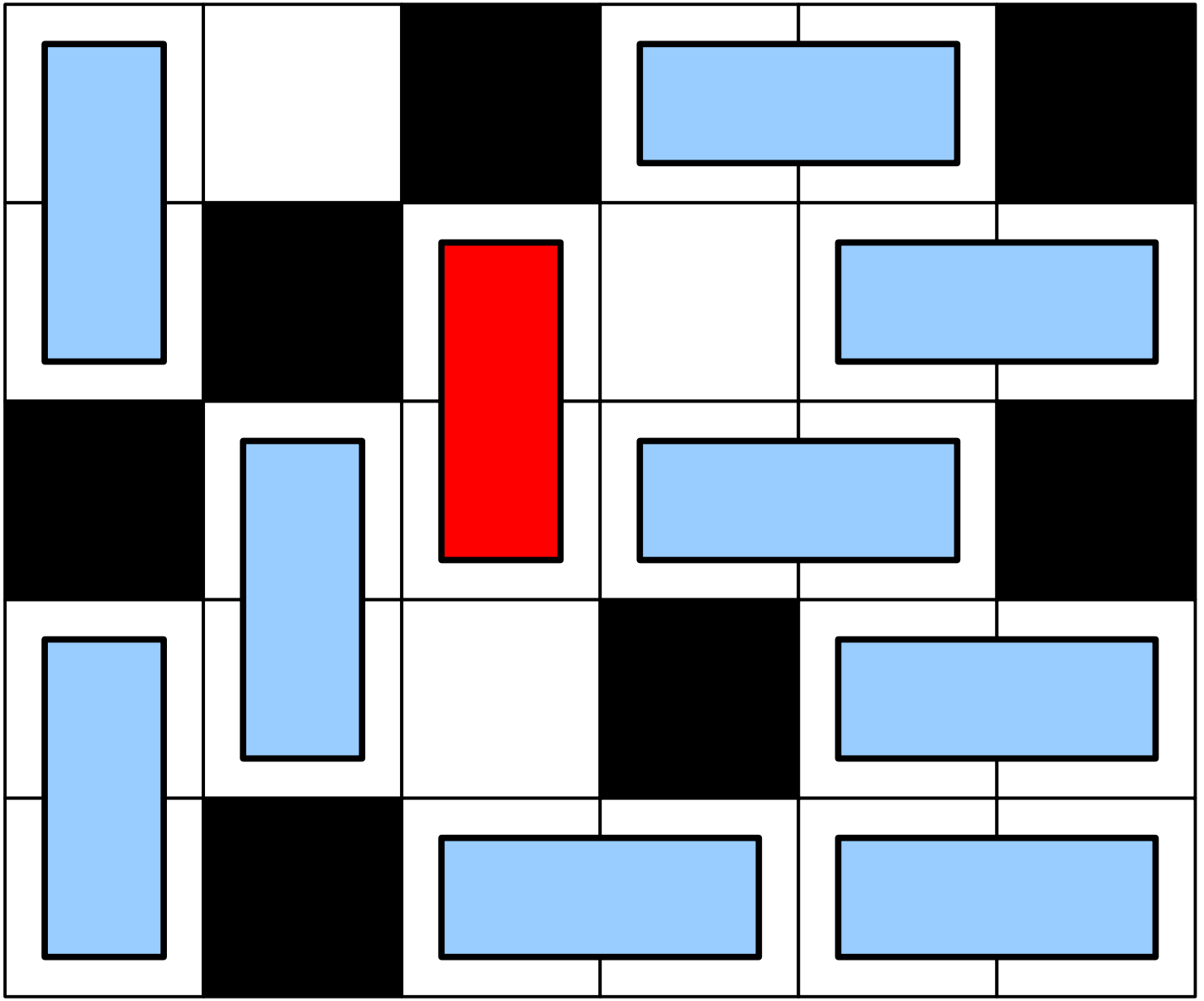

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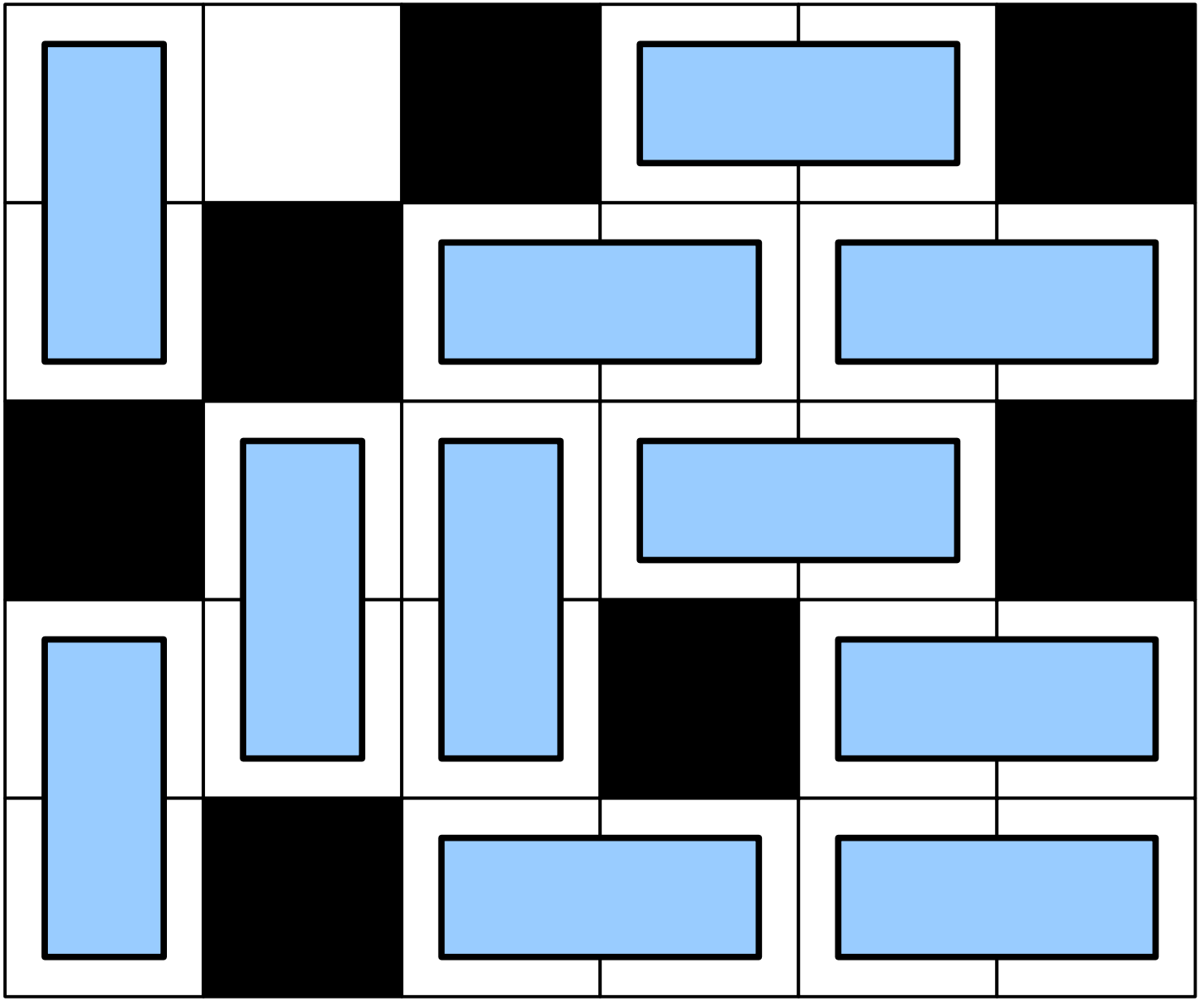
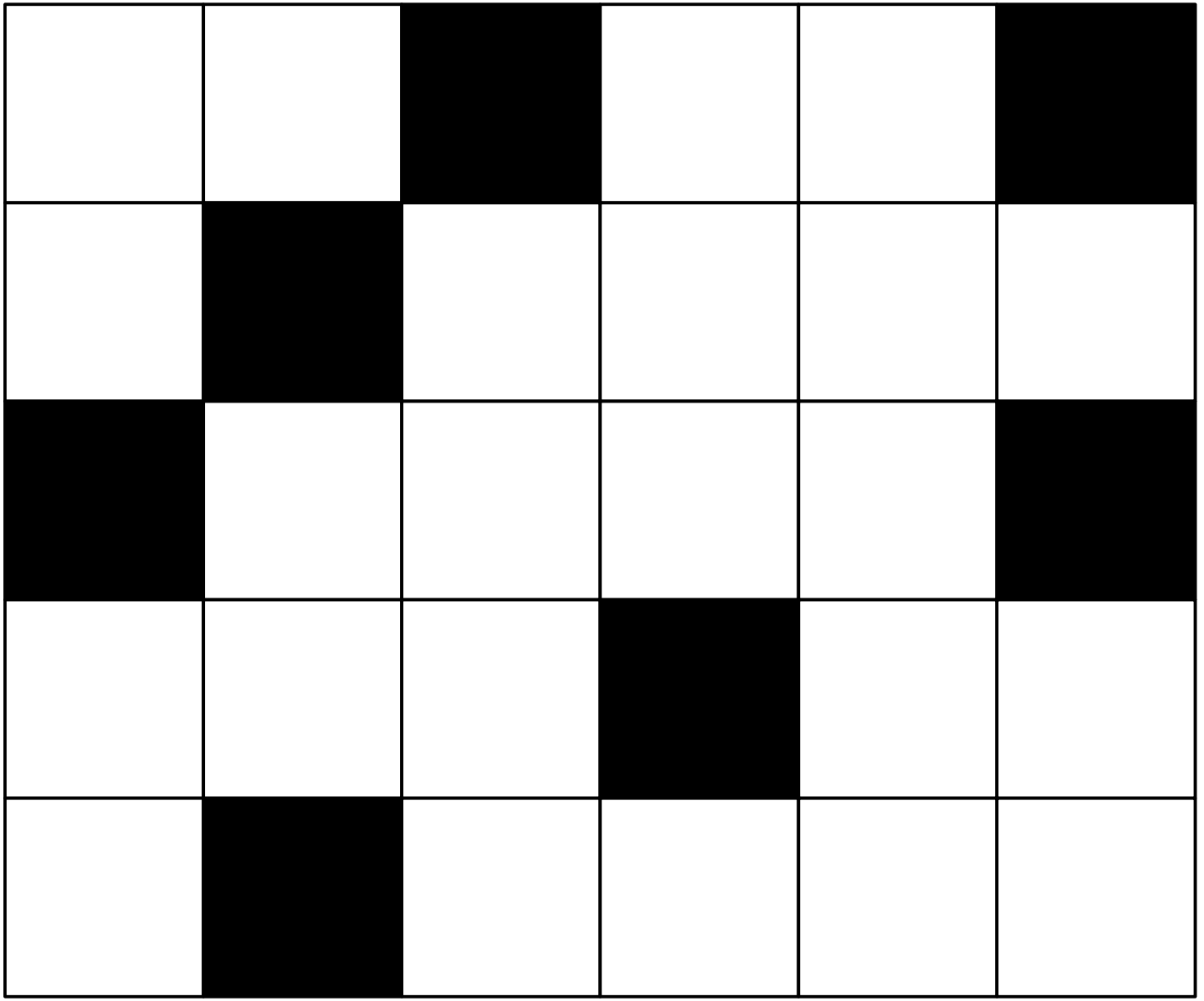


Solving Domino Tiling



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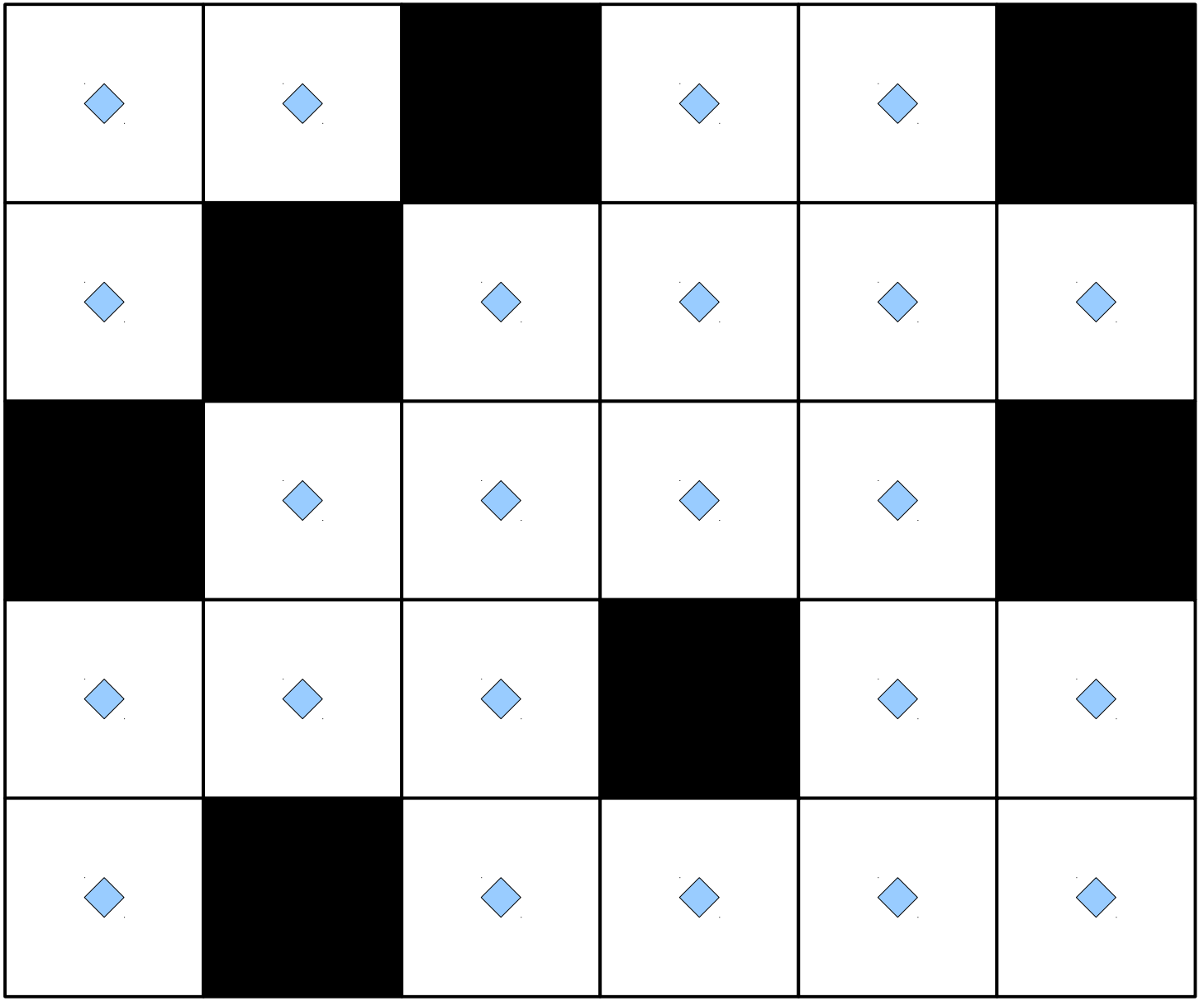

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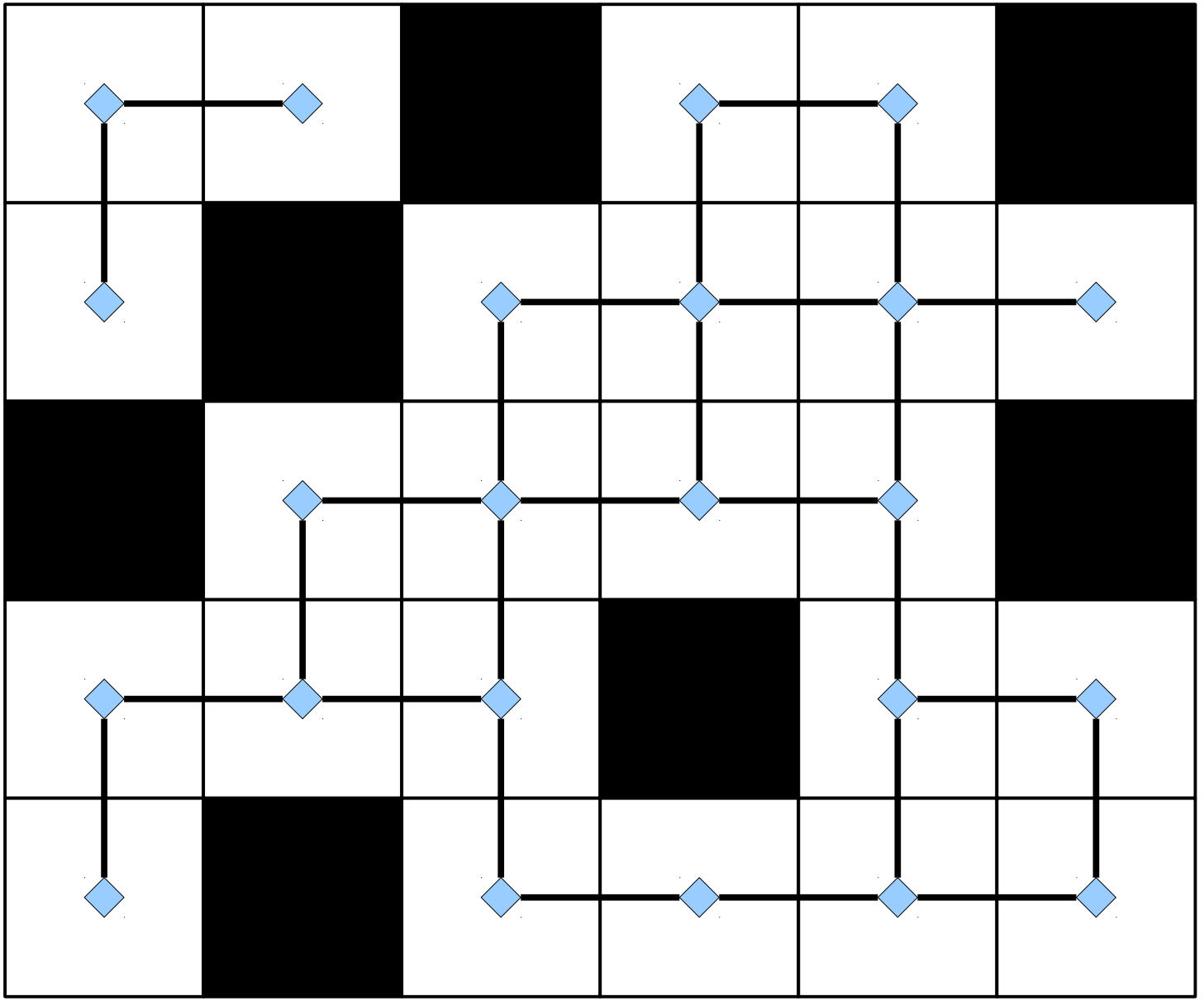

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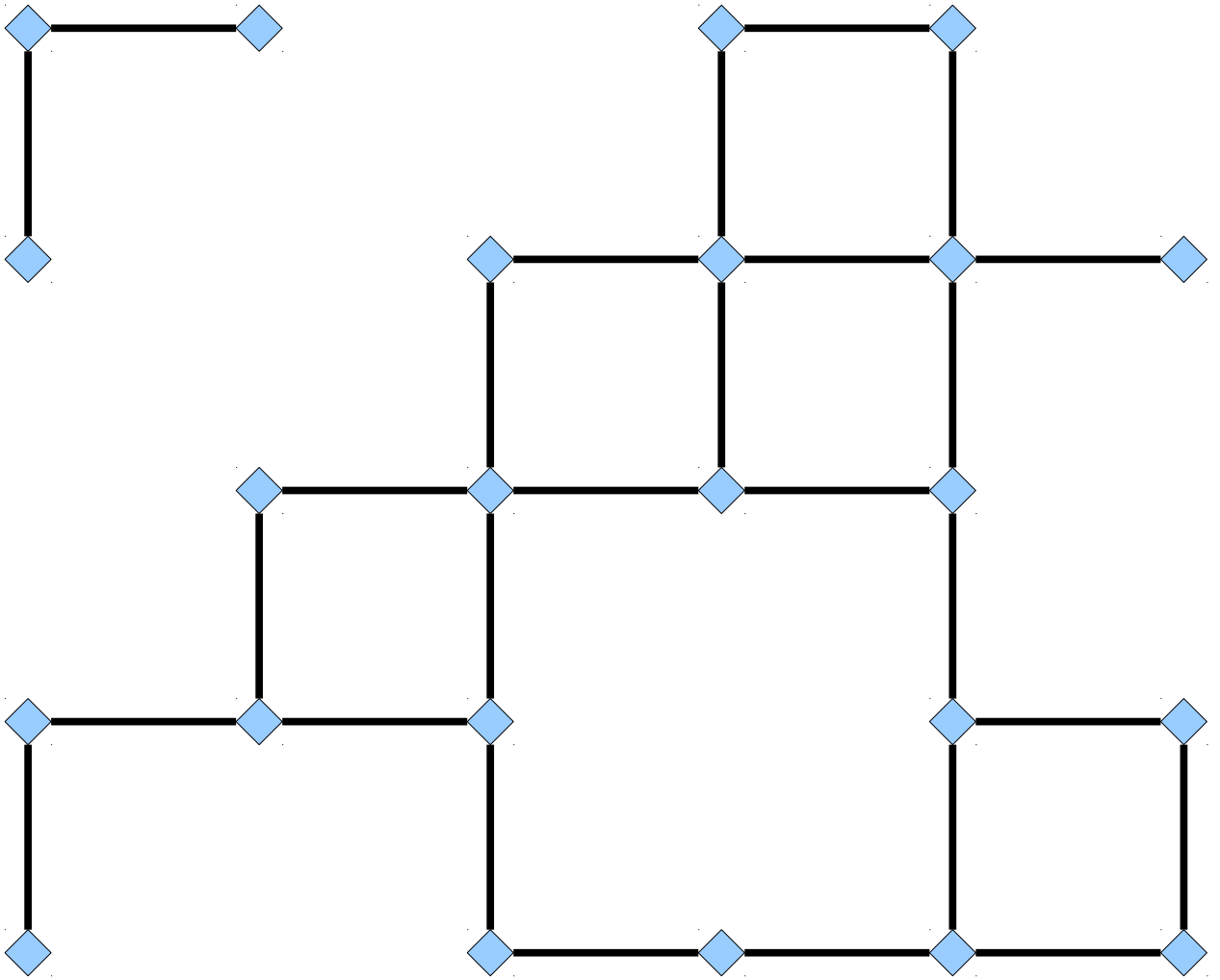

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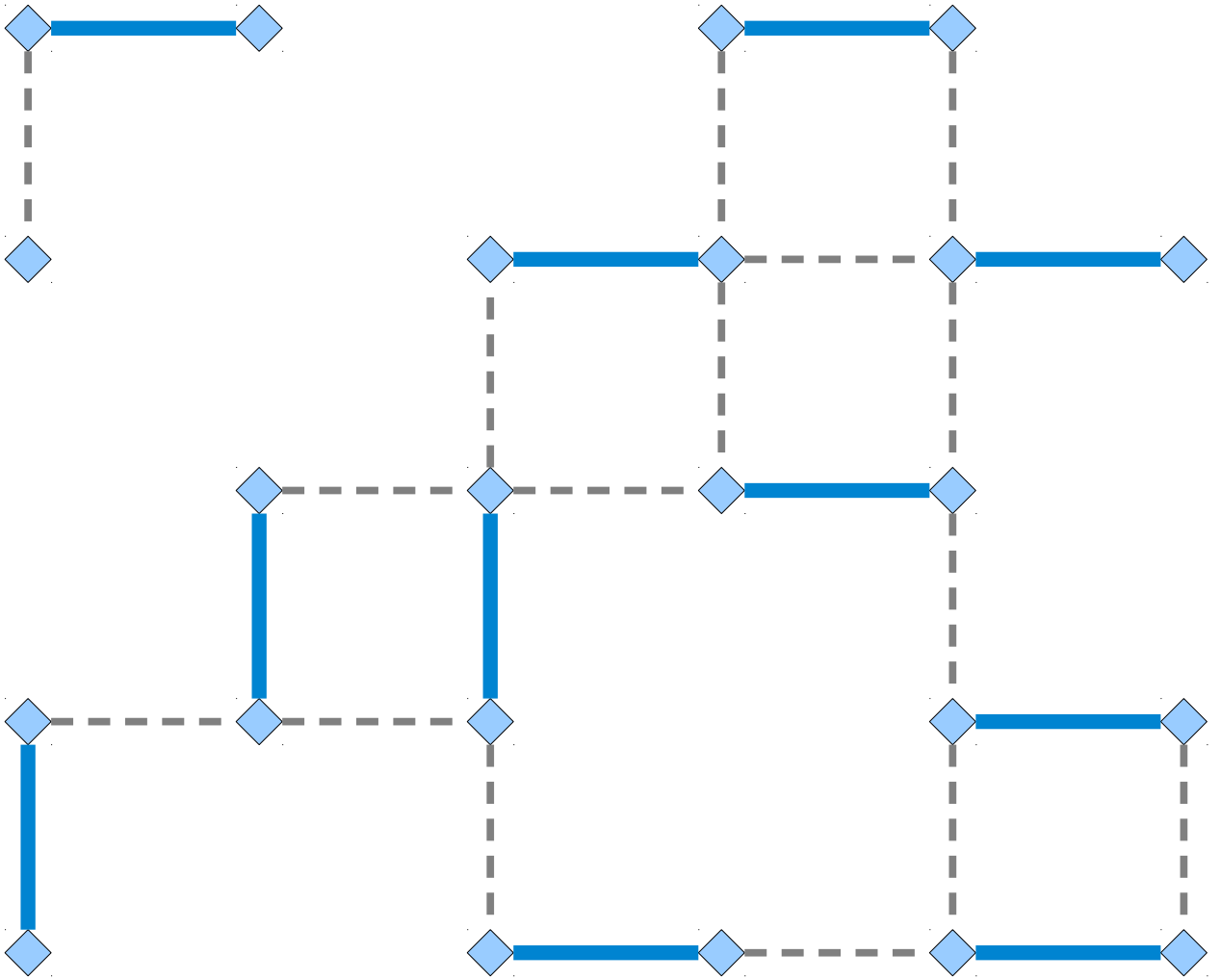

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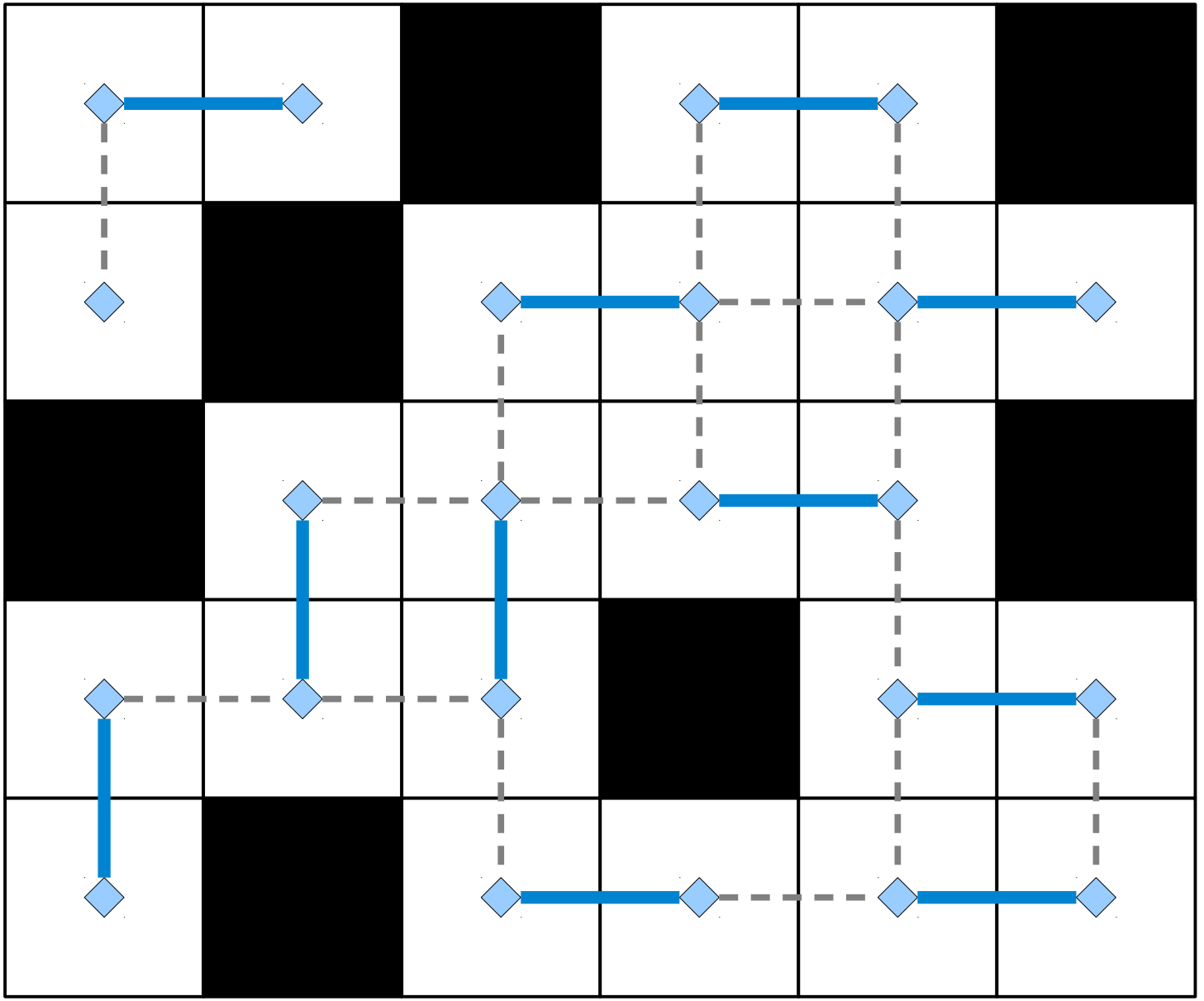

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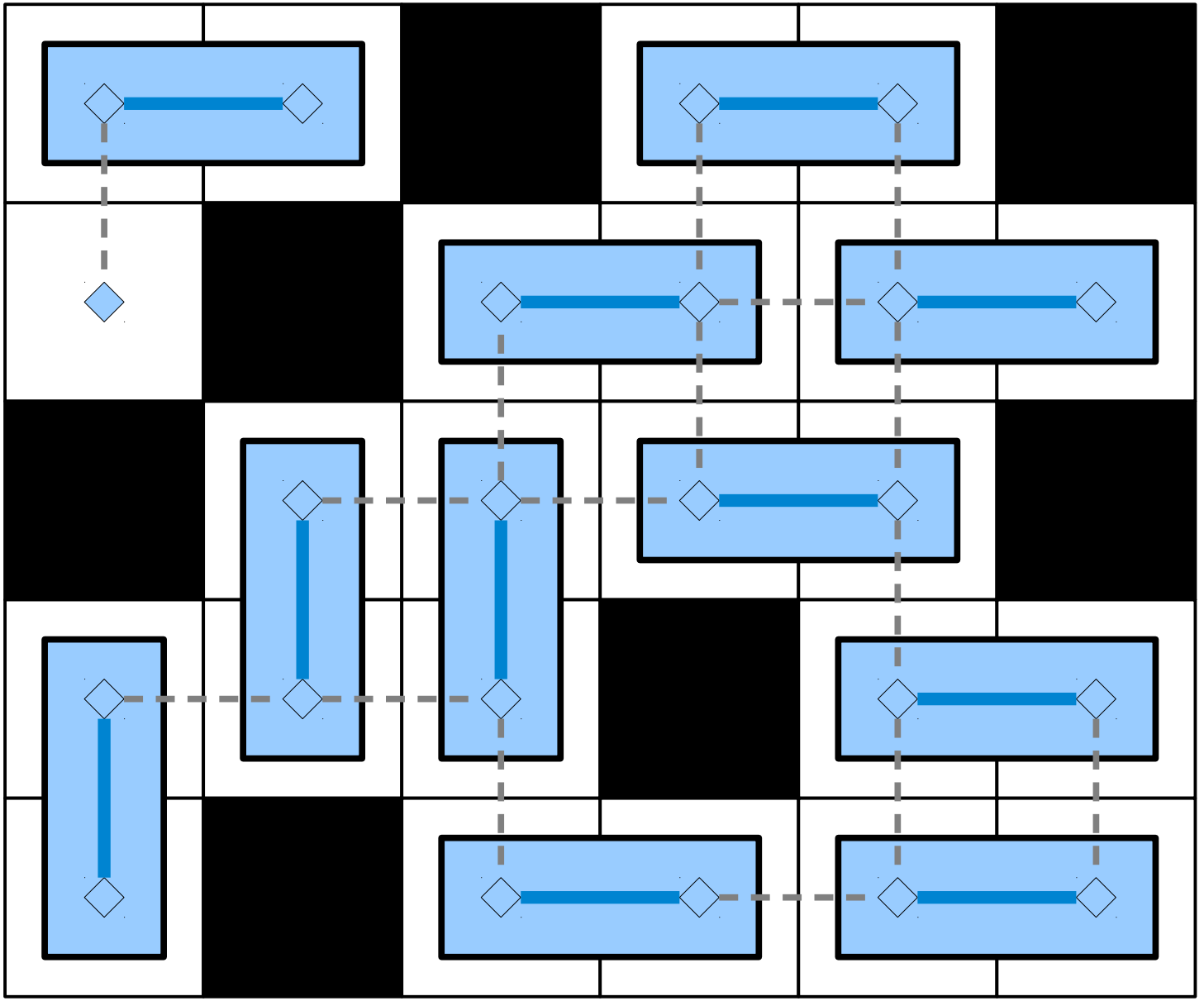

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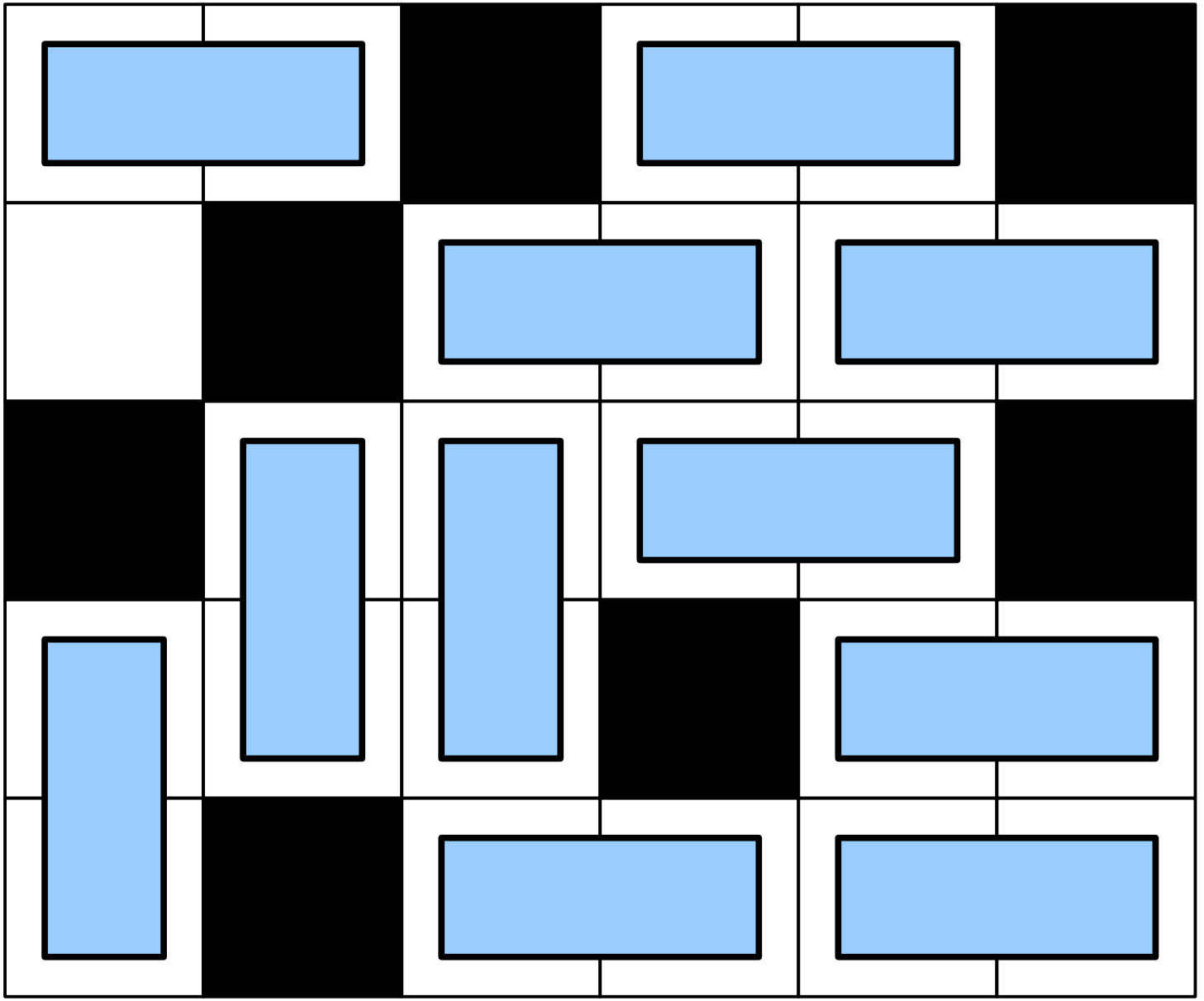

In Pseudocode

```
boolean canPlaceDominos(Grid G, int k) {
   return hasMatching(gridToGraph(G), k);
}
```

Intuition:

Tiling a grid with dominoes can't be “harder” than solving maximum matching, because if we can solve maximum matching efficiently, we can solve domino tiling efficiently.

Another Example

Reachability

- Consider the following problem:

  Given an directed graph *G* and nodes *s* and *t* in *G*, is there a path from *s* to *t*?

- It's known that this problem can be solved in polynomial time (use DFS or BFS).
- Given that we can solve the reachability problem in polynomial time, what other problems can we solve in polynomial time?

Converter Conundrums

- Suppose that you want to plug your laptop into a projector.
- Your laptop only has a VGA output, but the projector needs HDMI input.
- You have a box of connectors that convert various types of input into various types of output (for example, VGA to DVI, DVI to DisplayPort, etc.)
- ***Question:*** Can you plug your laptop into the projector?

Converter Conundrums

Connectors

RGB to USB
VGA to DisplayPort
DB13W3 to CATV
DisplayPort to RGB
DB13W3 to HDMI
DVI to DB13W3
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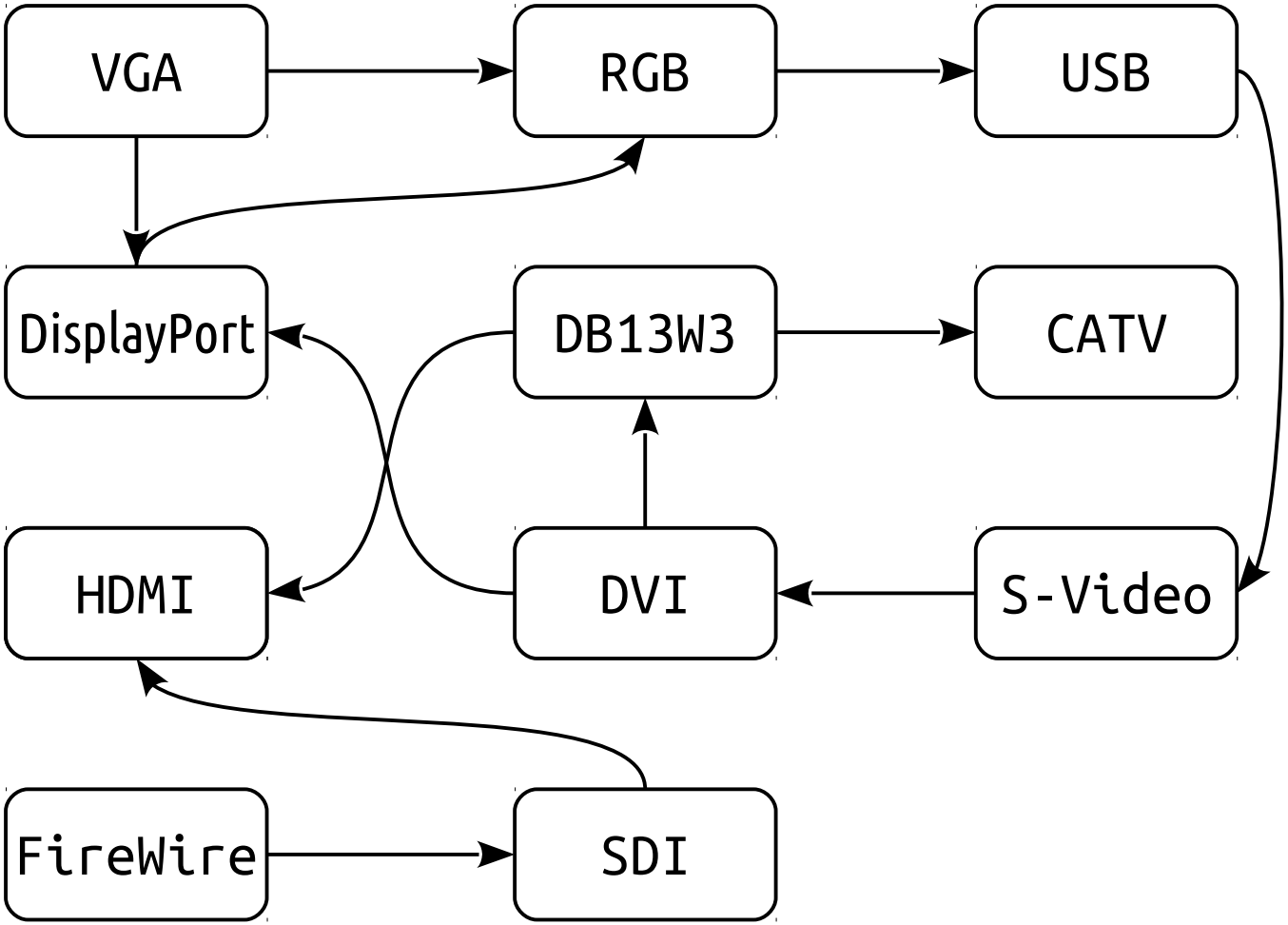

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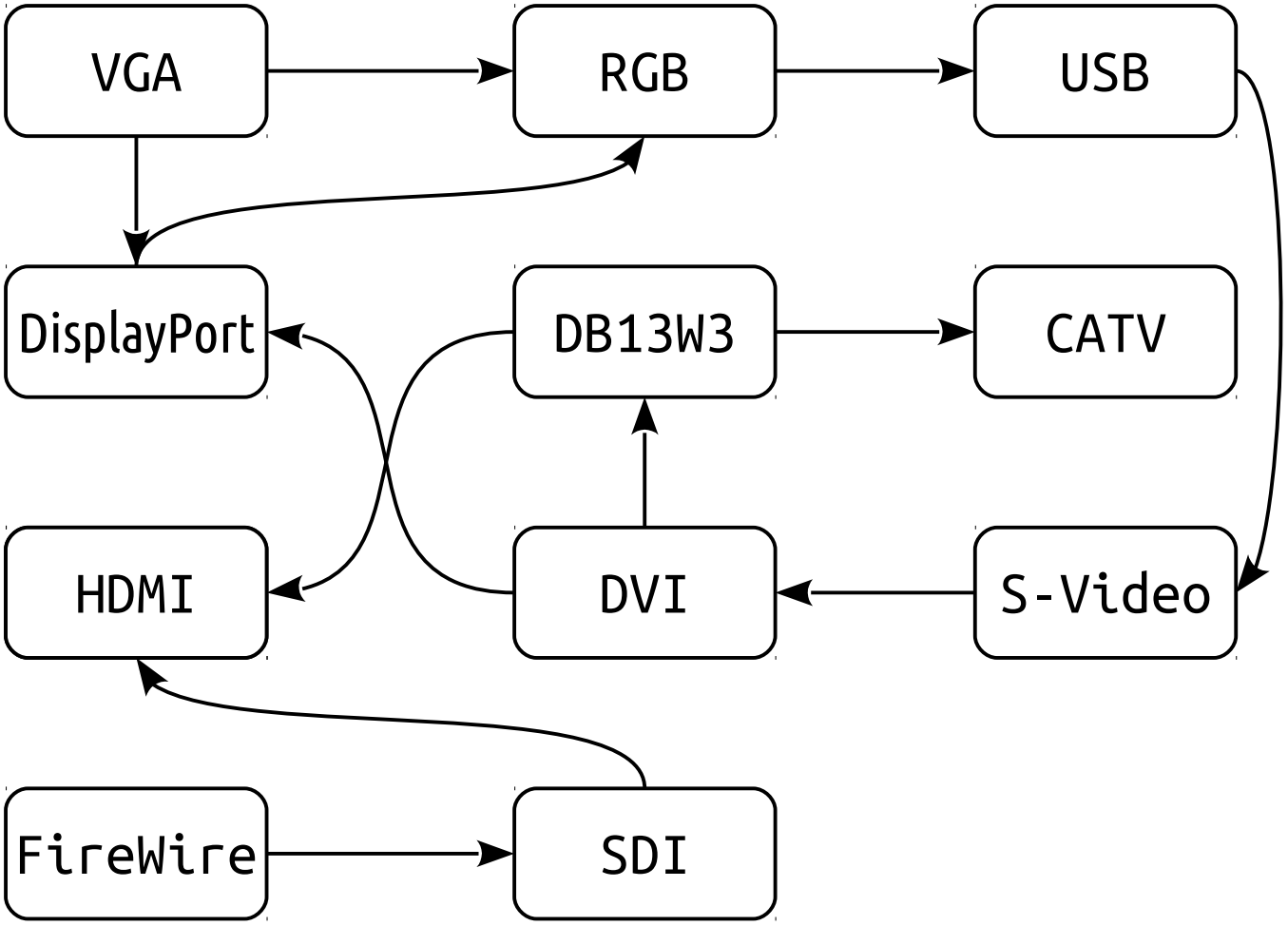

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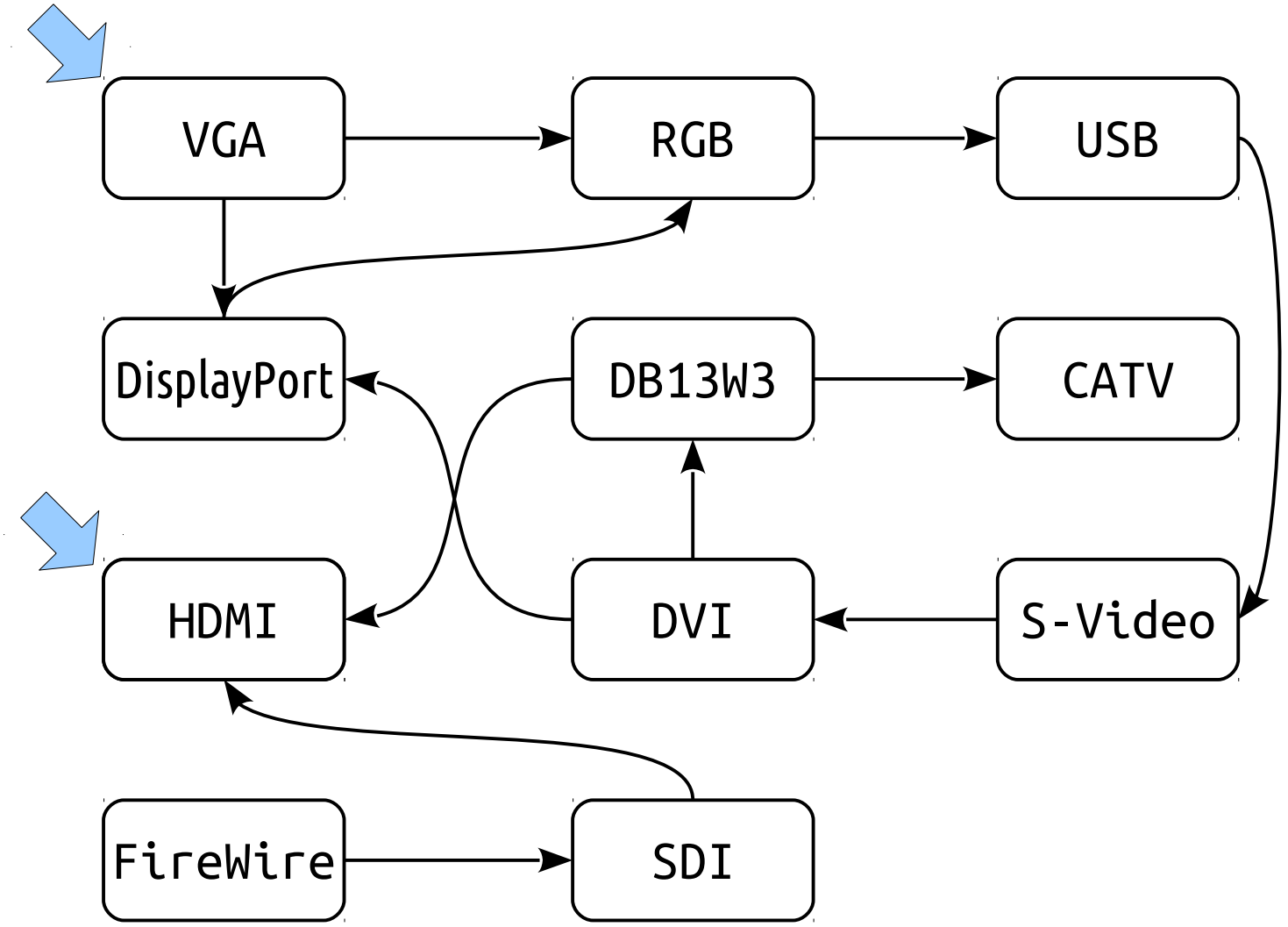

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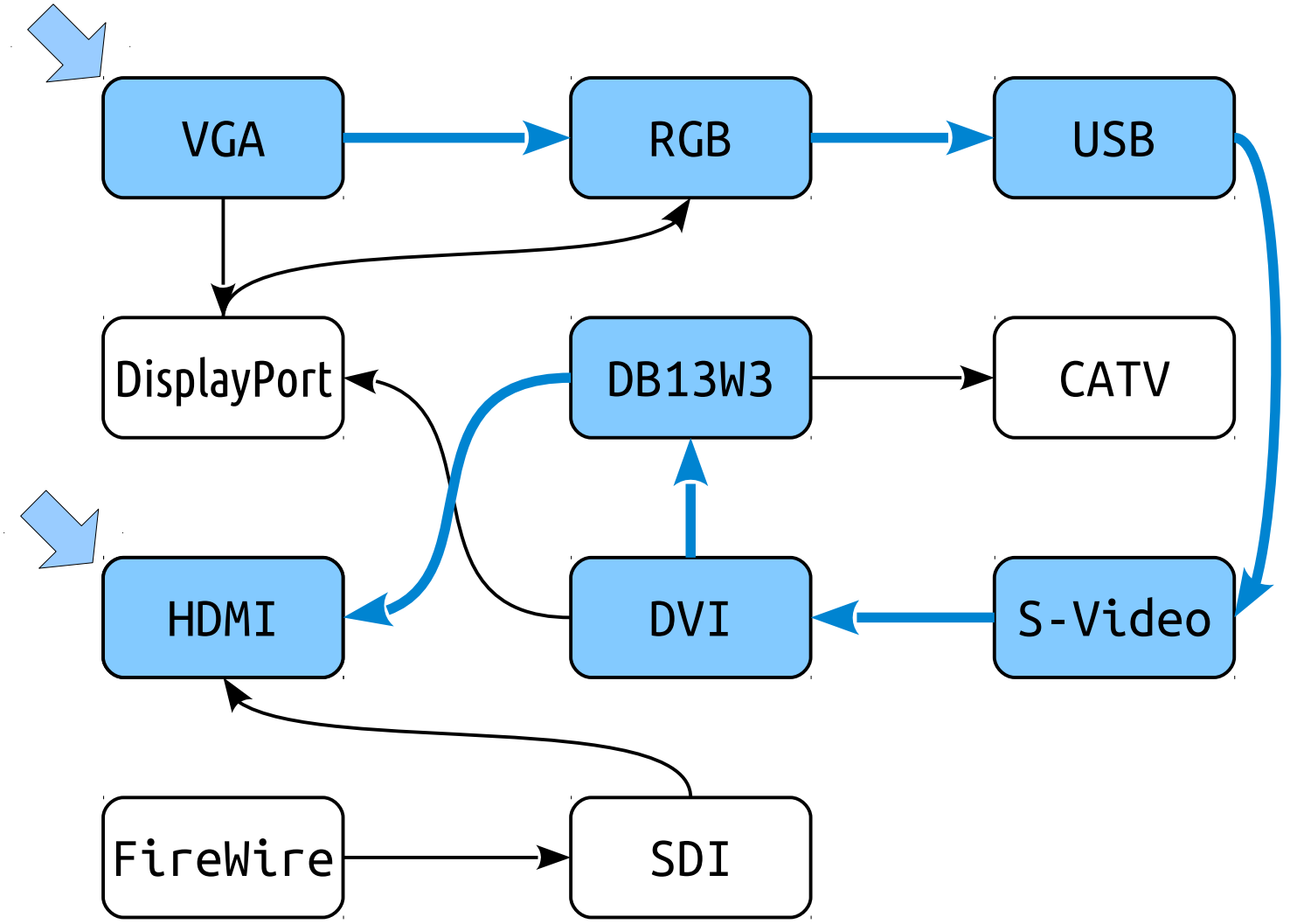

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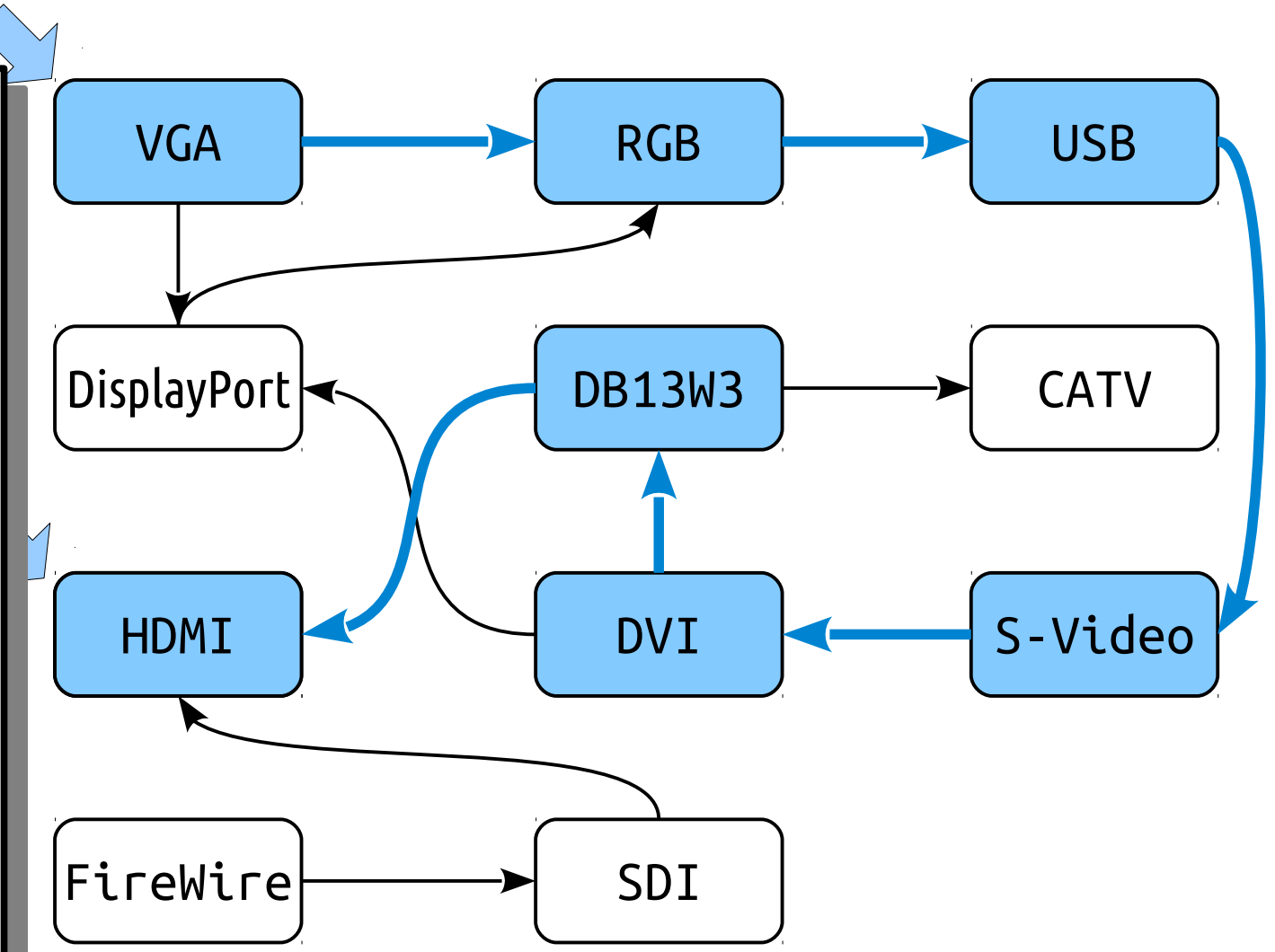


In Pseudocode

```
boolean canPlugIn(List<Plug> plugs) {
   return isReachable(plugsToGraph(plugs),
                      VGA, HDMI);
}
```

Intuition:

Finding a way to plug a computer into a projector can't be “harder” than determining reachability in a graph, since if we can determine reachability in a graph, we can find a way to plug a computer into a projector.

```
bool solveProblemA(string input) {
    return solveProblemB(transform(input));
}
```

Intuition:

Problem *A* can't be "harder" than problem *B*, because solving problem *B* lets us solve problem *A*.

```
bool solveProblemA(string input) {
    return solveProblemB(transform(input));
}
```

- If A and B are problems where it's possible to solve problem A using the strategy shown above*, we write

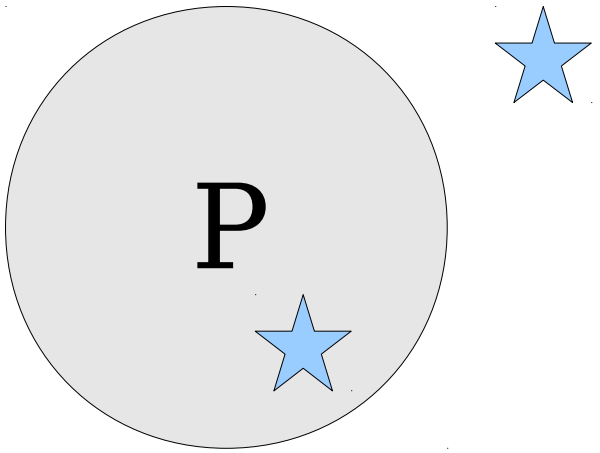
$$A \leq_{\mathrm{p}} B.$$

- We say that ***A is polynomial-time reducible to B***.

* Assuming that `transform` runs in polynomial time.

Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.



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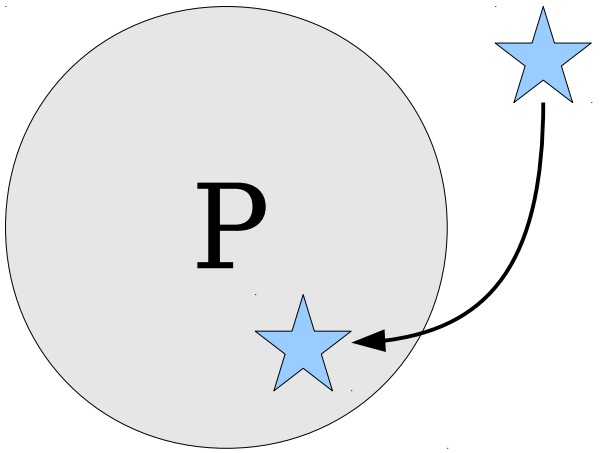

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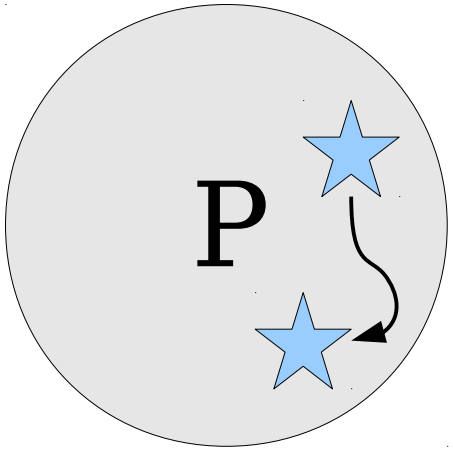

Polynomial-Time Reductions

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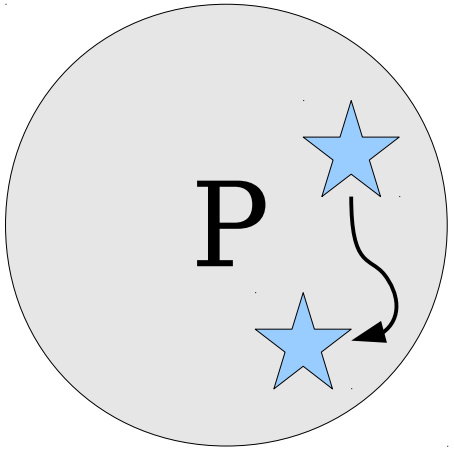

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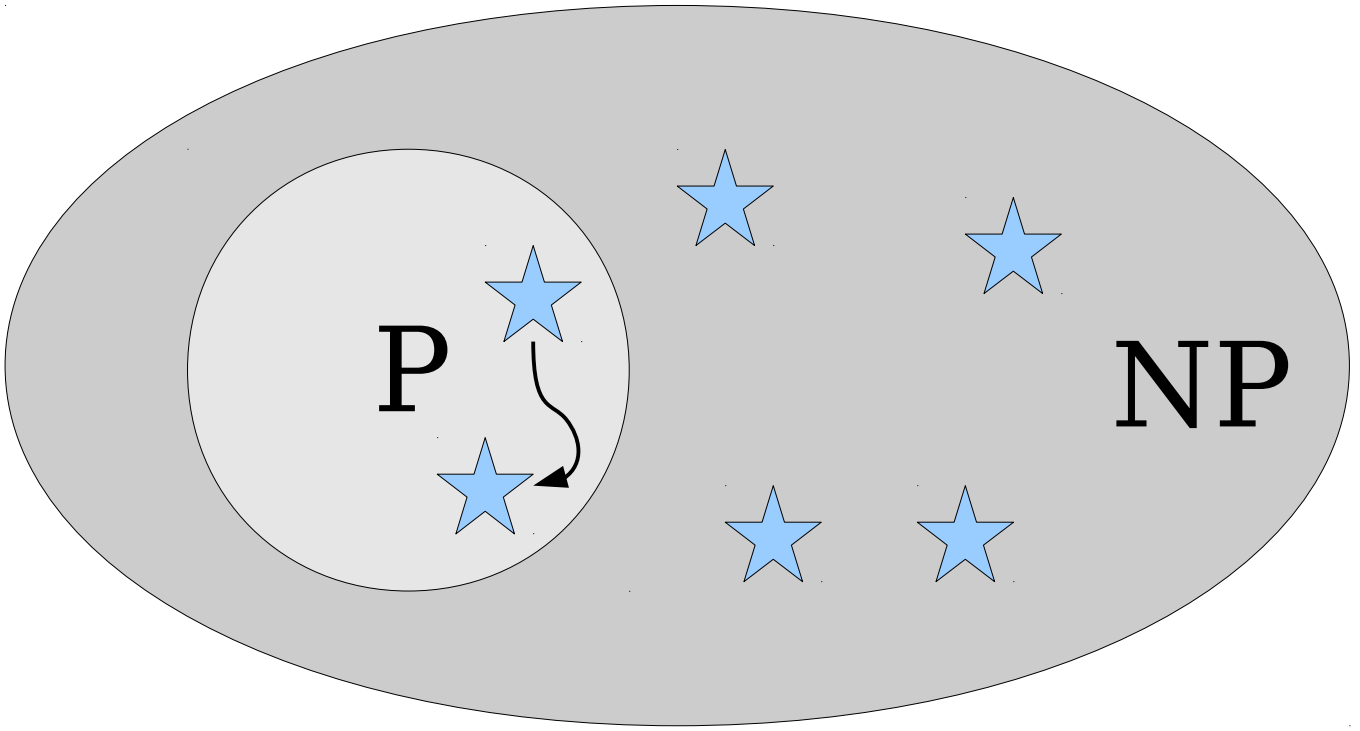

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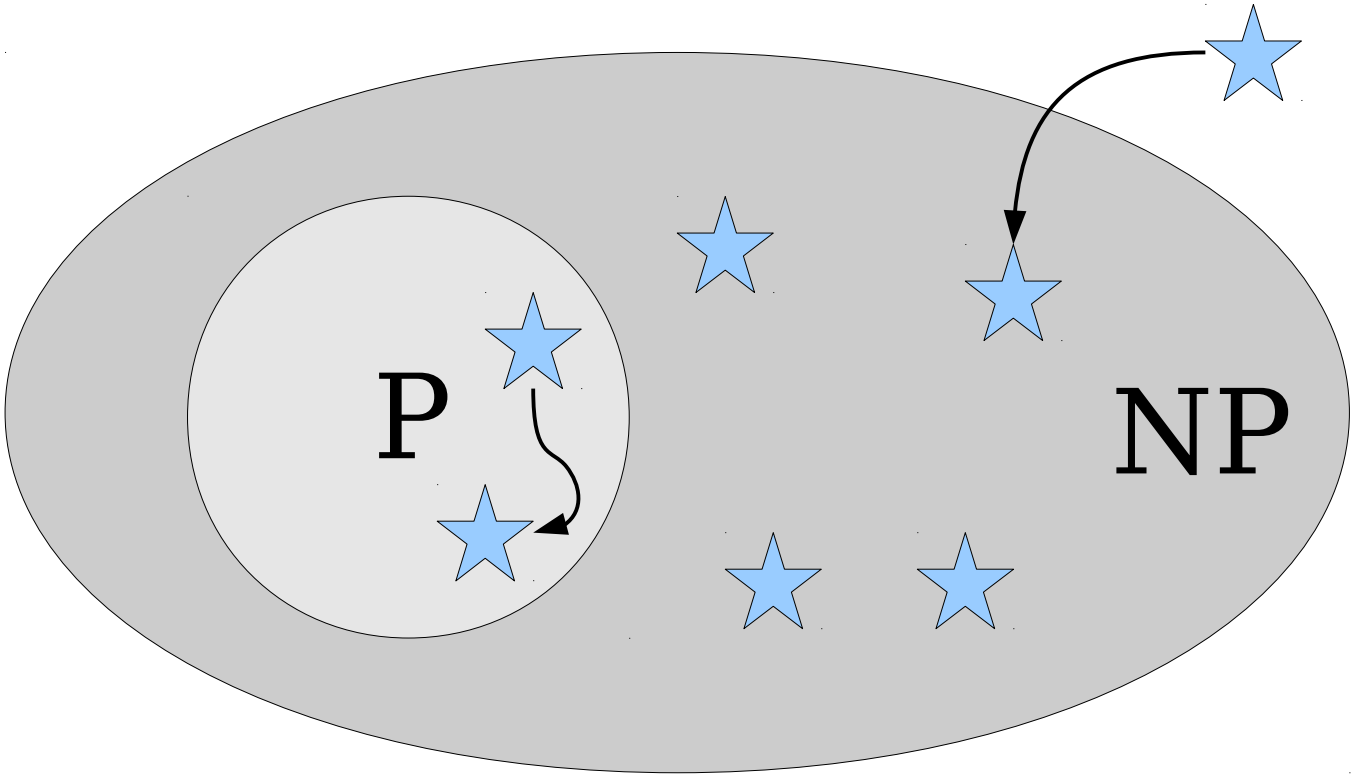

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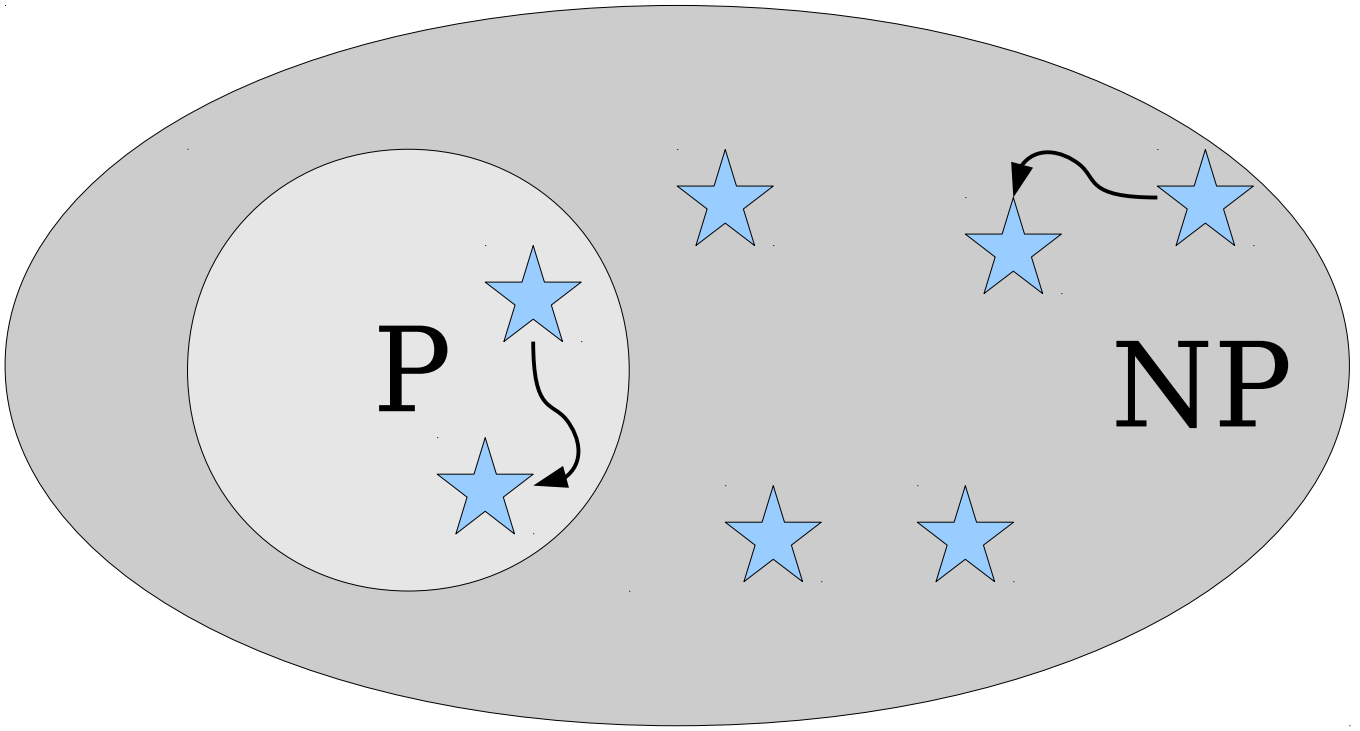


This $\leq_p$ relation lets us rank the relative difficulties of problems in **P** and **NP**.

What else can we do with it?

Time-Out for Announcements!

Please evaluate this course on Axess.

Your feedback makes a difference.

Problem Set Nine

- Problem Set Nine is due this Friday at 2:30PM.
  - ***No late submissions can be accepted***. This is university policy – sorry!
- As always, if you have questions, stop by office hours or ask on Piazza!

Final Exam Logistics

- Our final exam is next Monday, December 11th from 3:30PM – 6:30PM. Locations are divvied up by last (family) name:
  - Abb – Ngu: Go to ***Cubberley Auditorium***.
  - Ogr – Zwa: Go to ***Cemex Auditorium***.
- The final exam is cumulative and covers topics from PS1 – PS9 and all lectures. Topics from this week are fair game but will be deemphasized.
- The exam is closed-computer, closed-book, and limited-note. You may bring one double-sided 8.5” × 11” sheet of notes with you to the exam, decorated however you’d like.

Preparing for the Final

- On the course website you'll find
  - ***five*** practice final exams, which are all real exams with minor modifications, with solutions, and
  - a giant set of 45 practice problems (EPP3), with solutions.
- Our recommendation: Look back over the exams and problem sets and redo any problems that you didn't really get the first time around.
- Keep the TAs in the loop: stop by office hours to have them review your answers and offer feedback.

Your Questions

“I feel like I 'get' the lectures, but PSets have been killing me and I don't know how to ask for help. Any tips for people who have trouble reaching out?”

- A ball of anxiety passing for a human

For starters, I'm sorry to hear that, and I hope that everything is going well. I see two questions that you might have asked here, and if neither of these are right, please feel free to email me directly!

If you're in a situation of the form "I know that I need help, but I'm not sure what to ask for," it's often helpful to pick a particular problem you've been struggling with and then ask to sit down with a TA to go over your thought process and your approach to the problem. Forcing yourself to articulate what you're thinking or doing - especially if you feel really lost - lets us see whether there's something you're missing, or a key concept that you haven't internalized yet, or whether what you're doing is reasonable and just unluckily doesn't pan out.

If the issue is "I know I need help, but I'm nervous about asking or afraid to do so," remember that there is difference between "what you are capable of doing" and "where you are right now." The fact that you're having trouble with some concept doesn't reflect at all on your character or who you are.

"I feel like a lot of people around me are dividing and conquering the psets and then killing the exams. How do I stay motivated to do things the right way?"

Ultimately, your job is to learn the material to the best of your abilities. In that sense, it doesn't really matter what people around you are doing - if they're somehow managing to pick up all the concepts without doing all the work, more power to them. That's the exception rather than the rule. Trust me - when I'm computing final grades, it is really obvious when people are splitting up the work, and not in a good way.

I'm a fan of the "get rich slowly" approach to life. Consistently putting in a good honest effort and taking the initiative to fix the gaps in your skills and your knowledge is one of the few reliable ways to get extremely good at something.

Back to CS103!

NP-Hardness and **NP**-Completeness

Question: What makes a problem hard to solve?

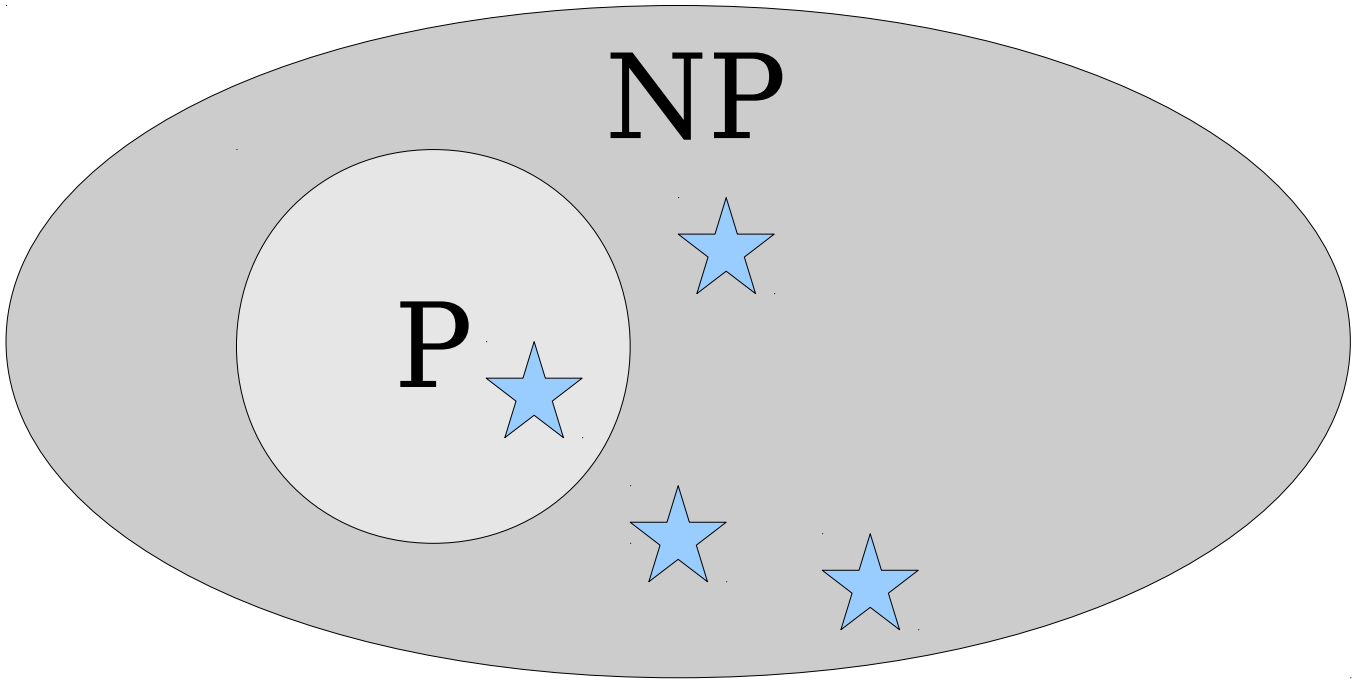
Intuition: If $A \leq_p B$, then problem B is at least as hard* as problem A.

* for some definition of “at least as hard as.”

Intuition: To show that some problem is hard, show that lots of other problems reduce to it.

NP-Hardness

- A language L is called ***NP-hard*** if for *every* $A \in \mathbf{NP}$, we have $A \leq_{\mathrm{P}} L$.



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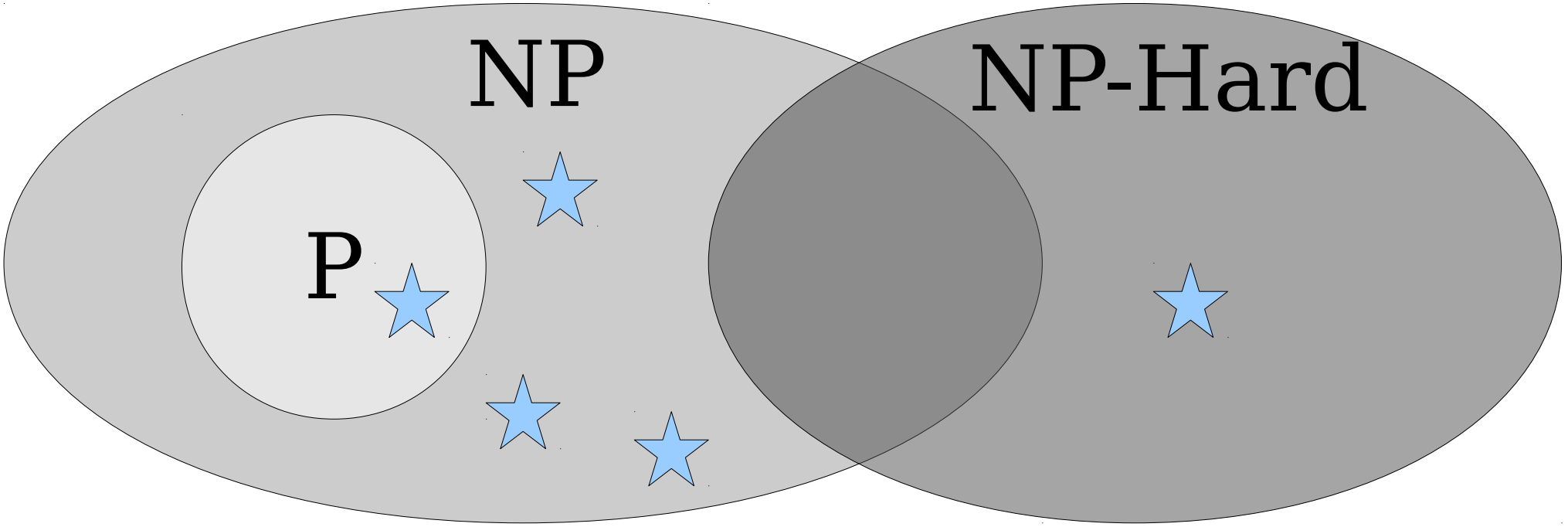

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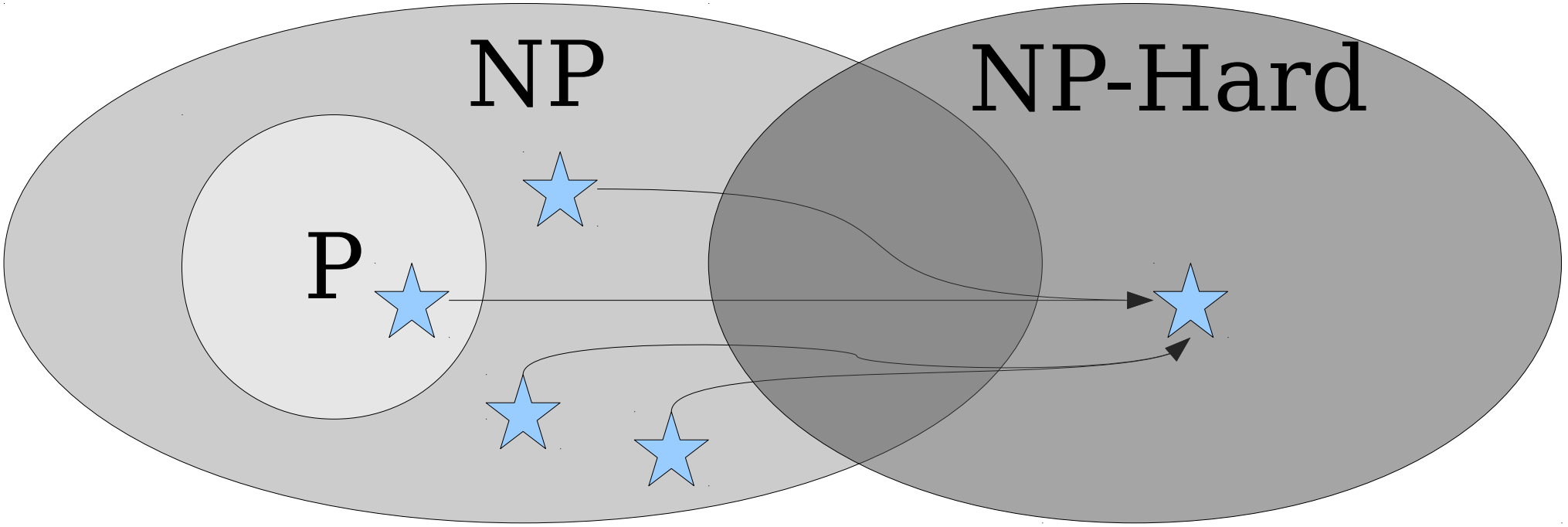
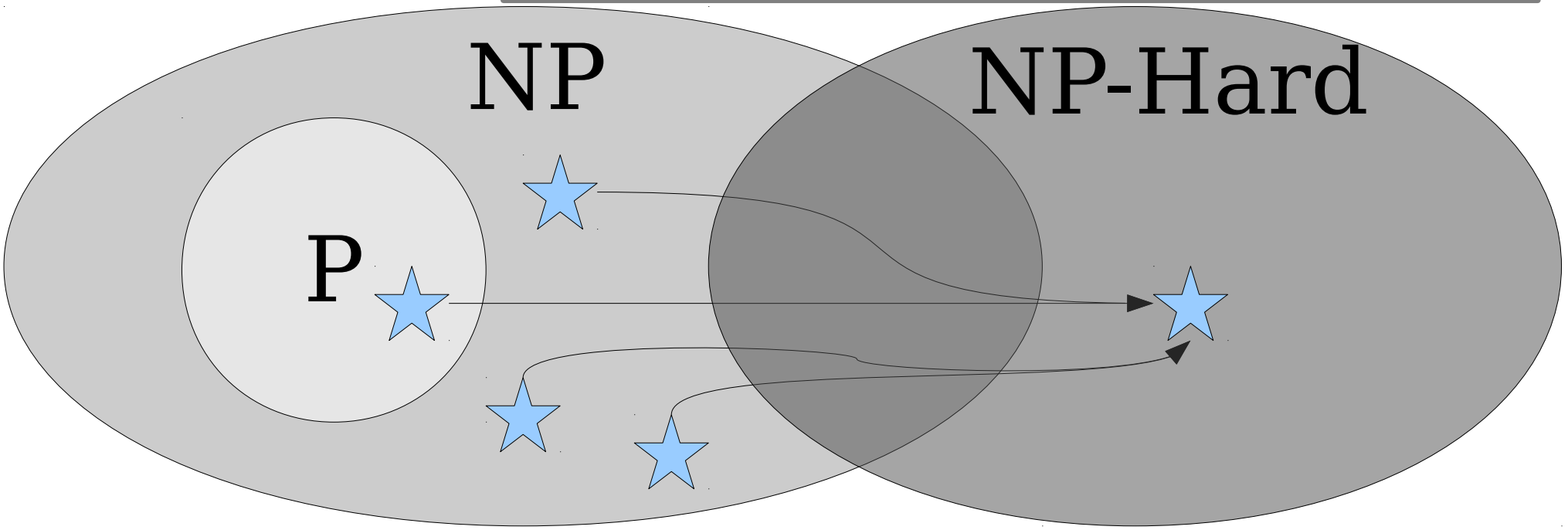


NP-Hardness

- A language L is called ***NP-hard*** if for *every* $A \in \mathbf{NP}$, we have $A \leq_P L$.

Intuitively: L has to be at least as hard as every problem in **NP**, since an algorithm for L can be used to decide all problems in **NP**.



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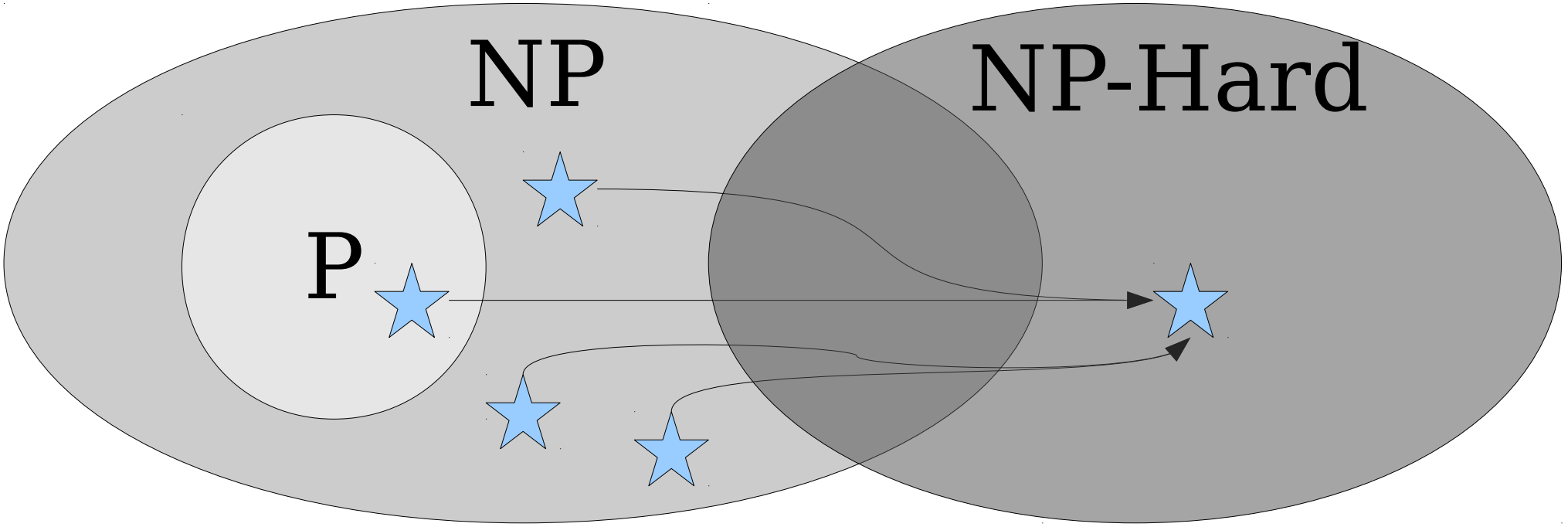

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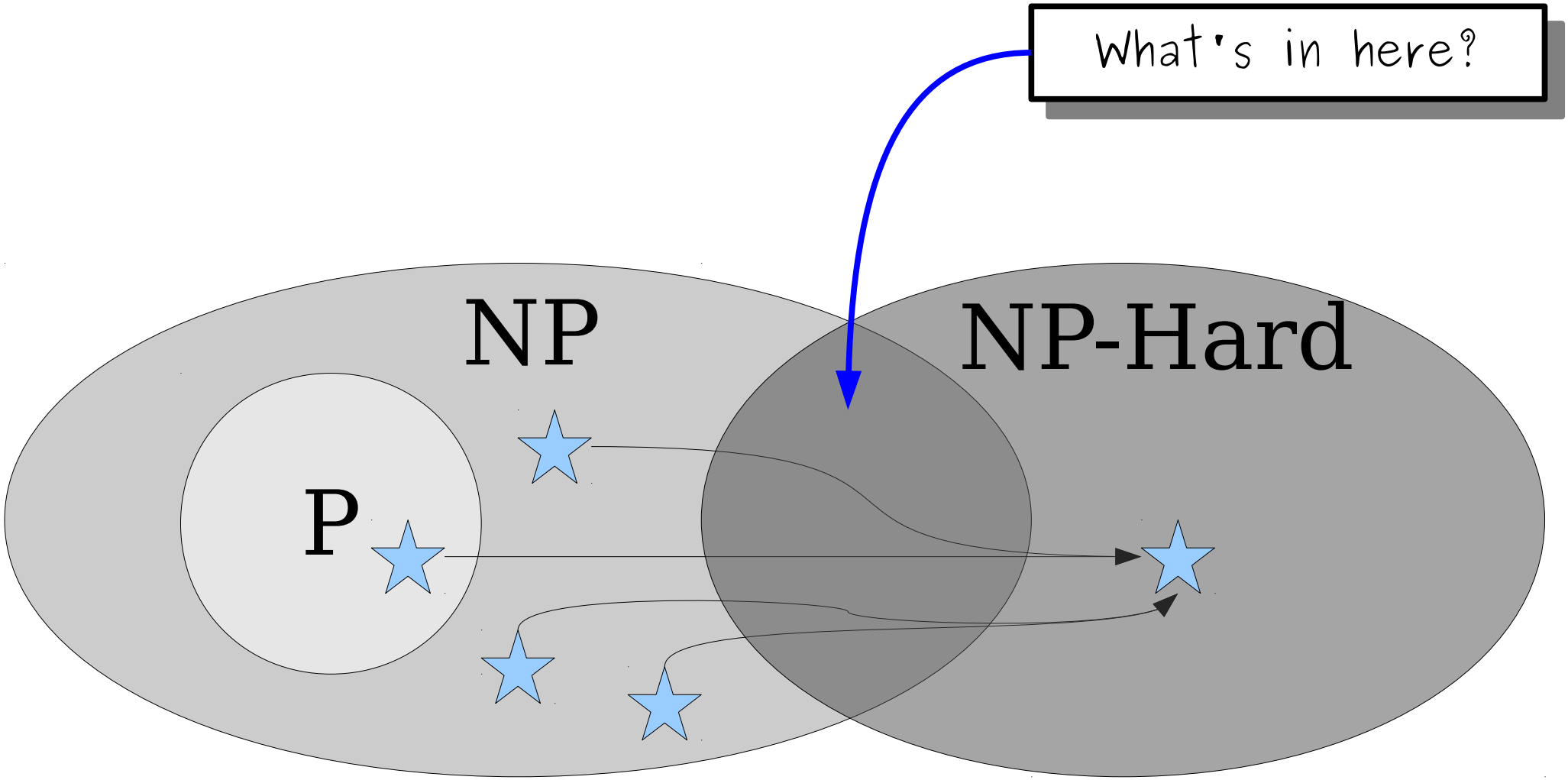
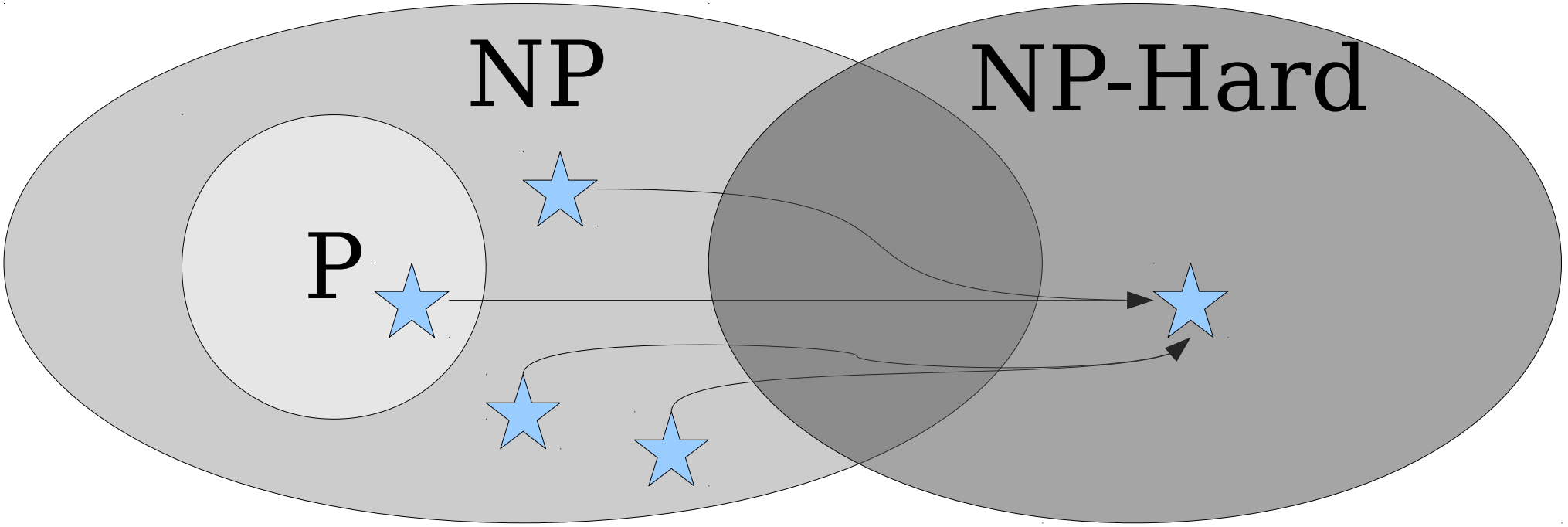


NP-Hardness

- A language L is called ***NP-hard*** if for *every* $A \in \mathbf{NP}$, we have $A \leq_{\mathrm{P}} L$.
- A language in L is called ***NP-complete*** if L is **NP**-hard and $L \in \mathbf{NP}$.
- The class ***NPC*** is the set of **NP**-complete problems.



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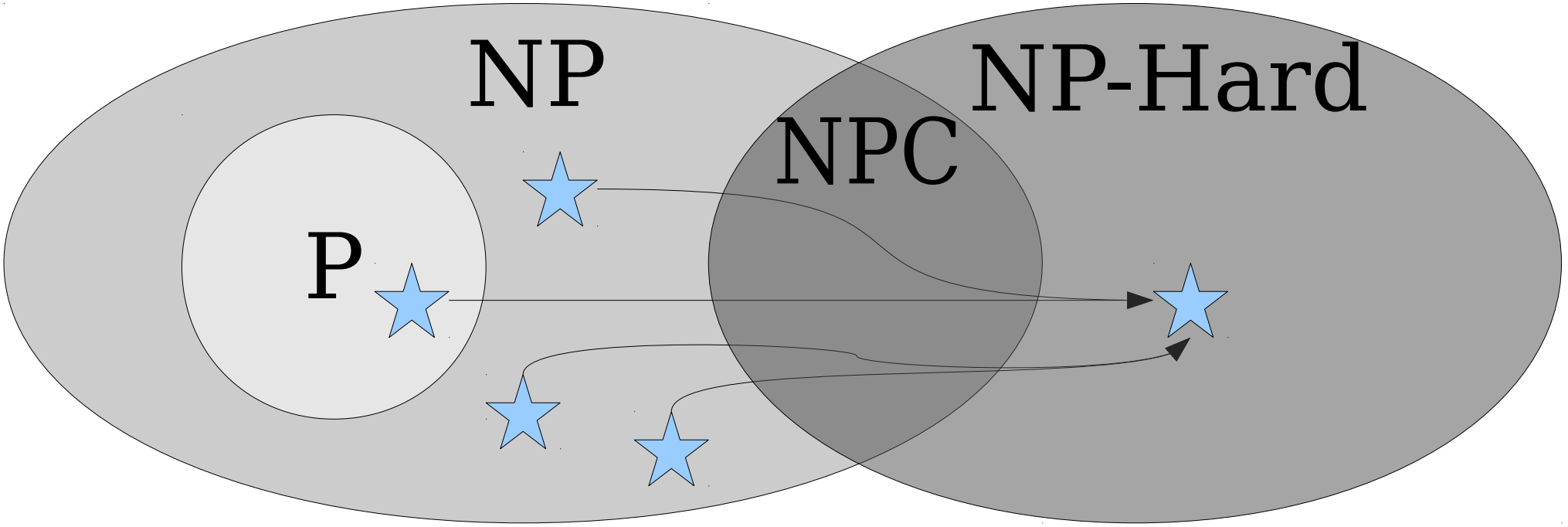

The Tantalizing Truth

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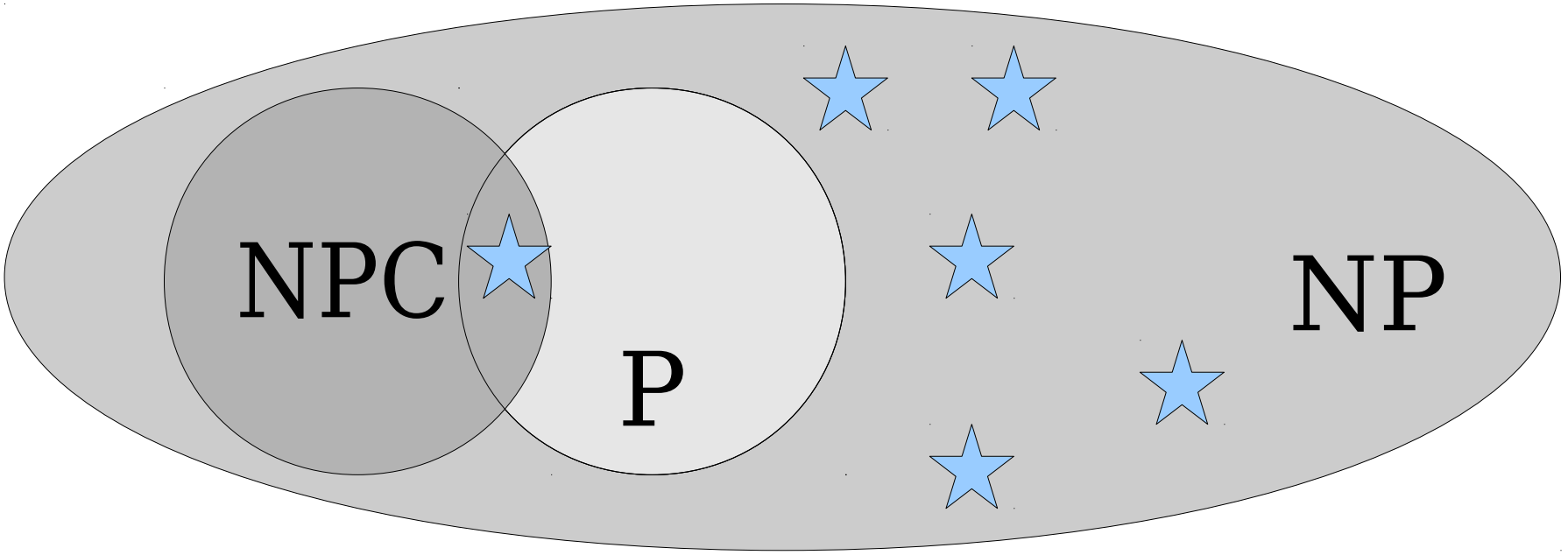

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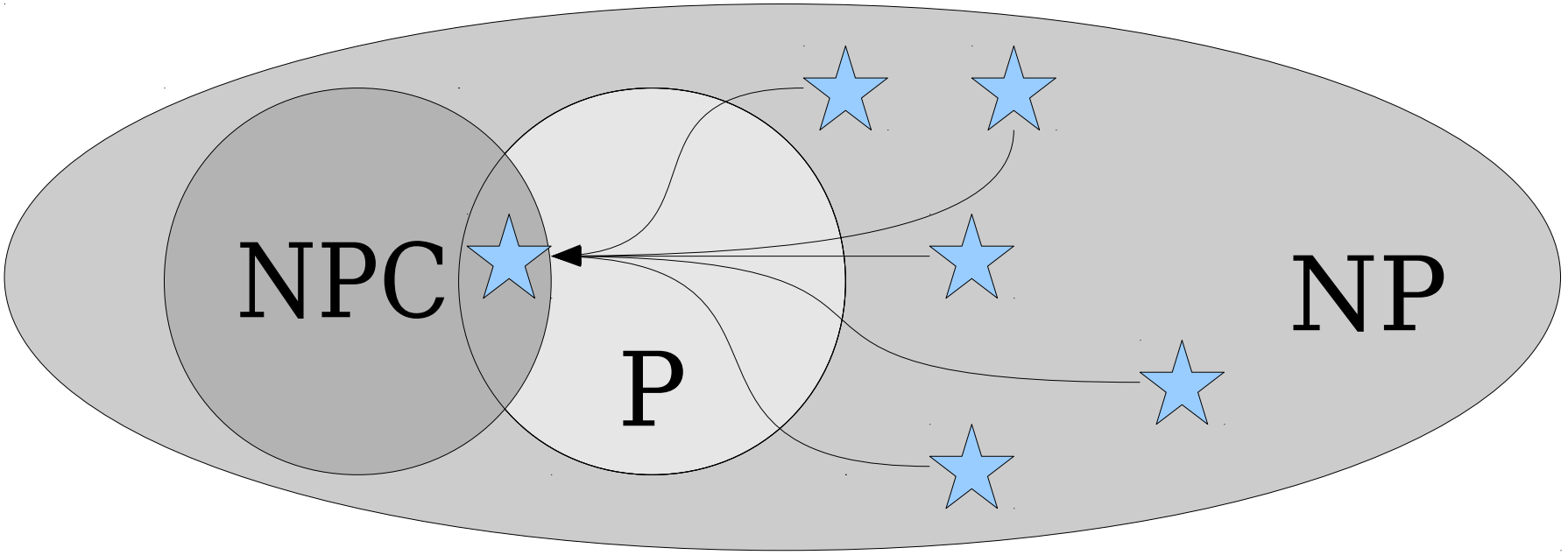

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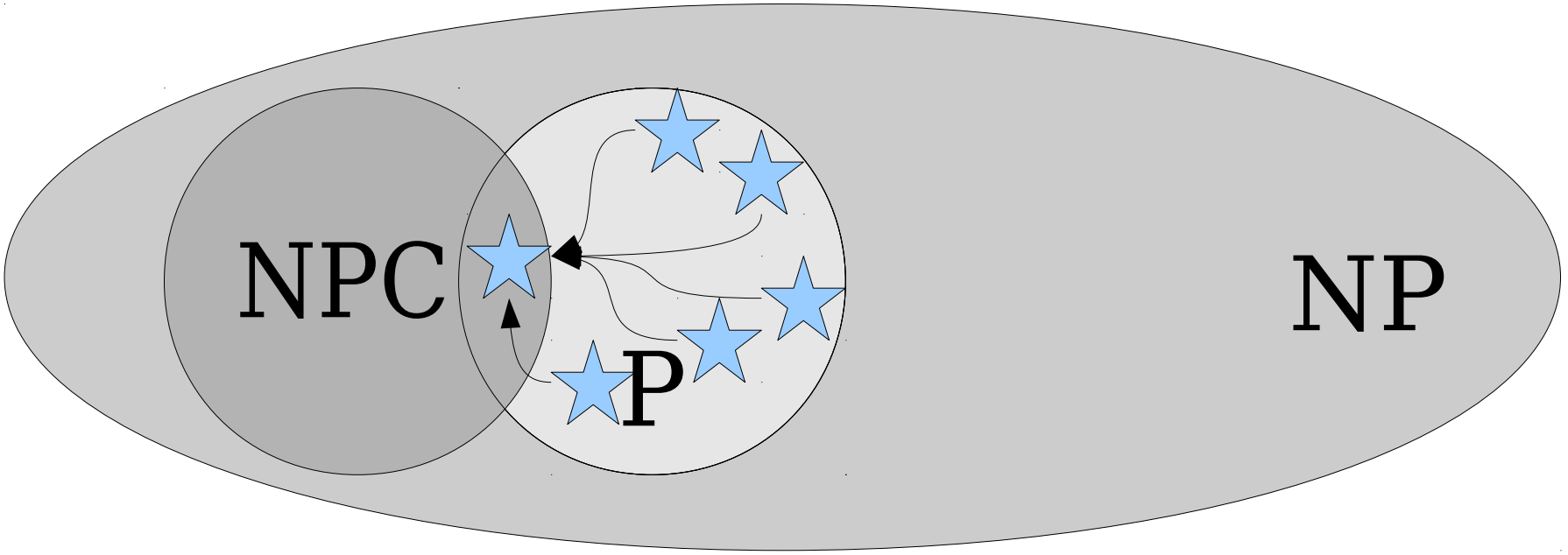

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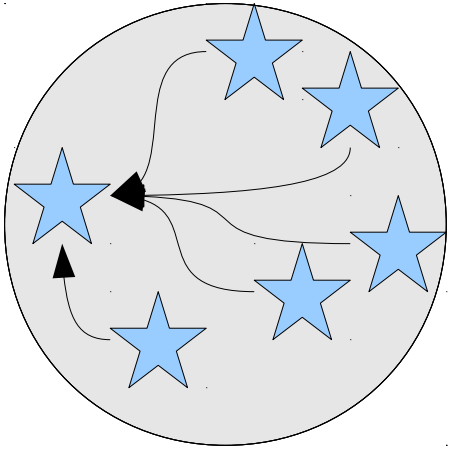
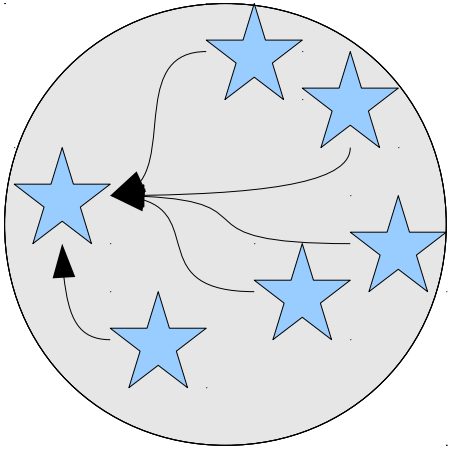


P = NP

The Tantalizing Truth

Theorem: If *any* **NP**-complete language is in **P**, then **P** = **NP**.

Proof: Suppose that L is **NP**-complete and $L \in \mathbf{P}$. Now consider any arbitrary **NP** problem A. Since L is **NP**-complete, we know that $A \leq_p L$. Since $L \in \mathbf{P}$ and $A \leq_p L$, we see that $A \in \mathbf{P}$. Since our choice of A was arbitrary, this means that $\mathbf{NP} \subseteq \mathbf{P}$, so **P** = **NP**. ■

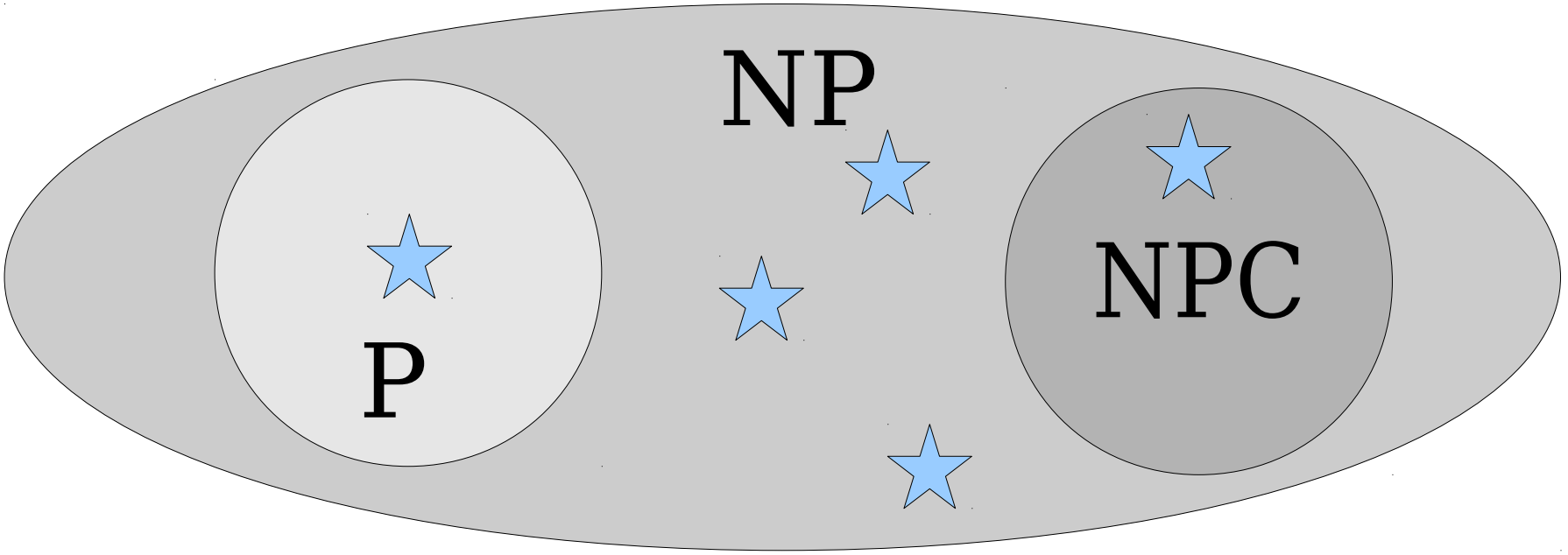


P = NP

The Tantalizing Truth

Theorem: If *any* **NP**-complete language is not in **P**, then **P** ≠ **NP**.

Proof: Suppose that L is an **NP**-complete language not in **P**. Since L is **NP**-complete, we know that $L \in$ **NP**. Therefore, we know that $L \in$ **NP** and $L \notin$ **P**, so **P** ≠ **NP**. ■



How do we even know NP-complete problems exist in the first place?

Satisfiability

- A propositional logic formula φ is called ***satisfiable*** if there is some assignment to its variables that makes it evaluate to true.
  - $p \wedge q$ is satisfiable.
  - $p \wedge \neg p$ is unsatisfiable.
  - $p \rightarrow (q \wedge \neg q)$ is satisfiable.
- An assignment of true and false to the variables of φ that makes it evaluate to true is called a ***satisfying assignment***.

SAT

- The ***boolean satisfiability problem*** (***SAT***) is the following:

  Given a propositional logic formula φ, is φ satisfiable?

- Formally:

  SAT **= { ⟨φ⟩ | φ is a satisfiable PL formula }**

Theorem (Cook-Levin): SAT is **NP**-complete.

Proof Idea: Given a polymomial-time verifier V for an arbitrary **NP** language L, for any string w you can construct a polynomially-sized formula $\varphi(w)$ that says "there is a certificate c where V accepts $\langle w, c \rangle$." This formula is satisfiable if and only if $w \in L$, so deciding whether the formula is satisfiable decides whether w is in L.

Proof: Take CS154!

Why All This Matters

- Resolving $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ is equivalent to just figuring out how hard SAT is.
  - If SAT $\in \mathbf{P}$, then $\mathbf{P} = \mathbf{NP}$.

    If SAT $\notin \mathbf{P}$, then $\mathbf{P} \neq \mathbf{NP}$.
- We've turned a huge, abstract, theoretical problem about solving problems versus checking solutions into the concrete task of seeing how hard one problem is.
- You can get a sense for how little we know about algorithms and computation given that we can't yet answer this question!

Why All This Matters

- You will almost certainly encounter **NP**-hard problems in practice – they're everywhere!
- If a problem is **NP**-hard, then there is no known algorithm for that problem that
  - is efficient on all inputs,
  - always gives back the right answer, and
  - runs deterministically.
- ***Useful intuition:*** If you need to solve an **NP**-hard problem, you will either need to settle for an approximate answer, an answer that's likely but not necessarily right, or have to work on really small inputs.

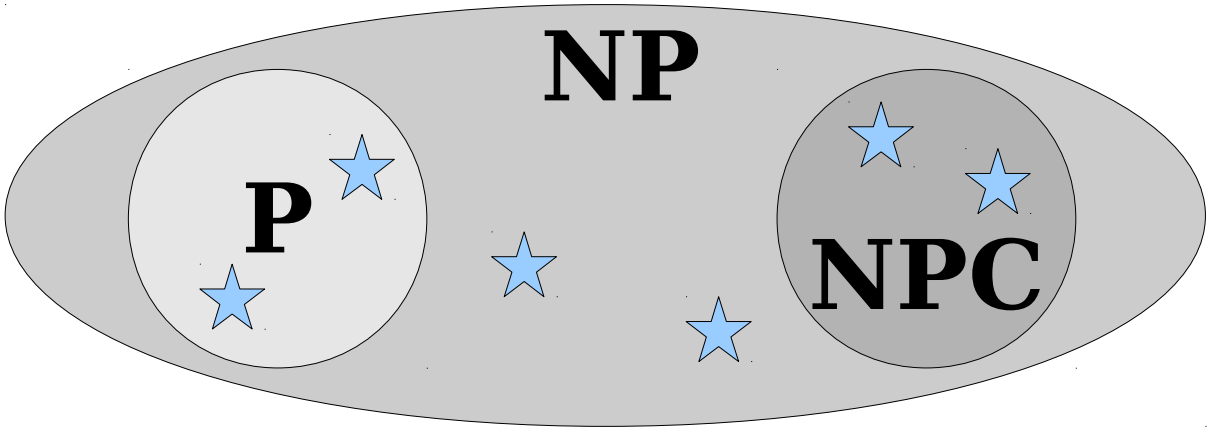
Sample **NP**-Hard Problems

- ***Computational biology:*** Given a set of genomes, what is the most probable evolutionary tree that would give rise to those genomes? *(Maximum parsimony problem)*
- ***Game theory:*** Given an arbitrary perfect-information, finite, twoplayer game, who wins? *(Generalized geography problem)*
- ***Operations research:*** Given a set of jobs and workers who can perform those tasks in parallel, can you complete all the jobs within some time bound? *(Job scheduling problem)*
- ***Machine learning:*** Given a set of data, find the simplest way of modeling the statistical patterns in that data *(Bayesian network inference problem)*
- ***Medicine:*** Given a group of people who need kidneys and a group of kidney donors, find the maximum number of people who can end up with kidneys *(Cycle cover problem)*
- ***Systems:*** Given a set of processes and a number of processors, find the optimal way to assign those tasks so that they complete as soon as possible *(Processor scheduling problem)*

Coda: What if $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ is resolved?

Intermediate Problems

- With few exceptions, every problem we've discovered in **NP** has either
  - definitely been proven to be in **P**, or
  - definitely been proven to be **NP**-complete.
- A problem that's **NP**, not in **P**, but not **NP**-complete is called ***NP-intermediate***.
- ***Theorem (Ladner):*** There are **NP**-intermediate problems if and only if **P** ≠ **NP**.



What if **P ≠ NP**?

A Good Read:

"A Personal View of Average-Case Complexity" by Russell Impagliazzo

What if **P = NP**?

And a Dismal Third Option

Next Time

- ***The Big Picture***
- ***Where to Go from Here***
- ***A Final "Your Questions"***
- ***Parting Words!***