

# Direct Proofs

# Outline for Today

- ***Mathematical Proof***
  - What is a mathematical proof? What does a proof look like?
- ***Direct Proofs***
  - A versatile, powerful proof technique.
- ***Universal and Existential Statements***
  - What exactly are we trying to prove?
- ***Proofs on Set Theory***
  - Formalizing our reasoning.

What is a Proof?

A *proof* is an argument that demonstrates why a conclusion is true, subject to certain standards of truth.

A ***mathematical proof*** is an argument that demonstrates why a mathematical statement is true, following the rules of mathematics.



# Modern Proofs

# Two Quick Definitions

- An integer  $n$  is **even** if there is some integer  $k$  such that  $n = 2k$ .
  - This means that 0 is even.
- An integer  $n$  is **odd** if there is some integer  $k$  such that  $n = 2k + 1$ .
  - This means that 0 is not odd.
- We'll assume the following for now:
  - Every integer is either even or odd.
  - No integer is both even and odd.



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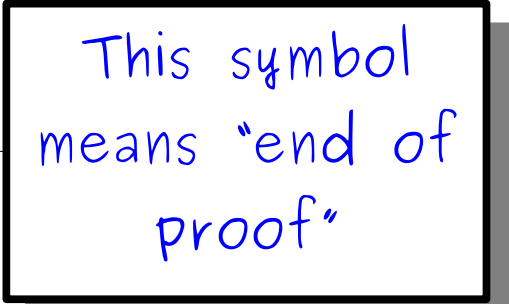
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This symbol means "end of proof"

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To prove a statement of the form

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Assume that  **$P$**  is true, then show that  **$Q$**  must be true as well.

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From this, we can write  $n^2$  as  $m$  (namely,  $2k$ )

Therefore,  $n^2$

This is the definition of an even integer. When writing a mathematical proof, it's common to call back to the definitions.

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Notice how we use the value of  $k$  that we obtained above. Giving names to quantities, even if we aren't fully sure what they are, allows us to manipulate them. This is similar to variables in programs.



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Our ultimate goal is to prove that  $n^2$  is even. This means that we need to find some  $m$  such that  $n^2 = 2m$ . Here, we're explicitly showing how we can do that.

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Hey, that's what we were trying to show! We're done now.

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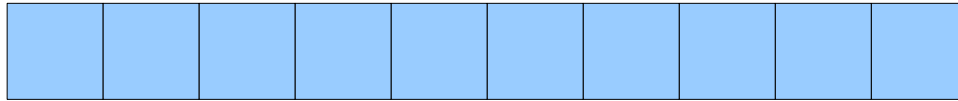
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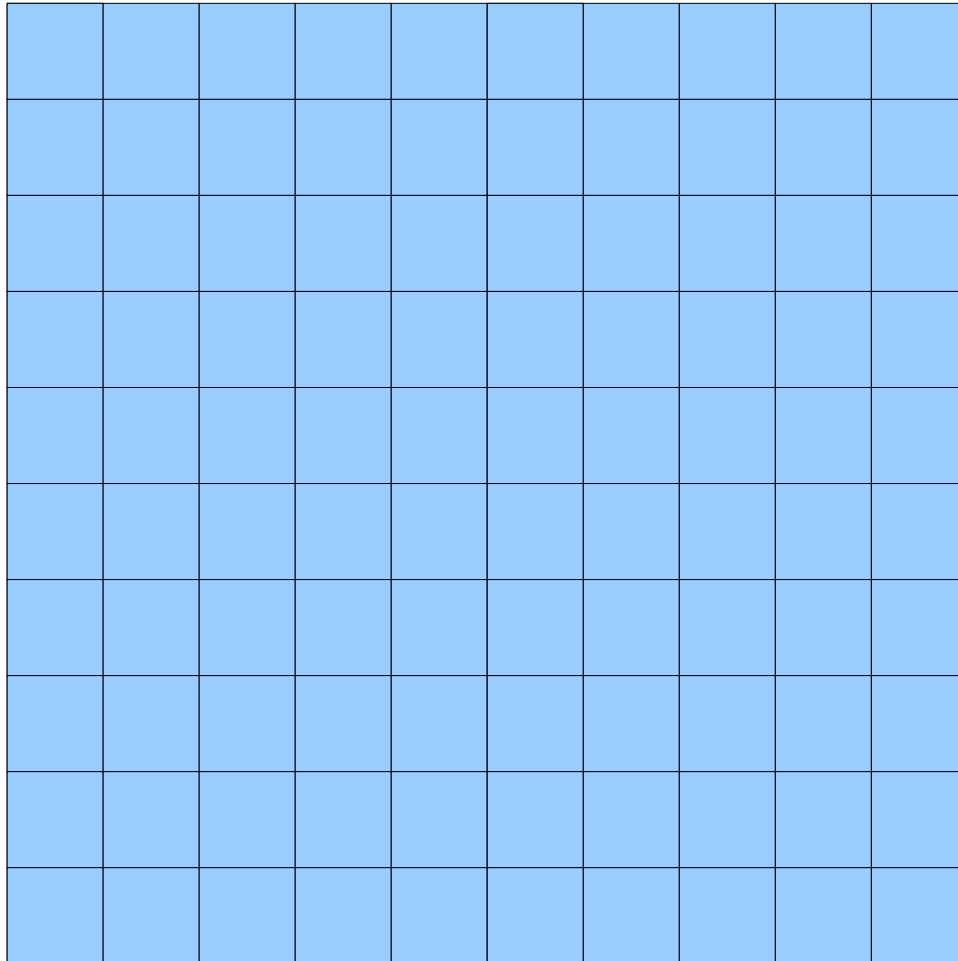
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# A Visual Intuition



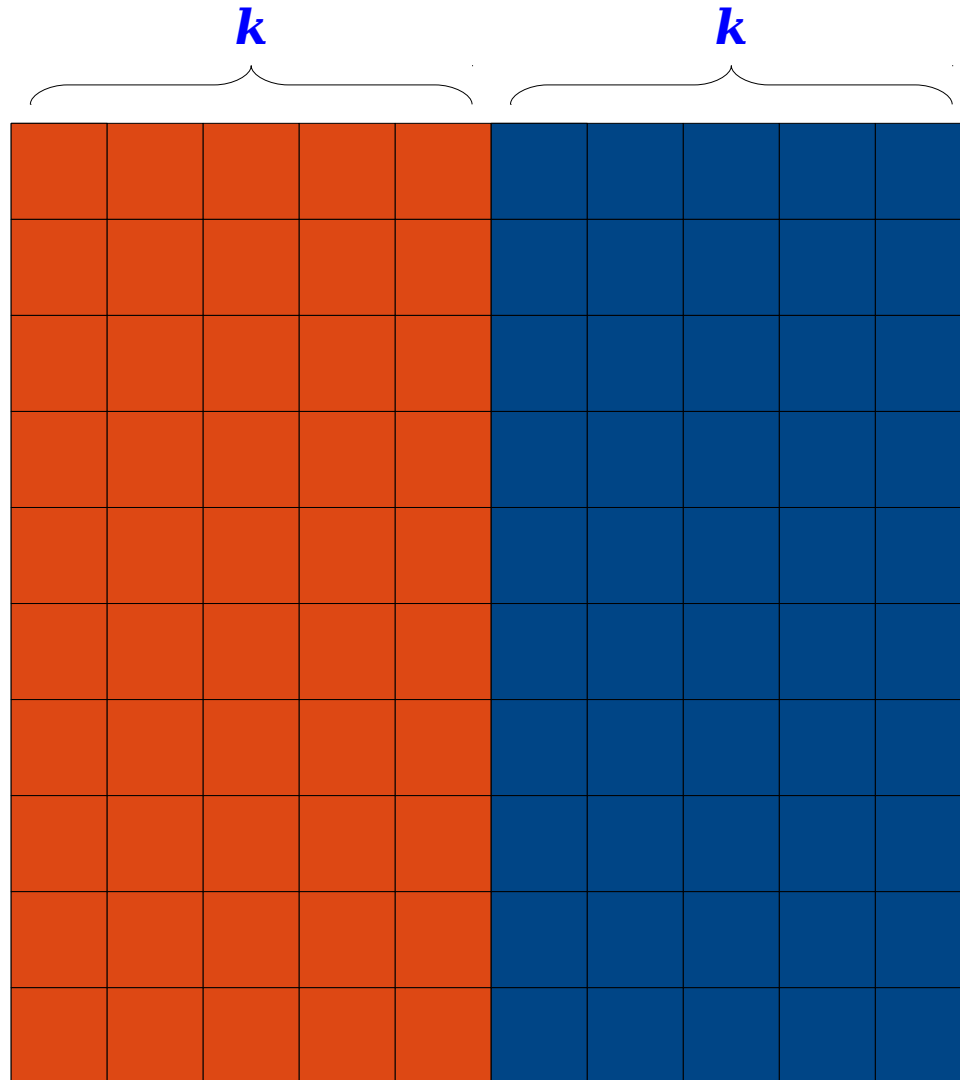
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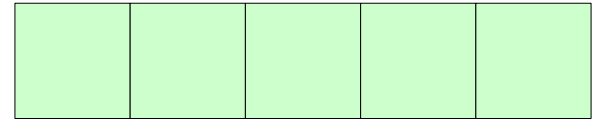
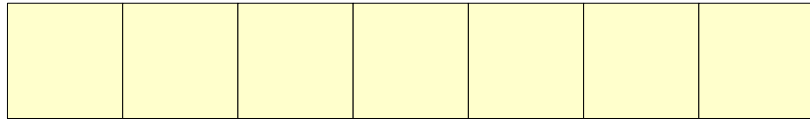
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That wasn't so bad! Let's do another one.

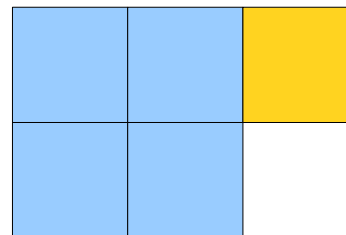
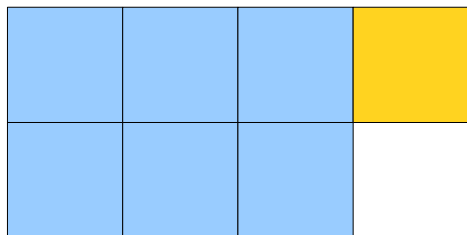
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How do we prove  
that this is true for  
*any* integers?

# Proving Something Always Holds

- Many statements have the form

**For any  $x$ , [some-property] holds of  $x$ .**

- Examples:

For all integers  $n$ , if  $n$  is even,  $n^2$  is even.

For any sets  $A$ ,  $B$ , and  $C$ , if  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

For all sets  $S$ ,  $|S| < |\wp(S)|$ .

Everything that drowns me makes me wanna fly.

- How do we prove these statements when there are (potentially) infinitely many cases to check?

# Arbitrary Choices

- To prove that some property holds true for all possible  $x$ , show that no matter what choice of  $x$  you make, that property must be true.
- Start the proof by choosing  $x$  *arbitrarily*:
  - “Let  $x$  be an arbitrary even integer.”
  - “Let  $x$  be any set containing 137.”
  - “Consider any  $x$ .”
  - “Pick an odd integer  $x$ .”
- Demonstrate that the property holds true for this choice of  $x$ .

# ar·bi·trar·y

adjective /'ärbi,trerē/

1. Based on random choice or personal whim, rather than any reason or system - *“his mealtimes were entirely arbitrary”*
2. *(of power or a ruling body)* Unrestrained and autocratic in the use of authority - *“arbitrary rule by King and bishops has been made impossible”*
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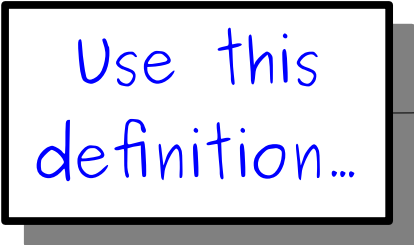
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By picking  $m$  and  $n$  arbitrarily, anything we prove about  $m$  and  $n$  will generalize to all possible choices we could have made.

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Numbering these equalities lets us refer back to them later on, making the flow of the proof a bit easier to understand.

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Notice that we use  $k$  in the first equality and  $r$  in the second equality. That's because we know that  $n$  is twice something plus one, but we can't say for sure that it's  $k$  specifically.

Equation (3) tells us that  $m + n = 2(k + r + 1)$ , which is even, as required. ■

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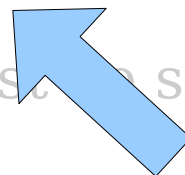
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This is a grammatically correct and complete sentence! Proofs are expected to be written in complete sentences, so you'll often use punctuation at the end of formulas.

We recommend using the "mugga mugga" test - if you read a proof and replace all the mathematical notation with "mugga mugga," what comes back should be a valid sentence.



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$$m = 2k + 1. \quad (1)$$

Similarly, because  $n$  is odd there must be some integer  $r$  such that

$$n = 2r + 1. \quad (2)$$

By adding equations (1) and (2) we learn that

$$\begin{aligned} m + n &= 2k + 1 + 2r + 1 \\ &= 2k + 2r + 2 \\ &= 2(k + r + 1). \end{aligned} \quad (3)$$

Equation (3) tells us that there is an integer  $s$  (namely,  $k + r + 1$ ) such that  $m + n = 2s$ . Therefore, we see that  $m + n$  is even, as required. ■

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Trace through this proof if  $m = 7$  and  $n = 9$ . What is the resulting value of  $s$ ?

- A. 3
- B. 8
- C. 17

Answer at [Pollevo.com/cs103](https://www.pollevo.com/cs103) or text **CS103** to **22333** once to join, then **A**, **B**, or **C**.



# Proof by Exhaustion

***Theorem:*** The product of any two consecutive integers is even.

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A horizontal number line is shown with integers from -3 to 11. The integers are spaced evenly. The integers 1 and 2 are enclosed in a yellow dashed rectangular box, highlighting them as a pair of consecutive integers.

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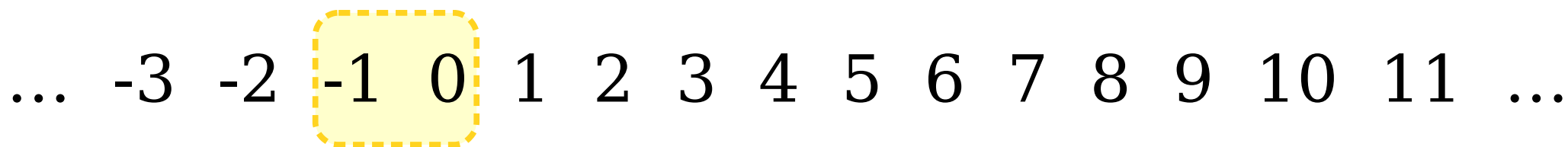
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A horizontal number line is shown with integers from -3 to 11. The numbers are spaced evenly. The integers -1 and 0 are enclosed in a yellow dashed rectangular box with rounded corners. Ellipses (...) are placed at both ends of the number line to indicate it continues in both directions.

***Theorem:*** The product of any two consecutive integers is even.

***Proof:***



***Theorem:*** The product of any two consecutive integers is even.

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This is called a **proof by cases** (alternatively, a **proof by exhaustion**) and works by showing that the theorem is true regardless of what specific outcome arises.

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$$\begin{aligned}n(n+1) &= (2k+1)(2k+2) \\ &= 2(2k+1)(k+1).\end{aligned}$$

After splitting into cases, it's a good idea to summarize what you just did so that the reader knows what to take away from it.

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# Some Little Exercises

- Here's a list of other theorems that are true about odd and even numbers:
  - **Theorem:** The sum and difference of any two even numbers is even.
  - **Theorem:** The sum and difference of an odd number and an even number is odd.
  - **Theorem:** The product of any integer and an even number is even.
  - **Theorem:** The product of any two odd numbers is odd.
- Feel free to use these results going forward.
- If you'd like to practice the techniques from today, try your hand at proving some of these results!

# Universal and Existential Statements

***Theorem:*** For any odd integer  $n$ , there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

**Theorem:** For any odd integer  $n$ , there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

**Proof:**

Which of the following should be the next sentence of this proof?

- A. "Pick any odd integer,  $n = 137$ ."
- B. "Pick any odd integer  $n$ ."
- C. "Pick any odd integer  $n$  and arbitrary integers  $r$  and  $s$  where  $r^2 - s^2 = n$ ."

Answer at [Pollev.com/cs103](https://pollev.com/cs103) or  
text **CS103** to **22333** once to join, then **A**, **B**, or **C**.



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**Proof:** Pick any odd integer  $n$ .

This is a very different sort of request than what we've seen in the past. How on earth do we go about proving something like this?

# Universal vs. Existential Statements

- A ***universal statement*** is a statement of the form  
**For all  $x$ , [some-property] holds for  $x$ .**
- We've seen how to prove these statements.
- An ***existential statement*** is a statement of the form  
**There is some  $x$  where [some-property] holds for  $x$ .**
- How do you prove an existential statement?

# Proving an Existential Statement

- Over the course of the quarter, we will see several different ways to prove an existential statement of the form

**There is an  $x$  where [some-property] holds for  $x$ .**

- ***Simplest approach:*** Search far and wide, find an  $x$  that has the right property, then show why your choice is correct.

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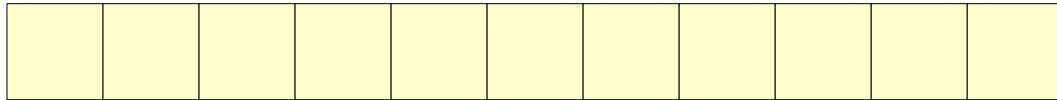
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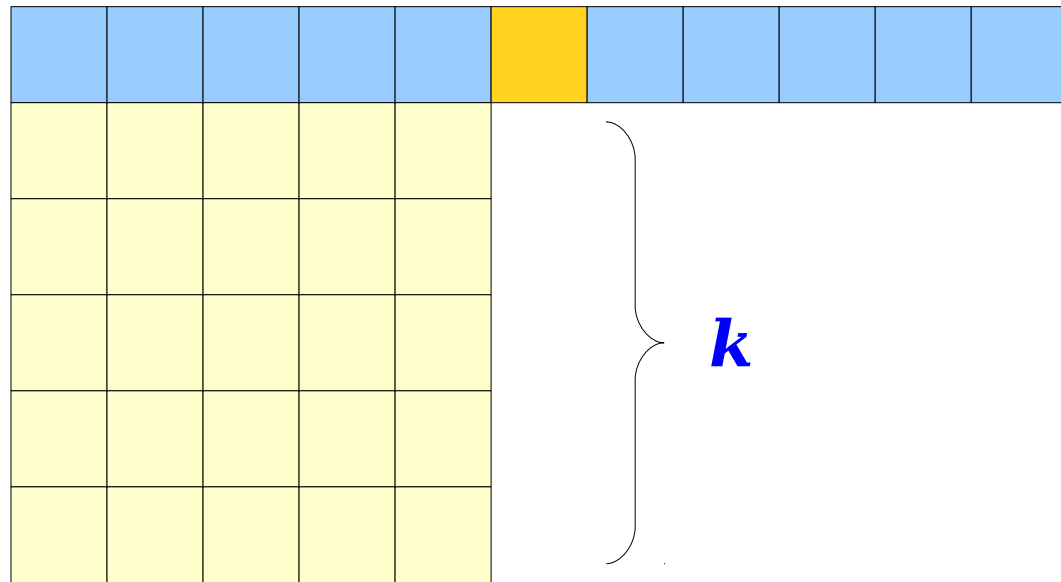
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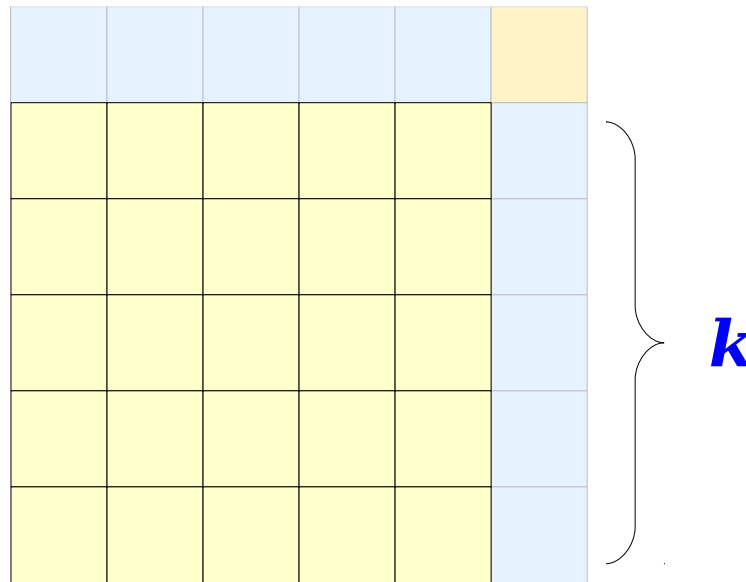
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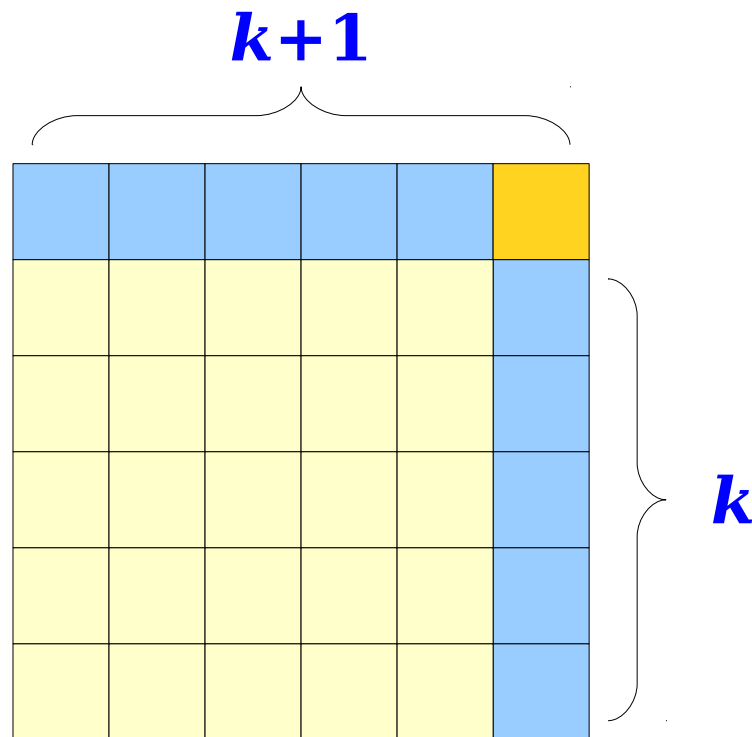
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**Proof:** Pick any odd integer  $n$ . Since  $n$  is odd, we know there is some integer  $k$  where  $n = 2k + 1$ .

Our guess:  
 $(k+1)^2 - k^2 = n$



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***Follow-Up Question:*** There are some integers that can't be written as  $r^2 - s^2$  for any integers  $r$  and  $s$ .

Can you prove that every integer can be formed by adding and subtracting some combination of at most *three* perfect squares?

**Time-Out for Announcements!**

Are you curious about computer science? Excited about the WiCS mission? Searching for a community on campus?

The Stanford Women in Computer Science Freshman Intern Application is **CURRENTLY LIVE:**

<https://goo.gl/forms/bhvnAFF1JqkEKFJc2>

The WiCS frosh intern program allows interns to rotate through different teams on WiCS, work on meaningful projects, and join the WiCS family. Interns will work on two different projects throughout the year (one in fall, one in winter/spring) with teams of current WiCS members, as well as participate in intern social and career development events. We organize company tours, social events, and more for our frosh interns exclusively.

Check out our [project descriptions & teams](#) for this year's program and make sure to [apply](#) by January 12 at 11:59pm!

If you have questions, please don't hesitate to reach out to Stephanie Campa ([scampa@stanford.edu](mailto:scampa@stanford.edu)) and Neehar Banerjee ([neehar@stanford.edu](mailto:neehar@stanford.edu)).

Get your resumes ready.... The Computer Forum Career Fair is held NEXT WEDNESDAY, January 17, from 11:00am - 4:00pm. Over sixty affiliated companies will gather on the lawn between the CS and EE Buildings.

Computer Forum Career Fair

Date: Wednesday, Jan. 17

Time: 11:00am - 4:00pm

Location: Lawn between the Gates and Packard Buildings

Please Register Via Handshake at

<https://stanford.joinhandshake.com/events/115612>

Stanford Students only and a valid Stanford Student ID will be required at check-in.

# Reading Recommendations

- We've released two handouts online that you should read over:
  - Handout 06: How to Succeed in CS103
  - Handout 07: Set Theory Definitions.
- Additionally, if you haven't yet read over the Guide to Elements and Subsets, we'd recommend doing so.
- Finally, we strongly recommend reading over Chapter 1 and Chapter 2 of the online course reader to get some more background with proofs and set theory.

# Piazza

- We have a Piazza site for CS103.
- Sign in to [www.piazza.com](http://www.piazza.com) and search for the course CS103 to sign in.
- Feel free to ask us questions!
- ***Use the site to find a partner for the problem sets!***



# Problem Set 0

- Problem Set 0 went out on Monday. It's due this Friday at 2:30PM.
  - Even though this just involves setting up your compiler and submitting things, please start this one early. If you start things on Friday morning, we can't help you troubleshoot Qt Creator issues!
  - There's a very detailed troubleshooting guide up on the CS103 website and a Piazza post detailing common fixes. If you're still having trouble, please feel free to ask on Piazza!

Back to CS103!

# Proofs on Sets

# Set Theory Review

- Recall from last time that we write  $x \in S$  if  $x$  is an element of set  $S$  and  $x \notin S$  if  $x$  is not an element of set  $S$ .
- If  $S$  and  $T$  are sets, we say that  $S$  is a subset of  $T$  (denoted  $S \subseteq T$ ) if the following statement is true:  
**For every object  $x$ , if  $x \in S$ , then  $x \in T$ .**
- Let's explore some properties of the subset relation.

***Theorem:*** For any sets  $A$ ,  $B$ , and  $C$ , if  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

**Theorem:** For any sets  $A$ ,  $B$ , and  $C$ , if  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

**Proof:** Let  $A$ ,  $B$ , and  $C$  be arbitrary sets where  $A \subseteq B$  and  $B \subseteq C$ .

**Theorem:** For any sets  $A$ ,  $B$ , and  $C$ , if  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

**Proof:** Let  $A$ ,  $B$ , and  $C$  be arbitrary sets where  $A \subseteq B$  and  $B \subseteq C$ .

We're showing here that regardless of what  $A$ ,  $B$ , and  $C$  you pick, the result will still be true.

**Theorem:** For any sets  $A$ ,  $B$ , and  $C$ , if  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

**Proof:** Let  $A$ ,  $B$ , and  $C$  be arbitrary sets where  $A \subseteq B$  and  $B \subseteq C$ .

To prove a statement of the form

**“If  $P$ , then  $Q$ ”**

Assume that  **$P$**  is true, then show that  **$Q$**  must be true as well.



**Theorem:** For any sets  $A$ ,  $B$ , and  $C$ , if  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

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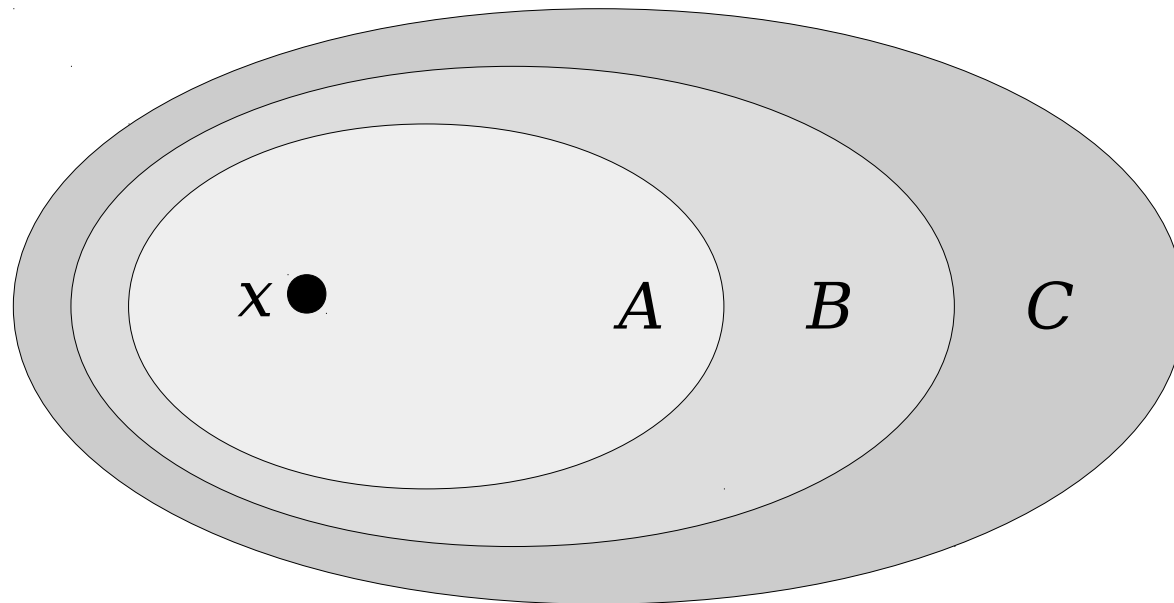
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This is, by definition, what it means for  **$A \subseteq C$**  to be true. Our job will be to prove this statement.

**Theorem:** For any sets  $A$ ,  $B$ , and  $C$ , if  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

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**Consider any  $x$**  where  $x \in A$ .

We're showing here that regardless of what  **$x$**  you pick, the result will still be true.

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Consider any  $x$  where  $x \in A$ . Since  $A \subseteq B$  and  $x \in A$ , we see that  $x \in B$ .



**Theorem:** For any sets  $A$ ,  $B$ , and  $C$ , if  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

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Consider any  $x$  where  $x \in A$ . Since  $A \subseteq B$  and  $x \in A$ , we see that  $x \in B$ . Similarly, since  $B \subseteq C$  and  $x \in B$ , we see that  $x \in C$ , which is what we needed to show. ■

This property of the subset relation is called *transitivity*. We'll revisit transitivity in a couple of weeks.

**Theorem:** For any sets  $A$ ,  $B$ , and  $C$ , if  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

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Consider any  $x$  where  $x \in A$ . Since  $A \subseteq B$  and  $x \in A$ , we see that  $x \in B$ . Similarly, since  $B \subseteq C$  and  $x \in B$ , we see that  $x \in C$ , which is what we needed to show. ■

Question to ponder: is this theorem still true if we replace  $\subseteq$  with  $\in$ ?

# Set Equality and Lemmas

# Set Equality

- As we mentioned on Monday, two sets  $A$  and  $B$  are equal when they have exactly the same elements.
- Here's a little theorem that's very useful for showing that two sets are equal:

***Theorem:*** If  $A$  and  $B$  are sets where  $A \subseteq B$   
and  $B \subseteq A$ , then  $A = B$ .

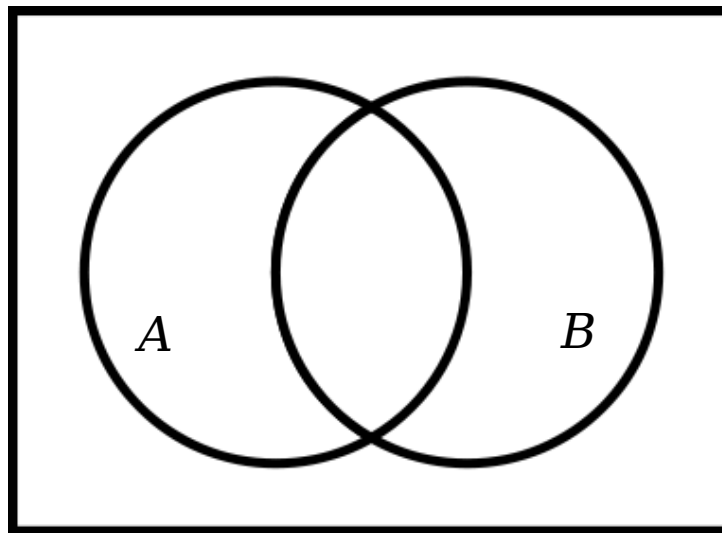
- We've included a proof of this result as an appendix to this slide deck. You should read over it on your own time.

# A Trickier Theorem

- Our last theorem for today is this one, which comes to us from the annals of set theory:

***Theorem:*** If  $A$  and  $B$  are sets and  
 $A \cup B \subseteq A \cap B$ , then  $A = B$ .

- Unlike our previous theorem, this one is a lot harder to see using Venn diagrams alone.



# Tackling our Theorem

***Theorem:*** If  $A$  and  $B$  are sets and  
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- Before we Flail and Panic, let's see if we can tease out some info about what this proof might look like.

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- We're going to pick arbitrary sets  $A$  and  $B$ .

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- We're going to assume  $A \cup B \subseteq A \cap B$ .

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# Tackling our Theorem

*Theorem:* If  $A$  and  $B$  are sets and  $A \cup B \subseteq A \cap B$ , then  $A = B$ .

Before we Flail and Panic, let's see if we can tease out some ideas that this proof might look like.

We're going to pick a direction.

We're going to assume  $A \cup B \subseteq A \cap B$ .

- We're going to prove that  $A = B$ .

Reasonable guess: let's try proving that  $A \subseteq B$  and that  $B \subseteq A$ .

***Lemma:*** If  $S$  and  $T$  are sets and  $S \cup T \subseteq S \cap T$ , then  $S \subseteq T$ .

A *lemma* is a smaller proof that's designed to build into a larger one. Think of it like program decomposition, except for proofs!

***Lemma:*** If  $S$  and  $T$  are sets and  $S \cup T \subseteq S \cap T$ , then  $S \subseteq T$ .

***Proof:***

**Lemma:** If  $S$  and  $T$  are sets and  $S \cup T \subseteq S \cap T$ , then  $S \subseteq T$ .

**Proof:** Let  $S$  and  $T$  be any sets where  $S \cup T \subseteq S \cap T$ .

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**Lemma:** If  $S$  and  $T$  are sets and  $S \cup T \subseteq S \cap T$ , then  $S \subseteq T$ .

**Theorem:** If  $A$  and  $B$  are sets and  $A \cup B \subseteq A \cap B$ , then  
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**Proof:** Let  $A$  and  $B$  be any sets where  $A \cup B \subseteq A \cap B$ . We will prove that  $A = B$  by showing  $A \subseteq B$  and  $B \subseteq A$ .



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First, notice that by our lemma, since  $A \cup B \subseteq A \cap B$ , we know that  $A \subseteq B$ .

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First, notice that by our lemma, since  $A \cup B \subseteq A \cap B$ , we know that  $A \subseteq B$ .

Next, since  $A \cup B = B \cup A$  and  $A \cap B = B \cap A$ , from  $A \cup B \subseteq A \cap B$  we learn that  $B \cup A \subseteq B \cap A$ .

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# What We've Covered

- ***What is a mathematical proof?***
  - An argument – mostly written in English – outlining a mathematical argument.
- ***What is a direct proof?***
  - It's a proof where you begin from some initial assumptions and reason your way to the conclusion.
- ***What are universal and existential statements?***
  - Universal statements make a claim about all objects of one type. Existential statements make claims about at least one object of some type.
- ***How do we write proofs about set theory?***
  - By calling back to definitions! Definitions are key.

# Next Time

- ***Indirect Proofs***
  - How do you prove something without actually proving it?
- ***Mathematical Implications***
  - What exactly does “if  $P$ , then  $Q$ ” mean?
- ***Proof by Contrapositive***
  - A helpful technique for proving implications.
- ***Proof by Contradiction***
  - Proving something is true by showing it can't be false.

## ***Appendix:*** Set Equality

# Set Equality

- If  $A$  and  $B$  are sets, we say that  $A = B$  precisely when the following statement is true:

**For any object  $x$ ,  $x \in A$  if and only if  $x \in B$ .**

- (This is called the *axiom of extensionality*.)
- In practice, this definition is tricky to work with.
- It's often easier to use the following result to show that two sets are equal:

**For any sets  $A$  and  $B$ ,  
if  $A \subseteq B$  and  $B \subseteq A$ , then  $A = B$ .**

***Theorem:*** For any sets  $A$  and  $B$ , if  $A \subseteq B$  and  $B \subseteq A$ , then  $A = B$ .

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**Theorem:** For any sets  $A$  and  $B$ , if  $A \subseteq B$  and  $B \subseteq A$ , then  $A = B$ .

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Since we've proven both directions of implication, we see that  $A = B$ .

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