Mathematical Logic

Part Three

Recap from Last Time

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects,
  - ***functions*** that map objects to one another, and
  - ***quantifiers*** that allow us to reason about many objects at once.

Some muggle is intelligent.

$$\exists m.\ (Muggle(m) \land Intelligent(m))$$

∃ is the ***existential quantifier*** and says "for some choice of m, the following is true."

"For any natural number n,
n is even iff n^2 is even"

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow Even(n^2)))$$

∀ is the ***universal quantifier***
and says "for any choice of n,
the following is true."

“All *A*'s are *B*'s”

translates as

$$\forall x.\ (A(x) \rightarrow B(x))$$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$$\forall x. (A(x) \to B(x))$$

If x is a counterexample, it *must* have property A but not have property B.

“Some *A* is a *B*”

translates as

$$\exists x.\ (A(x) \land B(x))$$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$$\exists x. (A(x) \land B(x))$$

If x is an example, it *must* have property A on top of property B.

The Aristotelian Forms

"All *A*s are *B*s"

$$\forall x.\ (A(x) \to B(x))$$

"Some *A*s are *B*s"

$$\exists x.\ (A(x) \land B(x))$$

"No *A*s are *B*s"

$$\forall x.\ (A(x) \to \neg B(x))$$

"Some *A*s aren't *B*s"

$$\exists x.\ (A(x) \land \neg B(x))$$

It is worth committing these patterns to memory. We'll be using them throughout the day and they form the backbone of many first-order logic translations.

The Art of Translation

Using the predicates

- *Person*(*p*), which states that *p* is a person, and
- *Loves*(*x*, *y*), which states that *x* loves *y*,

write a sentence in first-order logic that means “everybody loves someone else.”

Everybody loves someone else

Every person loves some other person

Every person p loves some other person

Every person p loves some other person

> "All *A*s are *B*s"
>
> $$\forall x.\ (A(x) \to B(x))$$

$\forall p.\ (Person(p) \rightarrow$

p loves some other person

$)$

"All As are Bs"

$\forall x.\ (A(x) \rightarrow B(x))$

$\forall p.\ (Person(p) \rightarrow$

p loves some other person

$)$

$\forall p. \; (Person(p) \rightarrow$

there is some other person that p loves

$)$

$\forall p. \; (Person(p) \rightarrow$

there is a person other than p that p loves

$)$

$\forall p. \ (Person(p) \rightarrow$

there is a person q, other than p, where p loves q

$)$

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"Some *A*s are *B*s"

$\exists x. \ (A(x) \land B(x))$

$\forall p.\ (Person(p) \rightarrow$

$\quad \exists q.\ (Person(q)\ \wedge$ *, other than p, where*

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$\quad)$

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“Some *A*s are *B*s”

$$\exists x.\ (A(x)\ \wedge\ B(x))$$

$\forall p.\ (Person(p) \rightarrow$

$\exists q.\ (Person(q)\ \wedge$, *other than p, where*

p loves q

$)$

$)$

$\forall p.\ (Person(p) \rightarrow$

$\quad \exists q.\ (Person(q) \wedge p \neq q \wedge$

p loves q

$\quad)$

$)$

$$
\begin{aligned}
&\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

How many of the following first-order logic statements are correct translations of "everyone loves someone else?"

$$\forall p.\ (Person(p) \to \\ \quad \exists q.\ (Person(q) \land \\ \quad\quad Loves(p, q) \\ \quad) \\ )$$

$$\forall p.\ (Person(p) \land \\ \quad \exists q.\ (Person(q) \land p \neq q \land \\ \quad\quad Loves(p, q) \\ \quad) \\ )$$

$$\forall p.\ (Person(p) \to \\ \quad \exists q.\ (Person(q) \land p \neq q \to \\ \quad\quad Loves(p, q) \\ \quad) \\ )$$

$$\exists p.\ (Person(p) \to \\ \quad \forall q.\ (Person(q) \land p \neq q \land \\ \quad\quad Loves(p, q) \\ \quad) \\ )$$

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **0**, **1**, **2**, **3**, or **4**.

Using the predicates

- *Person*(p), which states that p is a person, and
- *Loves*(x, y), which states that x loves y,

write a sentence in first-order logic that means “there is a person that everyone else loves.”

There is a person that everyone else loves

There is a person p where everyone else loves p

There is a person p where everyone else loves p

“Some *A*s are *B*s”

$\exists x.\ (A(x) \land B(x))$

$\exists p. \, (Person(p) \wedge$

everyone else loves p

$)$

"Some *A*s are *B*s"

$\boldsymbol{\exists x. \, (A(x) \wedge B(x))}$

$\exists p.\ (Person(p)\ \wedge$

everyone else loves p

$)$

$\exists p.\ (Person(p)\ \wedge$

every other person q loves p

$)$

$\exists p.\ (Person(p)\ \wedge$

every person q, other than p, loves p

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"All *A*s are *B*s"

$\forall \boldsymbol{x}.\ (\boldsymbol{A}(\boldsymbol{x}) \rightarrow \boldsymbol{B}(\boldsymbol{x}))$

$\exists p. \; (Person(p) \land$

$\quad \forall q. \; (Person(q) \land p \neq q \rightarrow$

$\quad\quad$ *q loves p*

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&\qquad Loves(q, p) \\
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Combining Quantifiers

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For Comparison

$$\forall p.\ (Person(p) \to \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

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$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \to Loves(q, p)))$$

There is some person

who everyone

who isn't them

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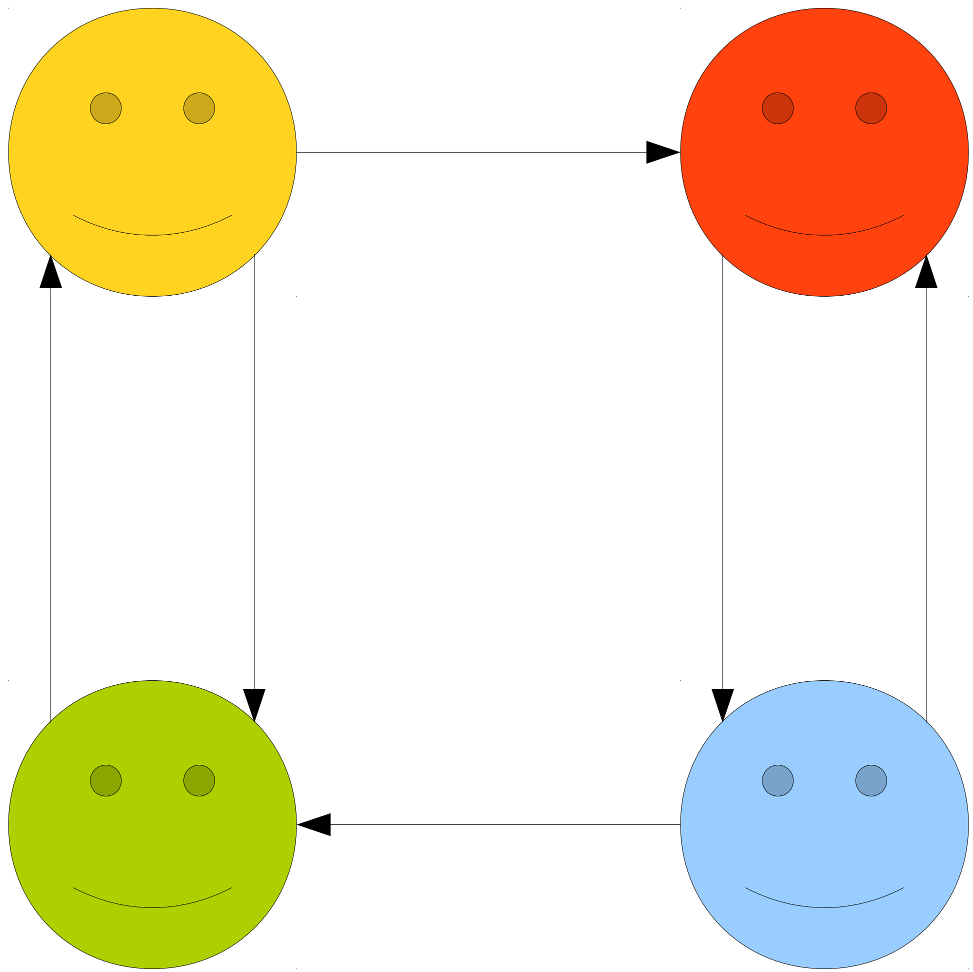
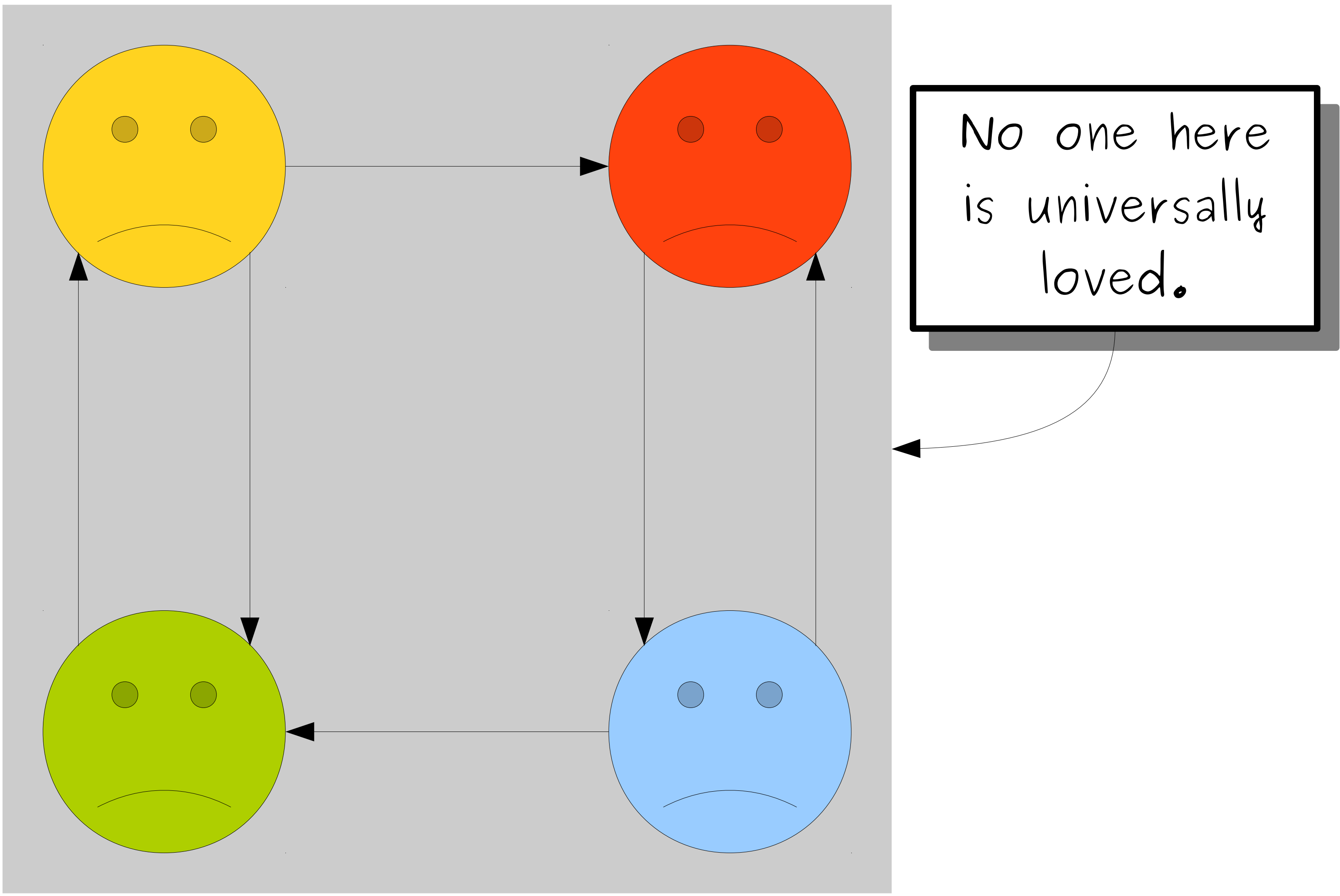
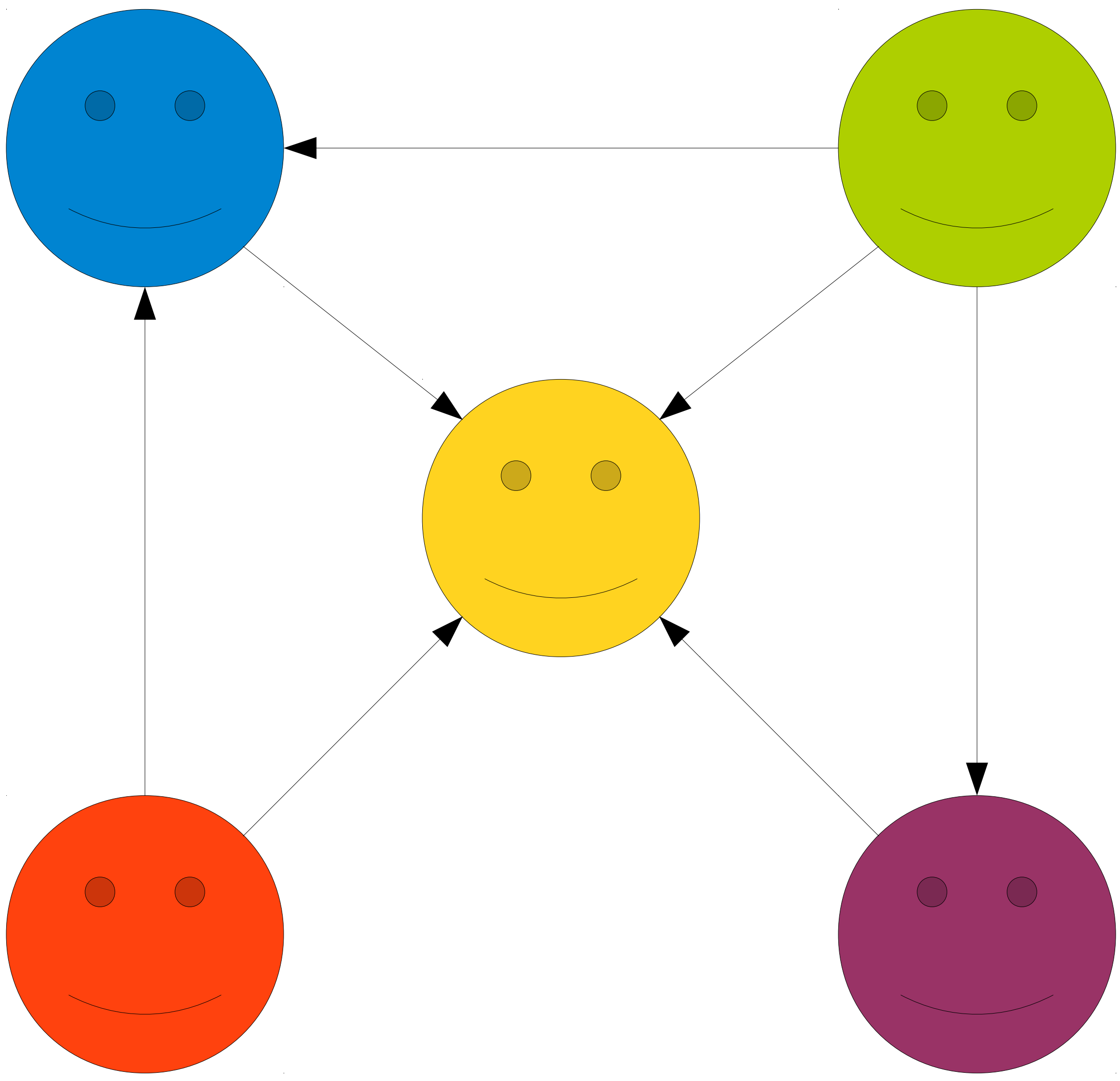
Everyone Loves Someone Else



Everyone Loves Someone Else



There is Someone Everyone Else Loves



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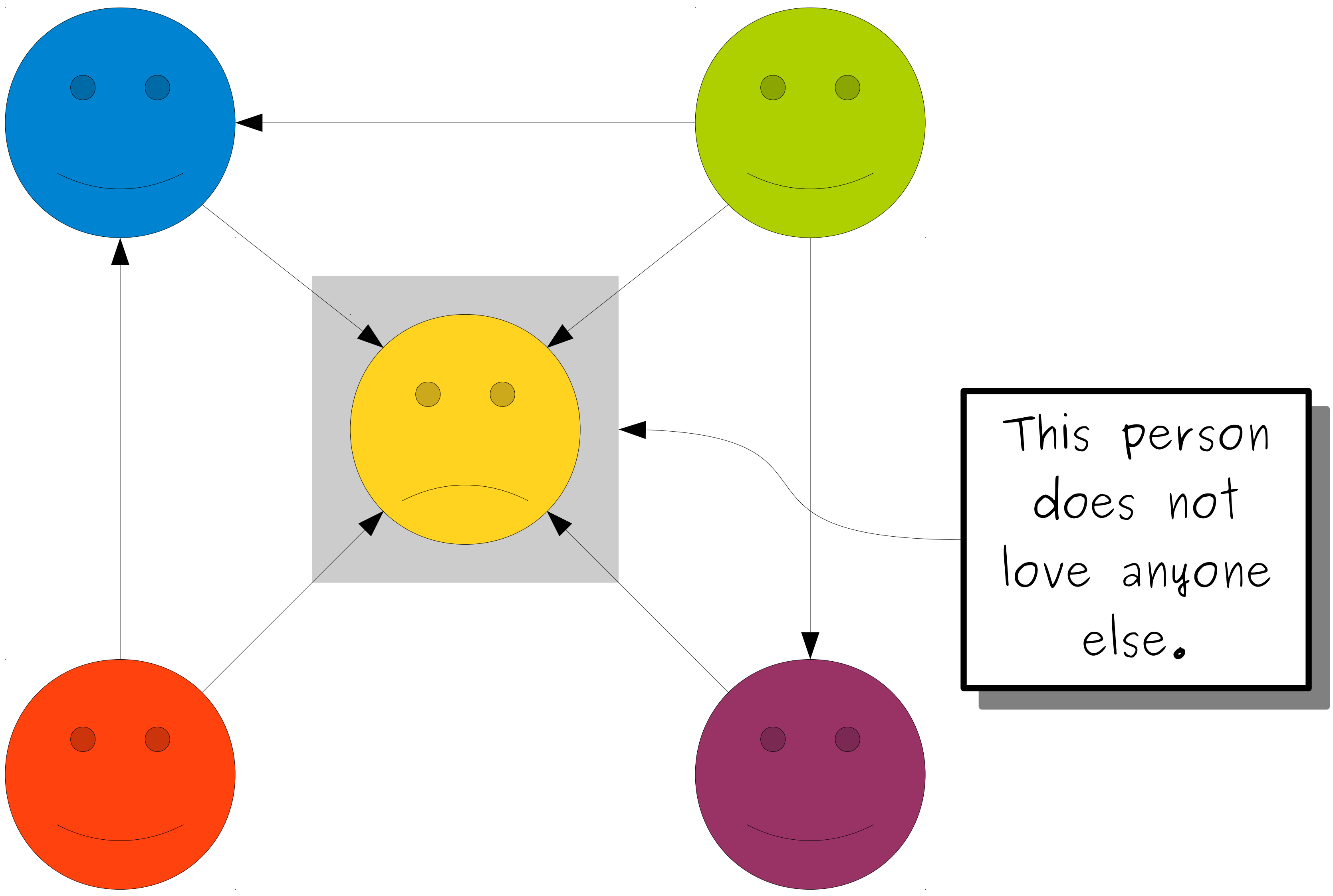
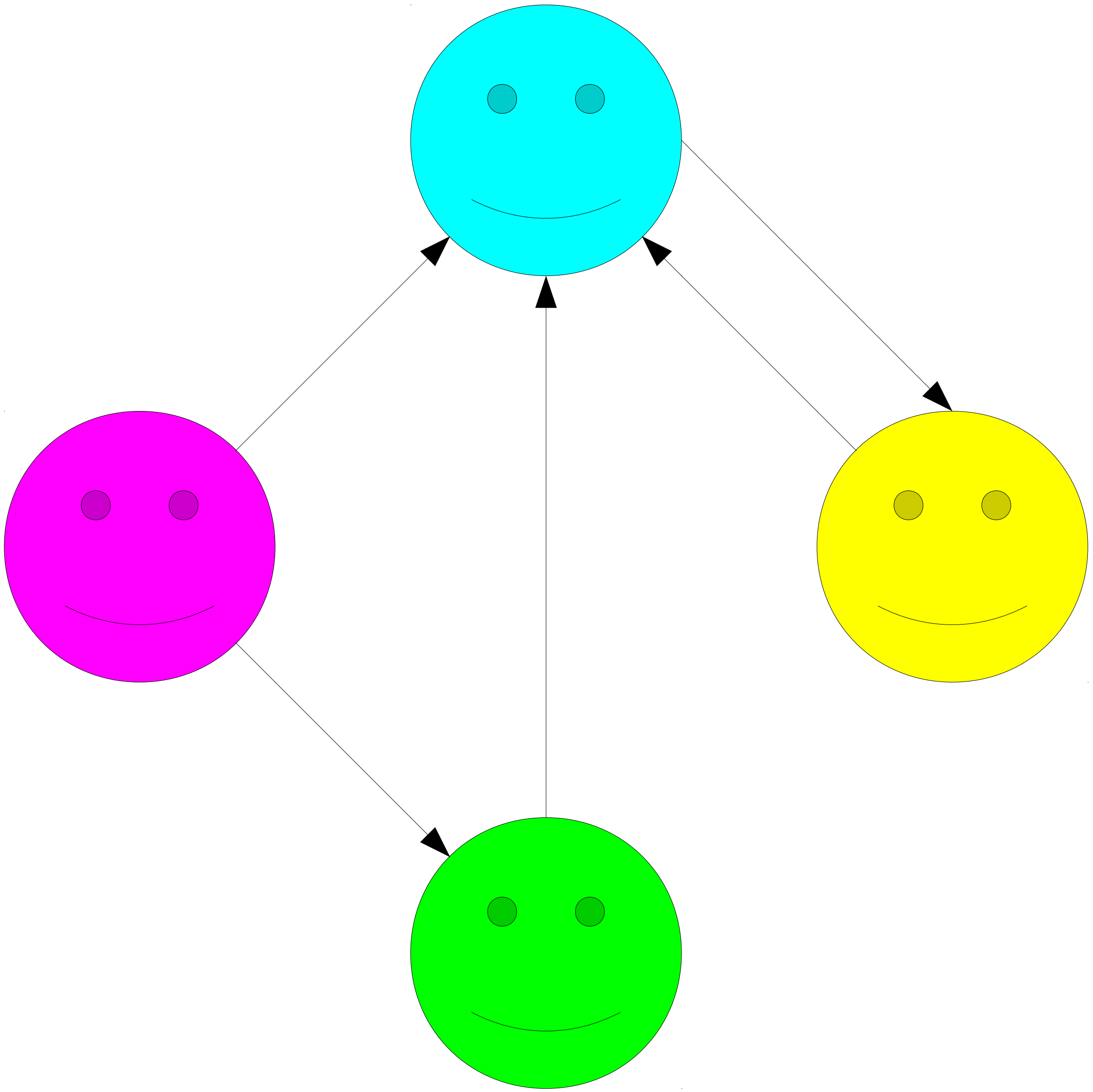


Everyone Loves Someone Else ***and*** There is Someone Everyone Else Loves



$$\forall p.\ (Person(p) \to \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

For every person,

there is some person

who isn't them

that they love.

$$\land$$

$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \to Loves(q, p)))$$

There is some person

who everyone

who isn't them

loves.

Quantifier Ordering

- The statement

$$\forall x.\ \exists y.\ P(x, y)$$

  means “for any choice of x, there's some choice of y where $P(x, y)$ is true.”
- The choice of y can be different every time and can depend on x.

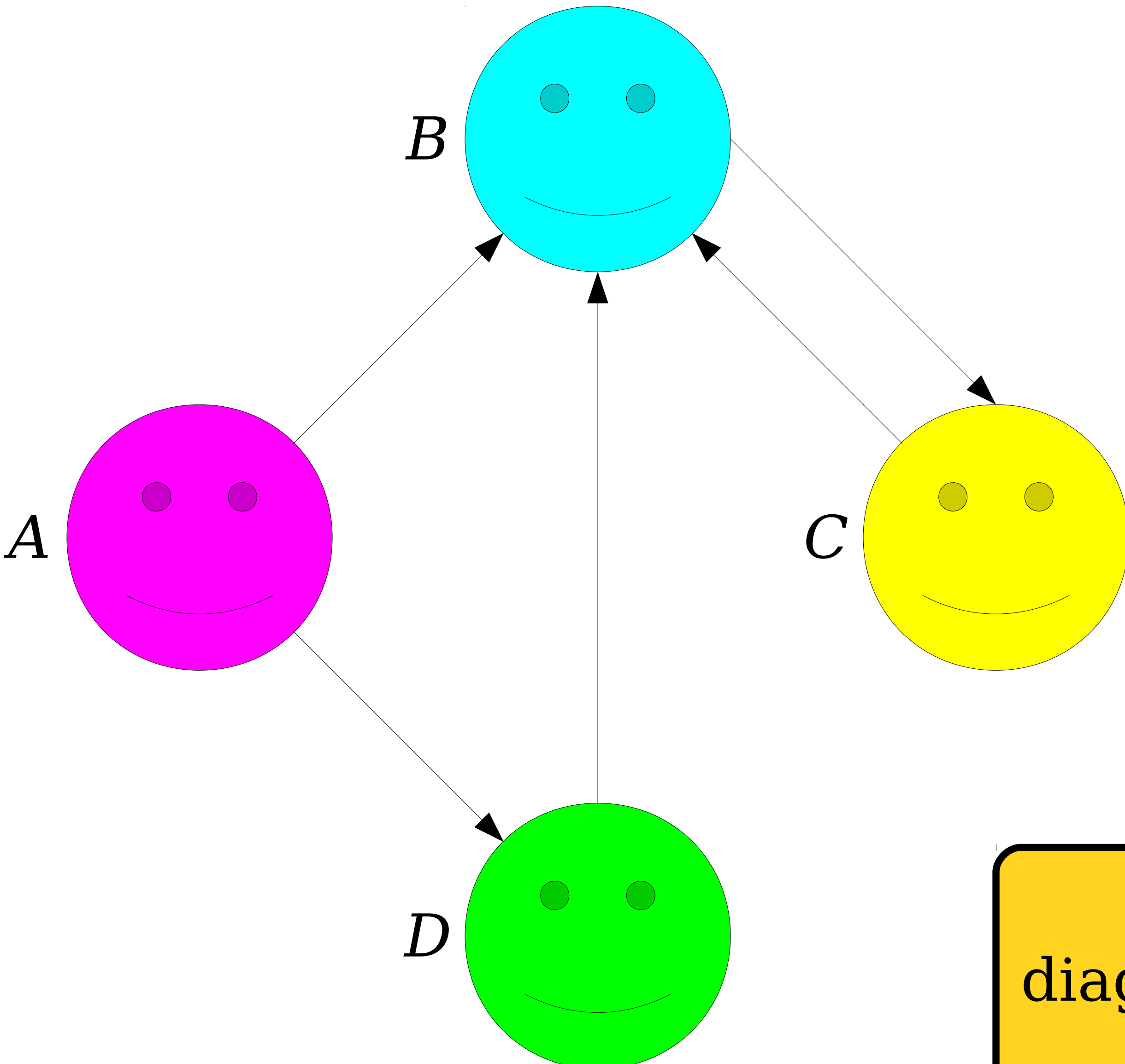
Quantifier Ordering

- The statement

  $$\exists x.\ \forall y.\ P(x, y)$$

  means "there is some x where for any choice of y, we get that $P(x, y)$ is true."
- Since the inner part has to work for any choice of y, this places a lot of constraints on what x can be.

Order matters when mixing existential and universal quantifiers!



Which person in this diagram do you most aspire to be?

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, **C**, or **D**.

Time-Out for Announcements!

Problem Set Two

- Problem Set Two is due this Friday at 2:30PM.
  - Once we're done with this lecture, you'll know everything you need to complete it!
  - Have questions? Feel free to stop by office hours or to ask on Piazza.
- Hopefully you've taken a few minutes to read over all the problems by now. If not, we'd strongly recommend doing so.
- ***Good idea:*** Aim to complete Q1 – Q5 by the end of the evening.

Problem Set One Solutions

- Problem Set One solutions are now available.
- ***Please take the time to read over these solutions***.
  - For non-proof questions, make sure that you understand the intuition behind the answers. If they match yours, great! If not, that would be a great question to ask us.
  - For proofs, look over the style and formatting. Compare them against yours. How do they compare?
  - Each question has a “Why We Asked This Question” section at the end. Make sure you read over it – it would be a shame if you did a problem and didn’t hit the key insight we wanted you to have.

Apply to Section Lead!

- Want to teach a CS106A/B/X section? Already completed CS106B or CS106X? Apply to section lead at

https://cs198.stanford.edu

- Application is due ***Thursday, February 1st***.
- There’s a second round of hiring later this quarter for folks currently in CS106B/X – stay tuned!
- ***This is an amazing program. Highly recommended!***

Back to CS103!

Set Translations

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means “the empty set exists.”

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means "the empty set exists."

First-order logic doesn't have set operators or symbols "built in." If we only have the predicates given above, how might we describe this?

The empty set exists.

There is some set S that is empty.

$\exists S.\ (Set(S)\ \land$

S is empty.

$)$

$\exists S.\ (Set(S)\ \wedge$

there are no elements in S

$)$

$\exists S.\ (Set(S)\ \wedge$

$\neg$*there is an element in S*

$)$

$\exists S. \ (Set(S) \ \land$

$\neg$*there is an element x in S*

$)$

$$\exists S.\ (Set(S)\ \land$$
$$\quad \neg\exists x.\ x \in S$$
$$)$$

$$\exists S.\ (Set(S) \wedge \neg \exists x.\ x \in S)$$

$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$

$\exists S.\ (Set(S) \land$

there are no elements in S

$)$

$\exists S.\ (Set(S) \wedge \neg \exists x.\ x \in S)$

$\exists S.\ (Set(S) \wedge$

every object does not belong to S

$)$

$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$

$\exists S.\ (Set(S) \land$

every object x does not belong to S

$)$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land$$
$$\quad \forall x.\ x \notin S$$
$$)$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

$$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

Both of these translations are correct. Just like in propositional logic, there are many different equivalent ways of expressing the same statement in first-order logic.

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means “two sets are equal if and only if they contain the same elements.”

Two sets are equal if and only if they have the same elements.

Any two sets are equal if and only if they have the same elements.

Any two sets S and T are equal if and only if they have the same elements.

$\forall S.\ (Set(S) \to$

$\quad \forall T.\ (Set(T) \to$

$\quad\quad$ *S and T are equal if and only if they have the same elements.*

$\quad)$

$)$

$\forall S. \; (Set(S) \rightarrow$

$\quad \forall T. \; (Set(T) \rightarrow$

$\quad\quad (S = T$ *if and only if they have the same elements.*$)$

$\quad)$

$)$

$\forall S.\ (Set(S) \rightarrow$

$\quad \forall T.\ (Set(T) \rightarrow$

$\quad\quad (S = T \leftrightarrow$ *they have the same elements.*)

$\quad)$

$)$

$\forall S.\ (Set(S) \rightarrow$

$\quad \forall T.\ (Set(T) \rightarrow$

$\quad\quad (S = T \leftrightarrow$ *S and T have the same elements.*$)$

$\quad)$

$)$

$\forall S.\ (Set(S) \rightarrow$

$\quad \forall T.\ (Set(T) \rightarrow$

$\quad\quad (S = T \leftrightarrow$ *every element of S is an element of T and vice-versa*)

$\quad)$

$)$

$\forall S.\ (Set(S) \rightarrow$

$\quad \forall T.\ (Set(T) \rightarrow$

$\quad\quad (S = T \leftrightarrow$ *x is an element of S if and only if x is an element of T*)

$\quad)$

$)$

$$
\begin{aligned}
&\forall S.\ (Set(S) \rightarrow \\
&\quad \forall T.\ (Set(T) \rightarrow \\
&\qquad (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T)) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
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&)
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$$

$$
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&\quad\quad (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T)) \\
&\quad) \\
&)
\end{aligned}
$$

You sometimes see the universal quantifier pair with the ↔ connective. This is especially common when talking about sets because two sets are equal when they have precisely the same elements.

$$
\begin{aligned}
&\forall S.\ (Set(S) \rightarrow \\
&\quad \forall T.\ (Set(T) \rightarrow \\
&\quad\quad (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T)) \\
&\quad) \\
&)
\end{aligned}
$$

Mechanics: Negating Statements

Which of the following is the negation of the statement
$\forall x.\ \exists y.\ Loves(x, y)$?

A. $\forall x.\ \forall y.\ \neg Loves(x, y)$
B. $\forall x.\ \exists y.\ \neg Loves(x, y)$
C. $\exists x.\ \forall y.\ \neg Loves(x, y)$
D. $\exists x.\ \exists y.\ \neg Loves(x, y)$
E. None of these.
F. Two or more of these.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, **C**, **D**, **E**, or **F**.

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	For some choice of x, $\neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For any choice of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
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$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	**For some choice of x, $\neg P(x)$**	For any choice of x, $P(x)$

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$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
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$\forall x.\ \neg P(x)$	**For any choice of x, $\neg P(x)$**	For some choice of x, $P(x)$
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$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
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$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	$\exists x.\ P(x)$
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$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	$\exists x.\ P(x)$
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$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	**For any choice of x, $P(x)$**

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	**For any choice of *x*, *P*(*x*)**	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	$\forall x.\ P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\exists x.\ \neg P(x)$
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$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	$\forall x.\ P(x)$

Negating First-Order Statements

- Use the equivalences

$$\neg \forall x.\ A \quad \equiv \quad \exists x.\ \neg A$$

$$\neg \exists x.\ A \quad \equiv \quad \forall x.\ \neg A$$

  to negate quantifiers.
- Mechanically:
  - Push the negation across the quantifier.
  - Change the quantifier from ∀ to ∃ or vice-versa.
- Use techniques from propositional logic to negate connectives.

Taking a Negation

$$\forall x.\ \exists y.\ Loves(x, y)$$

("Everyone loves someone.")

$$\neg\forall x.\ \exists y.\ Loves(x, y)$$

$$\exists x.\ \neg\exists y.\ Loves(x, y)$$

$$\exists x.\ \forall y.\ \neg Loves(x, y)$$

("There's someone who doesn't love anyone.")

Two Useful Equivalences

- The following equivalences are useful when negating statements in first-order logic:

$$\neg(p \land q) \equiv p \to \neg q$$

$$\neg(p \to q) \equiv p \land \neg q$$

- These identities are useful when negating statements involving quantifiers.
  - $\land$ is used in existentially-quantified statements.
  - $\to$ is used in universally-quantified statements.
- When pushing negations across quantifiers, we *strongly recommend* using the above equivalences to keep $\to$ with $\forall$ and $\land$ with $\exists$.

Negating Quantifiers

- What is the negation of the following statement, which says "there is a cute puppy"?

$$\exists x. \, (\mathit{Puppy}(x) \land \mathit{Cute}(x))$$

- We can obtain it as follows:

$$\neg \exists x. \, (\mathit{Puppy}(x) \land \mathit{Cute}(x))$$

$$\forall x. \, \neg(\mathit{Puppy}(x) \land \mathit{Cute}(x))$$

$$\forall x. \, (\mathit{Puppy}(x) \to \neg \mathit{Cute}(x))$$

- This says "no puppy is cute."
- Do you see why this is the negation of the original statement from both an intuitive and formal perspective?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

("There is a set with no elements.")

$$\neg\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$
$$\forall S.\ \neg(Set(S) \land \forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \to \neg\forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \to \exists x.\ \neg\neg(x \in S))$$
$$\forall S.\ (Set(S) \to \exists x.\ x \in S)$$

("Every set contains at least one element.")

These two statements are *not* negations of one another. Can you explain why?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

("There is a set that doesn't contain anything")

$$\forall S.\ (Set(S) \land \exists x.\ (x \in S))$$

("Everything is a set that contains something")

Remember: ∀ usually goes with →, not ∧

Restricted Quantifiers

Quantifying Over Sets

- The notation

$$\forall x \in S.\ P(x)$$

  means "for any element x of set S, $P(x)$ holds." (It's vacuously true if S is empty.)

- The notation

$$\exists x \in S.\ P(x)$$

  means "there is an element x of set S where $P(x)$ holds." (It's false if S is empty.)

Quantifying Over Sets

- The syntax

$$\forall \boldsymbol{x} \in \boldsymbol{S}.\ \boldsymbol{\varphi}$$

$$\exists \boldsymbol{x} \in \boldsymbol{S}.\ \boldsymbol{\varphi}$$

  is allowed for quantifying over sets.

- In CS103, feel free to use these restricted quantifiers, but please do not use variants of this syntax.
- For example, don't do things like this:

⚠ $\forall x \text{ with } P(x).\ Q(x)$ ⚠

⚠ $\forall y \text{ such that } P(y) \land Q(y).\ R(y).$ ⚠

⚠ $\exists P(x).\ Q(x)$ ⚠

Expressing Uniqueness

Using the predicate

- $Level(l)$, which states that l is a level,

write a sentence in first-order logic that means “there is only one level.”

A fun diversion:

http://www.onemorelevel.com/game/there_is_only_one_level

There is only one level.

Something is a level, and nothing else is.

Some thing l is a level, and nothing else is.

Some thing l is a level, and nothing besides l is a level

$\exists l.\ (Level(l)\ \wedge$

nothing besides l is a level.

$)$

$\exists l.\ (Level(l)\ \wedge$

anything that isn't l isn't a level

$)$

$\exists l. \; (Level(l) \;\wedge$

any thing x that isn't l isn't a level

$)$

$$
\begin{aligned}
&\exists l.\ (Level(l)\ \land \\
&\quad \forall x.\ (x \neq l \rightarrow \textit{x isn't a level}) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists l.\ (Level(l)\ \land \\
&\quad \forall x.\ (x \neq l \rightarrow \neg Level(x)) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists l.\ (Level(l)\ \wedge \\
&\quad \forall x.\ (x \neq l \rightarrow \neg Level(x)) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists l.\ (Level(l)\ \land \\
&\quad \forall x.\ (Level(x) \rightarrow x = l) \\
&)
\end{aligned}
$$

Expressing Uniqueness

- To express the idea that there is exactly one object with some property, we write that
  - there exists at least one object with that property, and that
  - there are no other objects with that property.
- You sometimes see a special "uniqueness quantifier" used to express this:

$$\exists! x.\ P(x)$$

- For the purposes of CS103, please do not use this quantifier. We want to give you more practice using the regular ∀ and ∃ quantifiers.

Next Time

- ***Binary Relations***
  - How do we model connections between objects?
- ***Equivalence Relations***
  - How do we model the idea that objects can be grouped into clusters?
- ***First-Order Definitions***
  - Where does first-order logic come into all of this?
- ***Proofs with Definitions***
  - How does first-order logic interact with proofs?