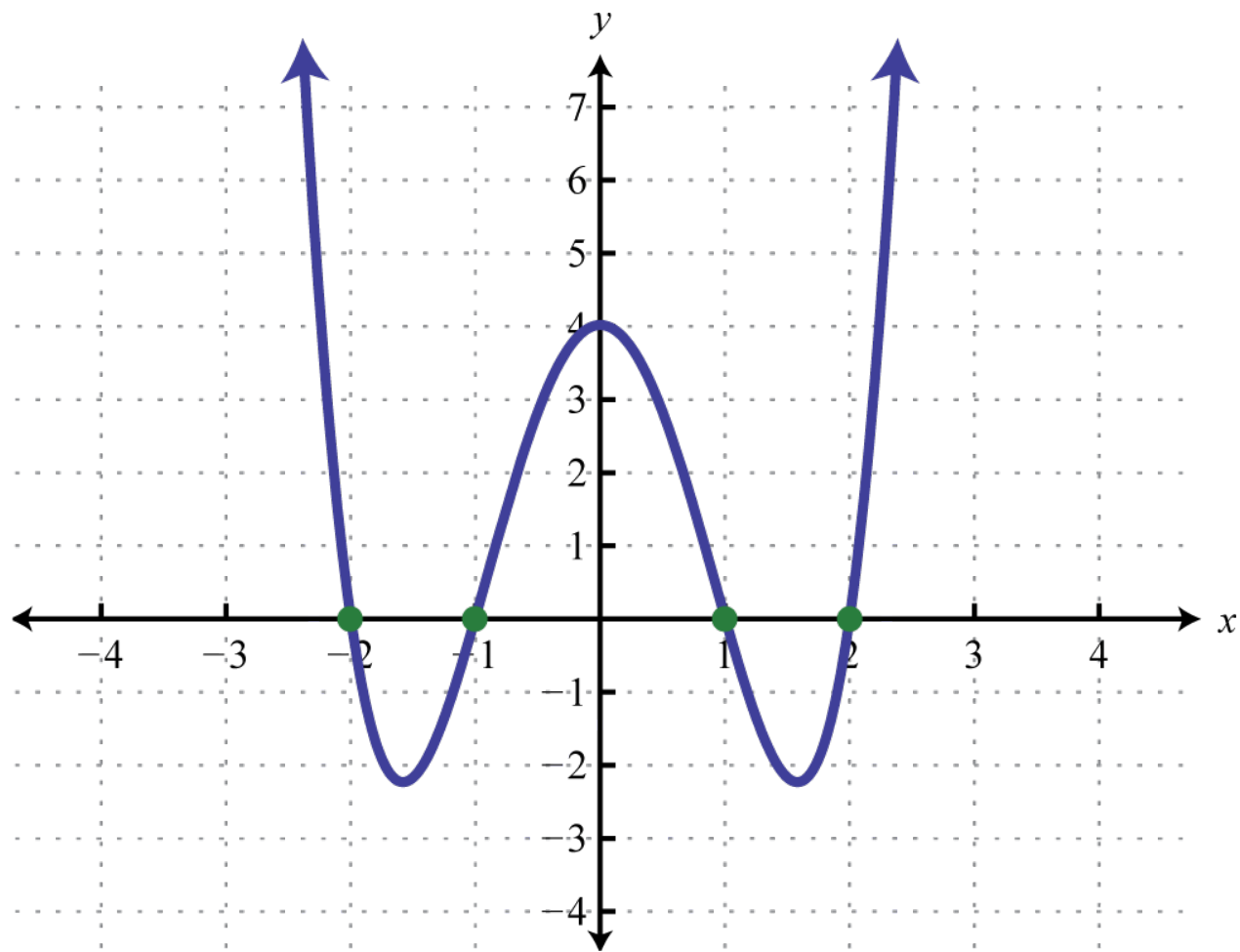


Functions

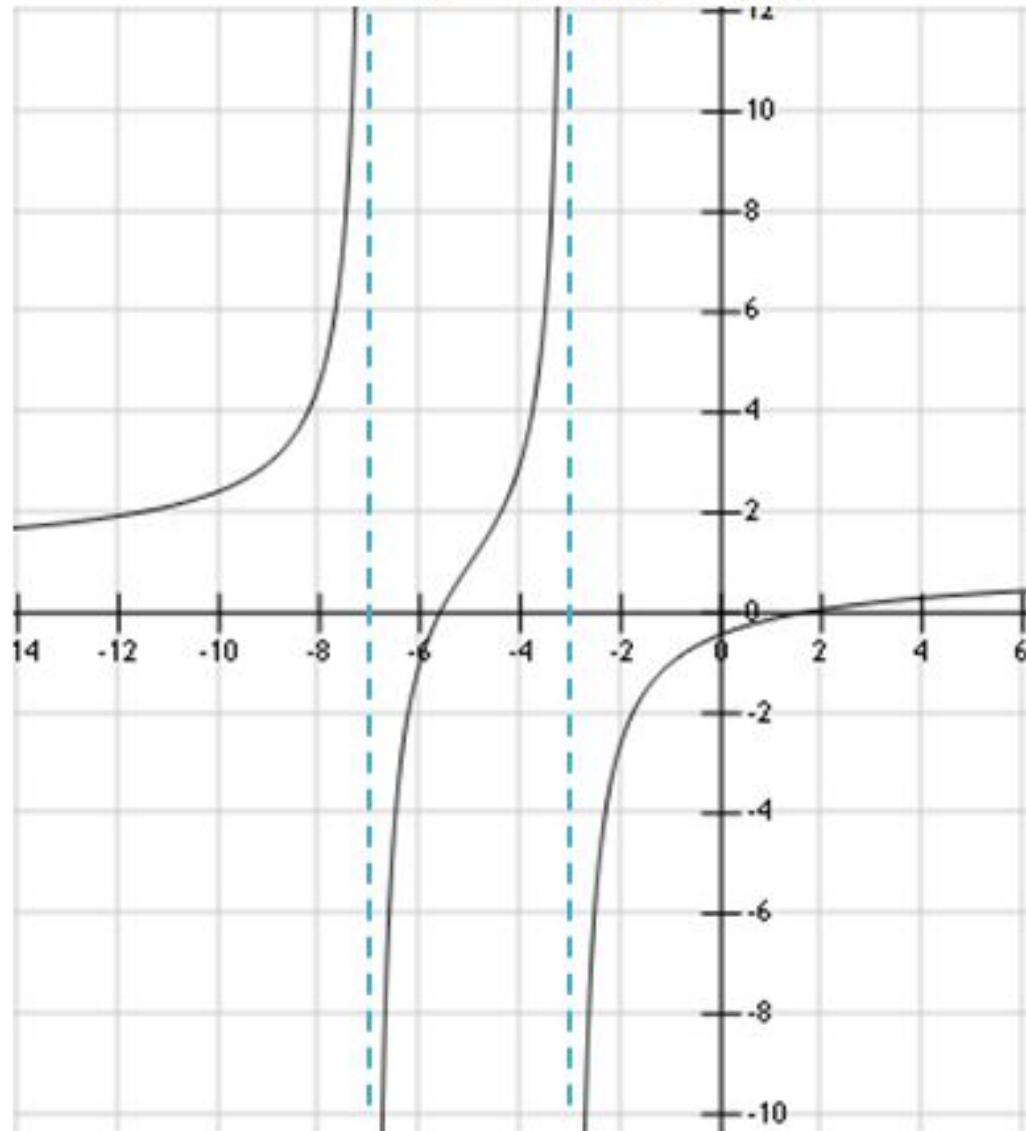
What is a function?

Functions, High-School Edition



$$f(x) = x^4 - 5x^2 + 4$$

$$f(x) = \frac{x^2 + 4x - 9}{x^2 + 10x + 21}$$



Functions, High-School Edition

- In high school, functions are usually given as objects of the form

$$f(x) = \frac{x^3 + 3x^2 + 15x + 7}{1 - x^{137}}$$

- What does a function do?
 - It takes in as input a real number.
 - It outputs a real number
 - ... except when there are vertical asymptotes or other discontinuities, in which case the function doesn't output anything.

Functions, CS Edition

```
int flipUntil(int n) {  
    int numHeads = 0;  
    int numTries = 0;  
  
    while (numHeads < n) {  
        if (randomBoolean()) numHeads++;  
  
        numTries++;  
    }  
  
    return numTries;  
}
```


Functions, CS Edition

- In programming, functions
 - might take in inputs,
 - might return values,
 - might have side effects,
 - might never return anything,
 - might crash, and
 - might return different values when called multiple times.

What's Common?

- Although high-school math functions and CS functions are pretty different, they have two key aspects in common:
 - They take in inputs.
 - They produce outputs.
- In math, we like to keep things easy, so that's pretty much how we're going to define a function.

Rough Idea of a Function:

A function is an object f that takes in an input and produces exactly one output.



(This is not a complete definition – we'll revisit this in a bit.)

High School versus CS Functions

- In high school, functions usually were given by a rule:

$$f(x) = 4x + 15$$

- In CS, functions are usually given by code:

```
int factorial(int n) {  
    int result = 1;  
    for (int i = 1; i <= n; i++) {  
        result *= i;  
    }  
    return result;  
}
```

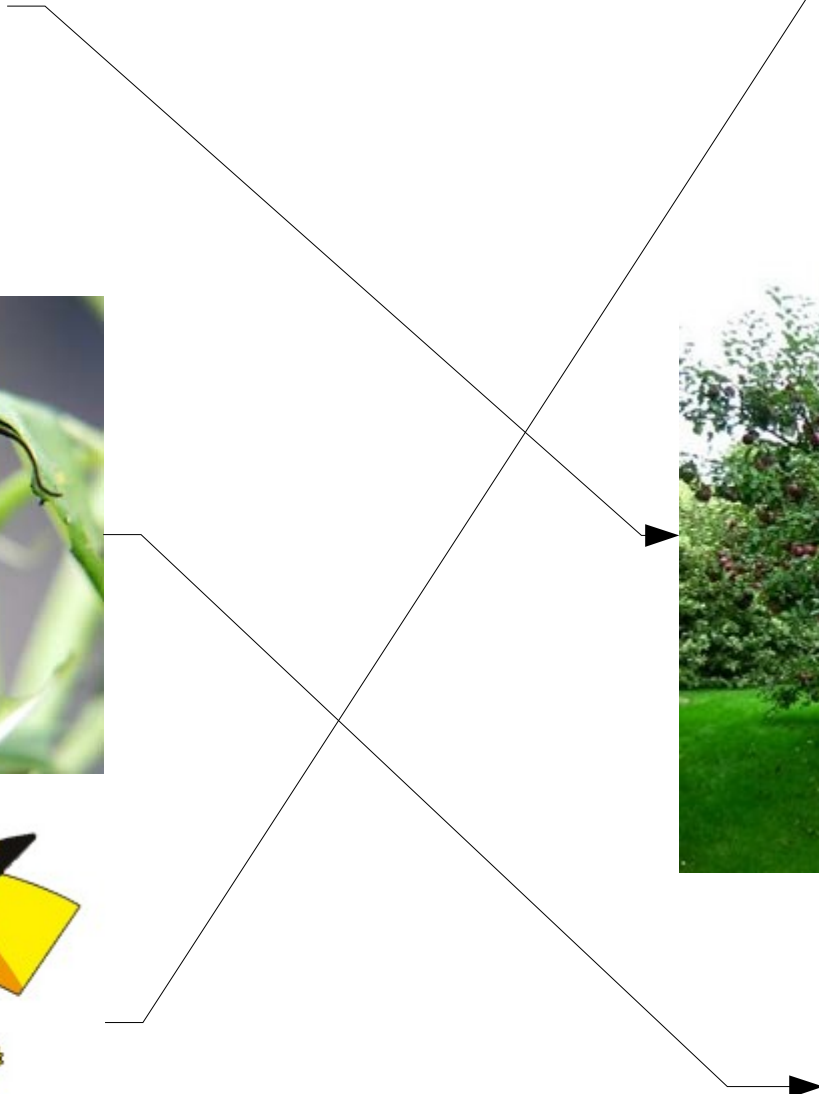
- What sorts of functions are we going to allow from a mathematical perspective?



Dikdik

Nubian
Ibex

Sloth



... but also ...

$$f(x) = x^2 + 3x - 15$$

$$f(n) = \begin{cases} -n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{otherwise} \end{cases}$$

Functions like these
are called ***piecewise
functions.***

To define a function, you will typically either

- draw a picture, or
- give a rule for determining the output.

In mathematics, functions are ***deterministic***.

That is, given the same input, a function must always produce the same output.

The following is a perfectly valid piece of C++ code, but it's not a valid function under our definition:

```
int randomNumber(int numOutcomes) {  
    return rand() % numOutcomes;  
}
```

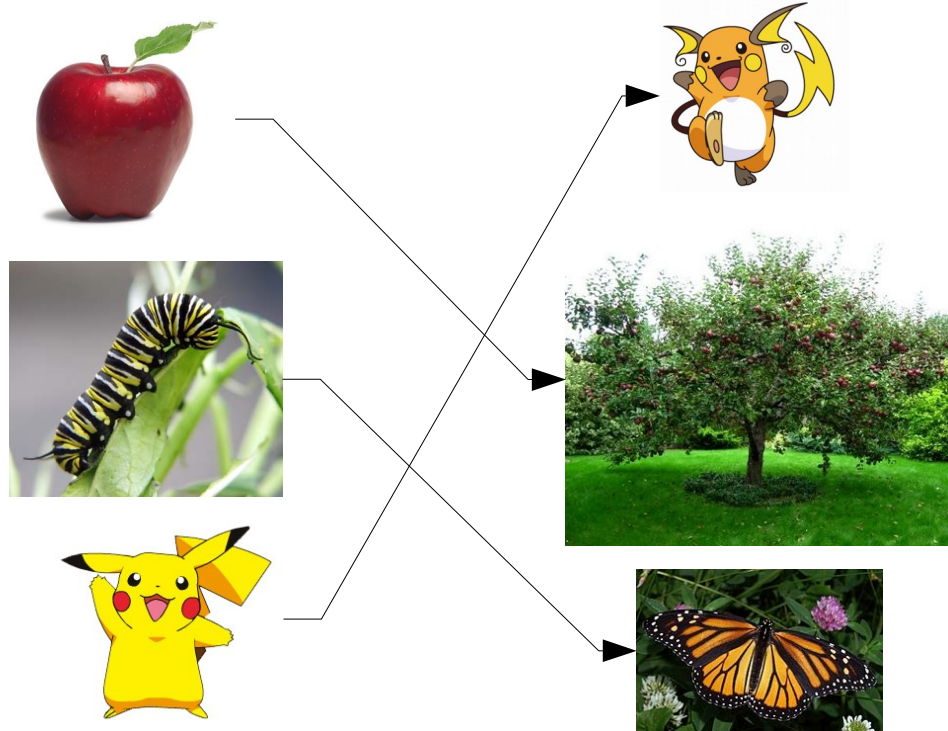
One Challenge

$$f(x) = x^2 + 2x + 5$$

$$f(3) = 3^2 + 3 \cdot 2 + 5 = 20$$

$$f(0) = 0^2 + 0 \cdot 2 + 5 = 5$$

$$f(\text{Pikachu}) = \dots ?$$



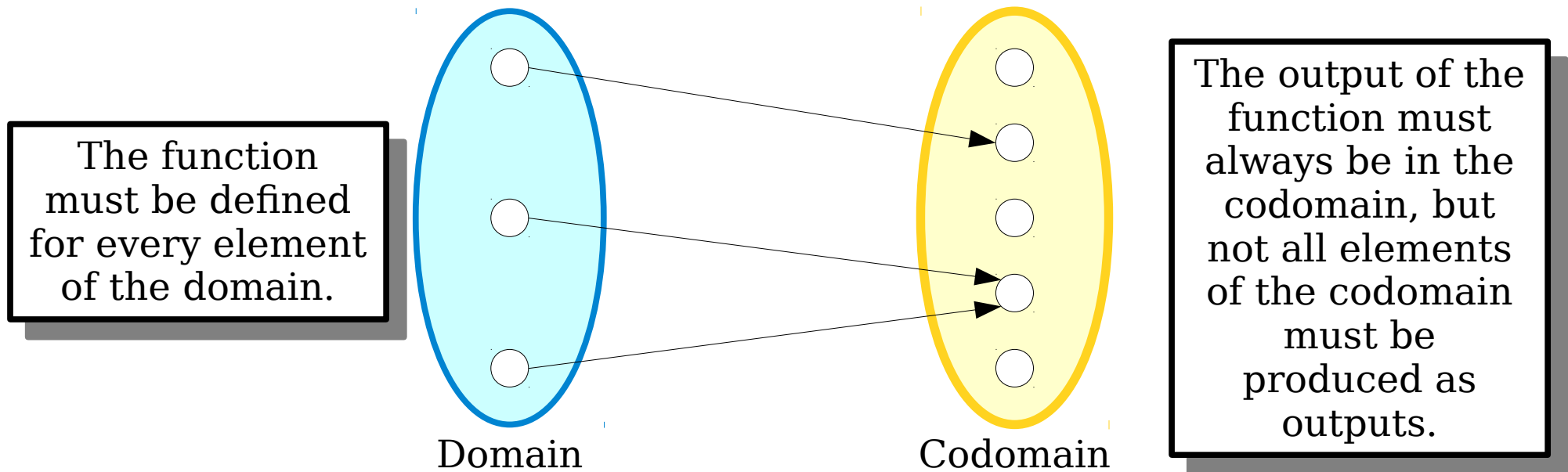
$$f(\text{Pikachu}) = \text{Flying Pikachu}$$

$$f(137) = \dots?$$

We need to make sure we can't apply functions to meaningless inputs.

Domains and Codomains

- Every function f has two sets associated with it: its **domain** and its **codomain**.
- A function f can only be applied to elements of its domain. For any x in the domain, $f(x)$ belongs to the codomain.



Domains and Codomains

- Every function f has two sets associated with it: its **domain** and its **codomain**.
- A function f can only be applied to elements of its domain. For any x in the domain, $f(x)$ belongs to the codomain.

The codomain of this function is \mathbb{R} . Everything produced is a real number, but not all real numbers can be produced.

The domain of this function is \mathbb{R} . Any real number can be provided as input.

```
private double absoluteValueOf(double x) {  
    if (x >= 0) {  
        return x;  
    } else {  
        return -x;  
    }  
}
```

Domains and Codomains

- If f is a function whose domain is A and whose codomain is B , we write $f : A \rightarrow B$.
- This notation just says what the domain and codomain of the function are. It doesn't say how the function is evaluated.
- Think of it like a “function prototype” in C or C++. The notation $f : ArgType \rightarrow RetType$ is like writing

$RetType$ $f(ArgType$ argument);

We know that f takes in an $ArgType$ and returns a $RetType$, but we don't know exactly which $RetType$ it's going to return for a given $ArgType$.

The Official Rules for Functions

- Formally speaking, we say that $f : A \rightarrow B$ if the following two rules hold.
- First, f must obey its domain/codomain rules:

$$\forall a \in A. \exists b \in B. f(a) = b$$

(“Every input in A maps to some output in B .”)

- Second, f must be deterministic:

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 = a_2 \rightarrow f(a_1) = f(a_2))$$

(“Equal inputs produce equal outputs.”)

- If you’re ever curious about whether something is a function, look back at these rules and check! For example:
 - Can a function have an empty domain?
 - Can a function with a nonempty domain have an empty codomain?

Defining Functions

- Typically, we specify a function by describing a rule that maps every element of the domain to some element of the codomain.
- Examples:
 - $f(n) = n + 1$, where $f : \mathbb{Z} \rightarrow \mathbb{Z}$
 - $f(x) = \sin x$, where $f : \mathbb{R} \rightarrow \mathbb{R}$
 - $f(x) = \lfloor x \rfloor$, where $f : \mathbb{R} \rightarrow \mathbb{Z}$
- Notice that we're giving both a rule and the domain/codomain.

Defining Functions

Typically, we specify a function by describing a rule that maps every element of the domain to some codomain.

Examples:

$$f(n) = n + 1, \text{ where } f : \mathbb{Z} \rightarrow \mathbb{Z}$$

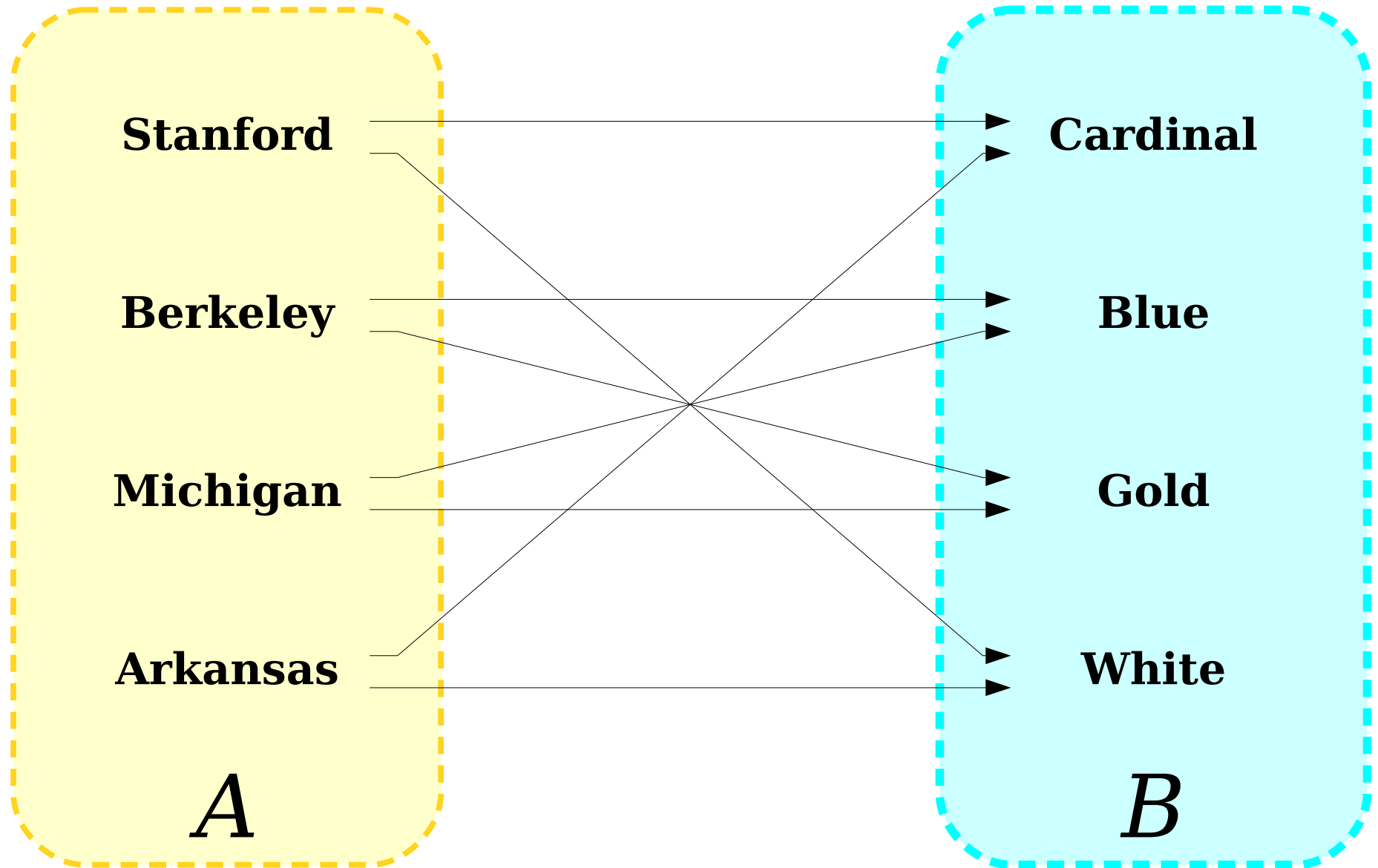
$$f(x) = \sin x, \text{ where } f : \mathbb{R} \rightarrow \mathbb{R}$$

- $f(x) = \lceil x \rceil, \text{ where } f : \mathbb{R} \rightarrow \mathbb{Z}$

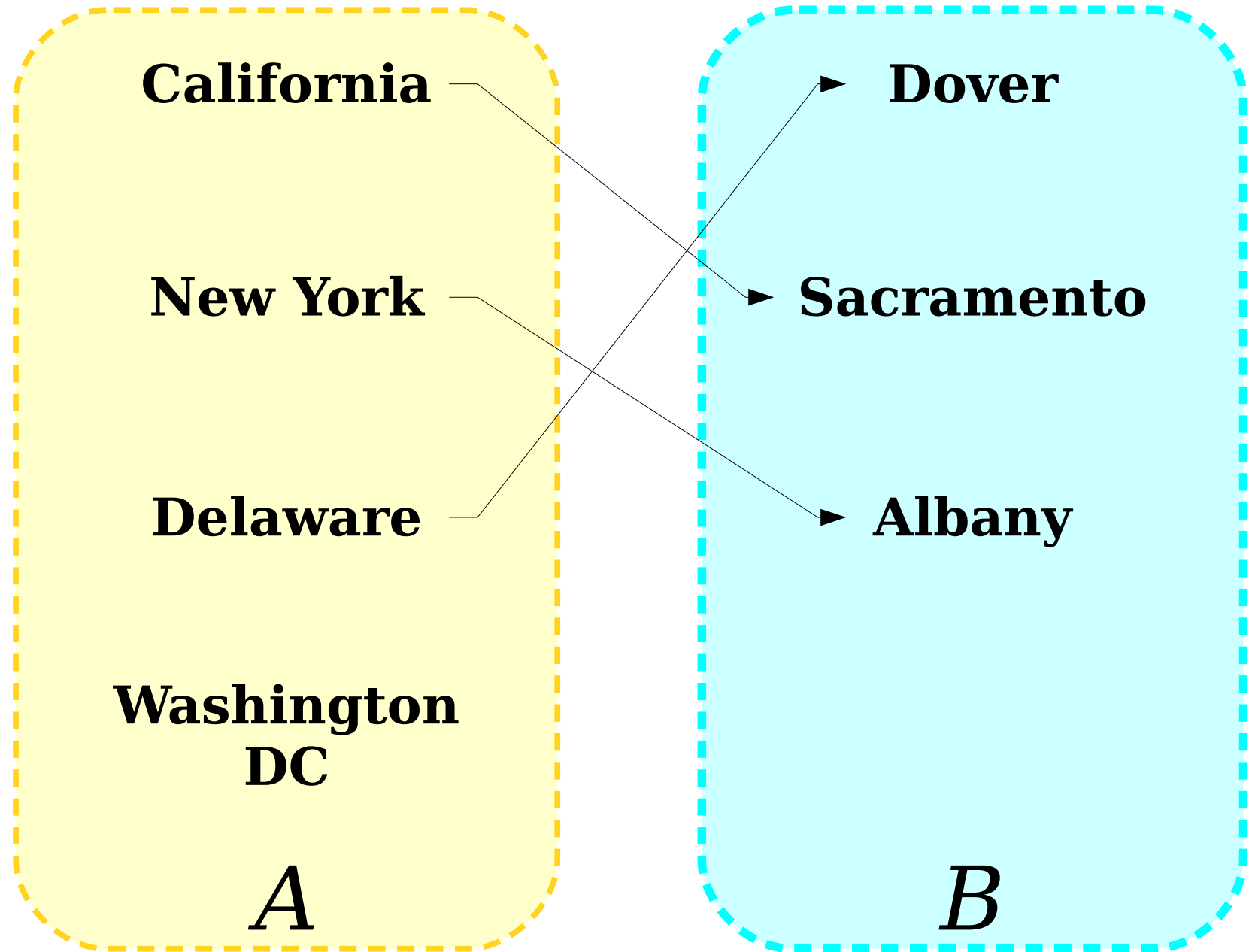
Notice that we're giving both a rule and the domain/codomain.

This is the ceiling function - the smallest integer greater than or equal to x . For example, $\lceil 1 \rceil = 1$, $\lceil 1.37 \rceil = 2$, and $\lceil \pi \rceil = 4$.

Is This a Function From A to B ?



Is This a Function From A to B ?



Is This a Function
From A to B ?

عيد الفطر

عيد الأضحى

A

مَحَرَّم

صَفَر

رَبِيعِ الأوَّل

رَبِيعِ الثَّانِي

جُمَادَى الأوَّلَى

جُمَادَى الآخِرَةَ

رَجَب

شَعْبَانَ

رَمَضَانَ

شَوَّال

ذُو القَعْدَةِ

ذُو الحِجَّةِ

B

Answer at PollEv.com/cs103 or
text **CS103** to **22333** once to join, then **Y** or **N**.

Combining Functions

f : People → Places

g : Places → Prices

Keith

Mountain View

Far Too Much

Guy

San Francisco

King's Ransom

Amy

Redding, CA

A Modest Amount

Chioma

Barrow, AK

Pocket Change

Shalom

Palo Alto

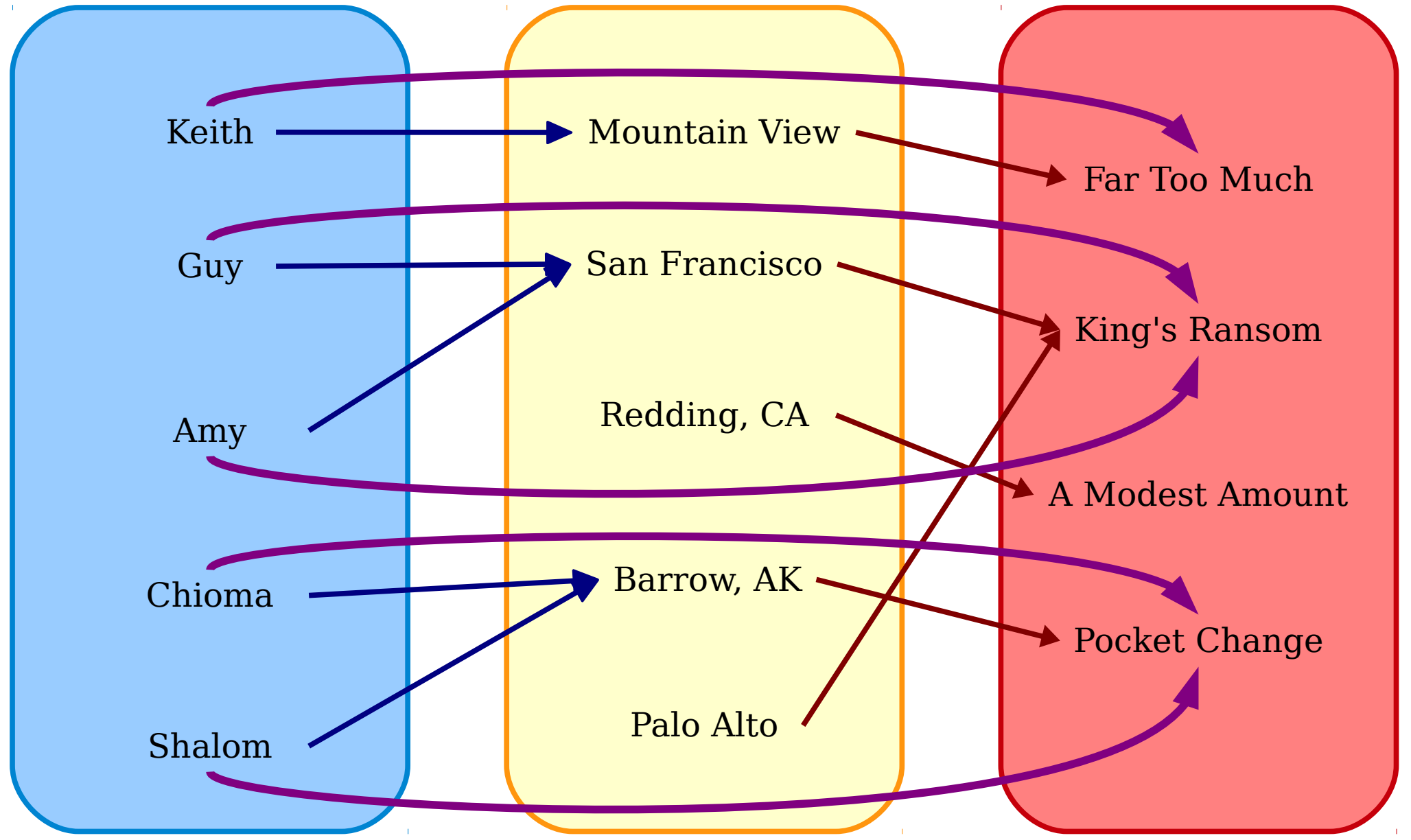
People

Places

Prices

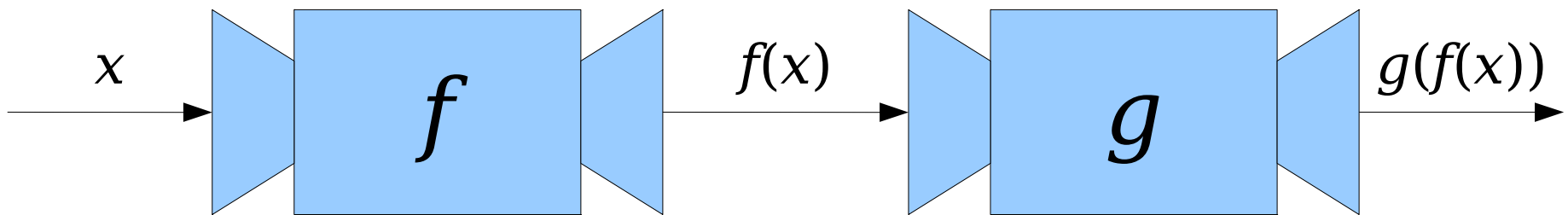
h : People → Prices

h(x) = g(f(x))



Function Composition

- Suppose that we have two functions $f : A \rightarrow B$ and $g : B \rightarrow C$.
- Notice that the codomain of f is the domain of g . This means that we can use outputs from f as inputs to g .



Function Composition

- Suppose that we have two functions $f : A \rightarrow B$ and $g : B \rightarrow C$.
- The **composition of f and g** , denoted $g \circ f$, is a function where
 - $g \circ f : A \rightarrow C$, and
 - $(g \circ f)(x) = g(f(x))$.
- A few things to notice:
 - The domain of $g \circ f$ is the domain of f . Its codomain is the codomain of g .
 - Even though the composition is written $g \circ f$, when evaluating $(g \circ f)(x)$, the function f is evaluated first.

The name of the function is $g \circ f$.
When we apply it to an input x ,
we write $(g \circ f)(x)$. I don't know
why, but that's what we do.

Time-Out for Announcements!

Problem Set Three

- The Problem Set Three checkpoint problem was due at 2:30PM today.
 - We'll aim to get feedback to you by Wednesday.
 - Solutions are now available.
- The remaining problems are due on Friday at 2:30PM.
- As always, feel free to ask questions on Piazza or to stop by office hours with questions!
- PS2 solutions are now available. We'll get your work graded and returned by Wednesday.



Info Session:
Friday, February 2nd
5PM - 6PM, at the **WCC**.
There will be boba!
[RSVP here.](#)



APPLY TO BE A
MENTOR BY 2/18

ONE DAY WORKSHOP
TEACH WHAT YOU KNOW AND LOVE
MINIMUM CO-REQUISITE: CS106A

APPLY HERE: [HTTP://BIT.LY/GTGTCMENTOR2018](http://bit.ly/gtgtcmentor2018)

Midterm Exam Logistics

- Our first midterm exam is next **Monday, February 5th**, from **7:00PM - 10:00PM**. Locations are divvied up by last (family) name:
 - A - H: Go to Cubberley Auditorium.
 - I - Z: Go to 320-105.
- You're responsible for Lectures 00 - 05 and topics covered in PS1 - PS2. Later lectures (relations forward) and problem sets (PS3 onward) won't be tested here.
- The exam is closed-book, closed-computer, and limited-note. You can bring a double-sided, 8.5" × 11" sheet of notes with you to the exam, decorated however you'd like.
- Students with OAE accommodations: please contact us **immediately** if you haven't yet done so. We'll ping you about setting up alternate exams.

Midterm Exam

- ***We want you to do well on this exam.*** We're not trying to weed out weak students. We're not trying to enforce a curve where there isn't one. We want you to show what you've learned up to this point so that you get a sense for where you stand and where you can improve.
- The purpose of this midterm is to give you a chance to show what you've learned in the past few weeks. It is not designed to assess your “mathematical potential” or “innate mathematical ability.”

Practice Midterm Exam

- To help you prepare for the midterm, we'll be holding a practice midterm exam on **Wednesday, January 31st** from **7PM - 10PM** in **Cemex Auditorium**.
 - The exam we'll use isn't one of the ones posted up on the course website, so feel free to use those as practice in the meantime.
- The practice midterm exam is an actual midterm we gave out in a previous quarter. It's probably the best indicator of what you should expect to see.
- Course staff will be on hand to answer your questions.
- Can't make it? We'll release that practice exam and solutions online. Set up your own practice exam time with a small group and work through it under realistic conditions!

Extra Practice Problems

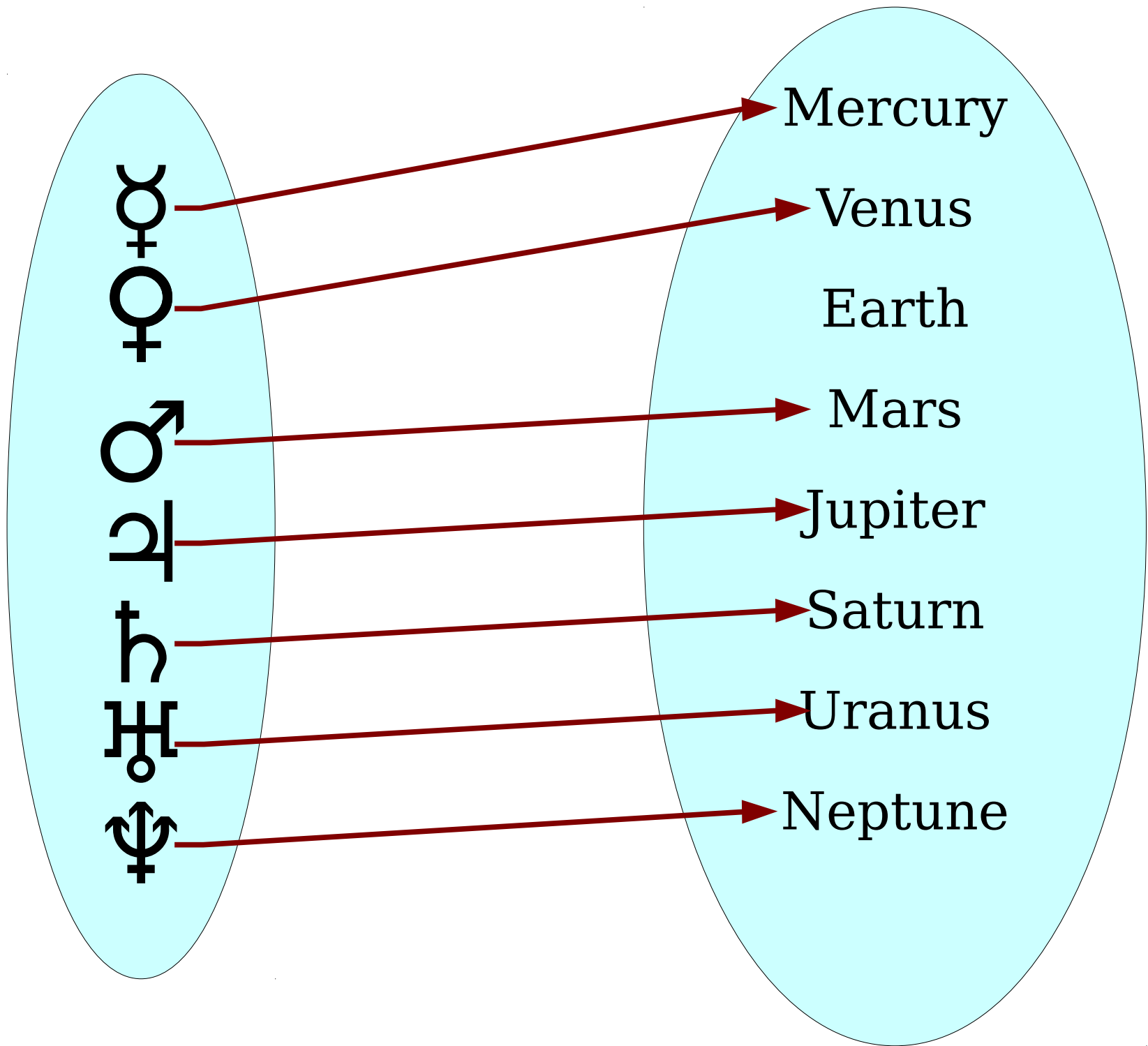
- Up on the course website, you'll find
 - Extra Practice Problems 1 (a set of cumulative review problems), and
 - three practice midterm exams, each of which is a (slightly modified) version of a real exam we've given out in a previous quarter.
- ***Use these resources strategically.*** Give these problems your best effort, and, importantly, ***have the course staff review your work.*** Ask for polite but honest feedback. 😊

Preparing for the Exam

- We've released a handout (Handout 21) containing advice about how to prepare for the exam, along with advice from previous CS103 students.
- Read over it... there's good advice there!

Back to CS103!

Special Types of Functions



Injective Functions

- A function $f : A \rightarrow B$ is called **injective** (or **one-to-one**) if the following statement is true about f :

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2))$$

(“If the inputs are different, the outputs are different.”)

- The following first-order definition is equivalent and is often useful in proofs.

$$\forall a_1 \in A. \forall a_2 \in A. (f(a_1) = f(a_2) \rightarrow a_1 = a_2)$$

(“If the outputs are the same, the inputs are the same.”)

- A function with this property is called an **injection**.
- How does this compare to our second rule for functions?

Injective Functions

Theorem: Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $f(n) = 2n + 7$.
Then f is injective.

Proof:

How many of the following are correct ways of starting off this proof?

Consider any $n_1, n_2 \in \mathbb{N}$ where $n_1 = n_2$. We will prove that $f(n_1) = f(n_2)$.

Consider any $n_1, n_2 \in \mathbb{N}$ where $n_1 \neq n_2$. We will prove that $f(n_1) \neq f(n_2)$.

Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$. We will prove that $n_1 = n_2$.

Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) \neq f(n_2)$. We will prove that $n_1 \neq n_2$.

Answer at [Pollevo.com/cs103](https://www.pollevo.com/cs103) or
text **CS103** to **22333** once to join, then a number between **0** and **4**.

Injective Functions

Theorem: Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $f(n) = 2n + 7$.
Then f is injective.

Proof:

What does it mean for the function f to be injective?

$$\forall n_1 \in \mathbb{N}. \forall n_2 \in \mathbb{N}. (f(n_1) = f(n_2) \rightarrow n_1 = n_2)$$

$$\forall n_1 \in \mathbb{N}. \forall n_2 \in \mathbb{N}. (n_1 \neq n_2 \rightarrow f(n_1) \neq f(n_2))$$

Therefore, we'll pick arbitrary $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$, then prove that $n_1 = n_2$.

Injective Functions

Theorem: Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $f(n) = 2n + 7$.
Then f is injective.

Proof: Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$. We will prove that $n_1 = n_2$.

Since $f(n_1) = f(n_2)$, we see that

$$2n_1 + 7 = 2n_2 + 7.$$

This in turn means that

$$2n_1 = 2n_2$$

so $n_1 = n_2$, as required. ■

Injective Functions

Theorem: Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $f(n) = 2n + 7$.
Then f is injective.

Proof: Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$. We will prove that $n_1 = n_2$.

How many of the following are correct ways of starting off this proof?

Consider any $n_1, n_2 \in \mathbb{N}$ where $n_1 = n_2$. We will prove that $f(n_1) = f(n_2)$.

Consider any $n_1, n_2 \in \mathbb{N}$ where $n_1 \neq n_2$. We will prove that $f(n_1) \neq f(n_2)$.

Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$. We will prove that $n_1 = n_2$.

Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) \neq f(n_2)$. We will prove that $n_1 \neq n_2$.

Good exercise: Repeat this proof using the other definition of injectivity!

Injective Functions

Theorem: Let $f : \mathbb{Z} \rightarrow \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof:

How many of the following are correct ways of starting off this proof?

Assume for the sake of contradiction that f is not injective.

Assume for the sake of contradiction that there are integers x_1 and x_2 where $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Consider arbitrary integers x_1 and x_2 where $x_1 \neq x_2$. We will prove that $f(x_1) = f(x_2)$.

Consider arbitrary integers x_1 and x_2 where $f(x_1) = f(x_2)$. We will prove that $x_1 \neq x_2$.

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Injective Functions

Theorem: Let $f : \mathbb{Z} \rightarrow \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof:

What does it mean for f to be injective?

$$\forall x_1 \in \mathbb{Z}. \forall x_2 \in \mathbb{Z}. (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$$

What is the negation of this statement?

$$\neg \forall x_1 \in \mathbb{Z}. \forall x_2 \in \mathbb{Z}. (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$$

$$\exists x_1 \in \mathbb{Z}. \neg \forall x_2 \in \mathbb{Z}. (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$$

$$\exists x_1 \in \mathbb{Z}. \exists x_2 \in \mathbb{Z}. \neg (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$$

$$\exists x_1 \in \mathbb{Z}. \exists x_2 \in \mathbb{Z}. (x_1 \neq x_2 \wedge \neg (f(x_1) \neq f(x_2)))$$

$$\exists x_1 \in \mathbb{Z}. \exists x_2 \in \mathbb{Z}. (x_1 \neq x_2 \wedge f(x_1) = f(x_2))$$

Therefore, we need to find $x_1, x_2 \in \mathbb{Z}$ such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

Can we do that?

Injective Functions

Theorem: Let $f : \mathbb{Z} \rightarrow \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof: We will prove that there exist integers x_1 and x_2 such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

Let $x_1 = -1$ and $x_2 = +1$.

$$f(x_1) = f(-1) = (-1)^4 = 1$$

and

$$f(x_2) = f(1) = 1^4 = 1,$$

so $f(x_1) = f(x_2)$ even though $x_1 \neq x_2$, as required. ■

Injective Functions

Theorem: Let $f : \mathbb{Z} \rightarrow \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof: We will prove that there exist integers x_1 and x_2 such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

How many of the following are correct ways of starting off this proof?

Assume for the sake of contradiction that f is not injective.

Assume for the sake of contradiction that there are integers x_1 and x_2 where $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Consider arbitrary integers x_1 and x_2 where $x_1 \neq x_2$. We will prove that $f(x_1) = f(x_2)$.

Consider arbitrary integers x_1 and x_2 where $f(x_1) = f(x_2)$. We will prove that $x_1 \neq x_2$.

Injections and Composition

Injections and Composition

- **Theorem:** If $f : A \rightarrow B$ is an injection and $g : B \rightarrow C$ is an injection, then the function $g \circ f : A \rightarrow C$ is an injection.
- Our goal will be to prove this result. To do so, we're going to have to call back to the formal definitions of injectivity and function composition.

Theorem: If $f : A \rightarrow B$ is an injection and $g : B \rightarrow C$ is an injection, then the function $g \circ f : A \rightarrow C$ is also an injection.

Proof: Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be arbitrary injections. We will prove that the function $g \circ f : A \rightarrow C$ is also injective.

There are two definitions of injectivity that we can use here:

$$\forall a_1 \in A. \forall a_2 \in A. ((g \circ f)(a_1) = (g \circ f)(a_2) \rightarrow a_1 = a_2)$$

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow (g \circ f)(a_1) \neq (g \circ f)(a_2))$$

Therefore, we'll choose an arbitrary $a_1, a_2 \in A$ where $a_1 \neq a_2$, then prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$.

Theorem: If $f : A \rightarrow B$ is an injection and $g : B \rightarrow C$ is an injection, then the function $g \circ f : A \rightarrow C$ is also an injection.

Proof: Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be arbitrary injections. We will prove that the function $g \circ f : A \rightarrow C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We will prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. Equivalently, we need to show that $g(f(a_1)) \neq g(f(a_2))$.

How is $(g \circ f)(x)$ defined?

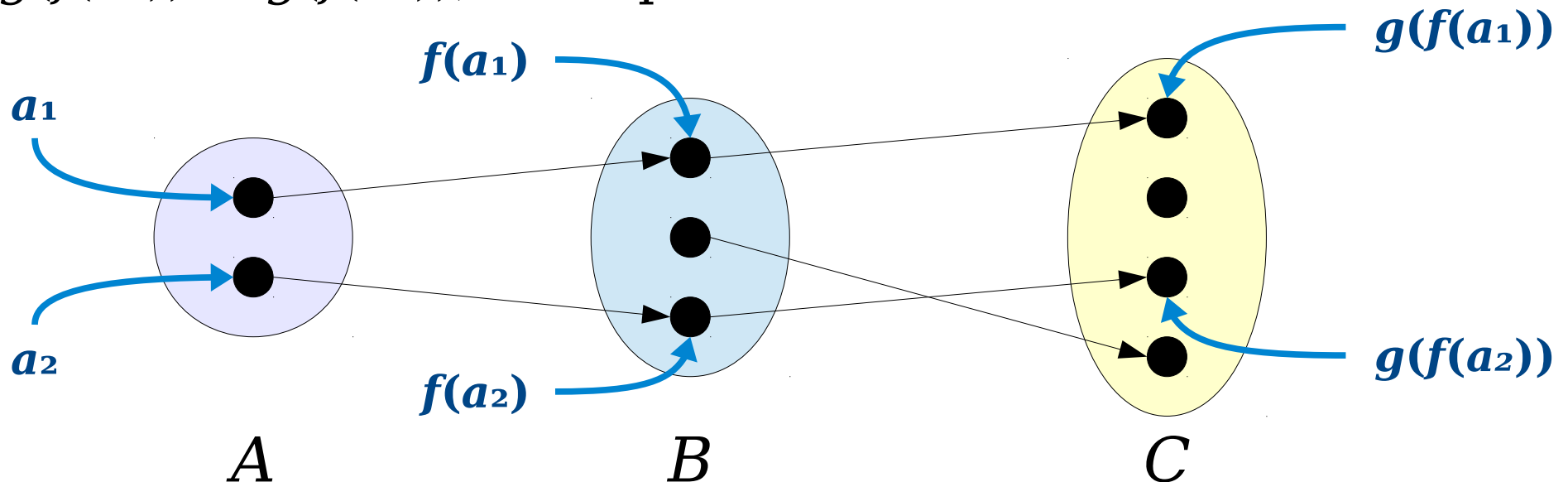
$$(g \circ f)(x) = g(f(x))$$

So we need to prove that $g(f(a_1)) \neq g(f(a_2))$.

Theorem: If $f : A \rightarrow B$ is an injection and $g : B \rightarrow C$ is an injection, then the function $g \circ f : A \rightarrow C$ is also an injection.

Proof: Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be arbitrary injections. We will prove that the function $g \circ f : A \rightarrow C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We will prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. Equivalently, we need to show that $g(f(a_1)) \neq g(f(a_2))$.

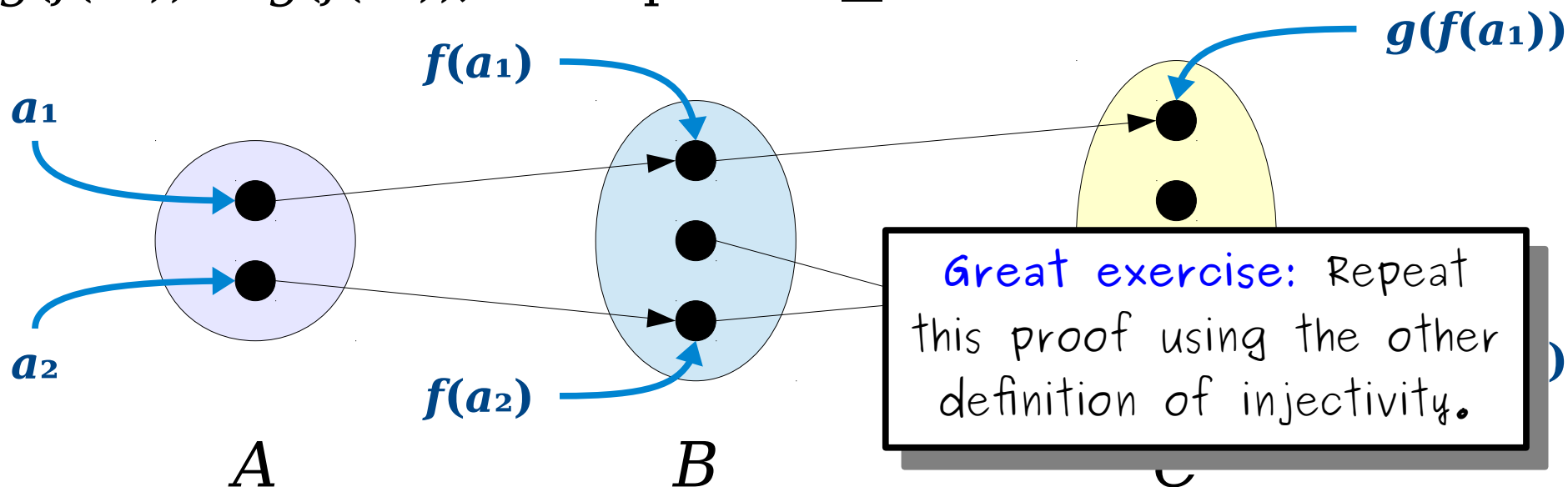
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■



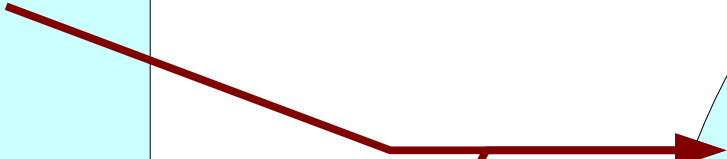
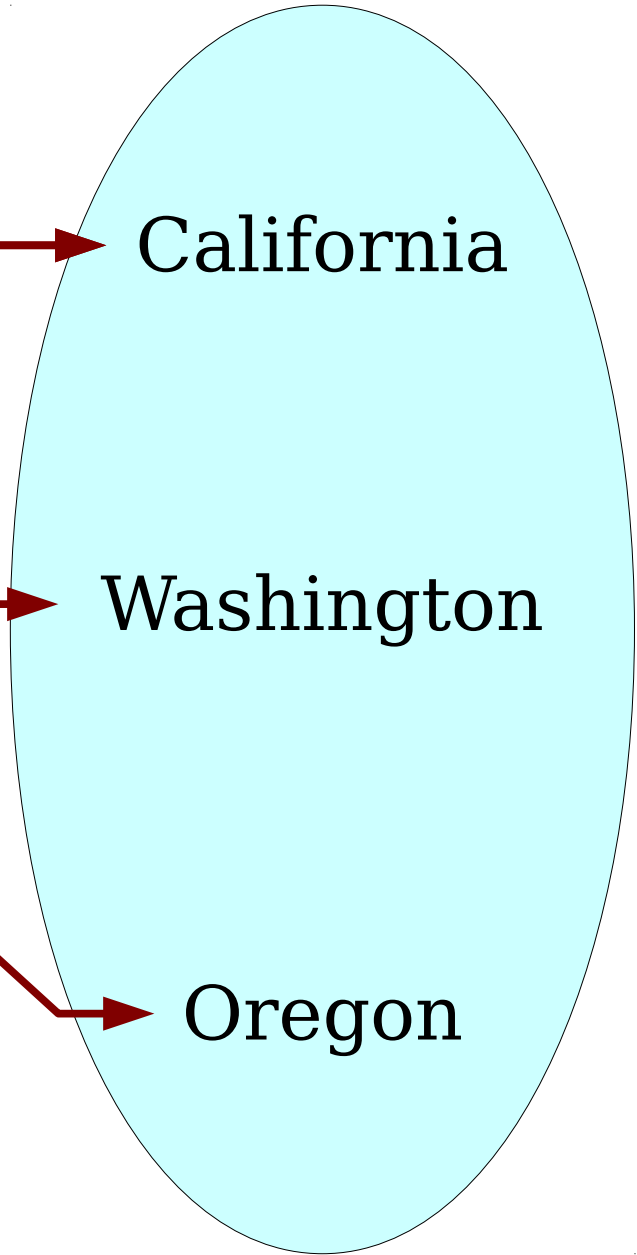
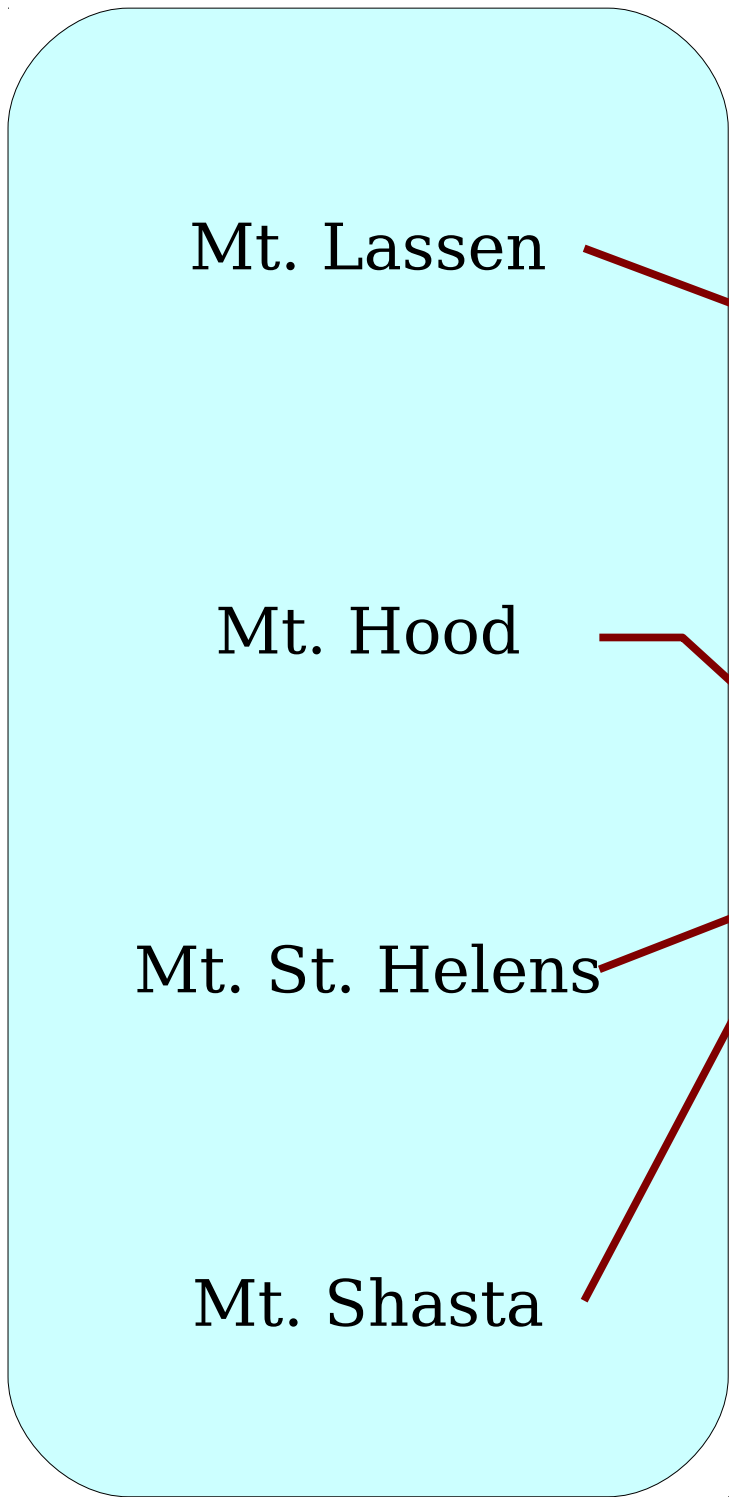
Theorem: If $f : A \rightarrow B$ is an injection and $g : B \rightarrow C$ is an injection, then the function $g \circ f : A \rightarrow C$ is also an injection.

Proof: Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be arbitrary injections. We will prove that the function $g \circ f : A \rightarrow C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We will prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. Equivalently, we need to show that $g(f(a_1)) \neq g(f(a_2))$.

Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■



Another Class of Functions



Surjective Functions

- A function $f : A \rightarrow B$ is called **surjective** (or **onto**) if this first-order logic statement is true about f :

$$\forall b \in B. \exists a \in A. f(a) = b$$

(“For every possible output, there's at least one possible input that produces it”)

- A function with this property is called a **surjection**.
- How does this compare to our first rule of functions?

Surjective Functions

Theorem: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = x / 2$. Then $f(x)$ is surjective.

Proof:

What does it mean for f to be surjective?

$$\forall y \in \mathbb{R}. \exists x \in \mathbb{R}. f(x) = y$$

Therefore, we'll choose an arbitrary $y \in \mathbb{R}$, then prove that there is some $x \in \mathbb{R}$ where $f(x) = y$.

Surjective Functions

Theorem: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = x / 2$. Then $f(x)$ is surjective.

Proof: Consider any $y \in \mathbb{R}$. We will prove that there is a choice of $x \in \mathbb{R}$ such that $f(x) = y$.

Let $x = 2y$. Then we see that

$$f(x) = f(2y) = 2y / 2 = y.$$

So $f(x) = y$, as required. ■

Composing Surjections

Theorem: If $f : A \rightarrow B$ is surjective and $g : B \rightarrow C$ is surjective, then $g \circ f : A \rightarrow C$ is also surjective.

Proof: Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be arbitrary surjections. We will prove that the function $g \circ f : A \rightarrow C$ is also surjective.

What does it mean for $g \circ f : A \rightarrow C$ to be surjective?

$$\forall c \in C. \exists a \in A. (g \circ f)(a) = c$$

Therefore, we'll choose arbitrary $c \in C$ and prove that there is some $a \in A$ such that $(g \circ f)(a) = c$.

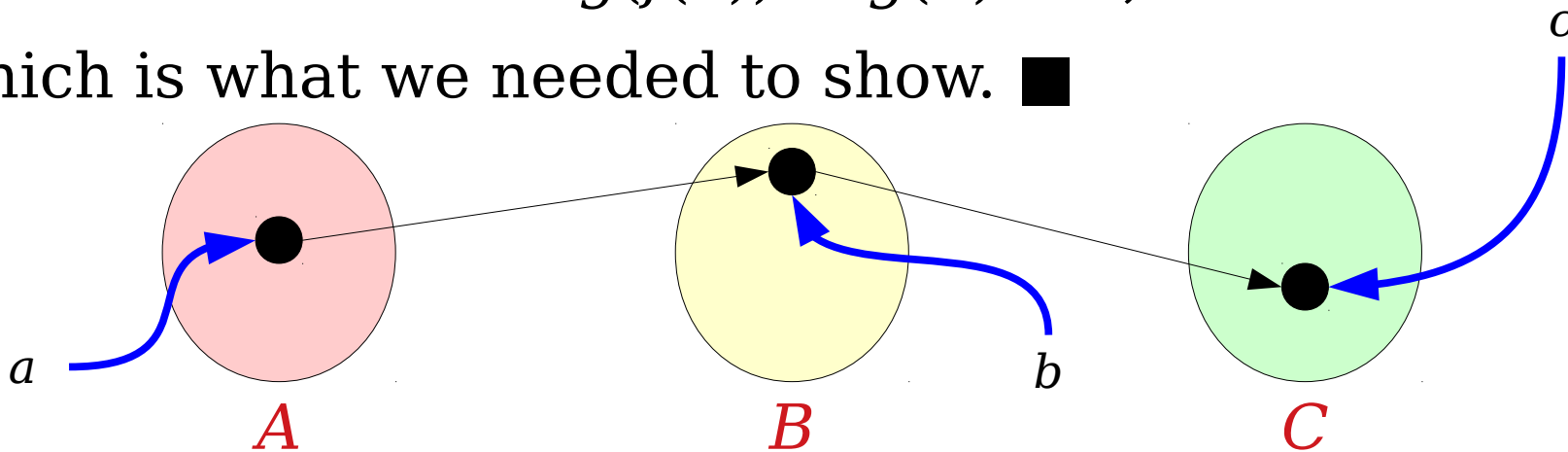
Theorem: If $f : A \rightarrow B$ is surjective and $g : B \rightarrow C$ is surjective, then $g \circ f : A \rightarrow C$ is also surjective.

Proof: Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be arbitrary surjections. We will prove that the function $g \circ f : A \rightarrow C$ is also surjective. To do so, we will prove that for any $c \in C$, there is some $a \in A$ such that $(g \circ f)(a) = c$. Equivalently, we will prove that for any $c \in C$, there is some $a \in A$ such that $g(f(a)) = c$.

Consider any $c \in C$. Since $g : B \rightarrow C$ is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since $f : A \rightarrow B$ is surjective, there is some $a \in A$ such that $f(a) = b$. This means that there is some $a \in A$ such that

$$g(f(a)) = g(b) = c,$$

which is what we needed to show. ■



Injections and Surjections

- An injective function associates *at most* one element of the domain with each element of the codomain.
- A surjective function associates *at least* one element of the domain with each element of the codomain.
- What about functions that associate *exactly one* element of the domain with each element of the codomain?



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Bijections

- A function that associates each element of the codomain with a unique element of the domain is called ***bijjective***.
 - Such a function is a ***bijection***.
- Formally, a bijection is a function that is both *injective* and *surjective*.
- Bijections are sometimes called ***one-to-one correspondences***.
 - Not to be confused with “one-to-one functions.”

Bijections and Composition

- Suppose that $f : A \rightarrow B$ and $g : B \rightarrow C$ are bijections.
- Is $g \circ f$ necessarily a bijection?
- **Yes!**
 - Since both f and g are injective, we know that $g \circ f$ is injective.
 - Since both f and g are surjective, we know that $g \circ f$ is surjective.
 - Therefore, $g \circ f$ is a bijection.

Inverse Functions



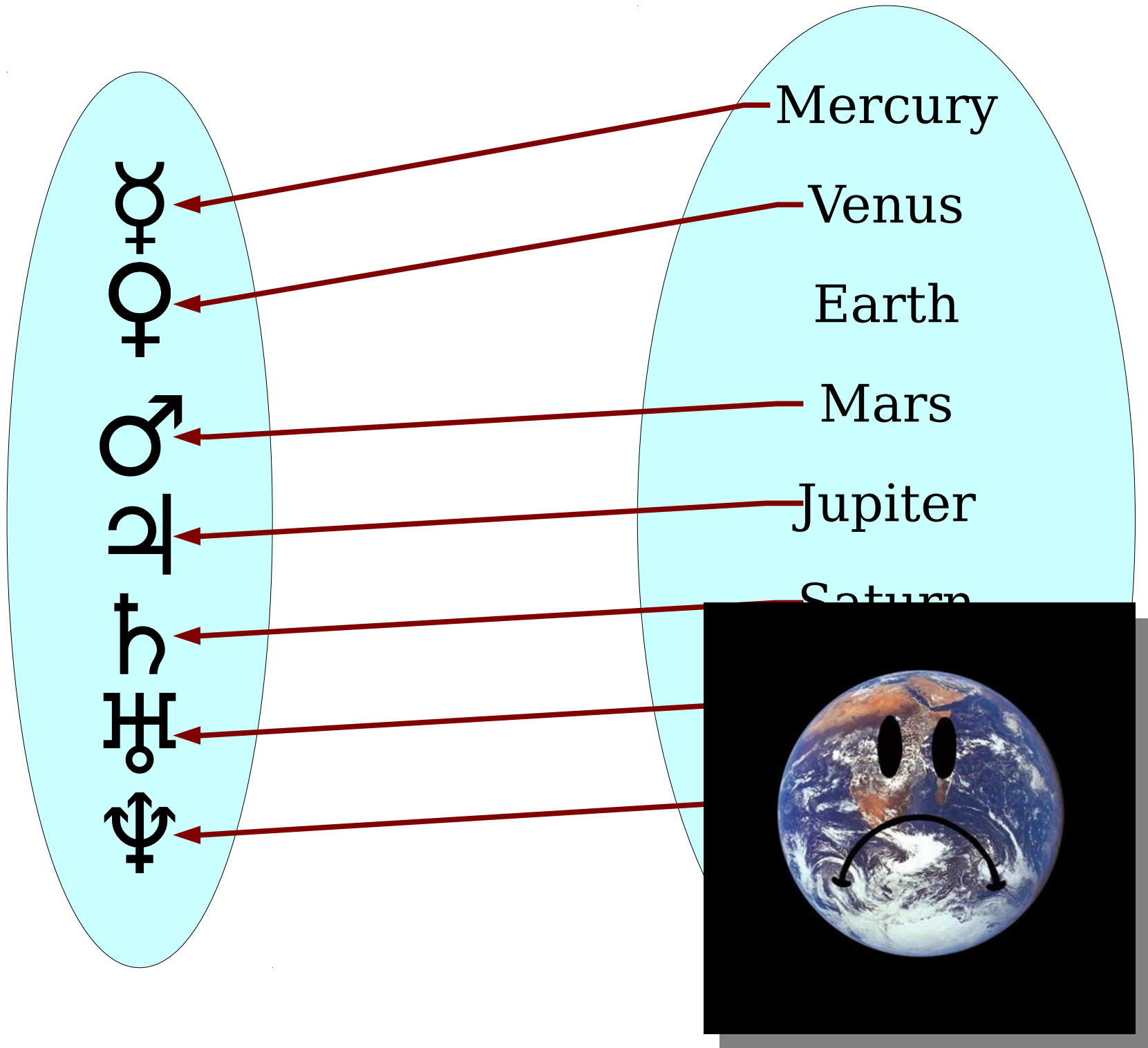
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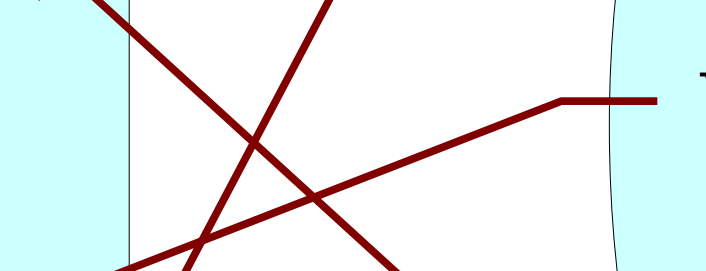
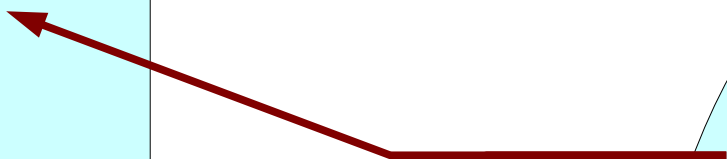
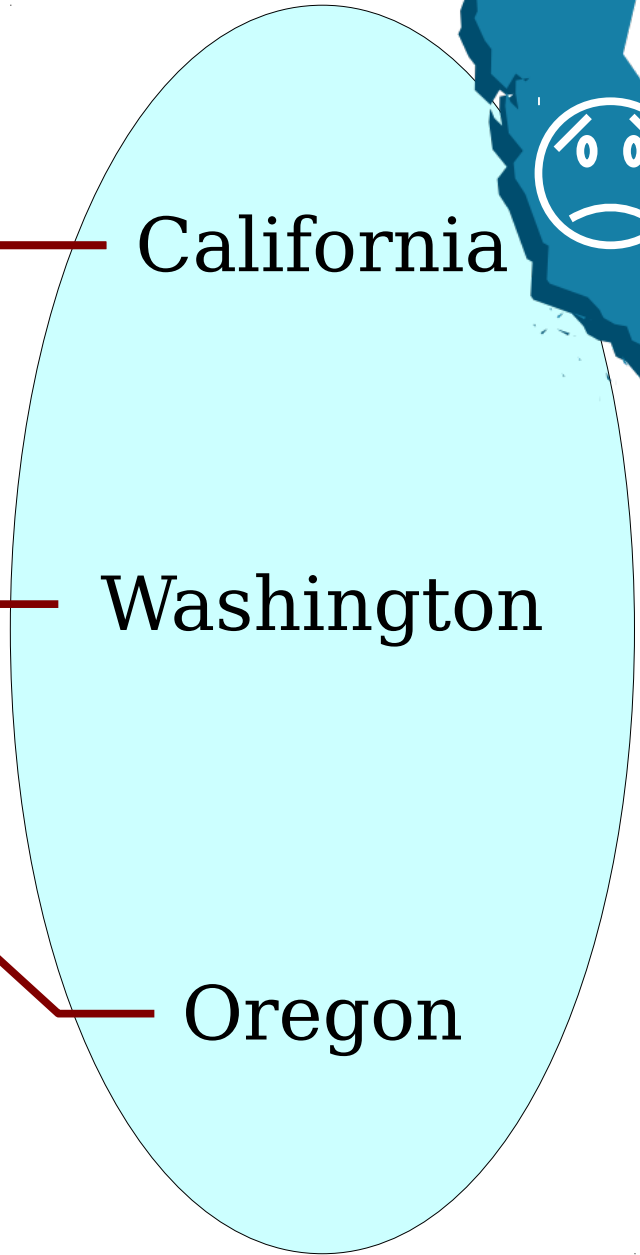
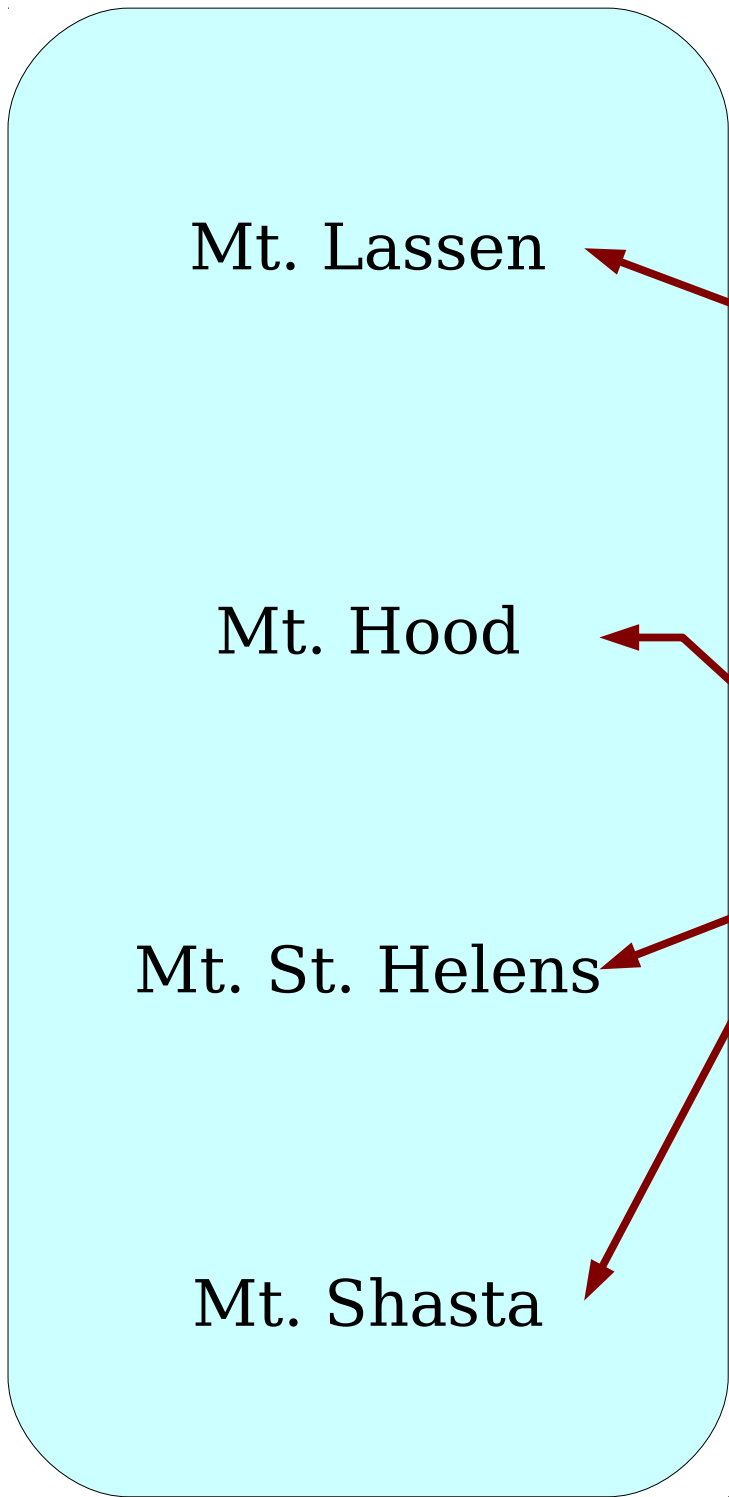


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Inverse Functions

- In some cases, it's possible to “turn a function around.”
- Let $f : A \rightarrow B$ be a function. A function $f^{-1} : B \rightarrow A$ is called an **inverse of f** if the following first-order logic statements are true about f and f^{-1}

$$\forall a \in A. (f^{-1}(f(a)) = a) \quad \forall b \in B. (f(f^{-1}(b)) = b)$$

- In other words, if f maps a to b , then f^{-1} maps b back to a and vice-versa.
- Not all functions have inverses (we just saw a few examples of functions with no inverses).
- If f is a function that has an inverse, then we say that f is **invertible**.

Inverse Functions

- ***Theorem:*** Let $f : A \rightarrow B$. Then f is invertible if and only if f is a bijection.
- These proofs are in the course reader. Feel free to check them out if you'd like!
- ***Really cool observation:*** Look at the formal definition of a function. Look at the rules for injectivity and surjectivity. Do you see why this result makes sense?

Where We Are

- We now know
 - what an injection, surjection, and bijection are;
 - that the composition of two injections, surjections, or bijections is also an injection, surjection, or bijection, respectively; and
 - that bijections are invertible and invertible functions are bijections.
- You might wonder why this all matters. Well, there's a good reason...

Next Time

- ***Cardinality, Formally***
 - How do we rigorously define the idea that two sets have the same size?
- ***The Nature of Infinity***
 - It's even weirder than you think!
- ***Cantor's Theorem Revisited***
 - A formal proof of a major result!