Mathematical Induction

Part One

Everybody – do the wave!

The Wave

- If done properly, everyone will eventually end up joining in.
- Why is that?
  - Someone (me!) started everyone off.
  - Once the person before you did the wave, you did the wave.

Let P be some predicate. The ***principle of mathematical induction*** states that if

If it starts true…

$$P(0) \text{ is true}$$

…and it stays true…

and

$$\forall k \in \mathbb{N}.\ (P(k) \to P(k+1))$$

then

$$\forall n \in \mathbb{N}.\ P(n)$$

…then it's always true.

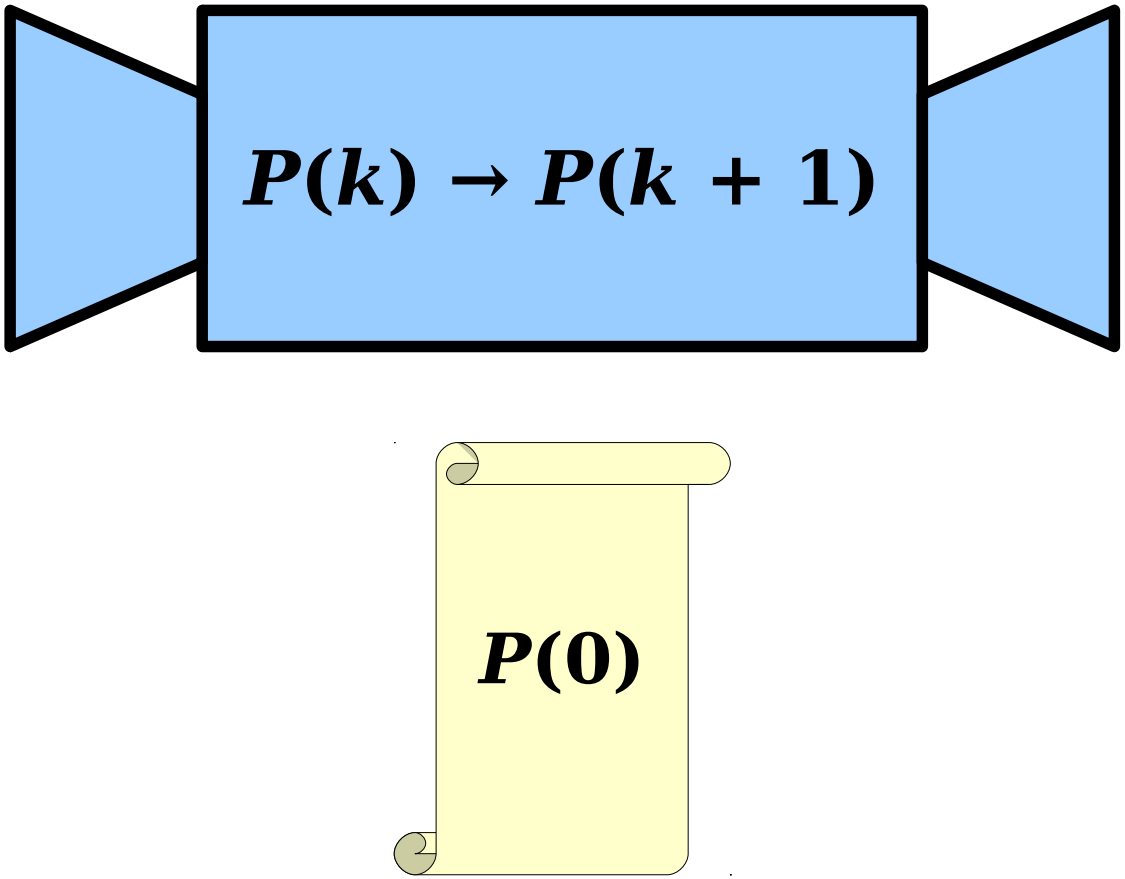
Induction, Intuitively

$$P(0)$$

$$\forall k \in \mathbb{N}.\ (P(k) \to P(k+1))$$

- It's true for 0.
- Since it's true for 0, it's true for 1.
- Since it's true for 1, it's true for 2.
- Since it's true for 2, it's true for 3.
- Since it's true for 3, it's true for 4.
- Since it's true for 4, it's true for 5.
- Since it's true for 5, it's true for 6.
- …

Why Induction Works



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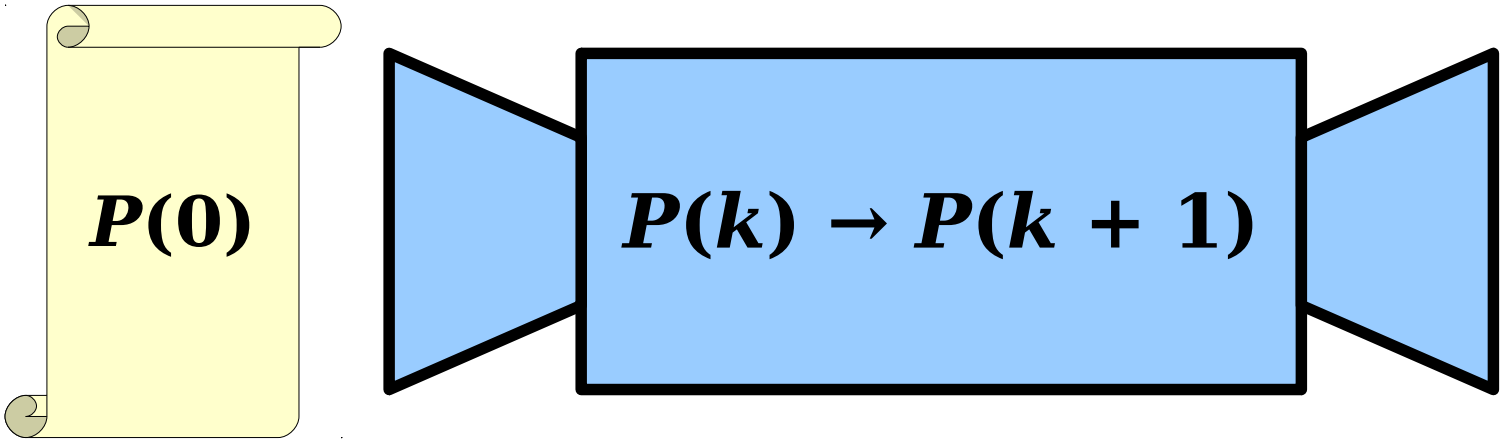

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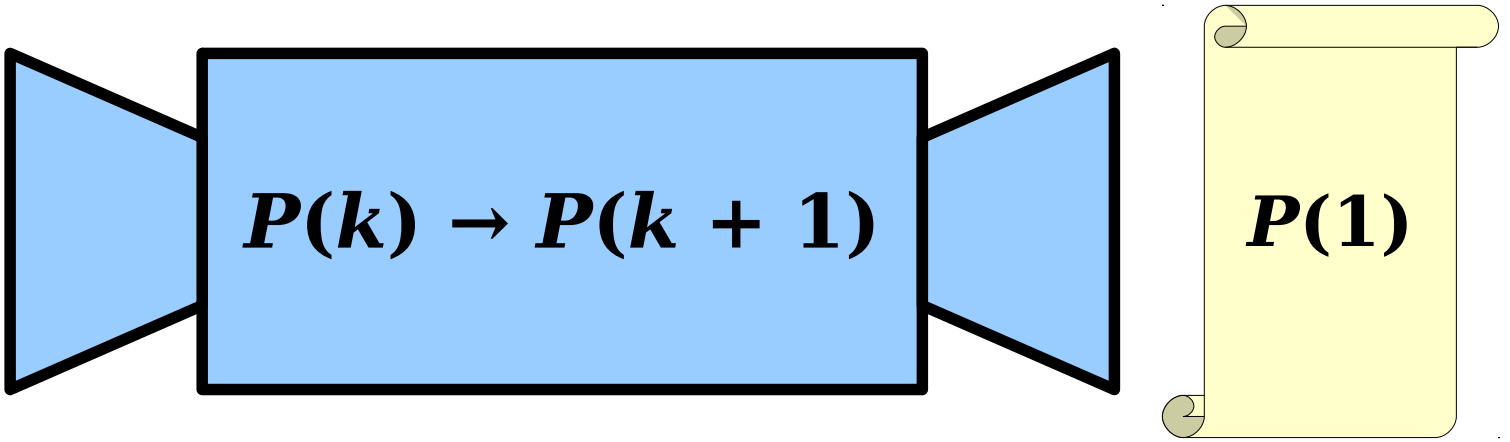

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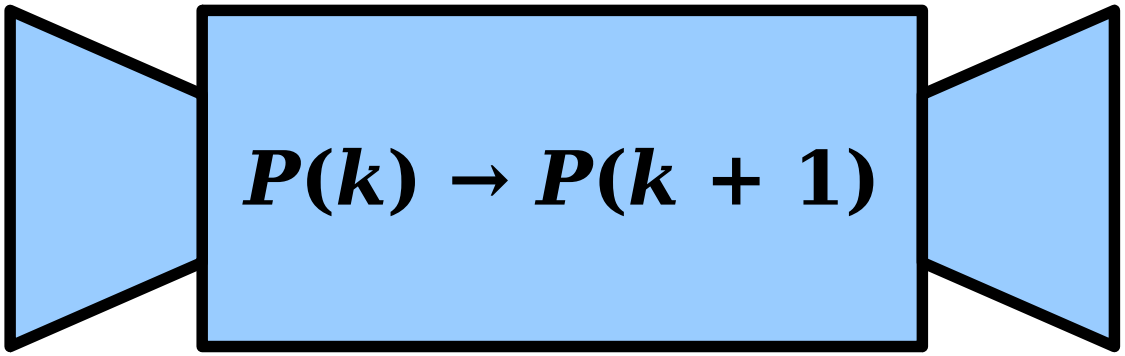

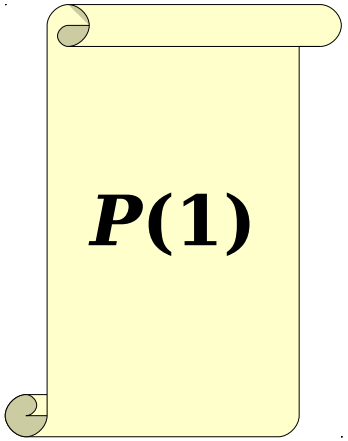

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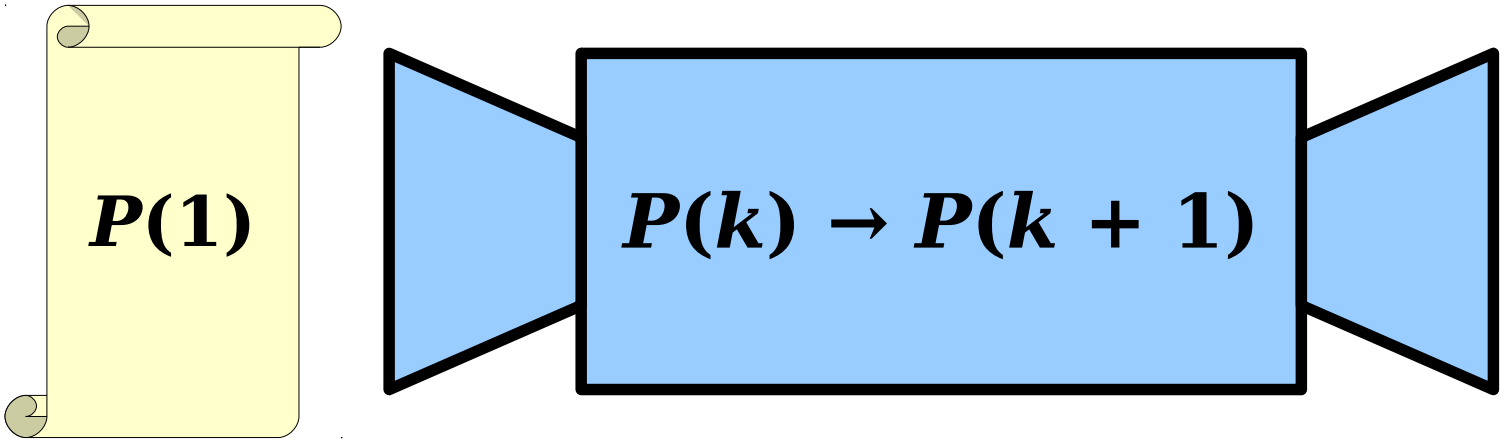

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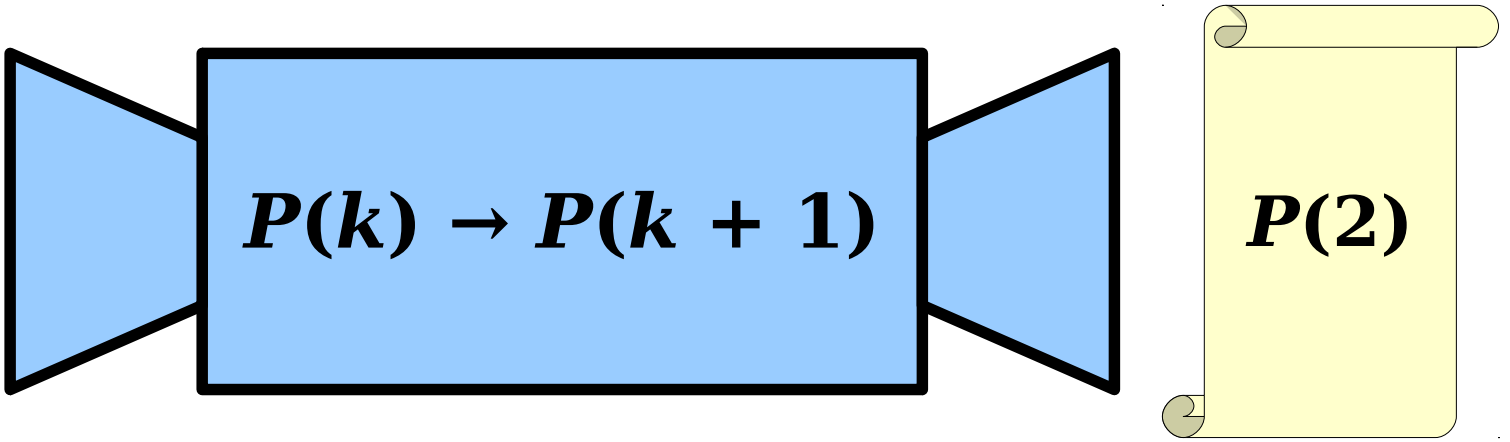

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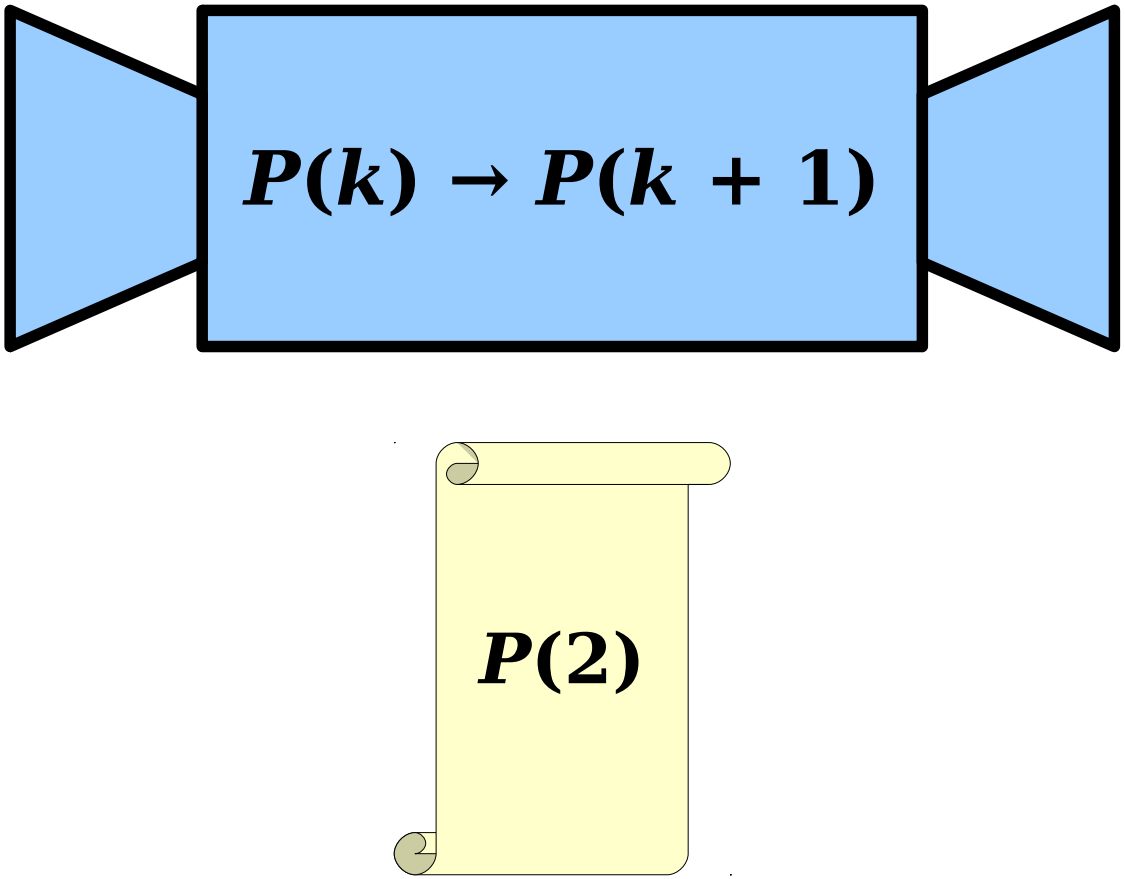

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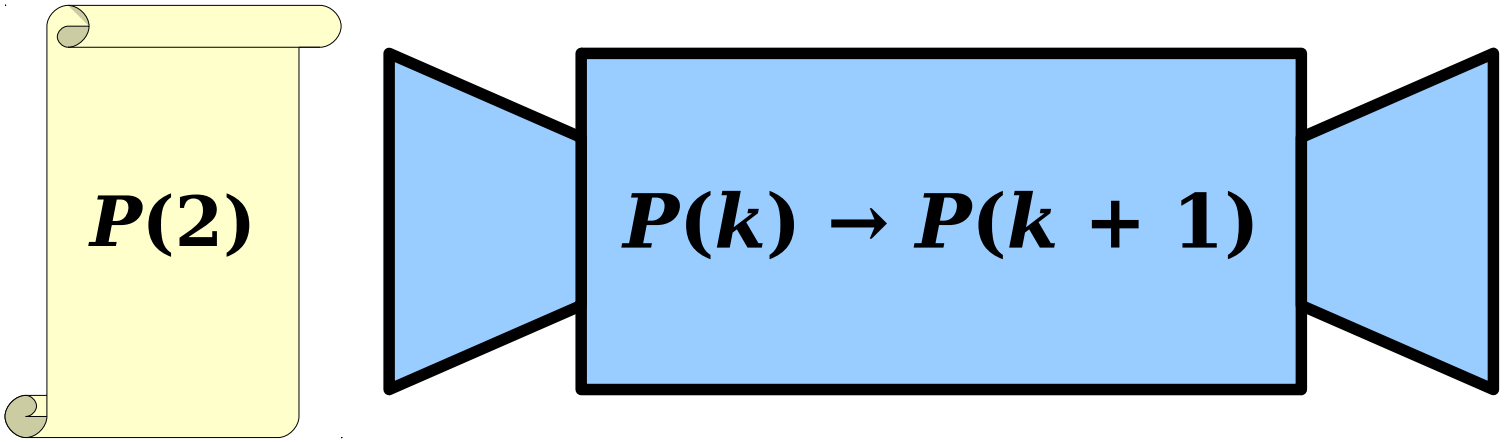

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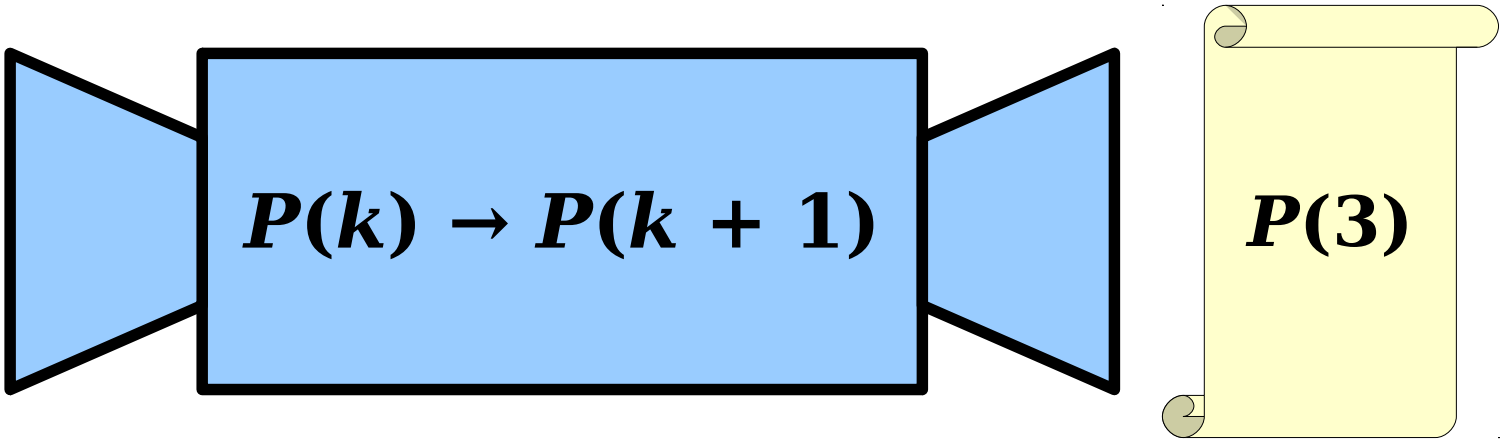

Proof by Induction

- A ***proof by induction*** is a way to use the principle of mathematical induction to show that some result is true for all natural numbers n.
- In a proof by induction, there are three steps:
  - Prove that $P(0)$ is true.
    - This is called the ***basis*** or the ***base case***.
  - Prove that if $P(k)$ is true, then $P(k+1)$ is true.
    - This is called the ***inductive step***.
    - The assumption that $P(k)$ is true is called the ***inductive hypothesis***.
  - Conclude, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$.

Some Sums

$$2^0$$

$$2^0 + 2^1$$

$$2^0 + 2^1 + 2^2$$

$$2^0 + 2^1 + 2^2 + 2^3$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4$$

$$2^0 = 1$$

$$2^0 + 2^1 = 1 + 2 = 3$$

$$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7$$

$$2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31$$

$$2^0 = 1 \qquad = 2^1 - 1$$

$$2^0 + 2^1 = 1 + 2 = 3 = 2^2 - 1$$

$$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7 = 2^3 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15 = 2^4 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31 = 2^5 - 1$$

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At the start of the proof, we tell the reader what predicate we're going to show is true for all natural numbers n, then tell them we're going to prove it by induction.

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Here, we state what $P(0)$ actually says. Now, can go prove this using any proof techniques we'd like!

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- ✓ $P(0)$ is true
- □ If $P(k)$ is true, then $P(k+1)$ is true.

What should the next step of this proof be?

A. Prove that, for any $k \in \mathbb{N}$, that $P(k)$ is true.
B. Assume for any $k \in \mathbb{N}$ that $P(k)$ and $P(k+1)$ are true.
C. Assume that $P(k)$ holds for all $k \in \mathbb{N}$.
D. Pick an arbitrary $k \in \mathbb{N}$, and prove $P(k+1)$.
E. Pick an arbitrary $k \in \mathbb{N}$, assume $P(k)$, and prove $P(k+1)$.
F. None of these, or more than one of these.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, …, or **E**.

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For the inductive step, assume that for some arbitrary $k \in \mathbb{N}$ that $P(k)$ holds, meaning that

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The goal of this step is to prove

"If $P(k)$ is true, then $P(k+1)$ is true."

To do this, we'll choose an arbitrary k, assume that $P(k)$ is true, then try to prove $P(k+1)$.

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Here, we explicitly state $P(k+1)$, which is what we want to prove. Now, we can use any proof technique we want to prove it.

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Here, we use our inductive hypothesis (the assumption that $P(k)$ is true) to simplify a complex expression. This is a common theme in inductive proofs.

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Therefore, $P(k + 1)$ is true, completing the induction.

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Proof: Let $P(n)$ be the statement "the sum of the first n powers of two is $2^n - 1$." We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

For our base case, we need to show $P(0)$ is true, meaning that the sum of the first zero powers of two is $2^0 - 1$. Since the sum of the first zero powers of two is zero and $2^0 - 1$ is zero as well, we see that $P(0)$ is true.

For the inductive step, assume that for some arbitrary $k \in \mathbb{N}$ that $P(k)$ holds, meaning that

$$2^0 + 2^1 + \cdots + 2^{k-1} = 2^k - 1 \qquad (1)$$

We n … um
of th…
notic…

$$2^0 \qquad \text{(via (1))}$$
$$= 2(2^k) - 1$$
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$$2^0 + 2^1 + \ldots + 2^{k-1} = 2^k - 1. \qquad (1)$$

We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers of two is $2^{k+1} - 1$. To see this, notice that

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \ldots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \qquad \textit{(via (1))} \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Therefore, $P(k + 1)$ is true, completing the induction. ■

A Quick Aside

- This result helps explain the range of numbers that can be stored in an `int`.
- If you have an unsigned 32-bit integer, the largest value you can store is given by $1 + 2 + 4 + 8 + \ldots + 2^{31} = 2^{32} - 1$.
- This formula for sums of powers of two has many other uses as well. If we have time, we'll see one more today.
  - If not, we'll see it next time!

Structuring a Proof by Induction

- Define some predicate P that you'll show, by induction, is true for all natural numbers.
- Prove the base case:
  - State that you're going to prove that $P(0)$ is true, then go prove it.
- Prove the inductive step:
  - Say that you're assuming $P(k)$ for some arbitrary natural number k, then write out exactly what that means.
  - Say that you're going to prove $P(k+1)$, then write out exactly what that means.
  - Prove that $P(k+1)$ using any proof technique you'd like!
- This is a rather verbose way of writing inductive proofs. As we get more experience with induction, we'll start leaving out some details from our proofs.

The Counterfeit Coin Problem

Problem Statement

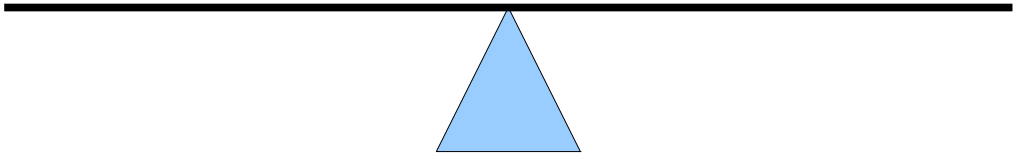
- You are given a set of three seemingly identical coins, two of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only one weighing on the balance, find the counterfeit coin.

Finding the Counterfeit Coin

1

2

3



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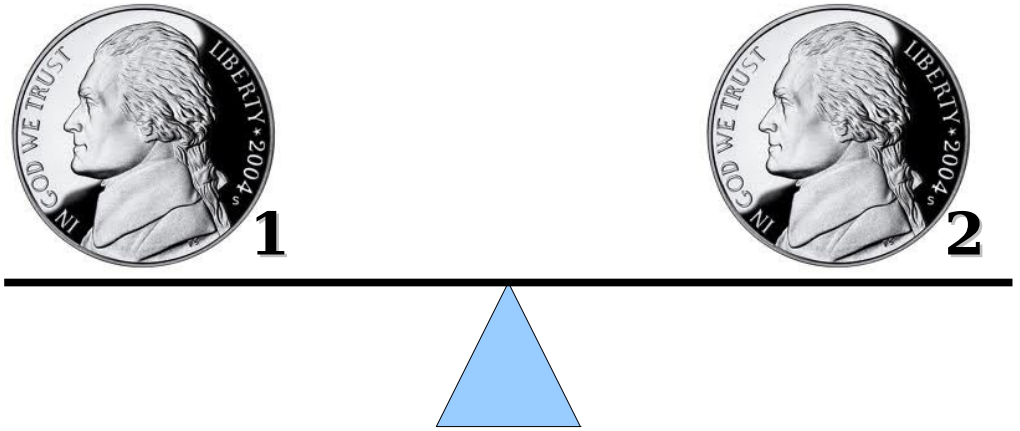

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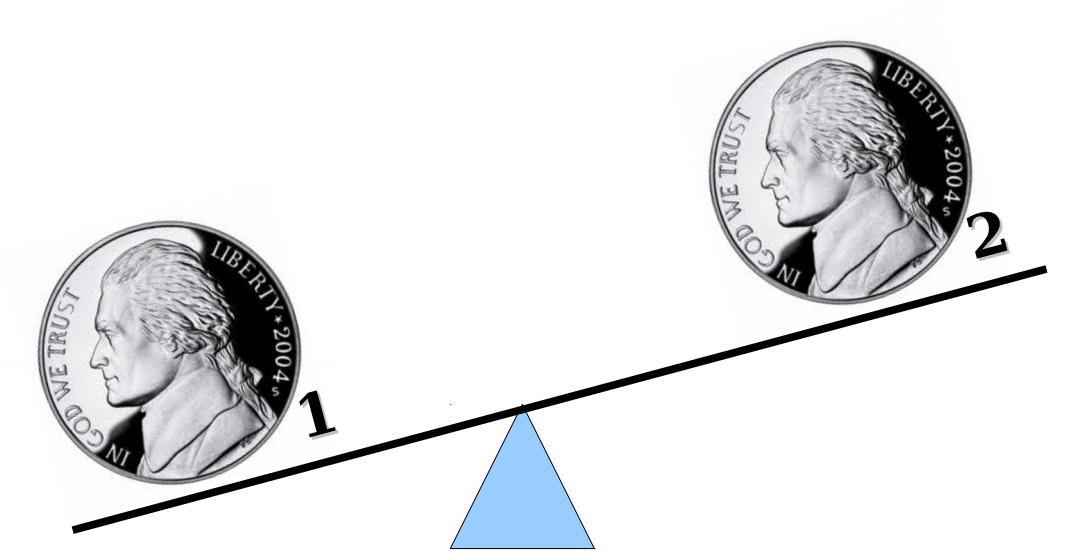

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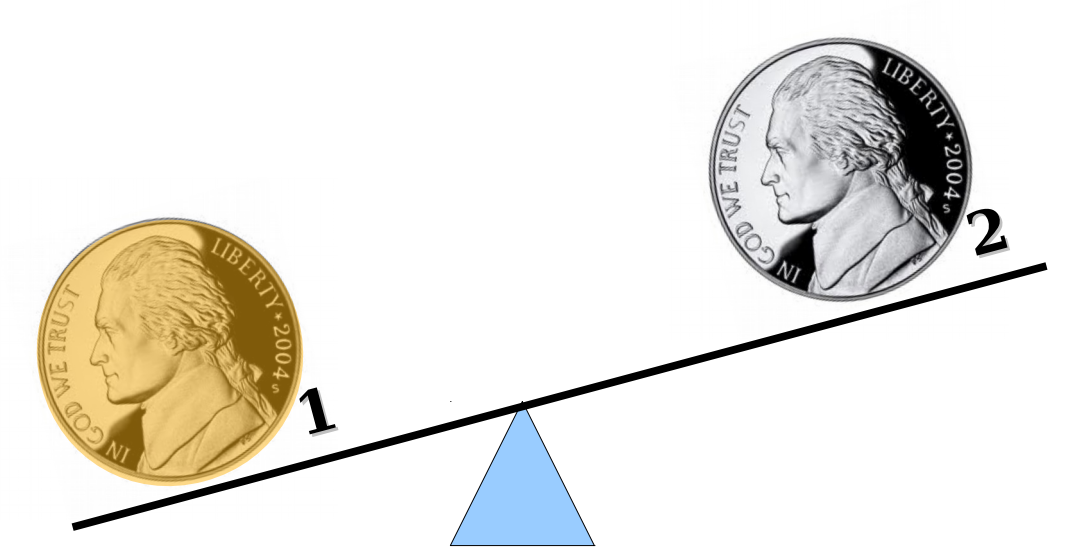

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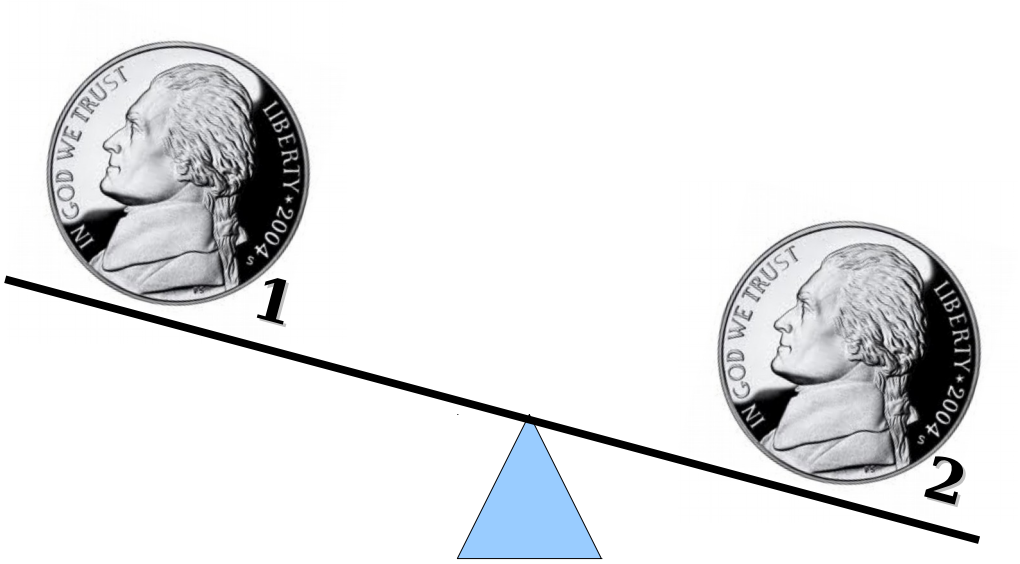

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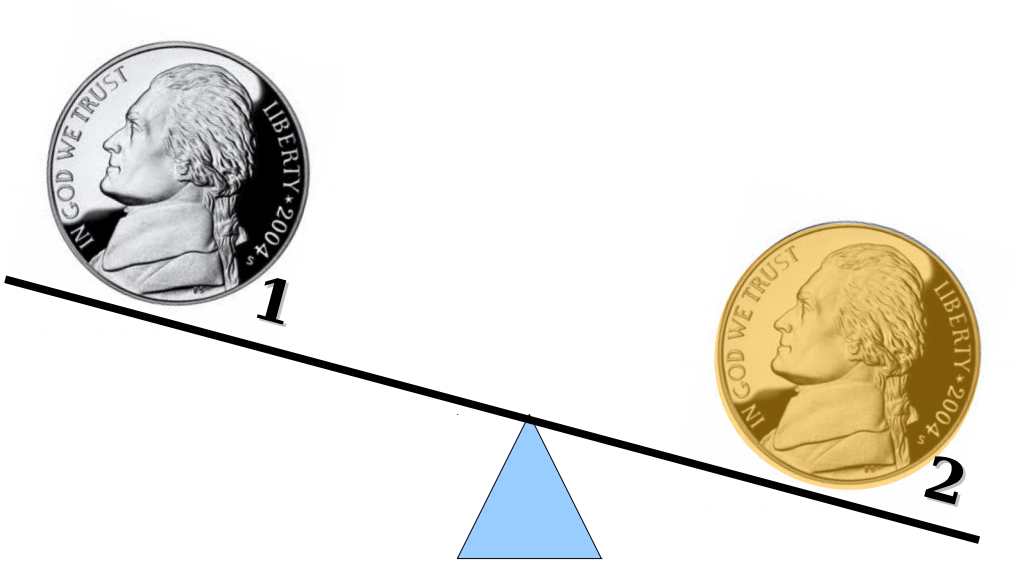

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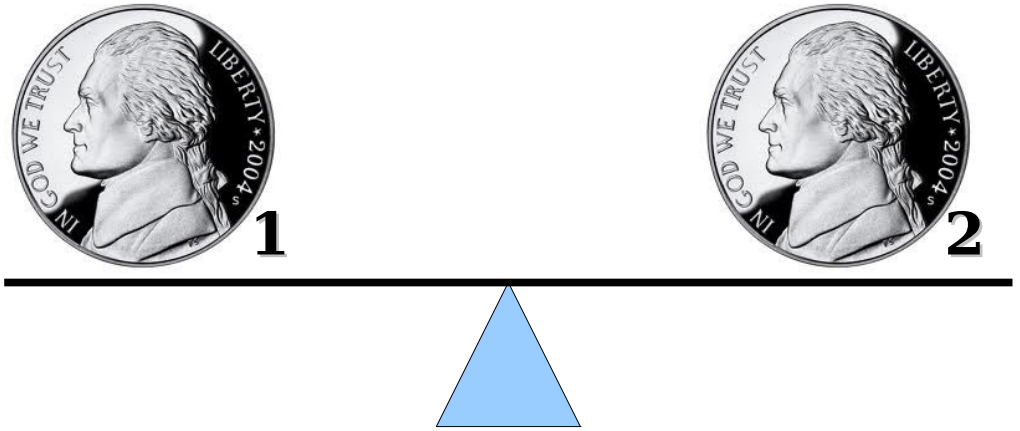

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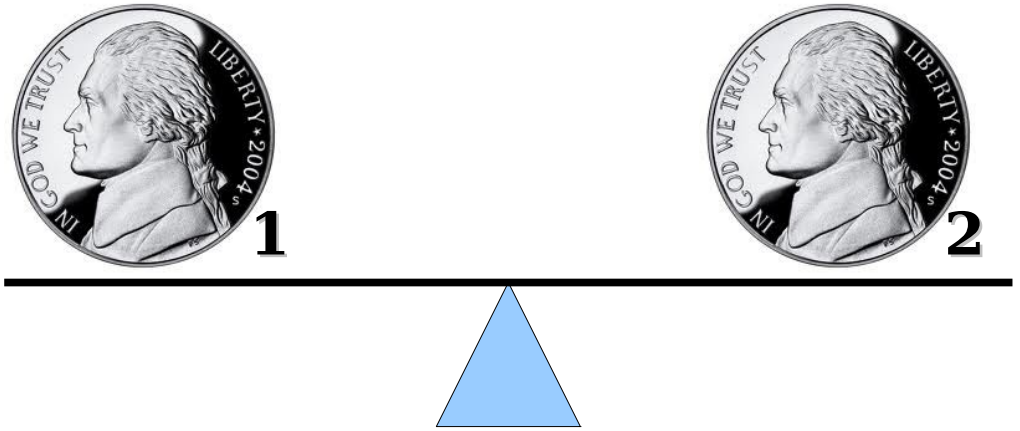
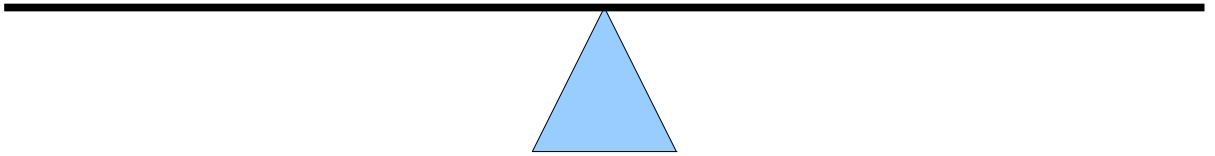


A Harder Problem

- You are given a set of ***nine*** seemingly identical coins, eight of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only ***two*** weighings on the balance, find the counterfeit coin.

Finding the Counterfeit Coin



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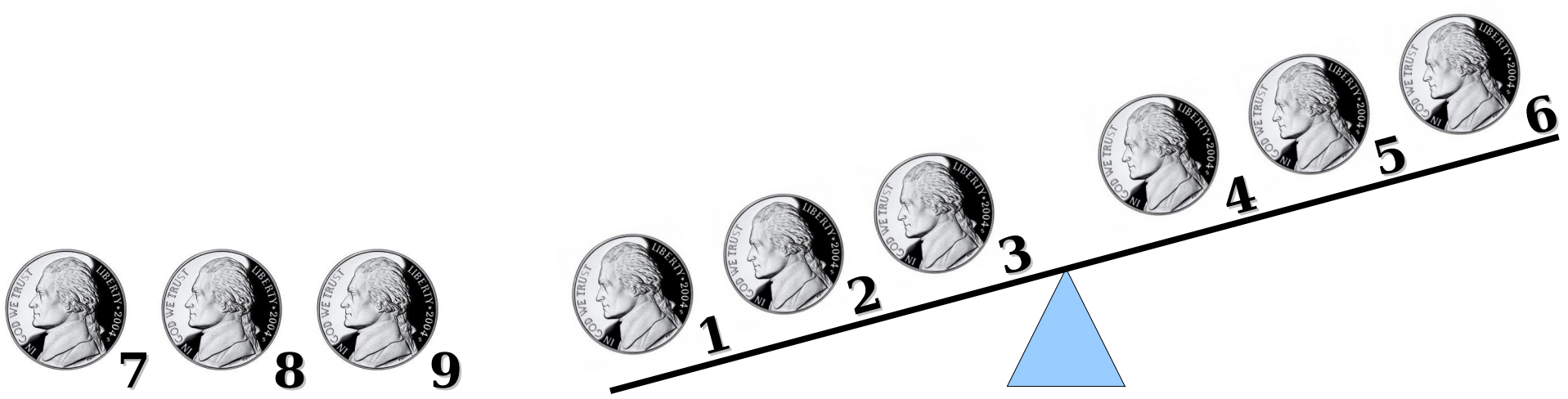

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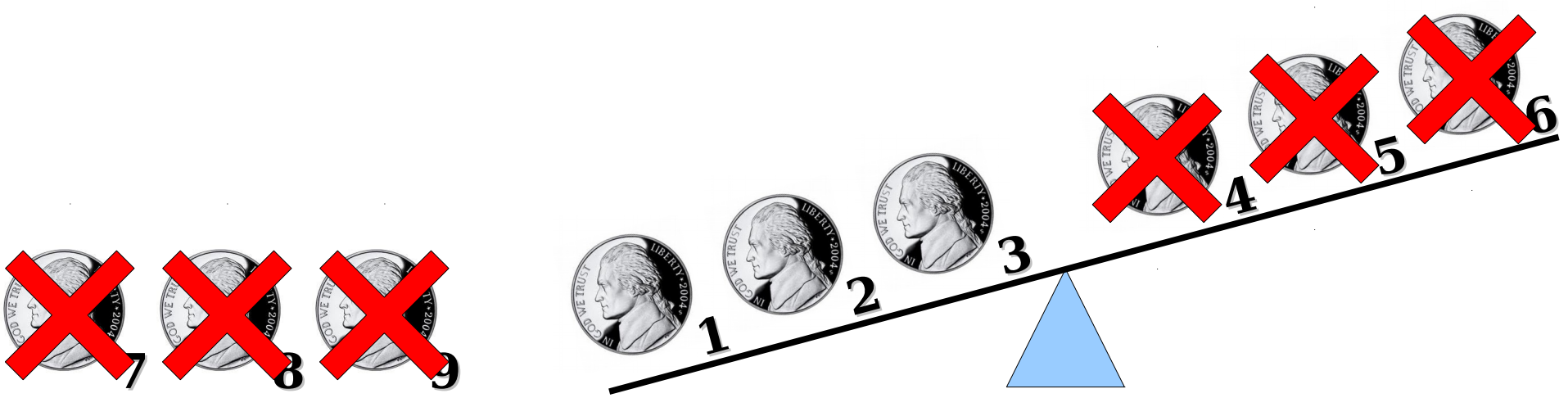

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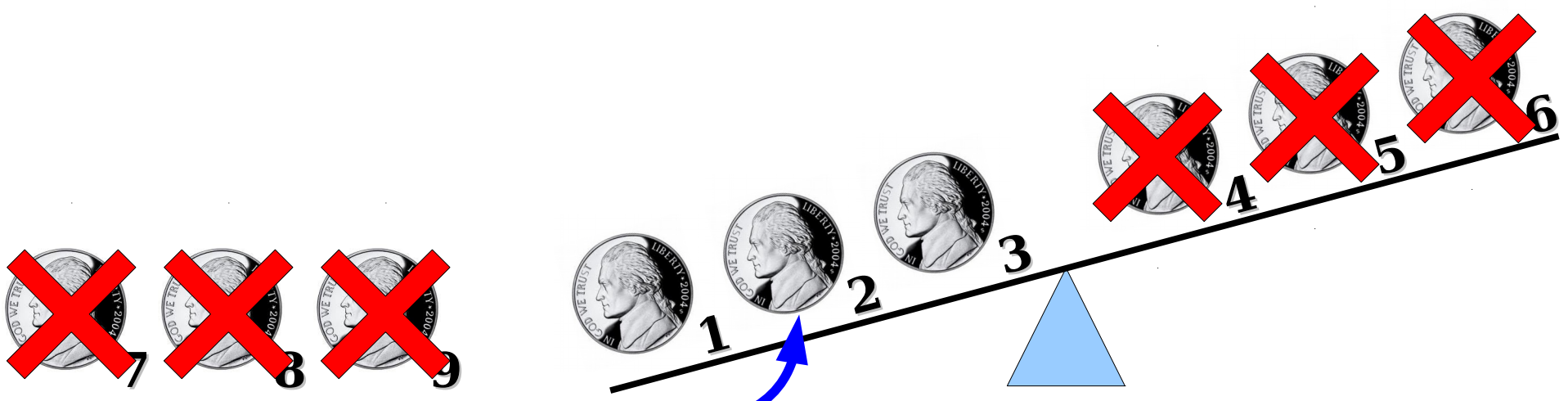
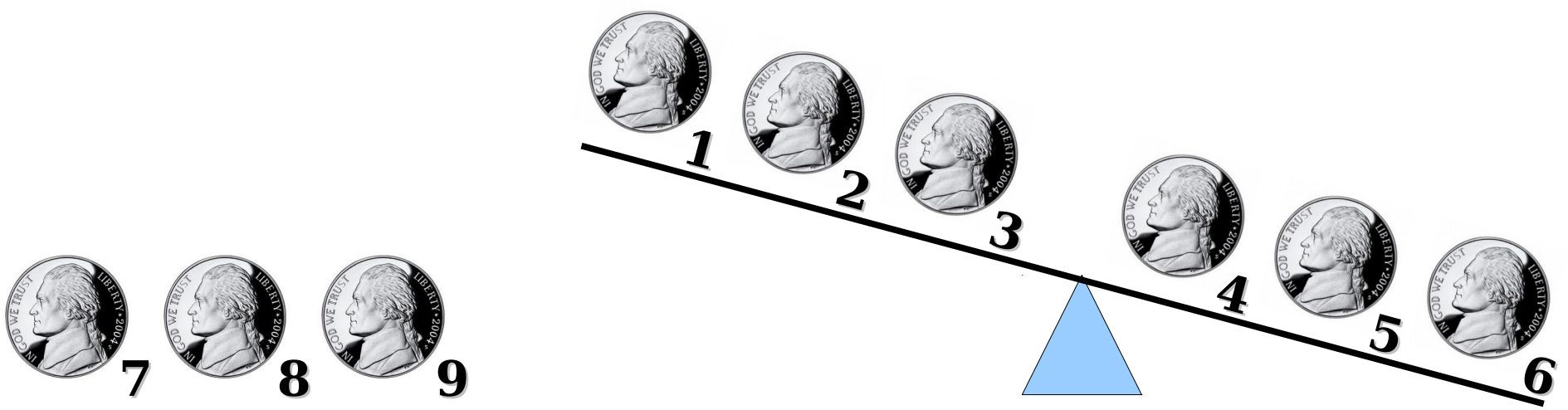


Now we have one weighing to find the counterfeit out of these three coins.

Finding the Counterfeit Coin



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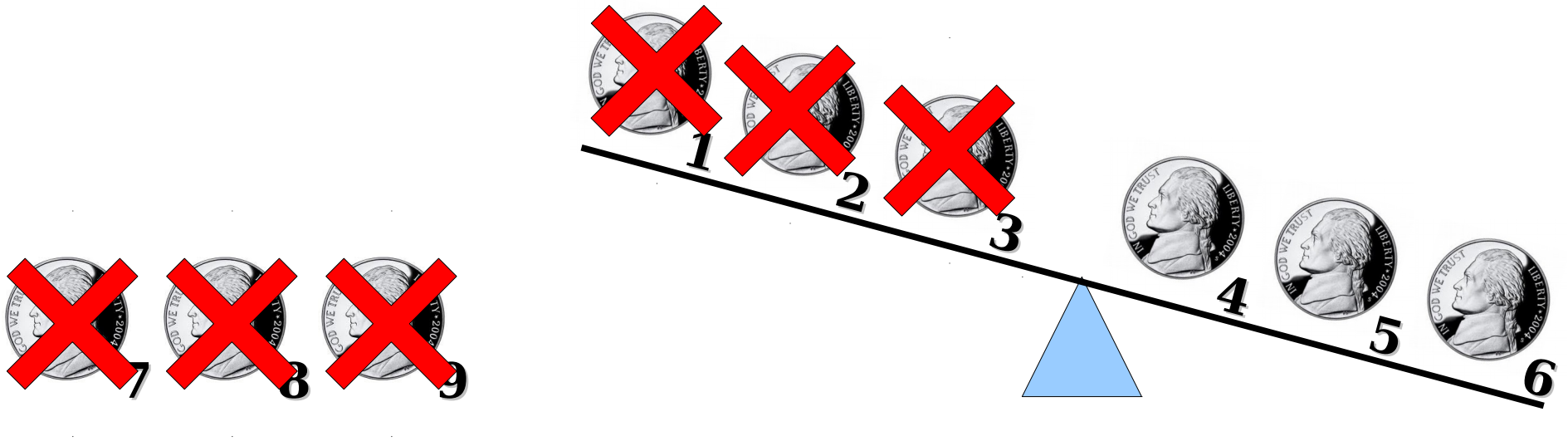

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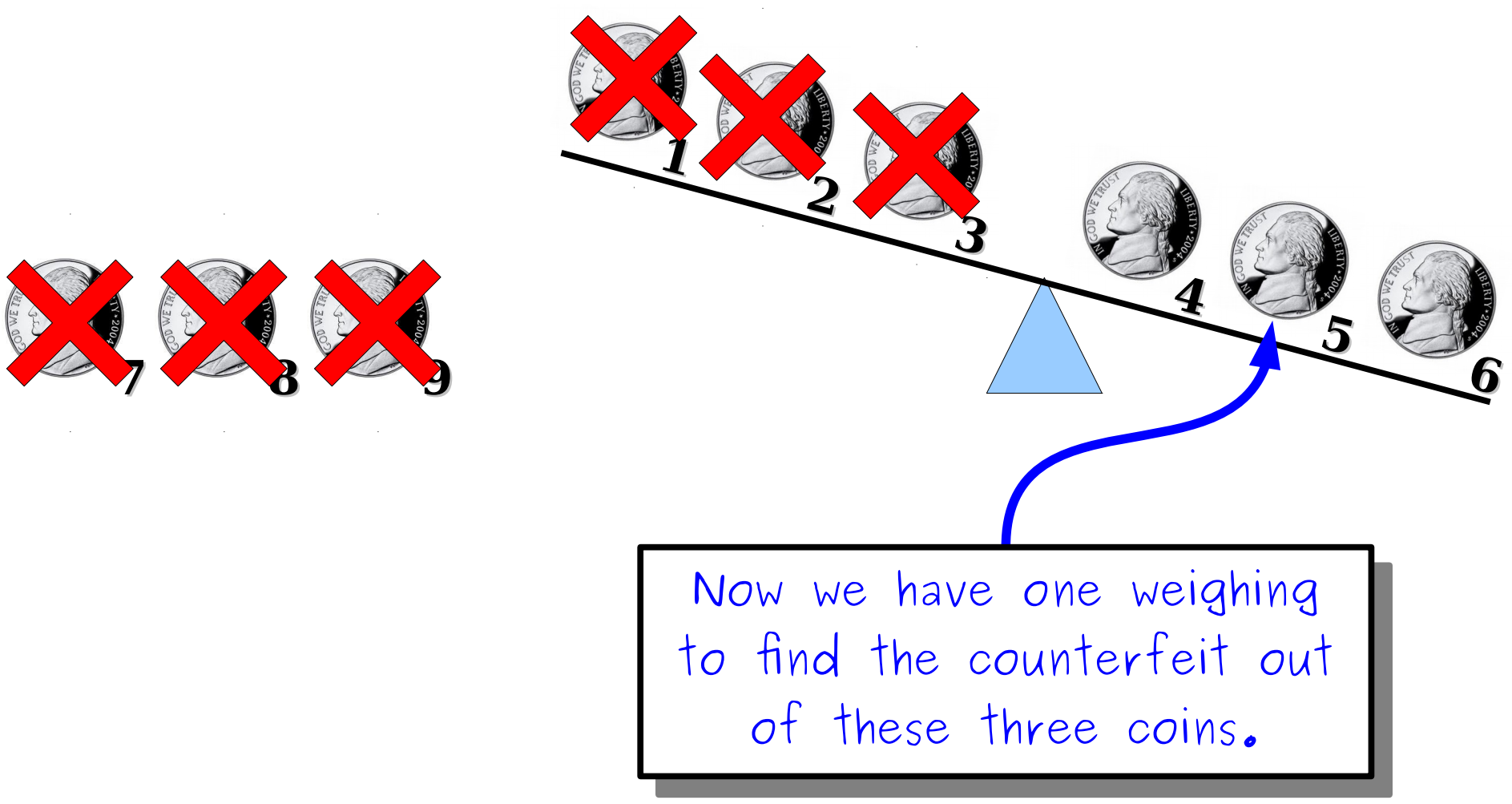

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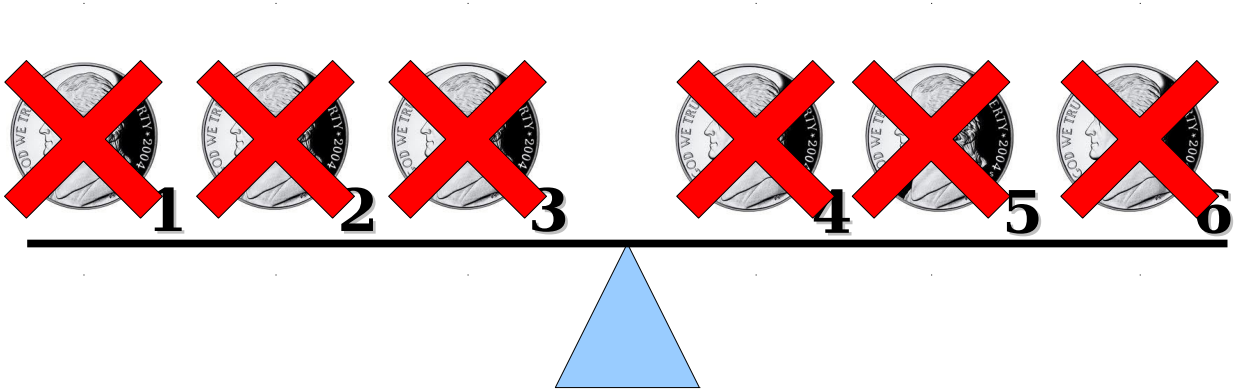

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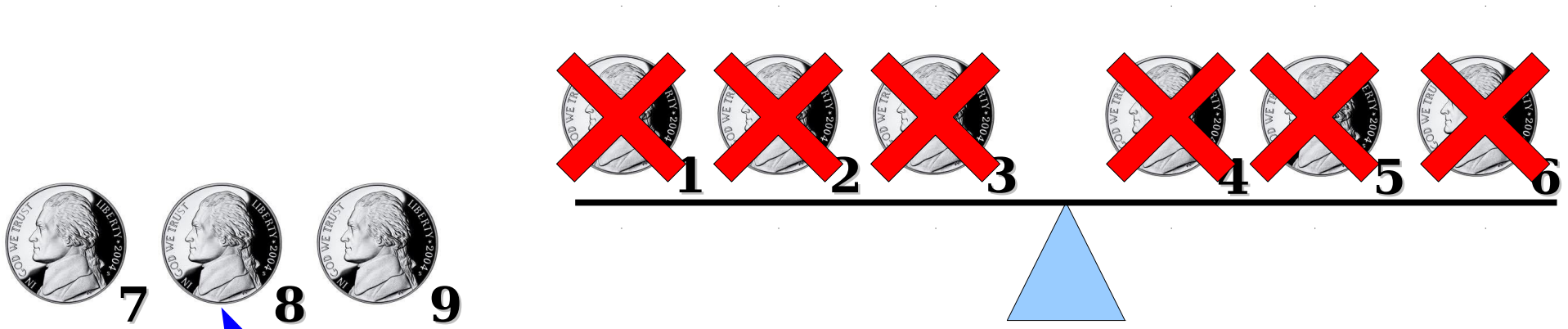

Now we have one weighing to find the counterfeit out of these three coins.

Can we generalize this?

A Pattern

- Assume out of the coins that are given, exactly one is counterfeit and weighs more than the other coins.
- If we have no weighings, how many coins can we have while still being able to find the counterfeit?
  - ***One*** coin, since that coin has to be the counterfeit!
- If we have one weighing, we can find the counterfeit out of ***three*** coins.
- If we have two weighings, we can find the counterfeit out of ***nine*** coins.

So far, we have

$1, 3, 9 = 3^0, 3^1, 3^2$

Does this pattern continue?

Theorem: If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

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Proof: Let $P(n)$ be ...

Which of these is a good choice for $P(n)$?

A. 3^n.

B. A group of 3^n coins.

C. For any $n \in \mathbb{N}$, if there are 3^n coins of which one is heavier than the rest, we can find it using n weighings on a balance.

D. If we can find the heavier coin out of a group of 3^n coins in n weighings, then we can find the heavier coin out of a group of 3^{n+1} coins in $n+1$ weighings.

E. None of these, or more than one of these.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, ..., or **E**.

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At the start of the proof, we tell the reader what predicate we're going to show is true for all natural numbers n, then tell them we're going to prove it by induction.

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Here, we state what $P(0)$ actually says. Now, can go prove this using any proof techniques we'd like!

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The goal of this step is to prove

"If $P(k)$ is true, then $P(k+1)$ is true."

To do this, we'll choose an arbitrary k, assume that $P(k)$ is true, then try to prove $P(k+1)$.

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Here, we explicitly state $P(k+1)$, which is what we want to prove. Now, we can use any proof technique we want to try to prove it.

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If exact… rest, that c… ce.

We'll use … rom which the theore…

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Here, we use our inductive hypothesis (the assumption that $P(k)$ is true) to solve this simpler version of the overall problem.

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For the…, so we can…1): that w…

In a proof by induction, we need to prove that

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- □ If $P(k)$ is true, then $P(k+1)$ is true.

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Some Fun Problems

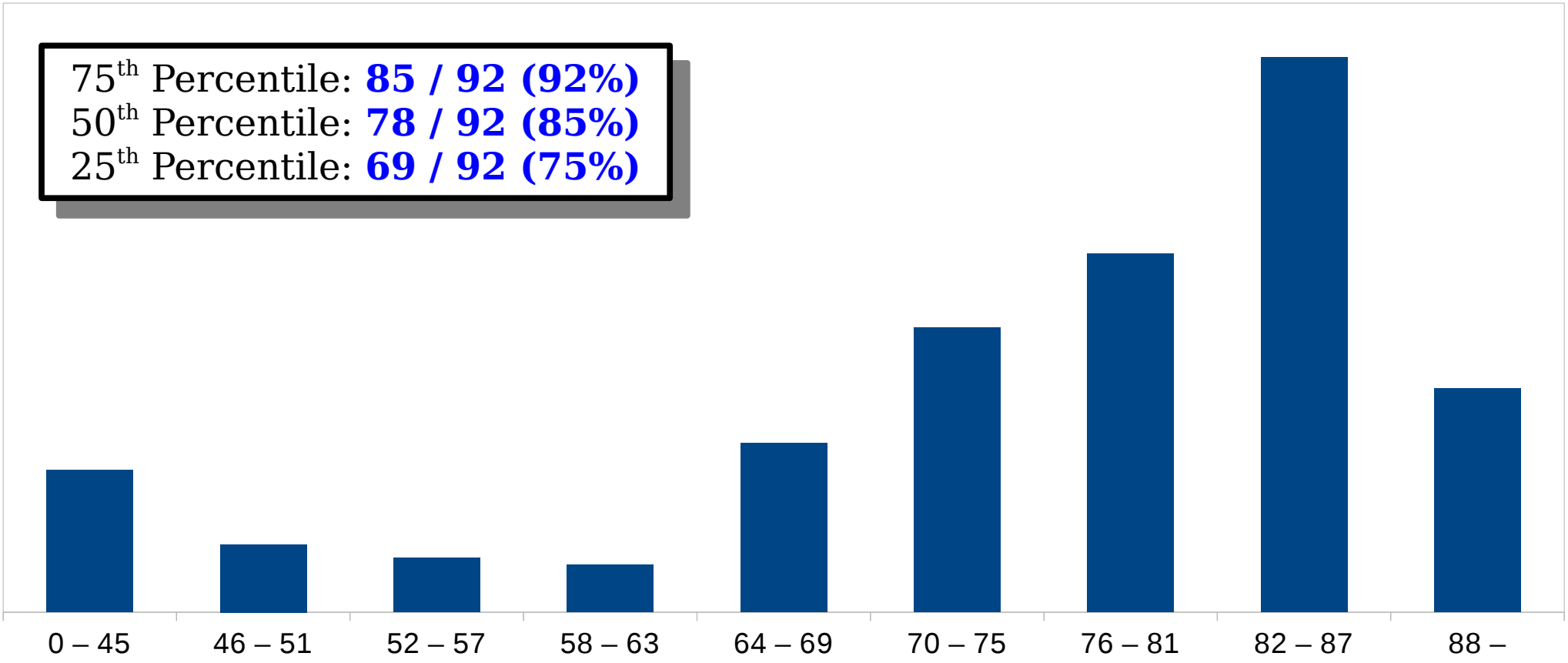
- Here's some nifty variants of this problem that you can work through:
  - Suppose that you have a group of coins where there's either exactly one heavier coin, or all coins weigh the same amount. If you only get k weighings, what's the largest number of coins where you can find the counterfeit or determine none exists?
  - What happens if the counterfeit can be either heavier or lighter than the other coins? What's the maximum number of coins where you can find the counterfeit if you have k weighings?
  - Can you find the counterfeit out of a group of more than 3^k coins with k weighings?
  - Can you find the counterfeit out of any group of at most 3^k coins with k weighings?

Time-Out for Announcements!

First Midterm Exam

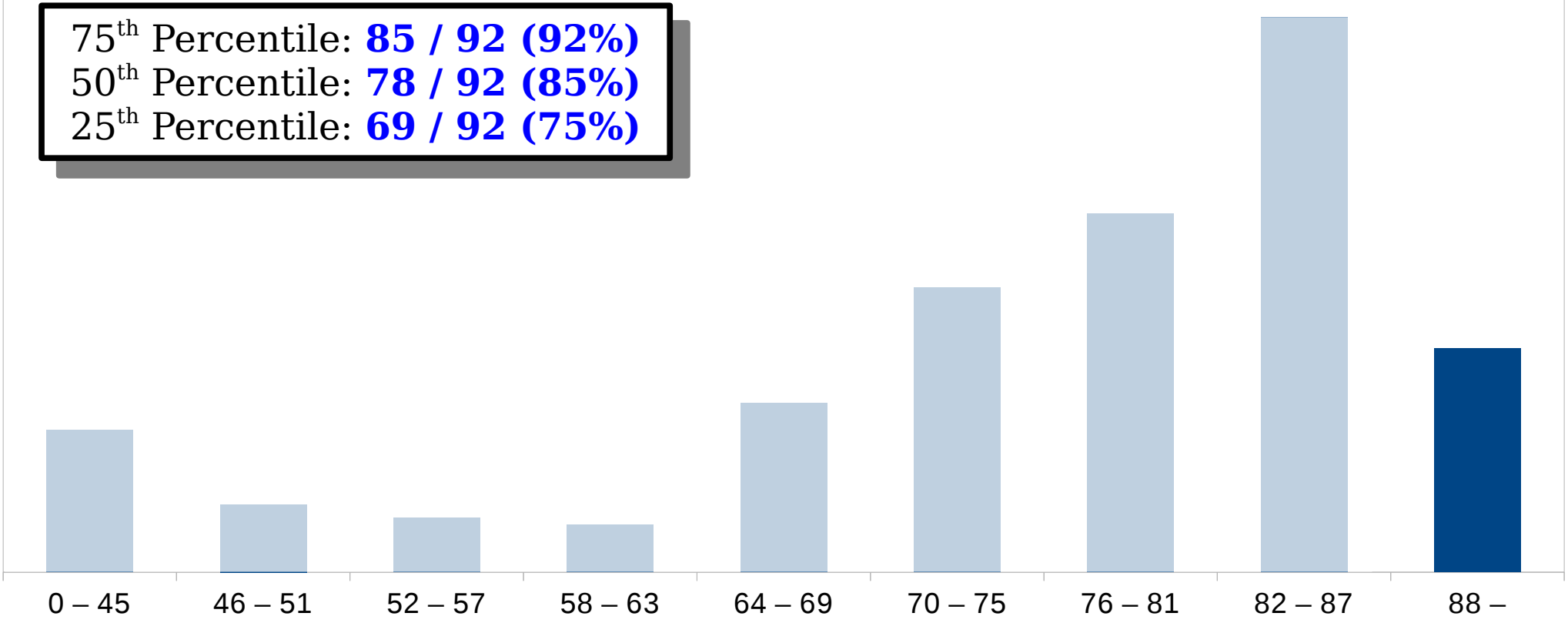
- You’re done with the first midterm! Woohoo!
- We’ll be grading it over the weekend. Solutions will go out when we have context for them (common errors, stats, etc.), but feel free to ask us if you’re curious about anything!

Problem Set Three Grades



Problem Set Three Grades

75th Percentile: **85 / 92 (92%)**
50th Percentile: **78 / 92 (85%)**
25th Percentile: **69 / 92 (75%)**



"Awesome job! Take a look at the feedback to see where you need to make final tweaks and adjustments."

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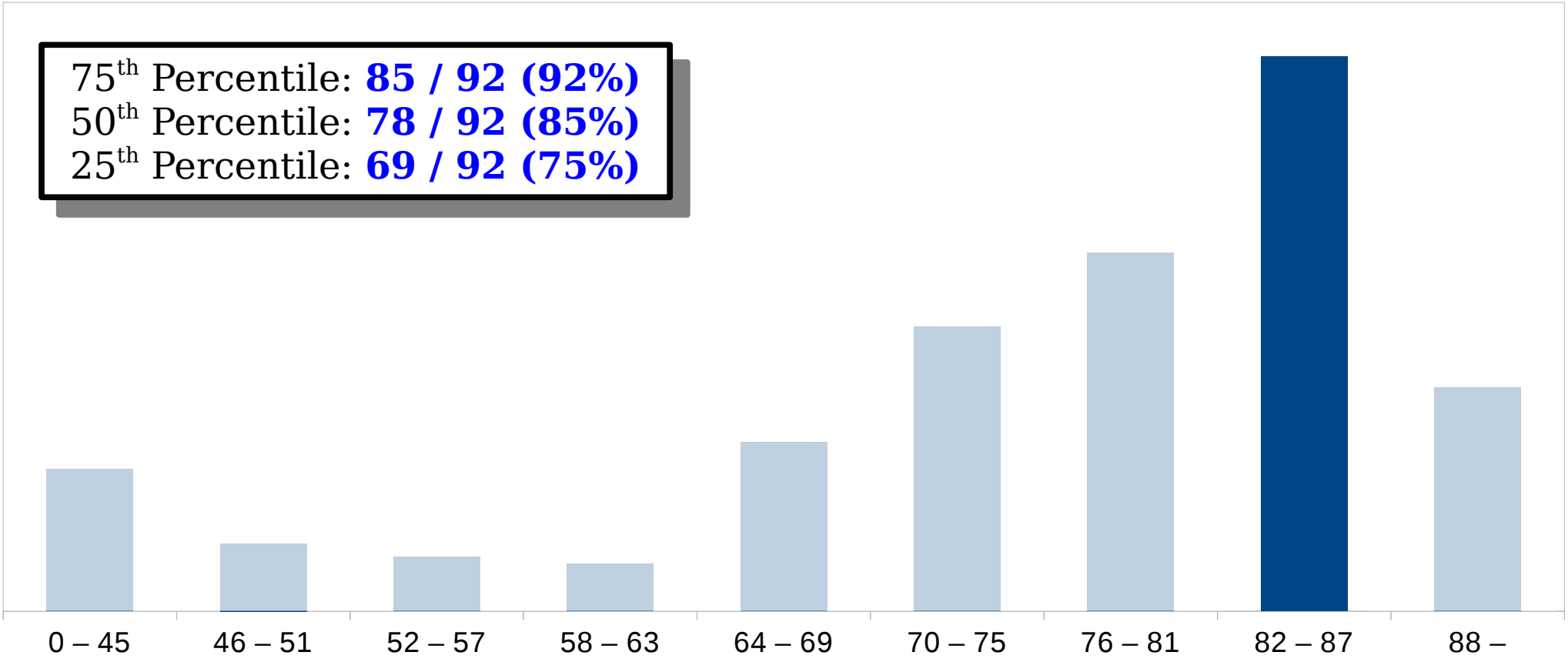

"Well done! There are few spots you may need to brush up on, so take a look to see what they are and keep them in mind going forward."

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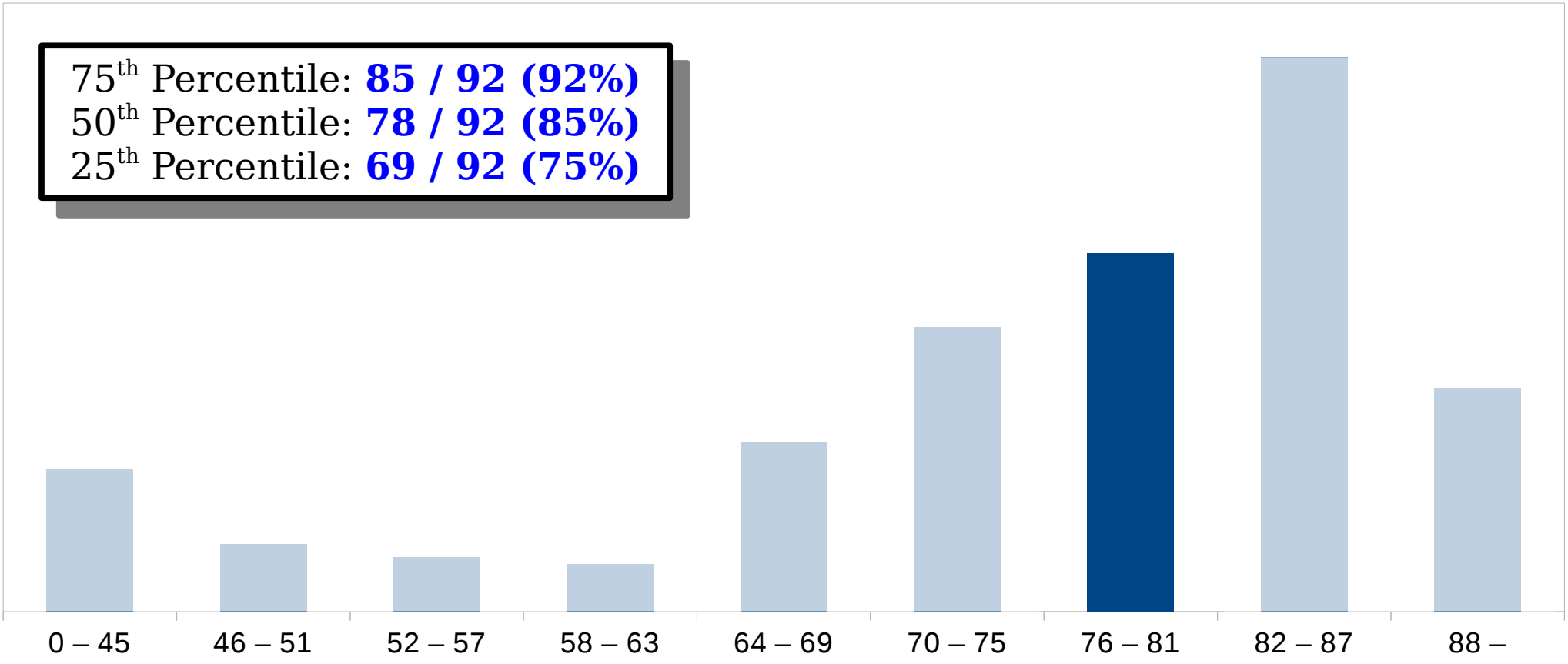

"Good job! You're doing well. Take a look at the feedback to see what concrete areas you need to focus on – chances are there's some skill you may need some more practice with – and keep up the good work!"

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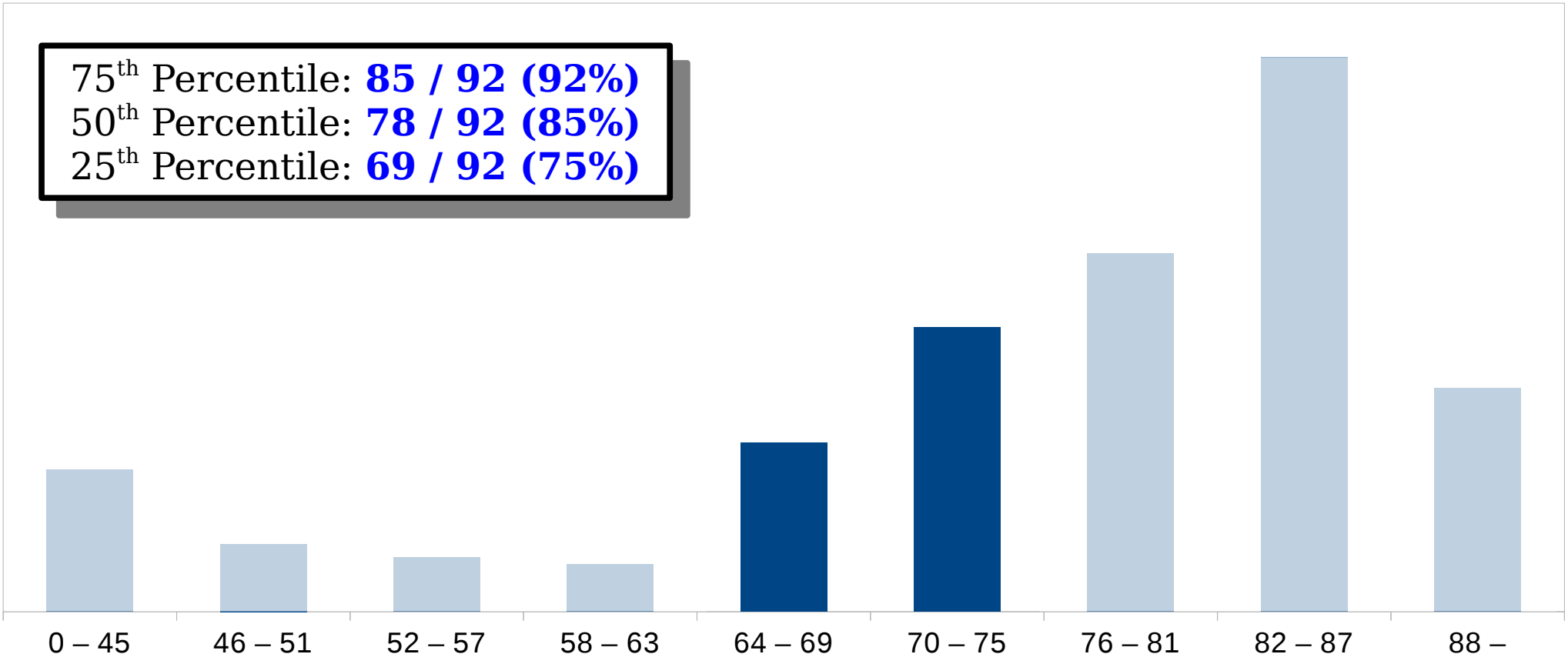

"You're definitely on the right track, and chances are there's some key skill you may need to get some more practice with. Take a look at your feedback, figure out where to focus your energy going forward, and let us know how we can help."

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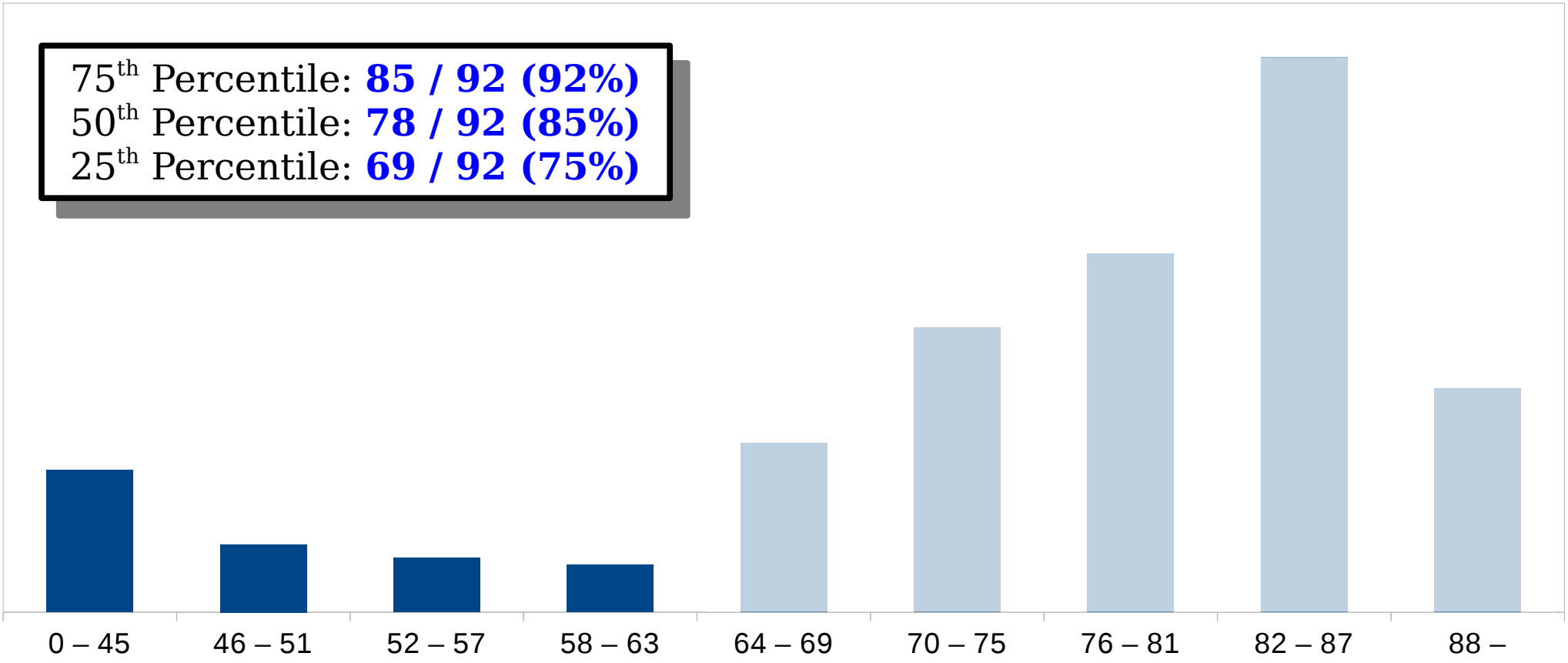

"Seems like there's some skill or technique that hasn't clicked yet. Feel free to chat with us in office hours or over Piazza about your feedback, and let us know how we can help you practice and improve."

Problem Set Four

- Problem Set Four is due this Friday at 2:30PM.
- ***Recommendation:*** As soon as you can, review all the feedback you got on PS3 and from the PS4 checkpoint. Ask yourself these questions:
  - Based on the proofwriting and style feedback you received, do you know what specific changes you'd make to your answers?
  - If you made any logic errors, do you understand what those errors are to the point that you could explain them to someone else?
- Feel free to stop by office hours or to visit Piazza if you have questions. We're happy to help out! You can do this!

Back to CS103!

How Not To Induct

Something's Wrong...

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Theorem: The sum of the first n powers of two is 2^n.

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We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers of two is 2^{k+1}.

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What's wrong with this proof?

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then your answer.

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When writing a proof by induction,

make sure to prove the base case!

Otherwise, your argument is invalid!

Why did this work?

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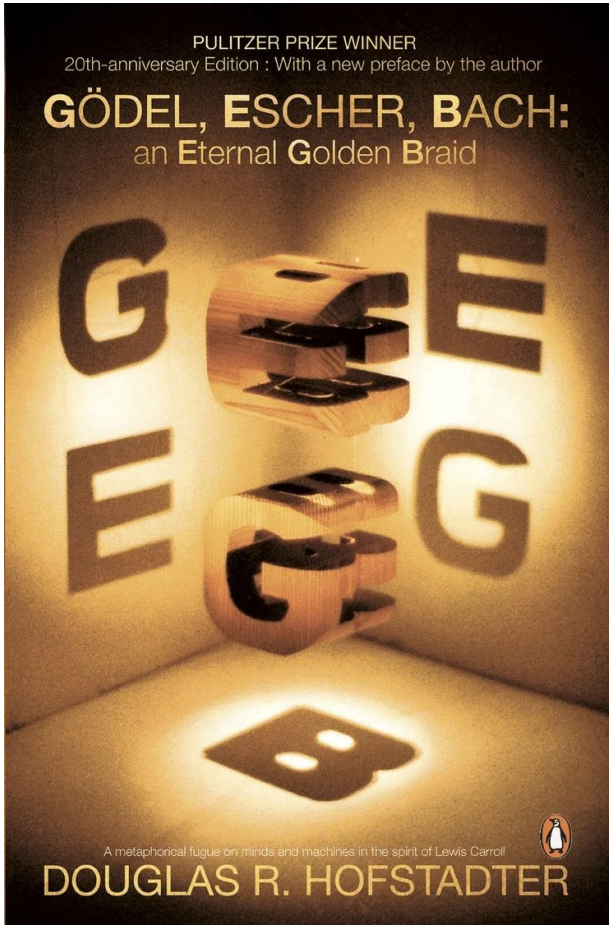
You can prove *anything* from a faulty assumption. This is called the **principle of explosion**. To see why, read "Animal, Vegetable, or Minister" for a silly example.

The MU Puzzle

Gödel, Escher Bach: An Eternal Golden Braid

- Douglas Hofstadter, cognitive scientist at the University of Indiana, wrote this Pulitzer-Prize-winning mind trip of a book.
- It's a great read after you've finished CS103 – you'll see so many of the ideas we'll cover presented in a totally different way!



The MU Puzzle

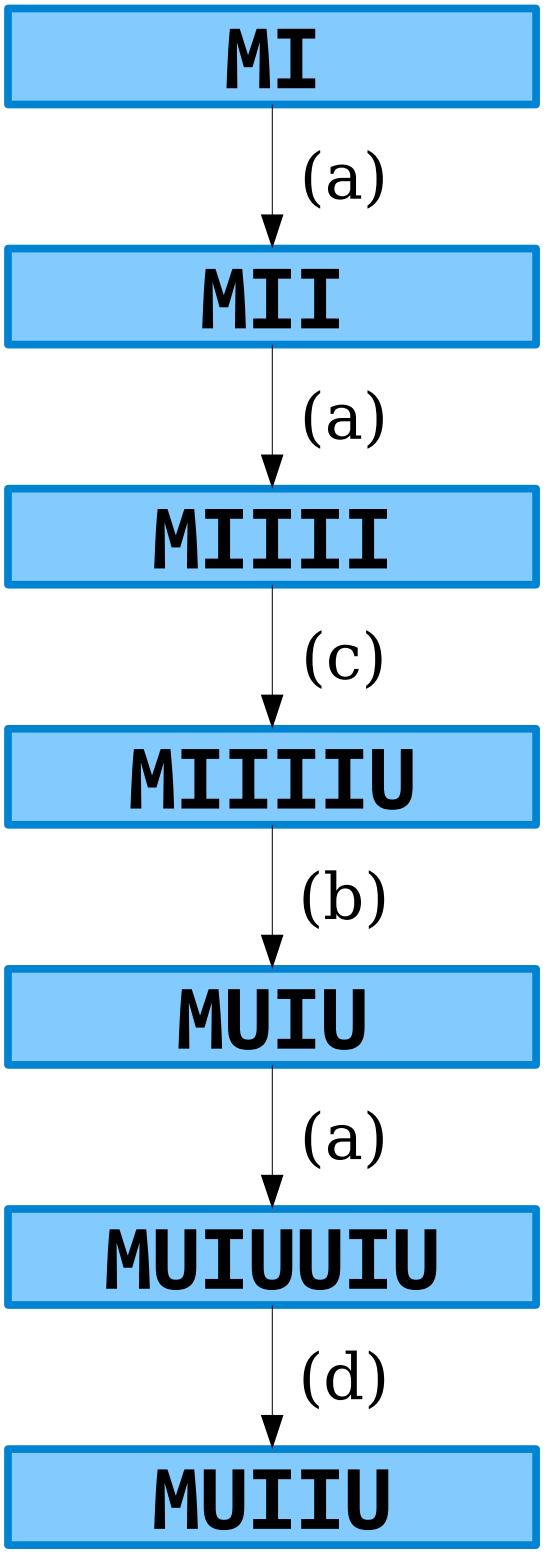
- Begin with the string `MI`.
- Repeatedly apply one of the following operations:
  - Double the contents of the string after the `M`: for example, `MIIU` becomes `MIIUIIU`, or `MI` becomes `MII`.
  - Replace `III` with `U`: `MIIII` becomes `MUI` or `MIU`.
  - Append `U` to the string if it ends in `I`: `MI` becomes `MIU`.
  - Remove any `UU`: `MUUU` becomes `MU`.
- ***Question***: How do you transform `MI` to `MU`?

(a) Double the string after an `M`.

(b) Replace `III` with `U`.

(c) Append `U`, if the string ends in `I`.

(d) Delete `UU` from the string.



Try It!

Starting with **MI**, apply these operations to make **MU**:

(a) Double the string after an **M**.

(b) Replace **III** with **U**.

(c) Append **U**, if the string ends in **I**.

(d) Delete **UU** from the string.

Not a single person in this room
was able to solve this puzzle.

Are we even sure that there is a solution?

Counting **I**'s


MI
1
MII
2
MIIII
4
MIIIIU
4
MIIIIUIIIIU
8
MIIIIUUIU
5
MIIIIUUIUIIIIUUIU
10
MUIUUIUIIIIUUIU
7

The Key Insight

- Initially, the number of `I`'s is *not* a multiple of three.
- To make `MU`, the number of `I`'s must end up as a multiple of three.
- Can we *ever* make the number of `I`'s a multiple of three?

Lemma 1: If n is an integer that is not a multiple of three, then $n - 3$ is not a multiple of three.

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Proof: By contrapositive; we'll prove that if $n - 3$ is a multiple of three, then n is also a multiple of three. Because $n - 3$ is a multiple of three, we can write $n - 3 = 3k$ for some integer k. Then $n = 3(k+1)$, so n is also a multiple of three, as required. ■

Lemma 2: If n is an integer that is not a multiple of three, then $2n$ is not a multiple of three.

Proof: Let n be a number that isn't a multiple of three. If n is congruent to one modulo three, then $n = 3k + 1$ for some integer k. This means $2n = 2(3k+1) = 6k + 2 = 3(3k) + 2$, so $2n$ is not a multiple of three. Otherwise, n must be congruent to two modulo three, so $n = 3k + 2$ for some integer k. Then $2n = 2(3k+2) = 6k+4 = 3(2k+1) + 1$, and so $2n$ is not a multiple of three. ■

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Therefore, no sequence of $k+1$ moves ends with a multiple of three **I**'s.

Lemma: No matter which moves are made, the number of **I**'s in the string never becomes multiple of three.

Proof: Let $P(n)$ be the statement "after any n moves, the number of **I**'s in the string will not be multiple of three." We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

As a base case, we'll prove $P(0)$, that the number of **I**'s after 0 moves is not a multiple of three. After no moves, the string is **MI**, which has one **I** in it. Since one isn't a multiple of three, $P(0)$ is true.

For our inductive step, suppose that $P(k)$ is true for some arbitrary $k \in \mathbb{N}$. We'll prove $P(k+1)$ is also true. Consider any sequence of $k+1$ moves. Let r be the number of **I**'s in the string after the kth move. By our inductive hypothesis (that is, $P(k)$), we know that r is not a multiple of three. Now, consider the four possible choices for the $k+1^{st}$ move:

Case 1: Double the string after the **M**. After this, we will have $2r$ **I**'s in the string, and from our lemma $2r$ isn't a multiple of three.

Case 2: Replace **III** with **U**. After this, we will have $r - 3$ **I**'s in the string, and by our lemma $r - 3$ is not a multiple of three.

Case 3: Either append **U** or delete **UU**. This preserves the number of **I**'s in the string, so we don't have a multiple of three **I**'s at this point.

Therefore, no sequence of $k+1$ moves ends with a multiple of three **I**'s. Thus $P(k+1)$ is true, completing the induction.

Lemma: No matter which moves are made, the number of **I**'s in the string never becomes multiple of three.

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Therefore, no sequence of $k+1$ moves ends with a multiple of three **I**'s. Thus $P(k+1)$ is true, completing the induction. ■

Theorem: The `MU` puzzle has no solution.

Proof: Assume for the sake of contradiction that the `MU` puzzle has a solution and that we can convert `MI` to `MU`. This would mean that at the very end, the number of `I`'s in the string must be zero, which is a multiple of three. However, we've just proven that the number of `I`'s in the string can never be a multiple of three.

We have reached a contradiction, so our assumption must have been wrong. Thus the `MU` puzzle has no solution. ■

Algorithms and Loop Invariants

- The proof we just made had the form
  - "If *P* is true before we perform an action, it is true after we perform an action."
- We could therefore conclude that after any series of actions of any length, if *P* was true beforehand, it is true now.
- In algorithmic analysis, this is called a ***loop invariant***.
- Proofs on algorithms often use loop invariants to reason about the behavior of algorithms.
  - Take CS161 for more details!

Next Time

- ***Variations on Induction***
  - Starting induction later.
  - Taking larger steps.
  - Complete induction.