Regular Expressions

Recap from Last Time

Regular Languages

- A language L is a ***regular language*** if there is a DFA D such that $\mathscr{L}(D) = L$.
- ***Theorem:*** The following are equivalent:
  - L is a regular language.
  - There is a DFA for L.
  - There is an NFA for L.

Language Concatenation

- If $w \in \Sigma^*$ and $x \in \Sigma^*$, then wx is the ***concatenation*** of w and x.
- If L_1 and L_2 are languages over Σ, the ***concatenation*** of L_1 and L_2 is the language L_1L_2 defined as

$$\boldsymbol{L_1L_2 = \{\ wx \mid w \in L_1 \text{ and } x \in L_2\ \}}$$

- Example: if $L_1 = \{$ **a**, **ba**, **bb** $\}$ and $L_2 = \{$ **aa**, **bb** $\}$, then

$L_1L_2 = \{$ **aaa**, **abb**, **baaa**, **babb**, **bbaa**, **bbbb** $\}$

Lots and Lots of Concatenation

- Consider the language L = { `aa`, `b` }
- LL is the set of strings formed by concatenating pairs of strings in L.

{ `aaaa`, `aab`, `baa`, `bb` }

- LLL is the set of strings formed by concatenating triples of strings in L.

{ `aaaaaa`, `aaaab`, `aabaa`, `aabb`, `baaaa`, `baab`, `bbaa`, `bbb`}

- $LLLL$ is the set of strings formed by concatenating quadruples of strings in L.

{ `aaaaaaaa`, `aaaaaab`, `aaaabaa`, `aaaabb`, `aabaaaa`, `aabaab`, `aabbaa`, `aabbb`, `baaaaaa`, `baaaab`, `baabaa`, `baabb`, `bbaaaa`, `bbaab`, `bbbaa`, `bbbb`}

Language Exponentiation

- We can define what it means to "exponentiate" a language as follows:
- $L^0 = \{\varepsilon\}$
  - The set containing just the empty string.
  - Idea: Any string formed by concatenating zero strings together is the empty string.
- $L^{n+1} = LL^n$
  - Idea: Concatenating $(n+1)$ strings together works by concatenating n strings, then concatenating one more.
- ***Question:*** Why define $L^0 = \{\varepsilon\}$?

The Kleene Closure

- An important operation on languages is the ***Kleene Closure***, which is defined as

$$\boldsymbol{L^* = \{ w \in \Sigma^* \mid \exists n \in \mathbb{N}.\ w \in L^n \}}$$

- Mathematically:

$$\boldsymbol{w \in L^* \quad \textbf{iff} \quad \exists n \in \mathbb{N}.\ w \in L^n}$$

- Intuitively, all possible ways of concatenating zero or more strings in L together, possibly with repetition.

The Kleene Closure

If L = { **a**, **bb** }, then L^* = {

ε,

a, **bb**,

aa, **abb**, **bba**, **bbbb**,

aaa, **aabb**, **abba**, **abbbb**, **bbaa**, **bbabb**, **bbbba**, **bbbbbb**,

…

}

Think of L* as the set of strings you can make if you have a collection of stamps – one for each string in L – and you form every possible string that can be made from those stamps.

Closure Properties

- ***Theorem:*** If L_1 and L_2 are regular languages over an alphabet Σ, then so are the following languages:
  - $\overline{L_1}$
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - L_1L_2
  - L_1*
- These properties are called ***closure properties of the regular languages***.

New Stuff!

Another View of Regular Languages

Rethinking Regular Languages

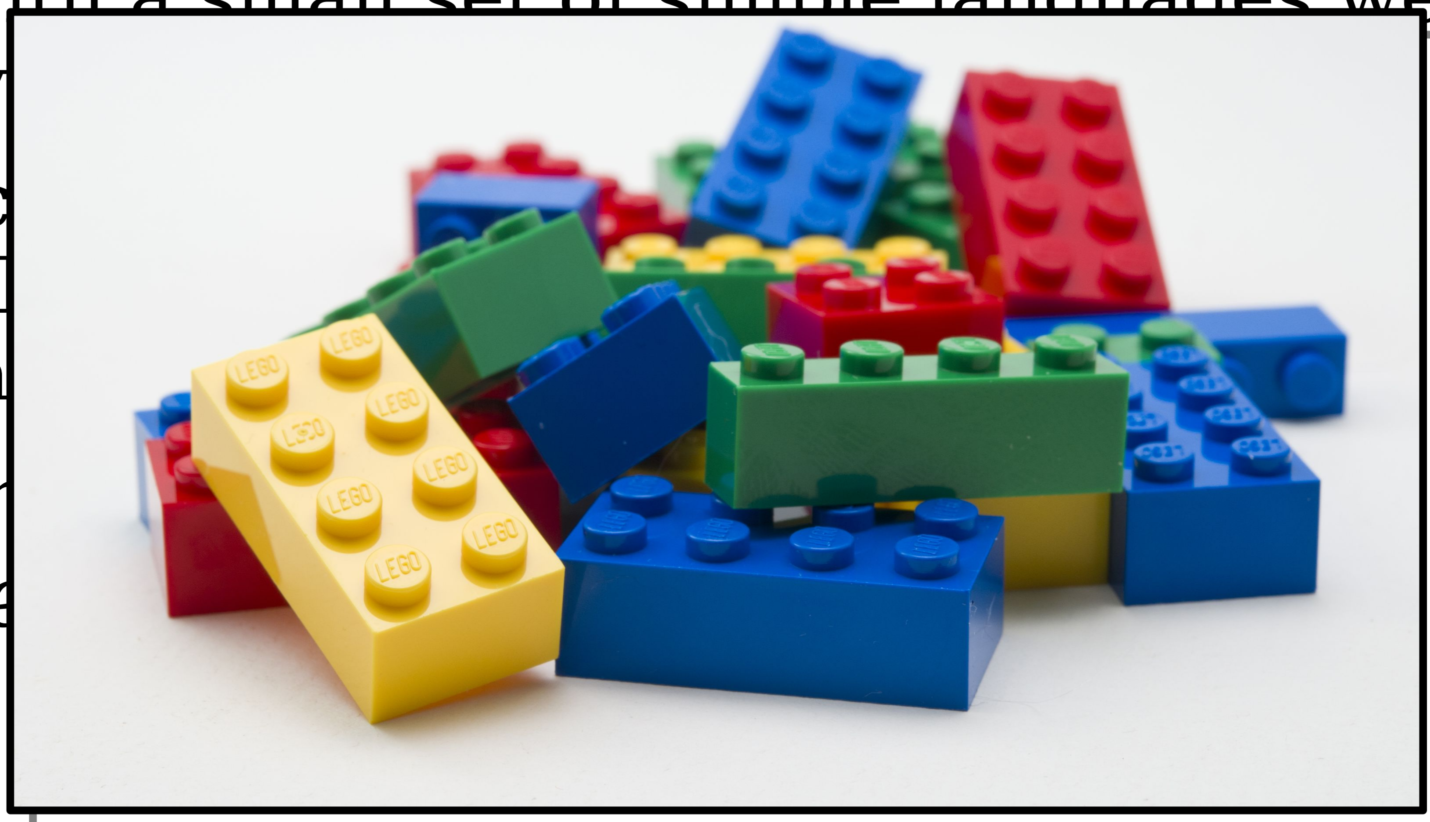
- We currently have several tools for showing a language L is regular:
  - Construct a DFA for L.
  - Construct an NFA for L.
  - Combine several simpler regular languages together via closure properties to form L.
- We have not spoken much of this last idea.

Constructing Regular Languages

- ***Idea:*** Build up all regular languages as follows:
  - Start with a small set of simple languages we already know to be regular.
  - Using closure properties, combine these simple languages together to form more elaborate languages.
- *A bottom-up approach to the regular languages.*

Constructing Regular Languages

- ***Idea:*** Build up all regular languages as follows:
  - Start with a small set of simple languages we already
  - Using c simple l elabora
- *A bottom language*



Regular Expressions

- ***Regular expressions*** are a way of describing a language via a string representation.
- They're used extensively in software systems for string processing and as the basis for tools like `grep` and `flex`.
- Conceptually, regular expressions are strings describing how to assemble a larger language out of smaller pieces.

Atomic Regular Expressions

- The regular expressions begin with three simple building blocks.
- The symbol **Ø** is a regular expression that represents the empty language Ø.
- For any **a** ∈ Σ, the symbol **a** is a regular expression for the language {**a**}.
- The symbol **ε** is a regular expression that represents the language {ε}.
  - ***Remember:* {ε} ≠ Ø!**
  - ***Remember:* {ε} ≠ ε!**

Compound Regular Expressions

- If R_1 and R_2 are regular expressions, $\boldsymbol{R_1 R_2}$ is a regular expression for the *concatenation* of the languages of R_1 and R_2.
- If R_1 and R_2 are regular expressions, $\boldsymbol{R_1 \cup R_2}$ is a regular expression for the *union* of the languages of R_1 and R_2.
- If R is a regular expression, $\boldsymbol{R^*}$ is a regular expression for the *Kleene closure* of the language of R.
- If R is a regular expression, $\boldsymbol{(R)}$ is a regular expression with the same meaning as R.

Operator Precedence

- Here's the operator precedence for regular expressions, from highest to lowest:

$$(R)$$

$$R^*$$

$$R_1 R_2$$

$$R_1 \cup R_2$$

Consider the regular expression

ab*c∪d

How many of the strings below are in the language described by this regular expression?

ababc
abd
ac
abcd

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **a number**.

Regular Expression Examples

- The regular expression **trick∪treat** represents the regular language { **trick**, **treat** }.
- The regular expression **booo*** represents the regular language { **boo**, **booo**, **boooo**, … }.
- The regular expression **candy!(candy!)*** represents the regular language { **candy!**, **candy!candy!**, **candy!candy!candy!**, … }.

Regular Expressions, Formally

- The ***language of a regular expression*** is the language described by that regular expression.
- Formally:
  - $\mathscr{L}(\boldsymbol{\varepsilon}) = \{\varepsilon\}$
  - $\mathscr{L}(\boldsymbol{\emptyset}) = \emptyset$
  - $\mathscr{L}(\mathbf{a}) = \{\mathbf{a}\}$
  - $\mathscr{L}(R_1R_2) = \mathscr{L}(R_1)\ \mathscr{L}(R_2)$
  - $\mathscr{L}(R_1 \cup R_2) = \mathscr{L}(R_1) \cup \mathscr{L}(R_2)$
  - $\mathscr{L}(R\mathbf{*}) = \mathscr{L}(R)^*$
  - $\mathscr{L}(\mathbf{(}R\mathbf{)}) = \mathscr{L}(R)$

Worthwhile activity: Apply this recursive definition to

a(b∪c)((d))

and see what you get.

Designing Regular Expressions

- Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.
- Let $L = \{\ w \in \Sigma^* \mid w$ contains `aa` as a substring $\}$.

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(a ∪ b)*aa(a ∪ b)*

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(a ∪ b)*aa(a ∪ b)*

bbabbbaabab

aaaa

bbbbbabbbbaabbbb

Designing Regular Expressions

- Let Σ = {**a**, **b**}.
- Let L = { $w \in \Sigma^*$ | w contains **aa** as a substring }.

(a ∪ b)*aa(a ∪ b)*

bbabbbaabab

aaaa

bbbbbabbbbaabbbbb

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains **aa** as a substring $\}$.

Σ*aaΣ*

bbabbbaabab
aaaa
bbbbbabbbbaabbbbb

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

Designing Regular Expressions

Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.

Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

The length of a string w is denoted $|w|$

Designing Regular Expressions

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$$\mathbf{\Sigma\Sigma\Sigma\Sigma}$$

Designing Regular Expressions

- Let $\Sigma = \{$**`a`**, **`b`**$\}$.
- Let $L = \{\ w \in \Sigma^* \mid |w| = 4\ \}$.

$\Sigma\Sigma\Sigma\Sigma$

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

ΣΣΣΣ

aaaa
baba
bbbb
baaa

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

$\Sigma\Sigma\Sigma\Sigma$

aaaa
baba
bbbb
baaa

Designing Regular Expressions

- Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

Σ^4

aaaa
baba
bbbb
baaa

Designing Regular Expressions

- Let $\Sigma = \{\texttt{a}, \texttt{b}\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

Σ^4

aaaa
baba
bbbb
baaa

Designing Regular Expressions

- Let Σ = {**a**, **b**}.
- Let L = { $w \in \Sigma^*$ | w contains at most one **a** }.

Which of the following is a regular expression for *L?*

A. **Σ*aΣ***
B. **b*ab* ∪ b***
C. **b*(a ∪ ε)b***
D. **b*a*b* ∪ b***
E. **b*(a* ∪ ε)b***
F. None of the above, or two or more of the above.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then ***A***, ***B***, ***C***, ***D***, ***E***, or ***F***.

Designing Regular Expressions

- Let Σ = {**a**, **b**}.
- Let L = { $w \in \Sigma^*$ | w contains at most one **a** }.

b*(a ∪ ε)b*

Designing Regular Expressions

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Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
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b*(a ∪ ε)b*

bbbbabbb
bbbbbb
abbb
a

Designing Regular Expressions

- Let Σ = {a, b}.
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b*(a ∪ ε)b*

bbbbabbb
bbbbbb
abbb
a

Designing Regular Expressions

- Let Σ = {**a**, **b**}.
- Let $L = \{ w \in \Sigma^* \mid w$ contains at most one **a** }.

b*a?b*

bbbbabbb
bbbbbb
abbb
a

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

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dot.at@dot.com

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`aa*`

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```
aa* (.aa*)*
```

```
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aa* (.aa*)* @ aa*.aa*

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```

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$$\mathtt{a}^{+}\,\mathtt{(.aa^*)^*}\;\mathtt{@}\;\mathtt{aa^*.aa^*}\,\mathtt{(.aa^*)^*}$$

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- Let Σ = { `a`, `.`, `@` }, where `a` represents "some letter."
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$$\texttt{a}^{+}\ (\texttt{.aa*})^{*}\ \texttt{@}\ \texttt{aa*.aa*}(\texttt{.aa*})^{*}$$

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- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
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$$\mathtt{a}^{+} \quad (\mathtt{.a}^{+})^{*} \quad \mathtt{@} \quad \mathtt{a}^{+} \qquad (\mathtt{.a}^{+})^{+}$$

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$$\mathtt{a}^+ (\mathtt{.a}^+)^* \ \mathtt{@} \ \mathtt{a}^+ (\mathtt{.a}^+)^+$$

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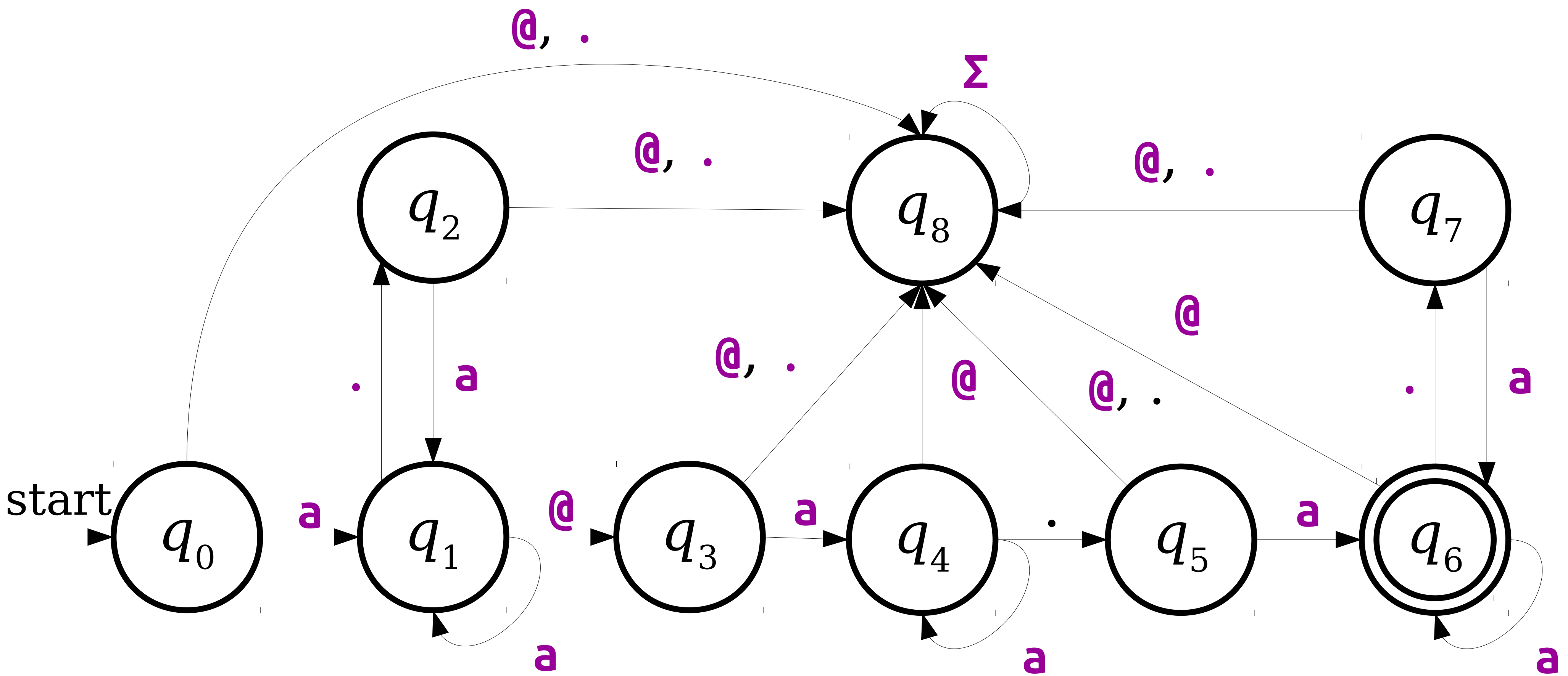
For Comparison

a⁺(.a⁺)*@a⁺(.a⁺)⁺



Shorthand Summary

- R^n is shorthand for $RR \ldots R$ (n times).
  - Edge case: define $R^0 = \varepsilon$.
- Σ is shorthand for "any character in Σ."
- R? is shorthand for $(R \cup \varepsilon)$, meaning "zero or one copies of R."
- R^+ is shorthand for RR^*, meaning "one or more copies of R."

Time-Out for Announcements!

Midterm Exam Logistics

- The next midterm is ***Monday, February 26th*** from ***7:00PM – 10:00PM***. Locations are divvied up by last (family) name:
  - `A-I`: Go to ***Cubberley Auditorium***.
  - `J-Z`: Go to ***Cemex Auditorium***.
- The exam focuses on Lecture 06 – 13 (binary relations through induction) and PS3 – PS5. Finite automata onward is *not* tested.
  - Topics from earlier in the quarter (proofwriting, first-order logic, set theory, etc.) are also fair game, but that's primarily because the later material builds on this earlier material.
- The exam is closed-book, closed-computer, and limited-note. You can bring a double-sided, 8.5” × 11” sheet of notes with you to the exam, decorated however you'd like.
- Students with OAE accommodations: please contact us ***immediately*** if you haven't yet done so. We'll ping you about setting up alternate exams.

Practice Midterm Exam

- We'll be holding a practice midterm exam ***tonight*** from ***7PM – 10PM*** in ***320-105***.
- The practice midterm exam is composed of what we think is a good representative sample of older midterm questions from across the years. It’s probably the best indicator of what you should expect to see.
- Course staff will be on hand to answer your questions.
- Can't make it? We'll release the practice exam and solutions online. Set up your own practice exam time with a small group and work through it under realistic conditions!

Other Practice Materials

- We’ve posted four practice midterms to the course website, with solutions.
  - We’ll post the practice exam from this evening a little bit later, bringing the total to five.
- There’s also Extra Practice Problems 2, plus all the CS103A materials.
- Need more practice? Let us know and we’ll see what we can do!

Problem Sets

- Problem Set Five solutions are now out.
  - Please read over them – there's a lot of good stuff in there!
  - We'll get PS5 graded and returned as soon as we can.
- Problem Set Six is out and is due this Friday at 2:30PM.
  - ***Be careful about using late days here***, since the exam is on Monday.

Back to CS103!

The Power of Regular Expressions

Theorem: If R is a regular expression, then $\mathscr{L}(R)$ is regular.

Proof idea: Use induction!

- The atomic regular expressions all represent regular languages.
- The combination steps represent closure properties.
- So anything you can make from them must be regular!

Thompson’s Algorithm

- In practice, many regex matchers use an algorithm called ***Thompson's algorithm*** to convert regular expressions into NFAs (and, from there, to DFAs).
  - Read Sipser if you’re curious!
- ***Fun fact:*** the “Thompson” here is Ken Thompson, one of the co-inventors of Unix!

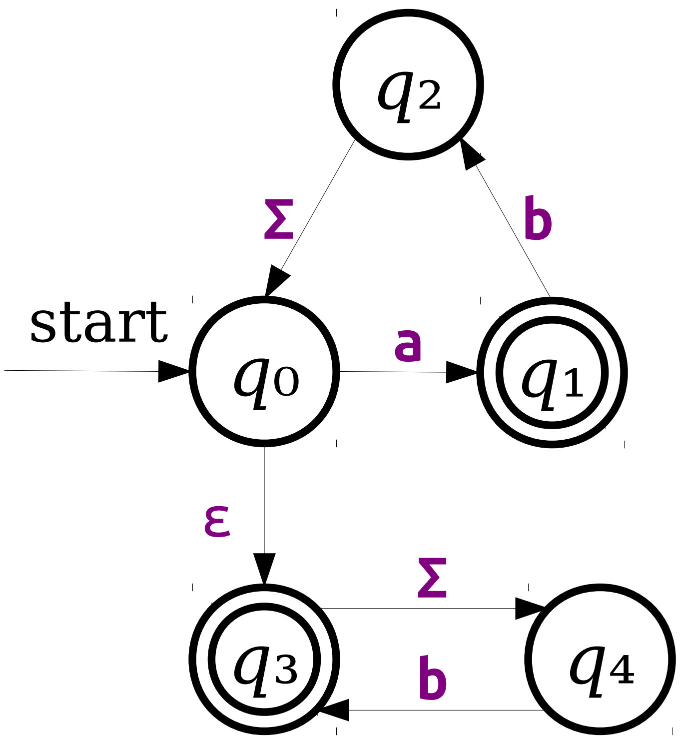
The Power of Regular Expressions

Theorem: If L is a regular language, then there is a regular expression for L.

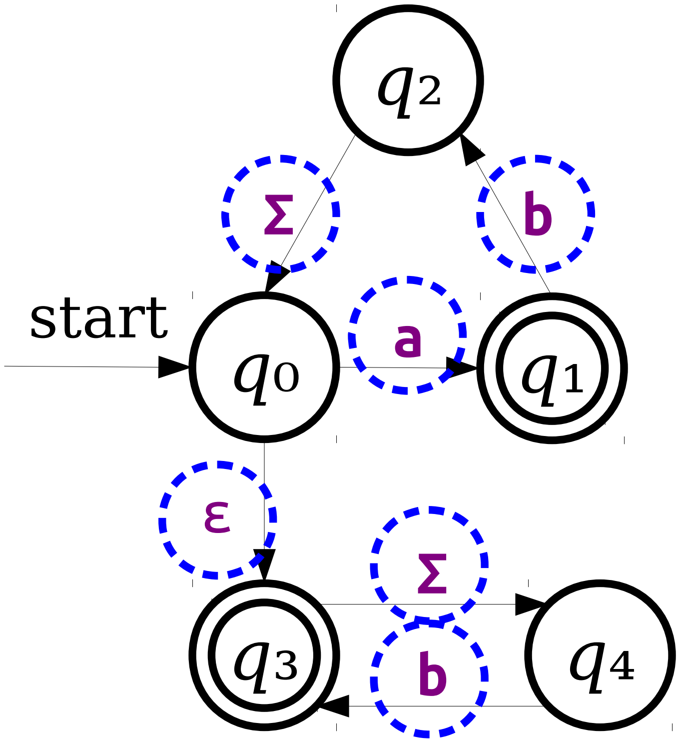
This is not obvious!

Proof idea: Show how to convert an arbitrary NFA into a regular expression.

Generalizing NFAs



Generalizing NFAs



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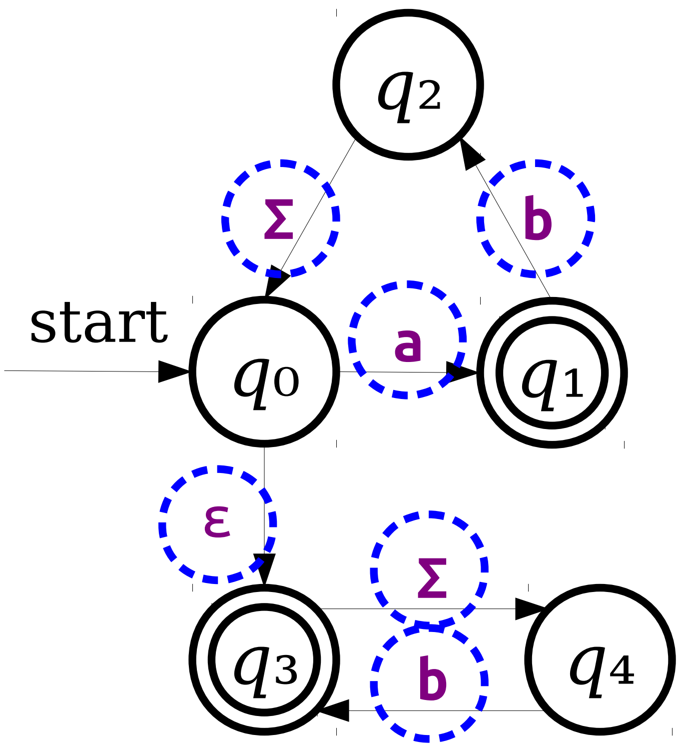
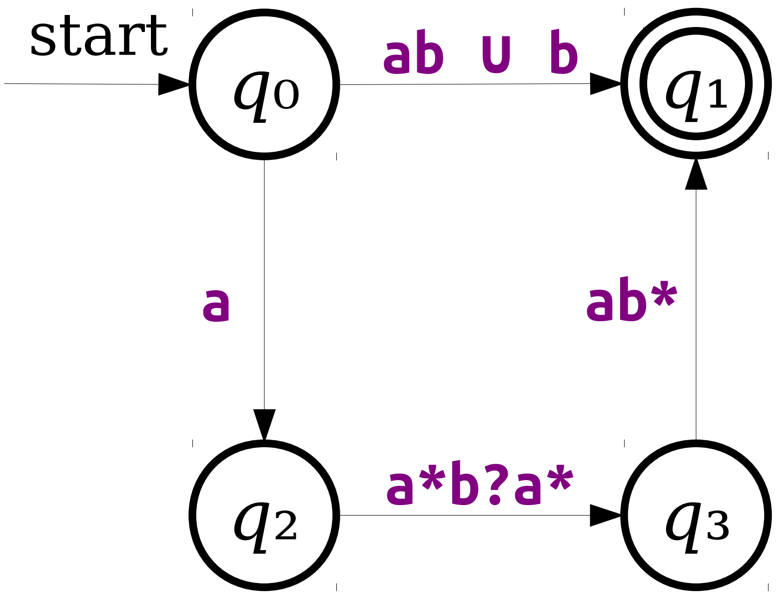


These are all regular expressions!

Generalizing NFAs



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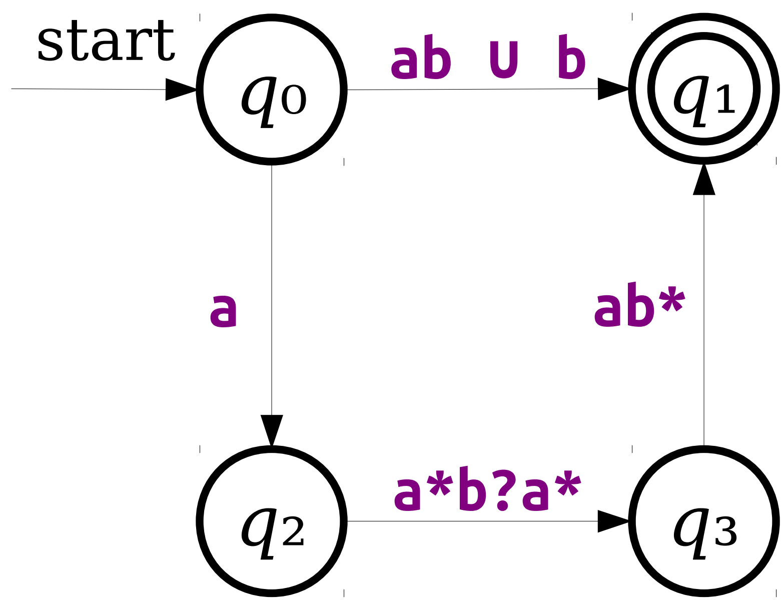
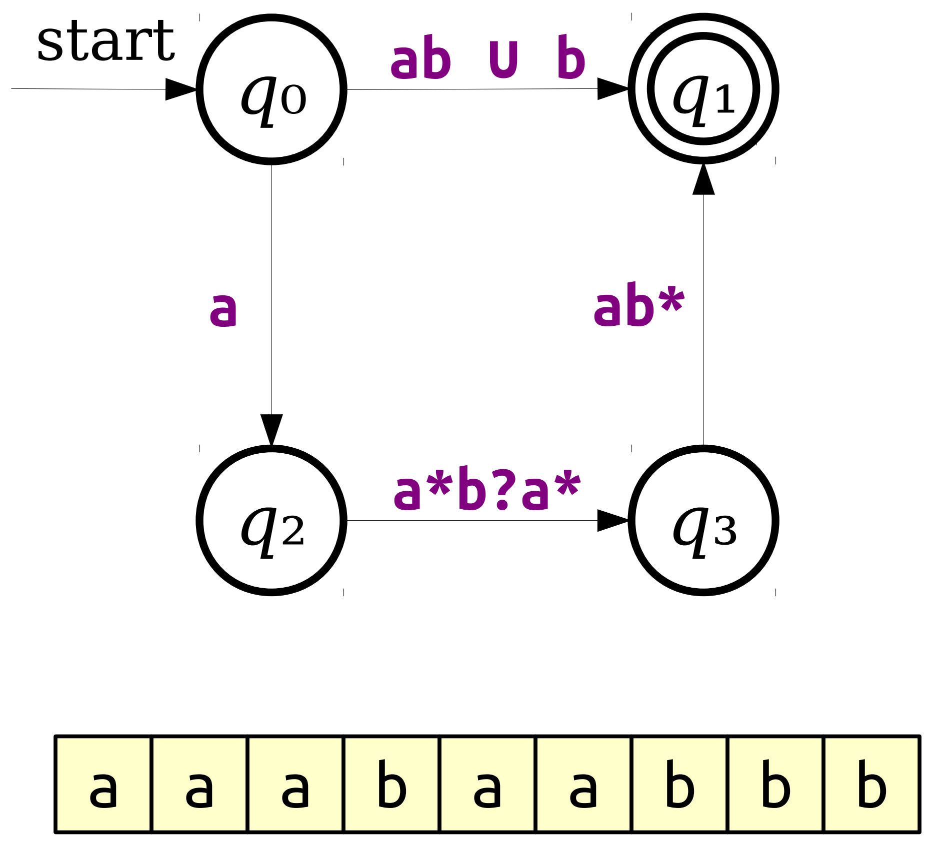
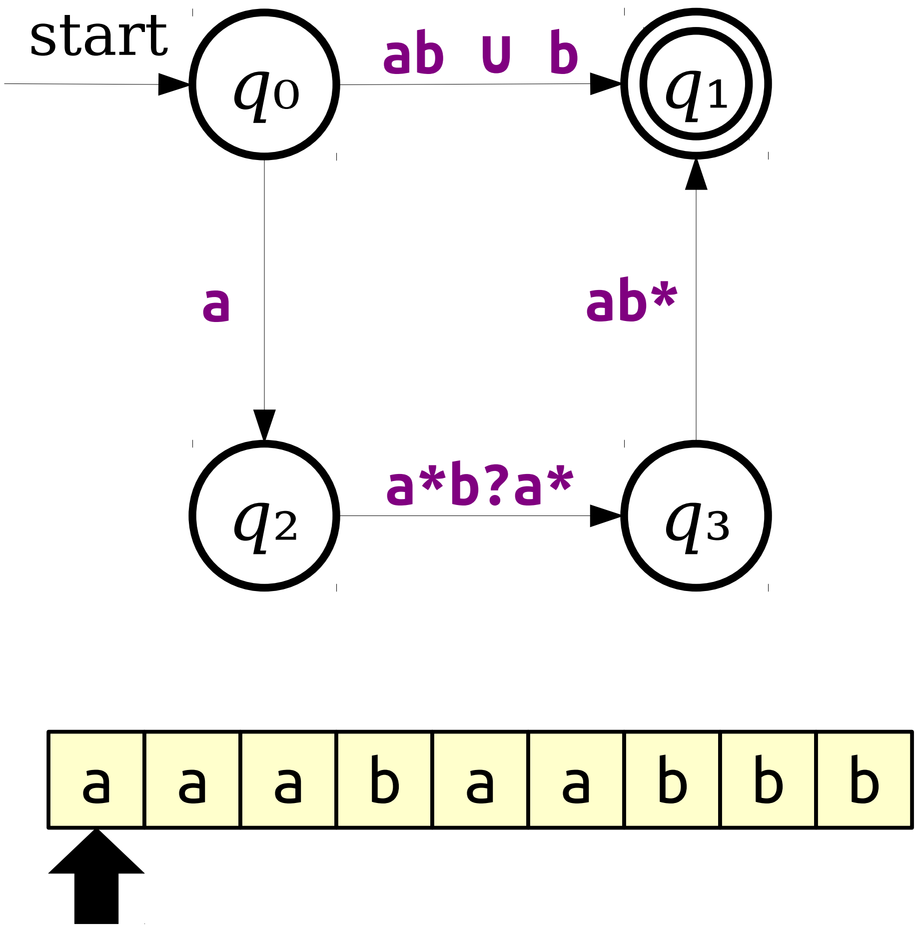


Note: Actual NFAs aren't allowed to have transitions like these. This is just a thought experiment.

Generalizing NFAs



Generalizing NFAs



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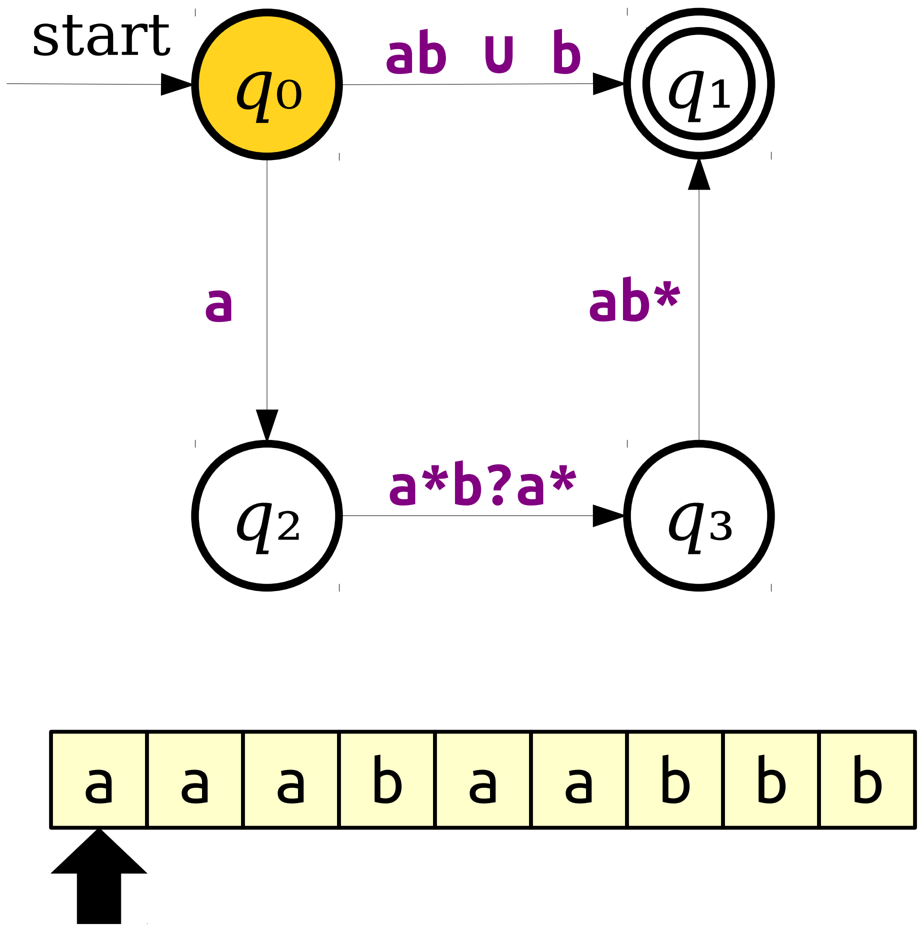

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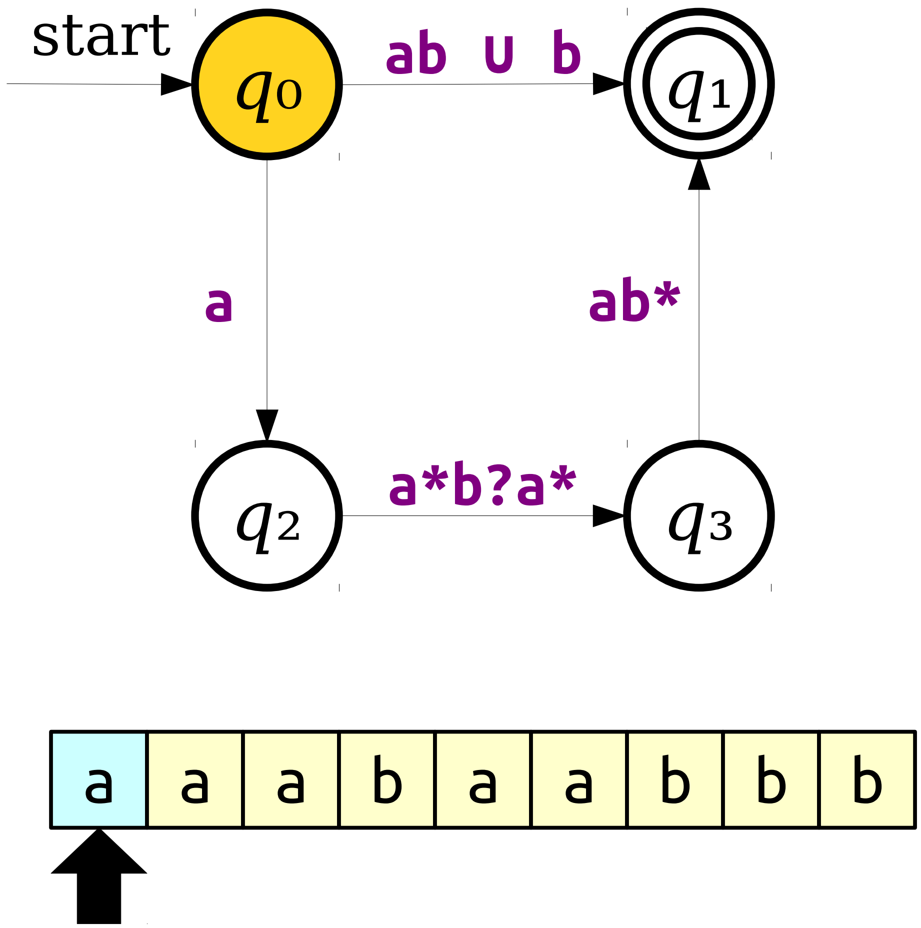
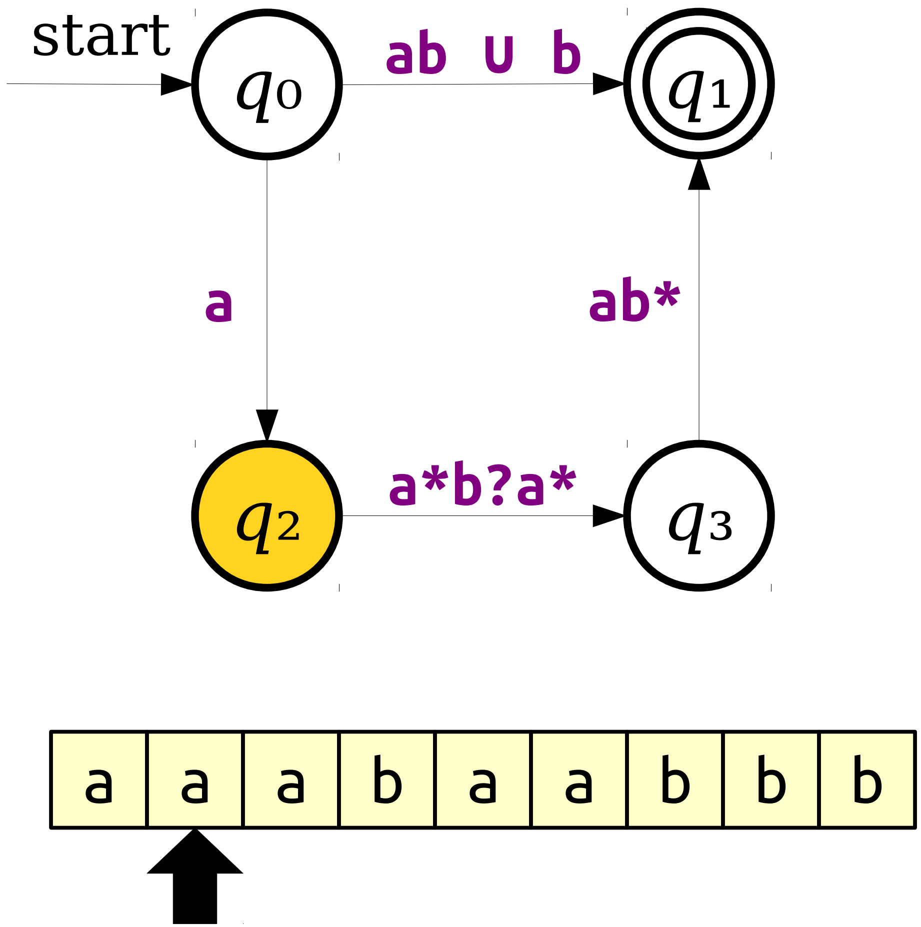
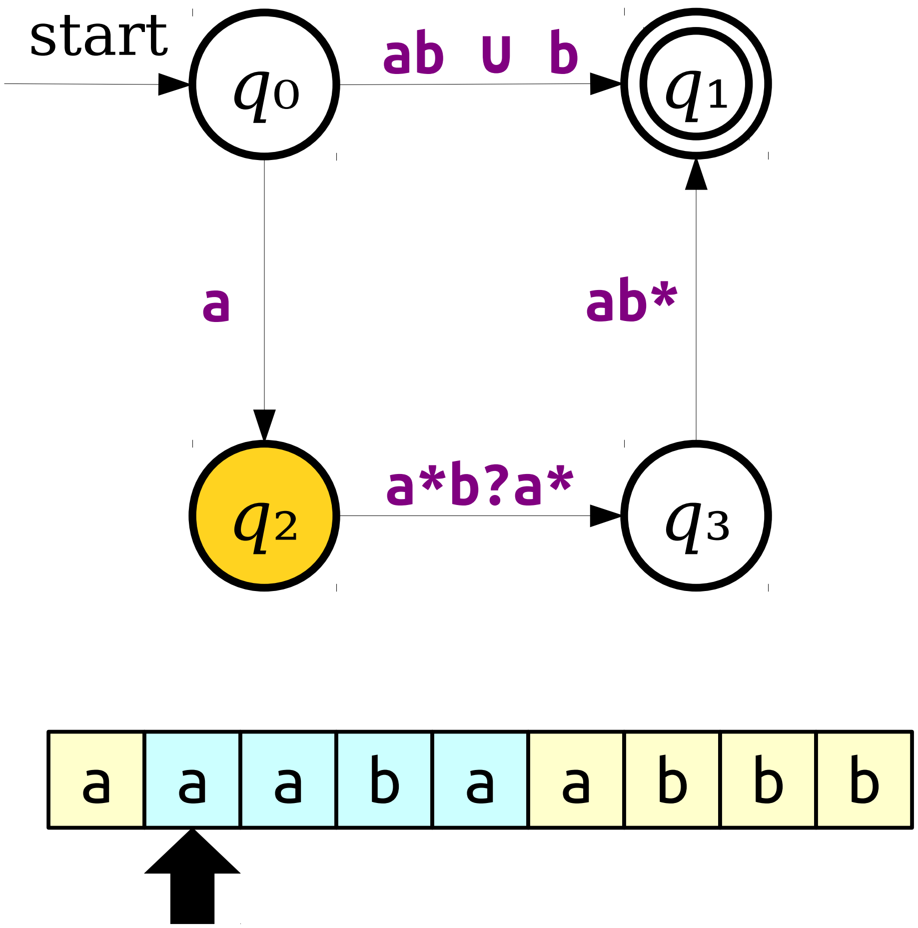


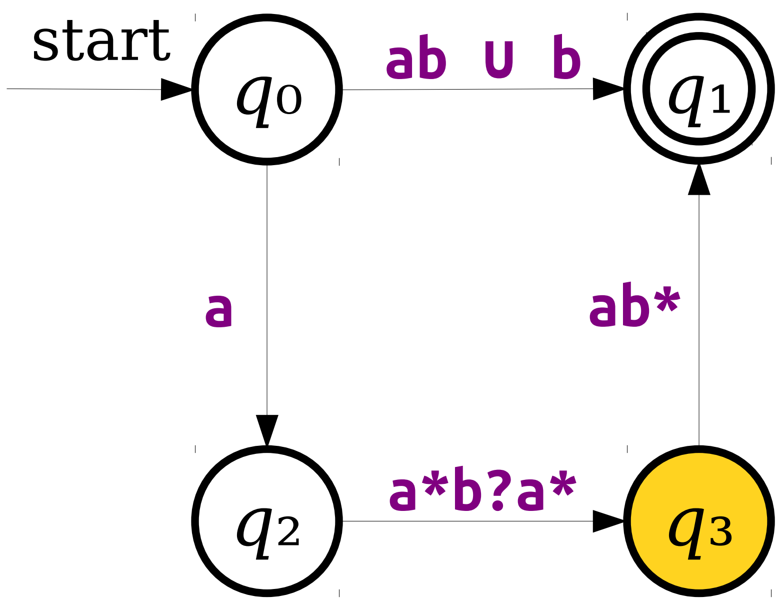
Generalizing NFAs

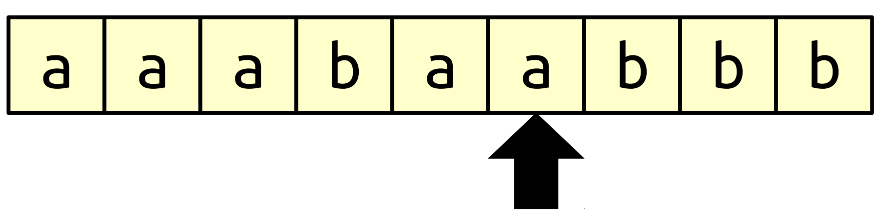


Generalizing NFAs

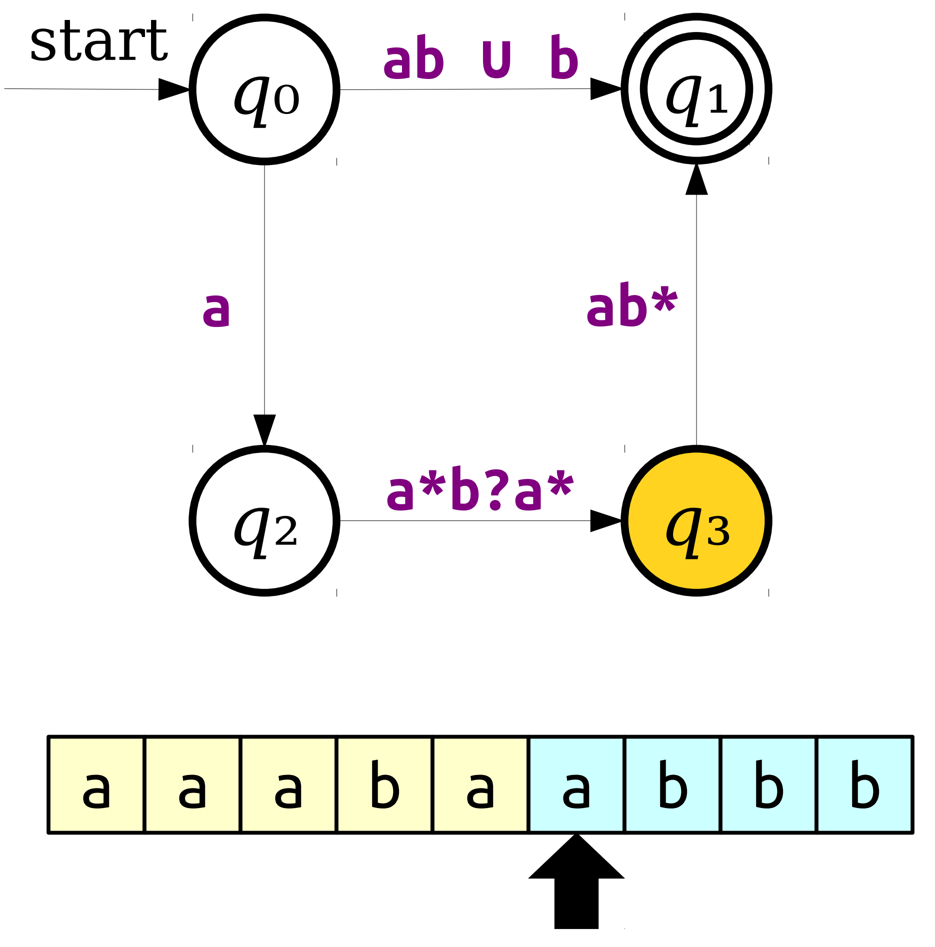


Generalizing NFAs

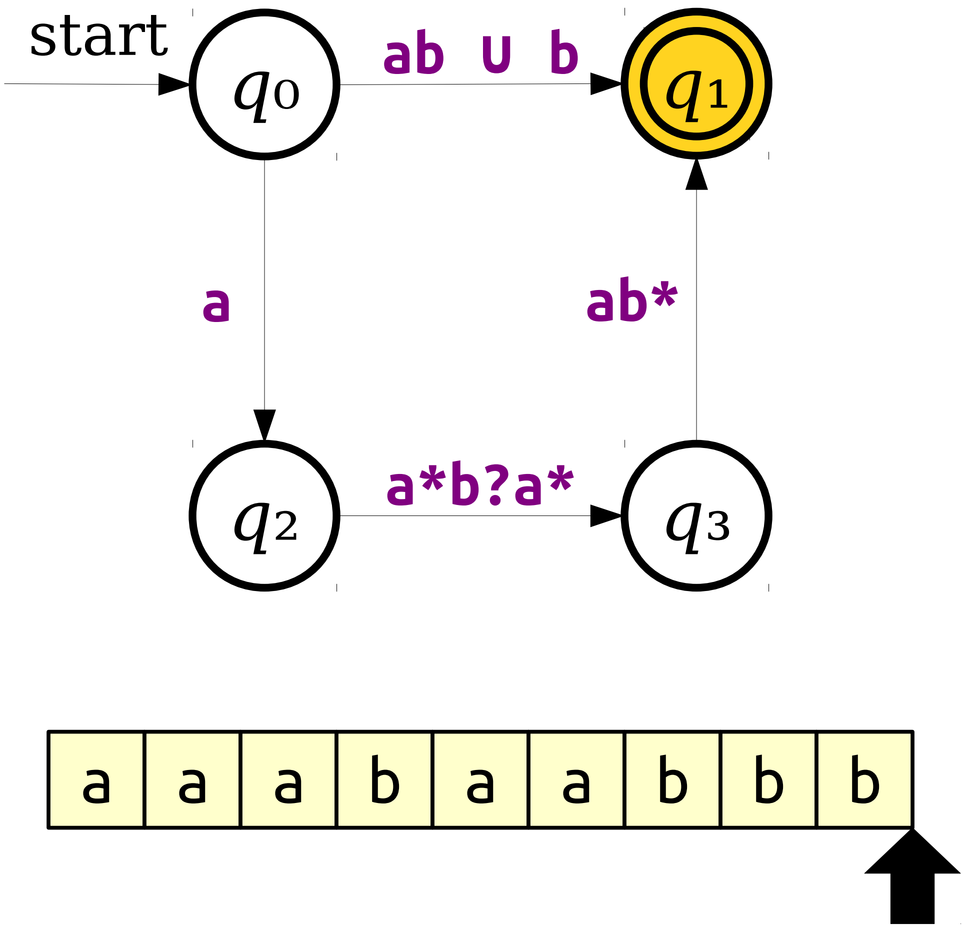




Generalizing NFAs



Generalizing NFAs



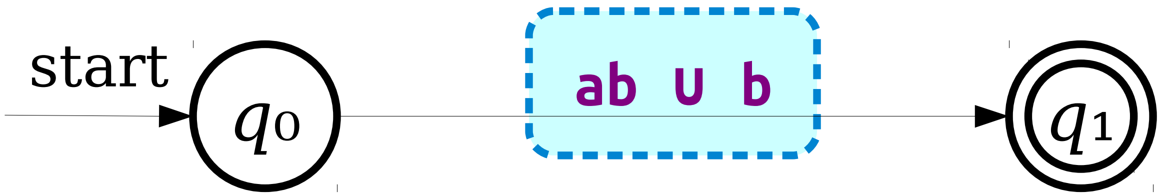
Key Idea 1: Imagine that we can label transitions in an NFA with arbitrary regular expressions.

Generalizing NFAs

Generalizing NFAs

Is there a simple regular expression for the language of this generalized NFA?

Generalizing NFAs



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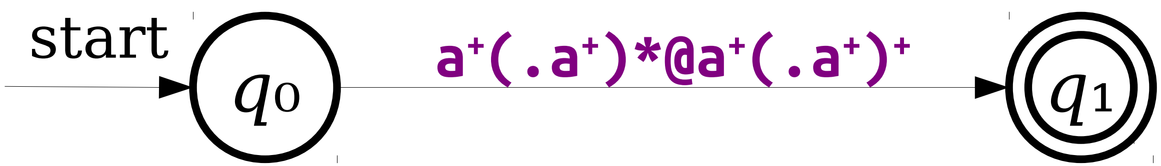
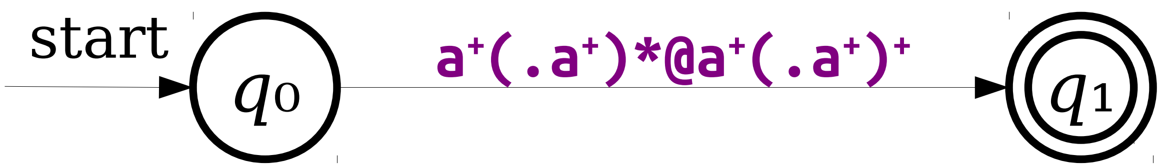
Generalizing NFAs

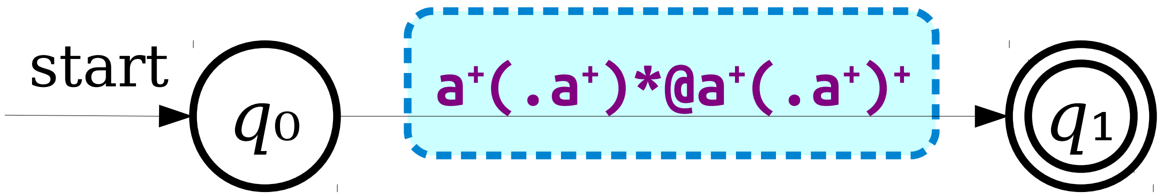


Generalizing NFAs



Is there a simple regular expression for the language of this generalized NFA?

Generalizing NFAs

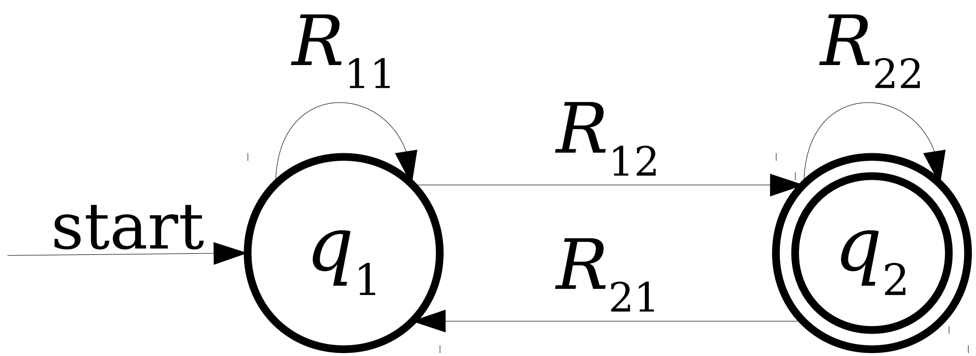


Is there a simple regular expression for the language of this generalized NFA?

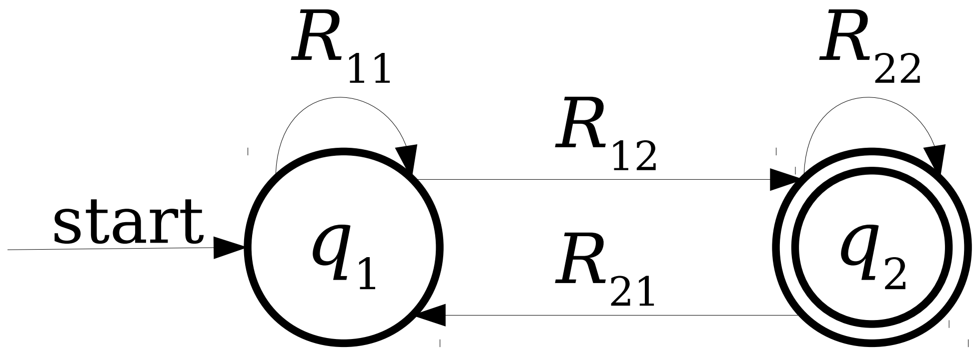
Key Idea 2: If we can convert an NFA into a generalized NFA that looks like this...

...then we can easily read off a regular expression for the original NFA.

From NFAs to Regular Expressions

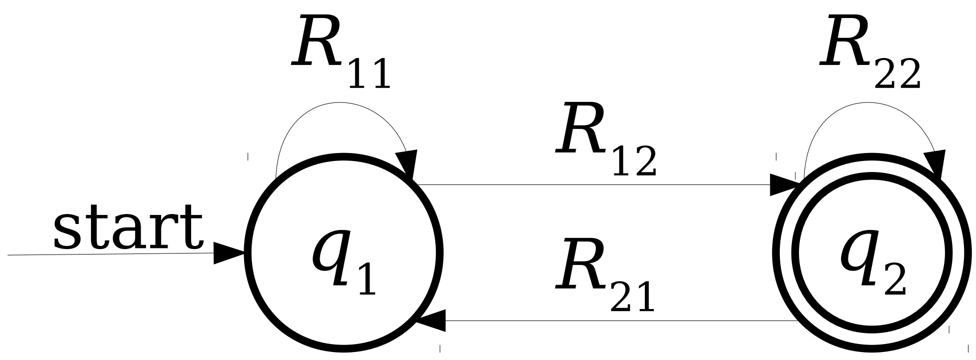


From NFAs to Regular Expressions



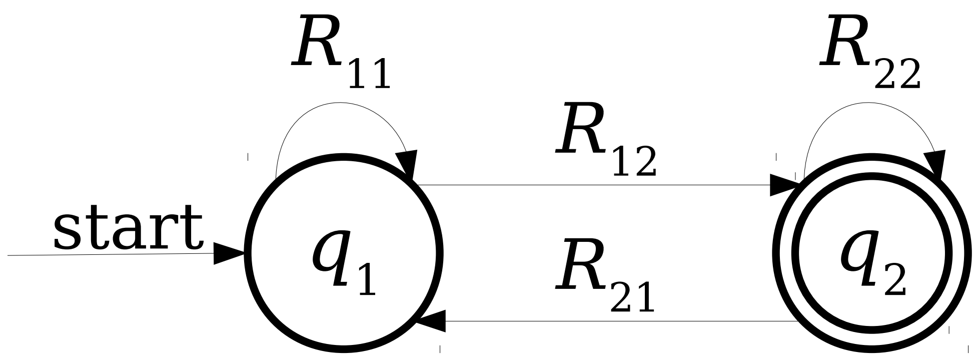
Here, R_{11}, R_{12}, R_{21}, and R_{22} are arbitrary regular expressions.

From NFAs to Regular Expressions



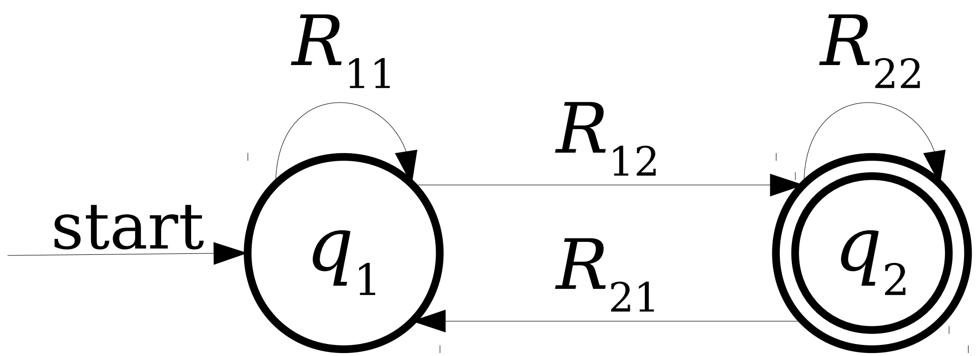
Question: Can we get a clean regular expression from this NFA?

From NFAs to Regular Expressions



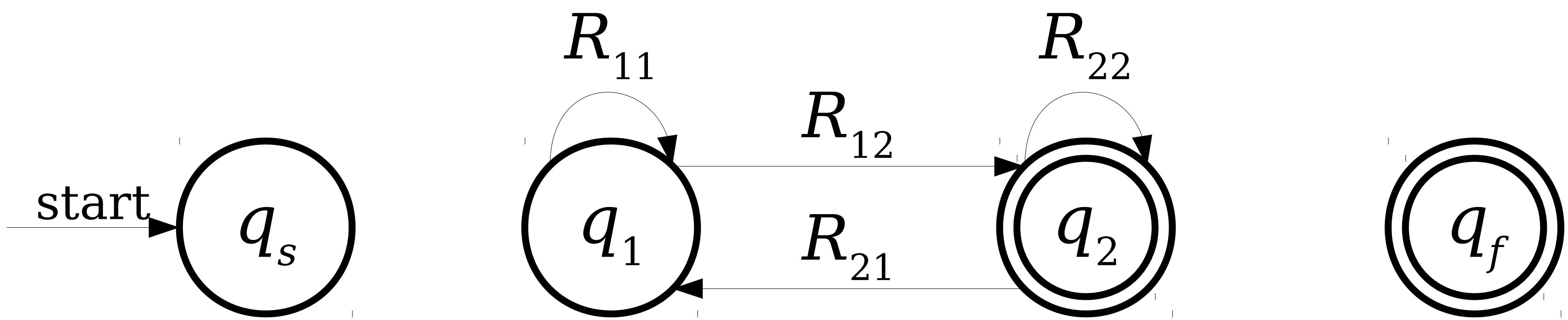
Key Idea 3: Somehow transform this NFA so that it looks like this:

From NFAs to Regular Expressions

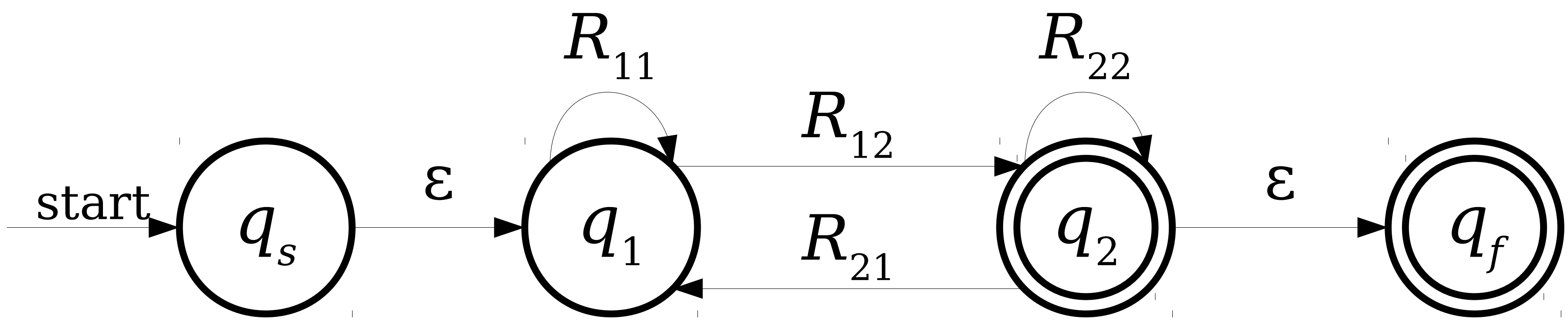


The first step is going to be a bit weird...

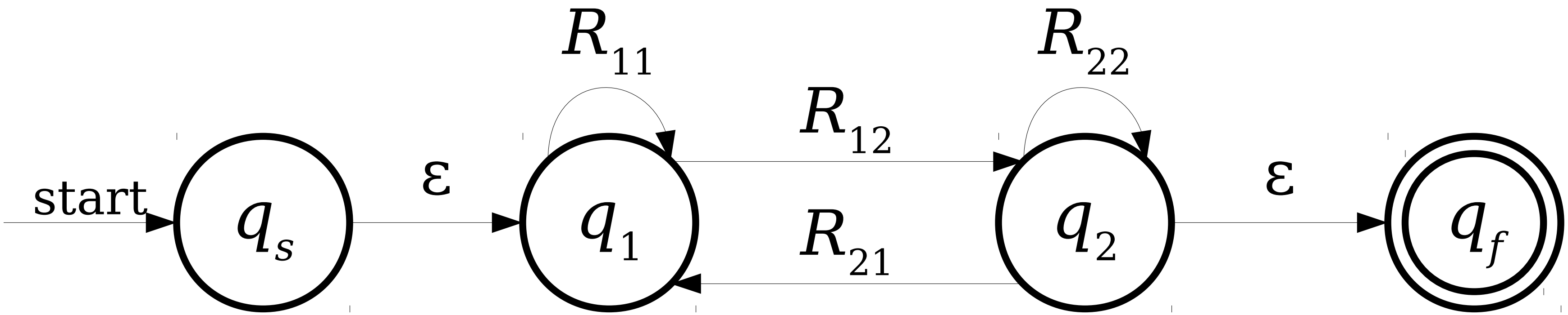
From NFAs to Regular Expressions



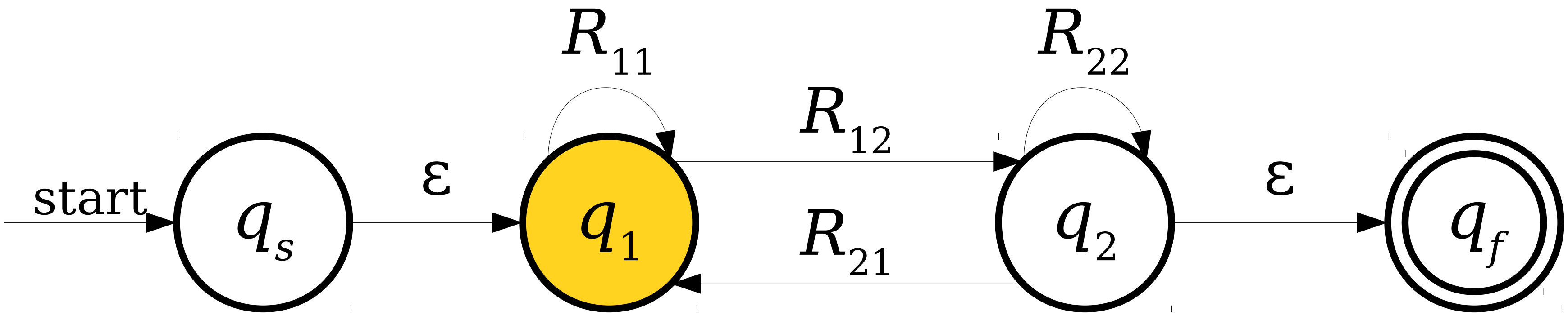
From NFAs to Regular Expressions



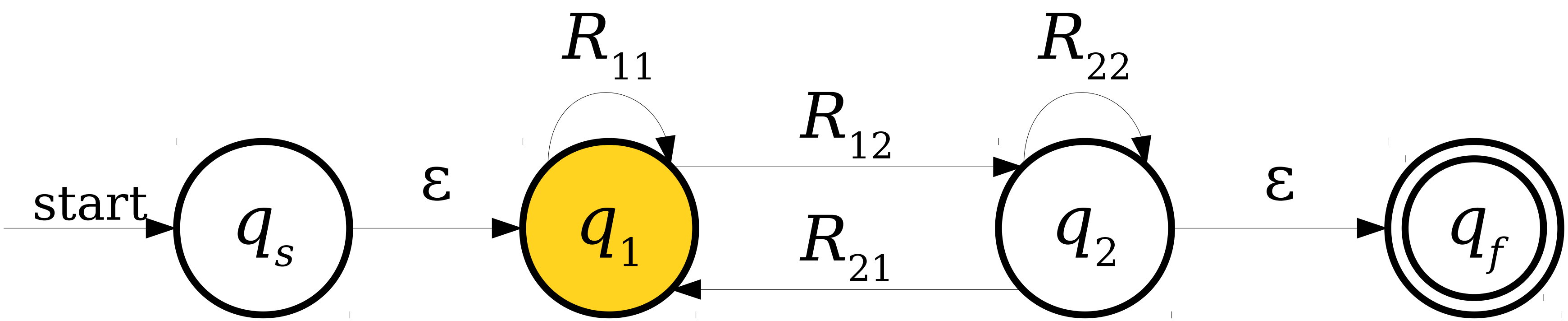
From NFAs to Regular Expressions



From NFAs to Regular Expressions

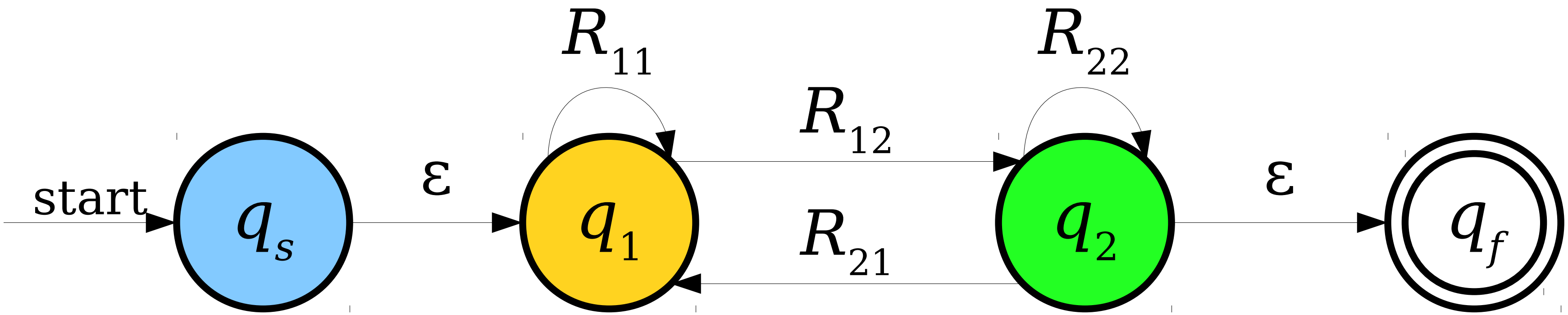


From NFAs to Regular Expressions

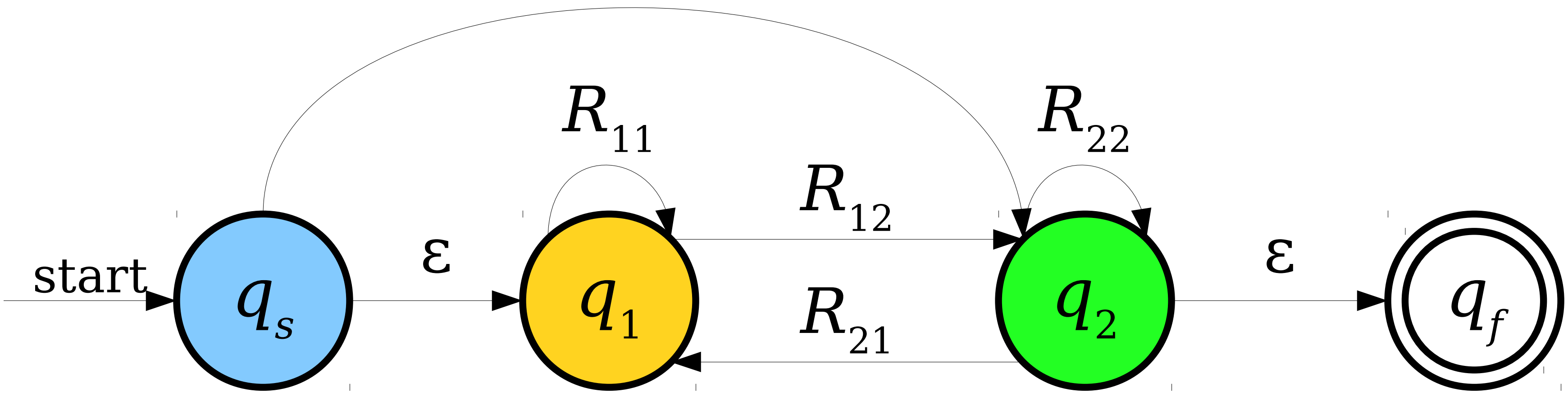


Could we eliminate this state from the NFA?

From NFAs to Regular Expressions



From NFAs to Regular Expressions



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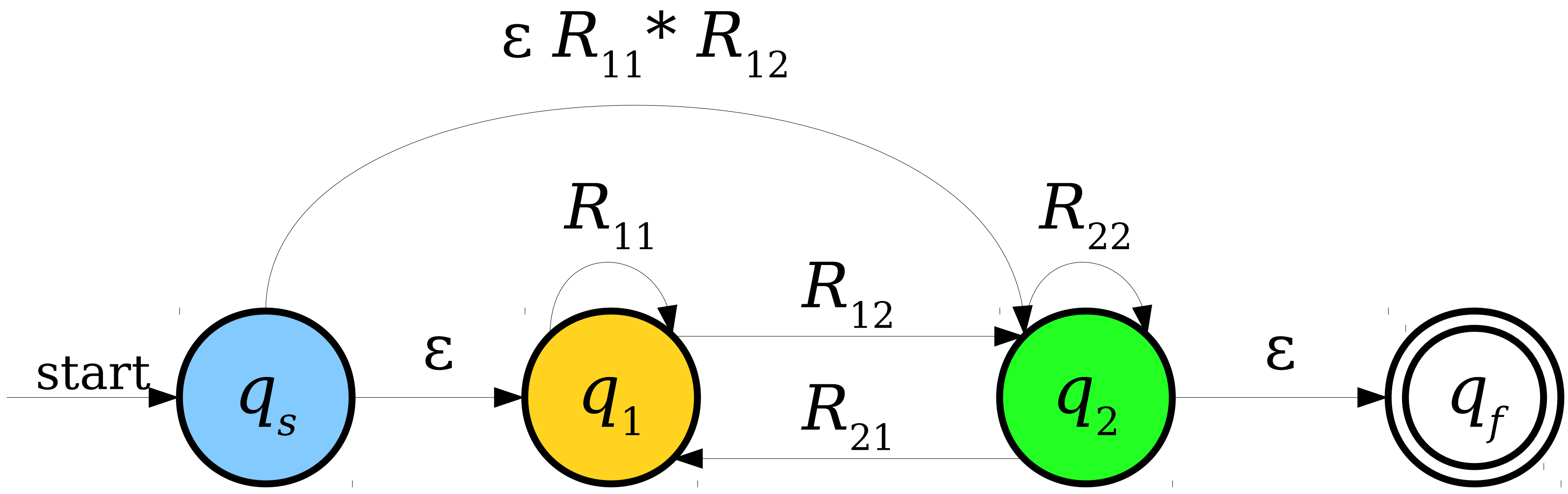
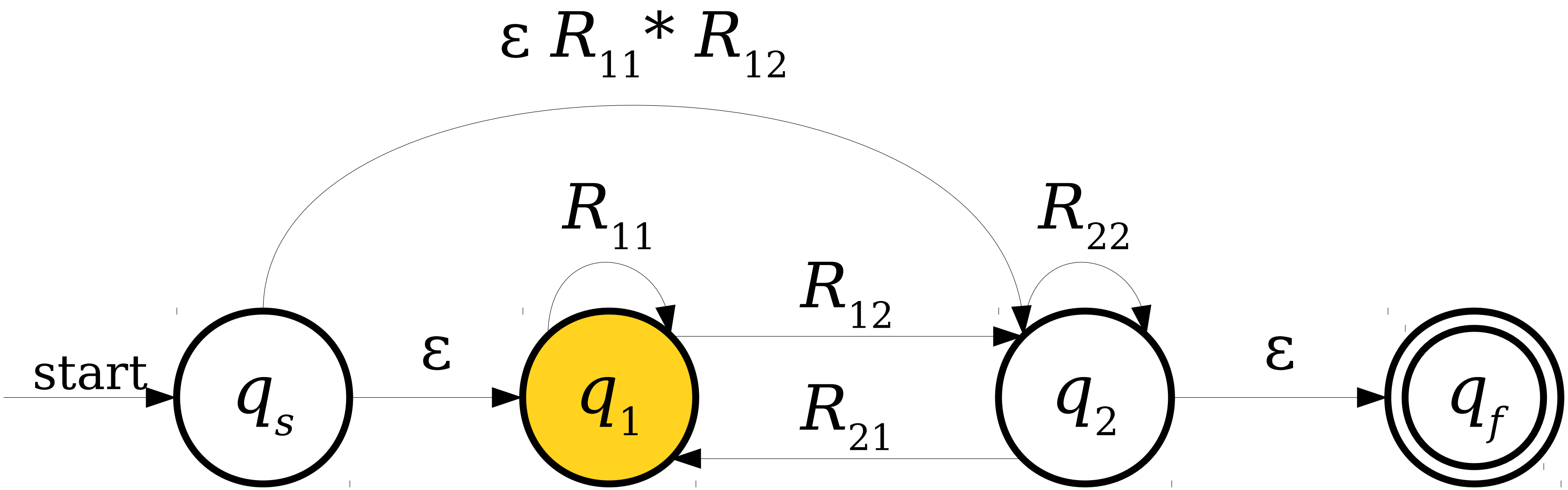
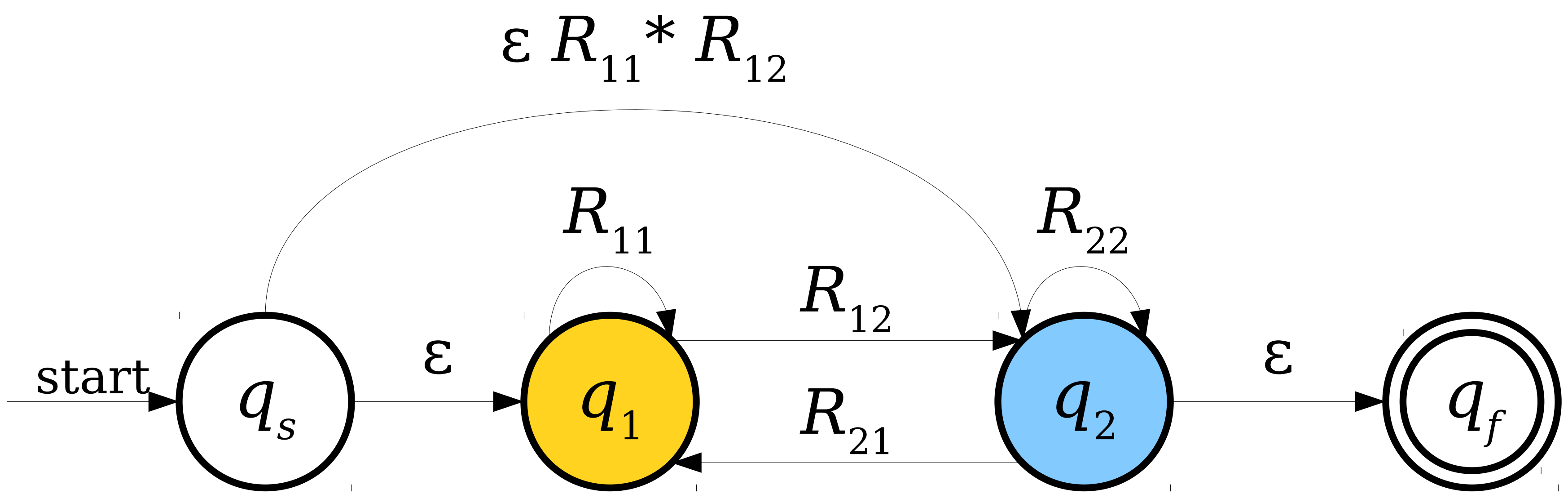


Note: We're using **concatenation** and **Kleene closure** in order to skip this state.

From NFAs to Regular Expressions



From NFAs to Regular Expressions



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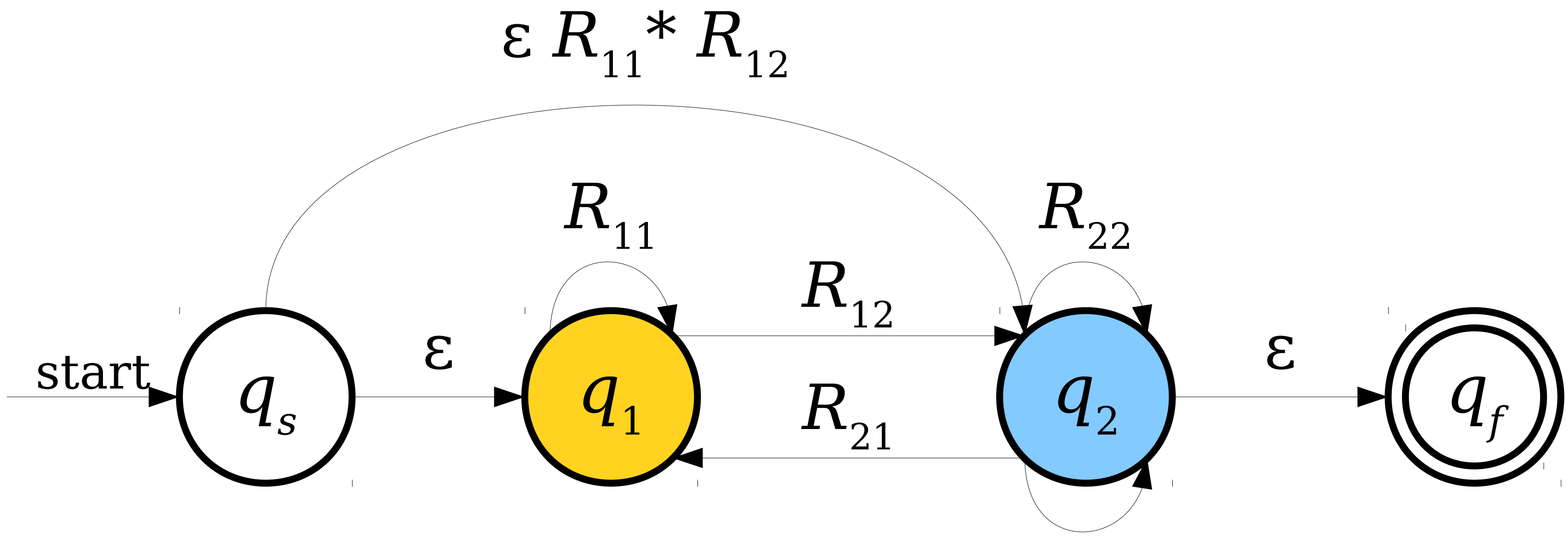
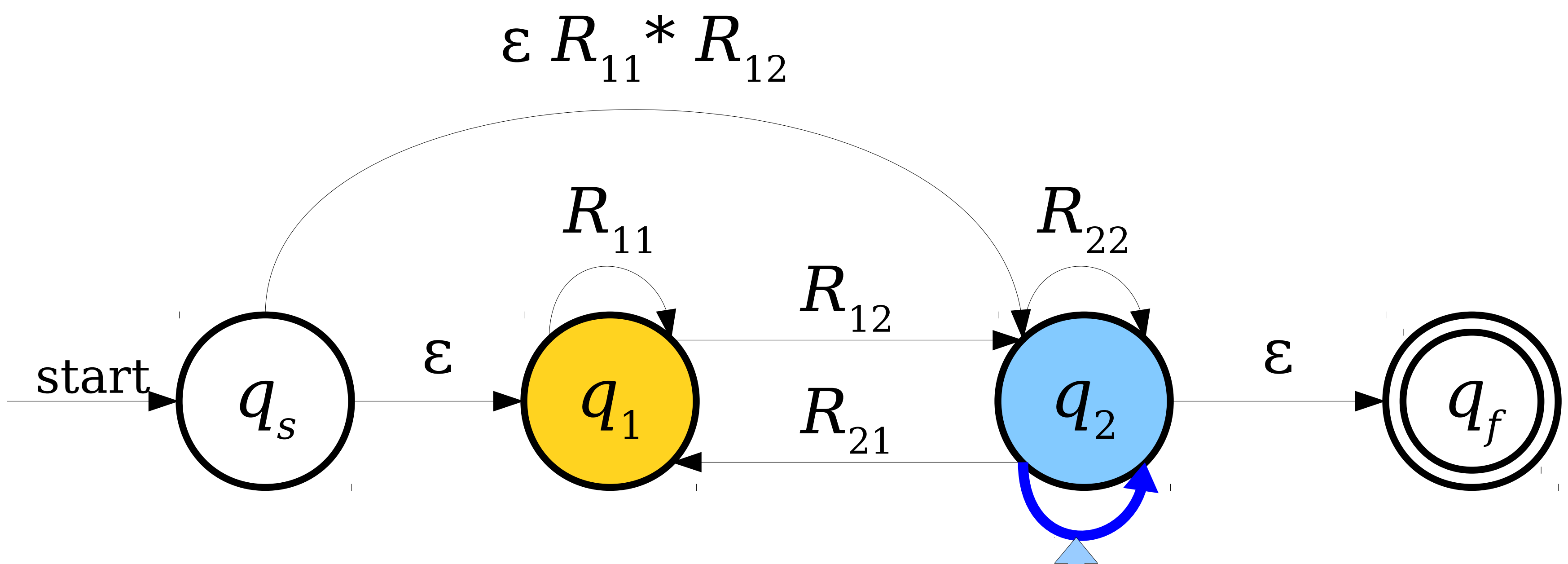


From NFAs to Regular Expressions

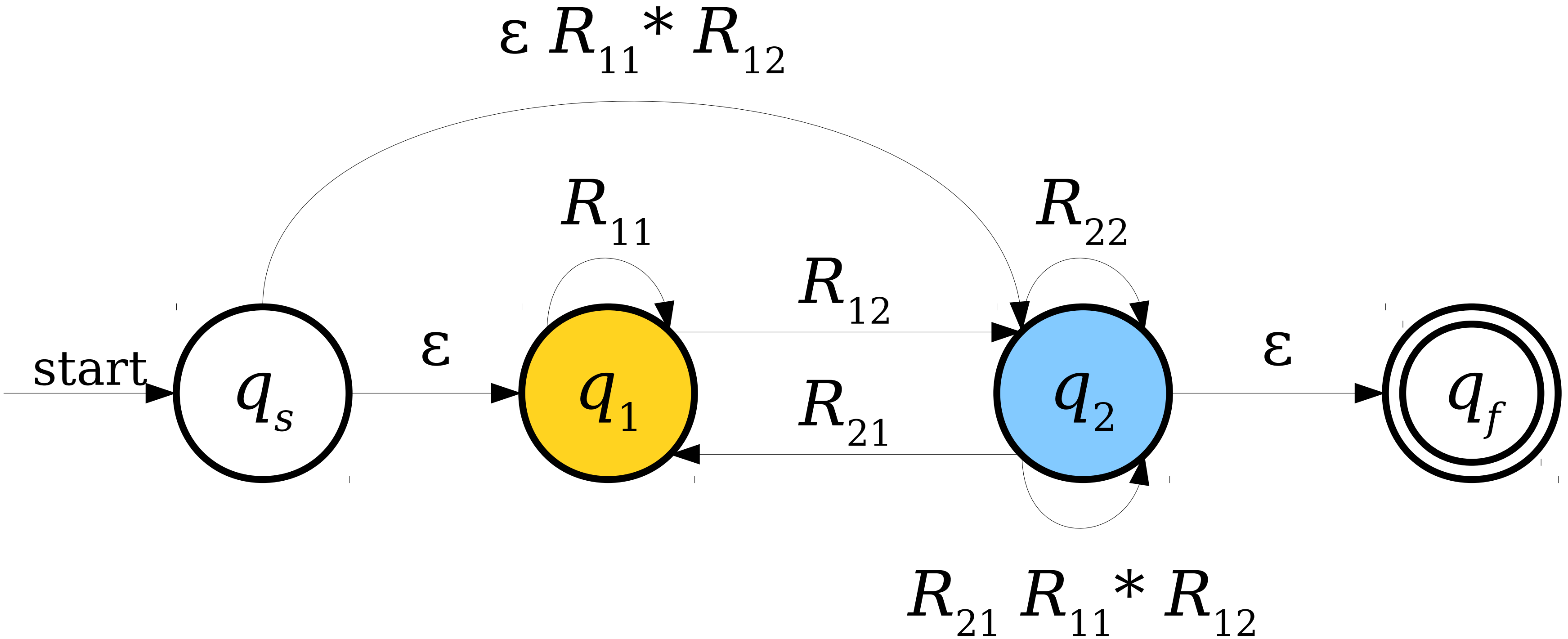


What regex should go on this edge?

A. $R_{12}\ R_{21}$ ***B.*** $R_{12}\ R_{22}^*\ R_{21}$ ***C.*** $R_{21}\ R_{12}$ ***D.*** $R_{21}\ R_{11}^*\ R_{12}$

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then ***A***, ***B***, ***C***, or ***D***.

From NFAs to Regular Expressions



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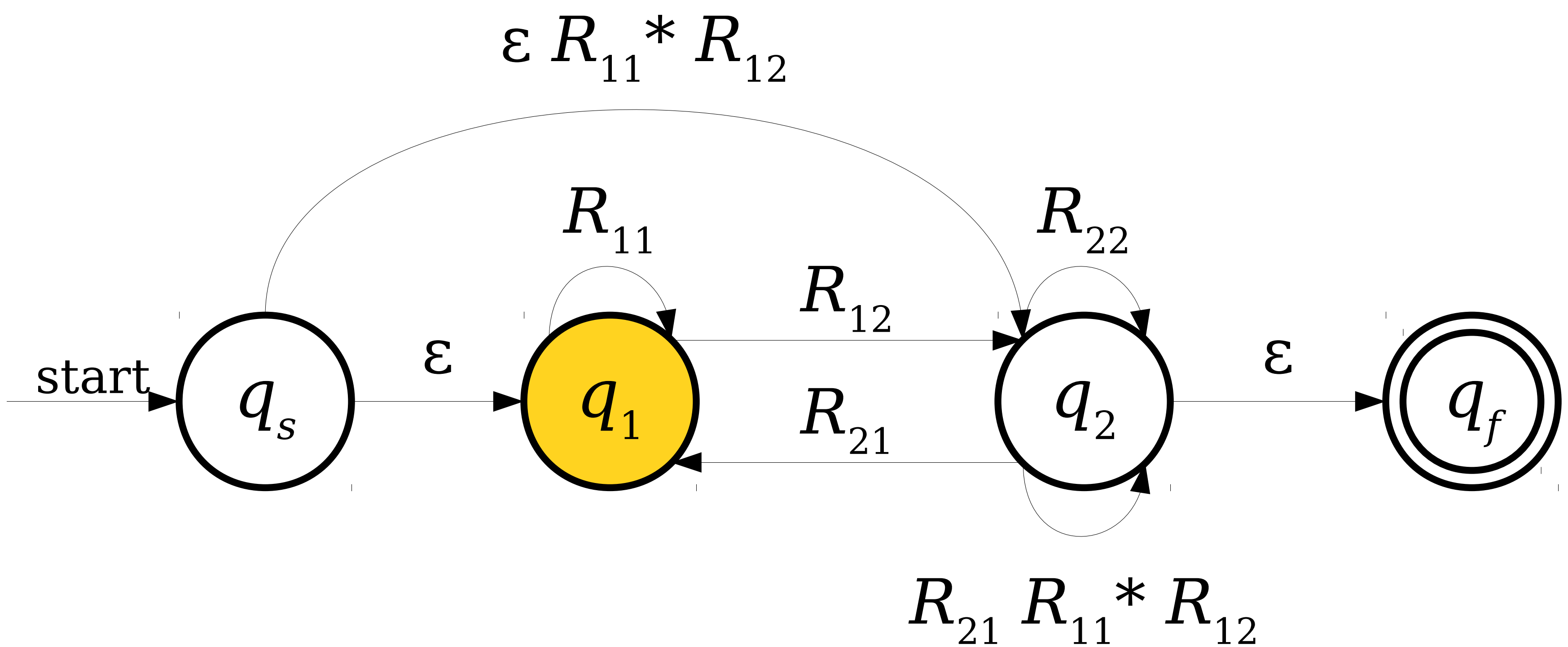

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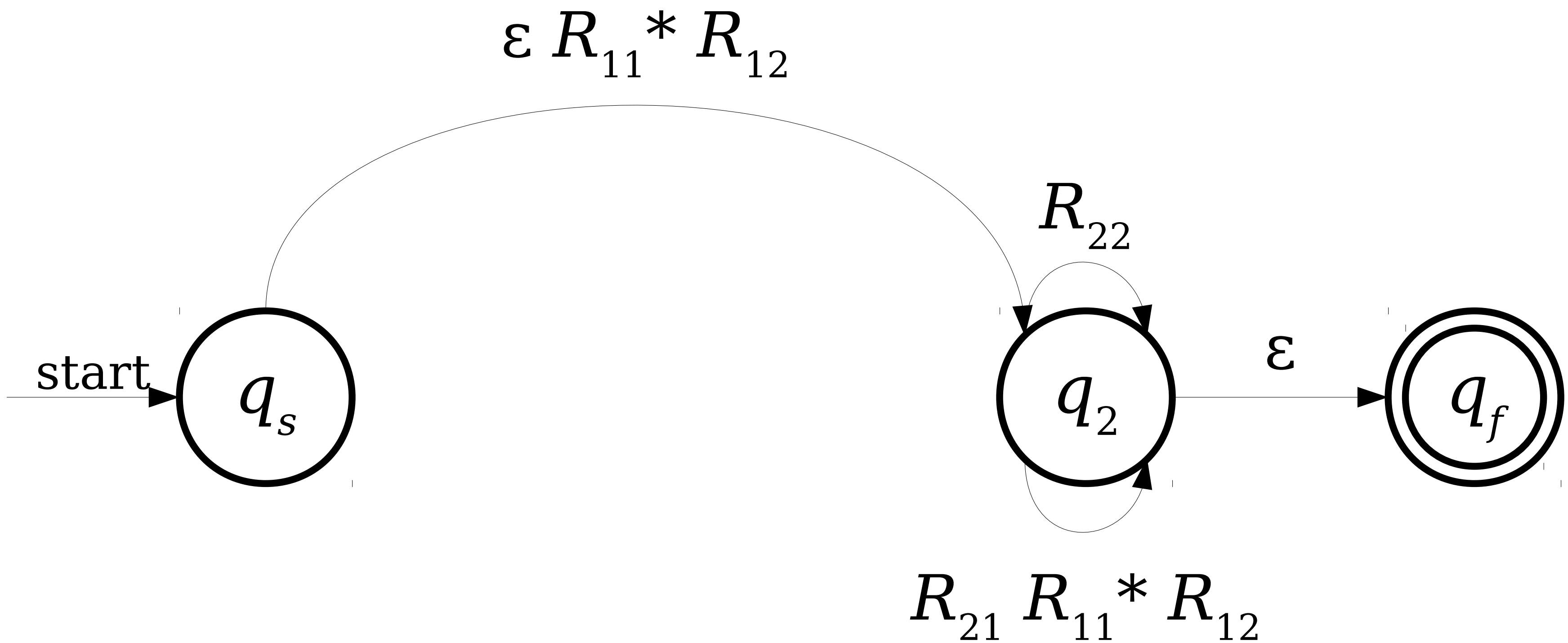
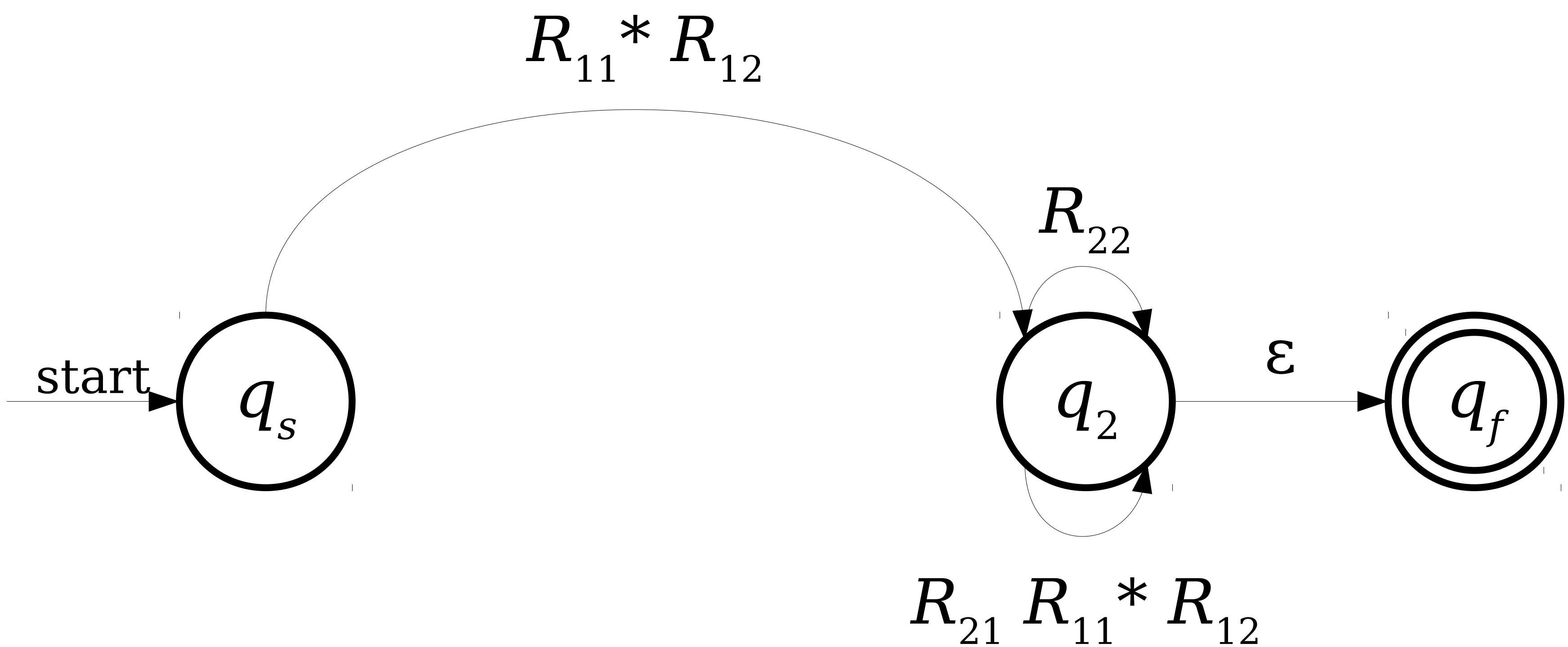


From NFAs to Regular Expressions



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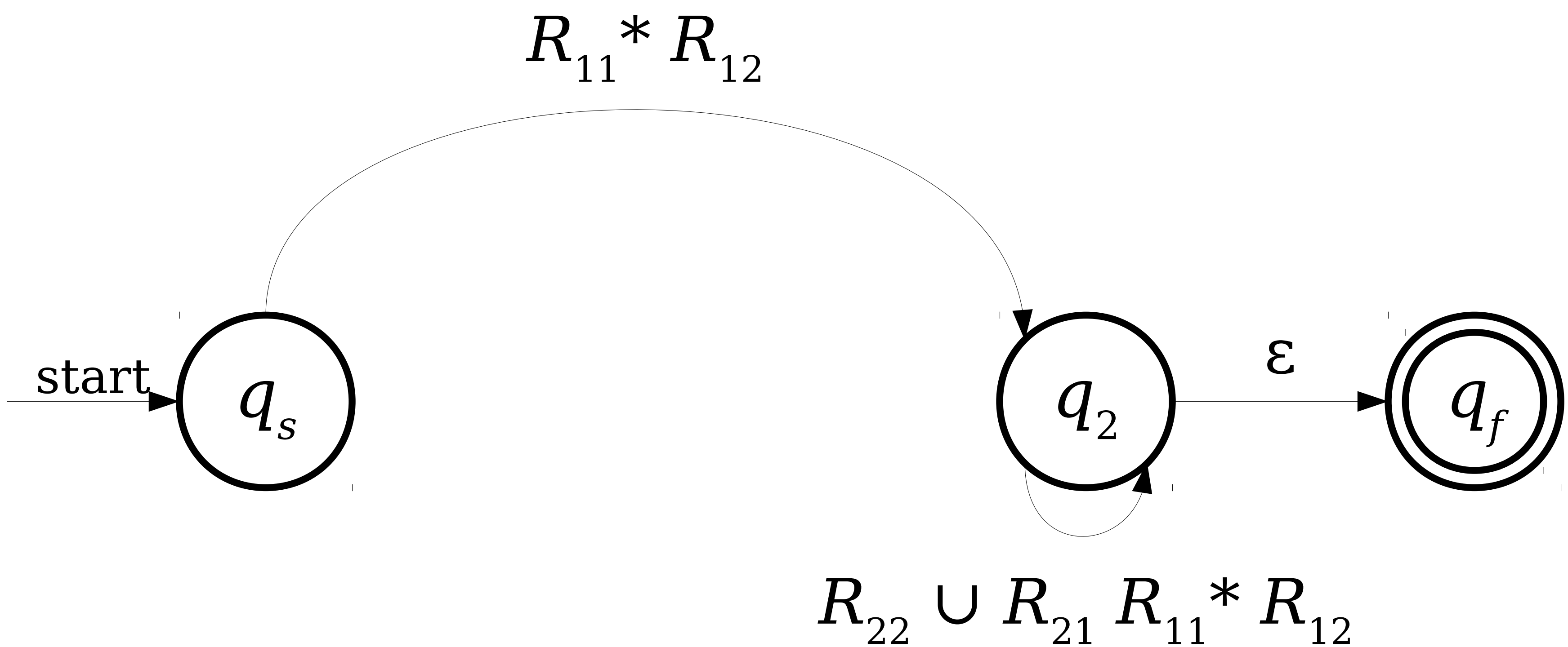
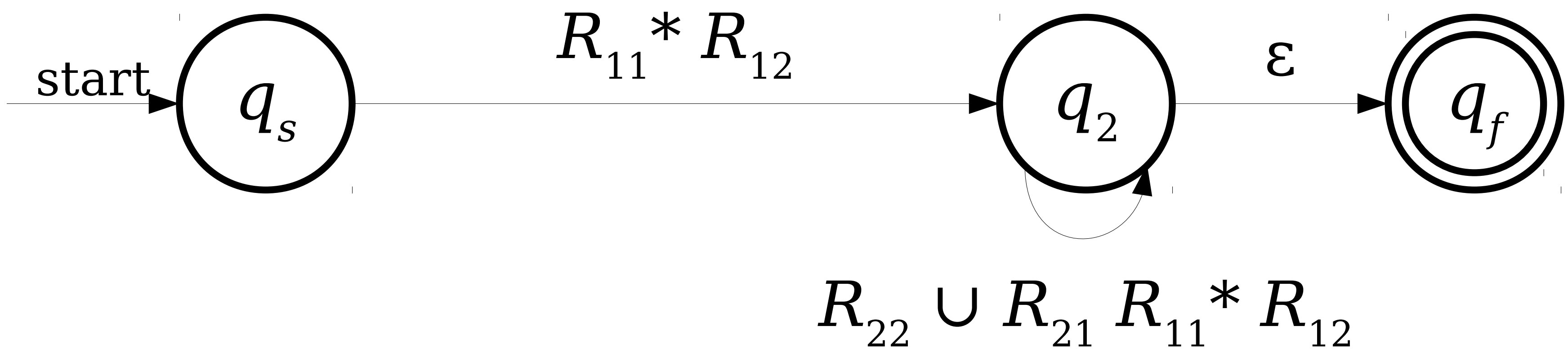


Note: We're using union to combine these transitions together.

From NFAs to Regular Expressions



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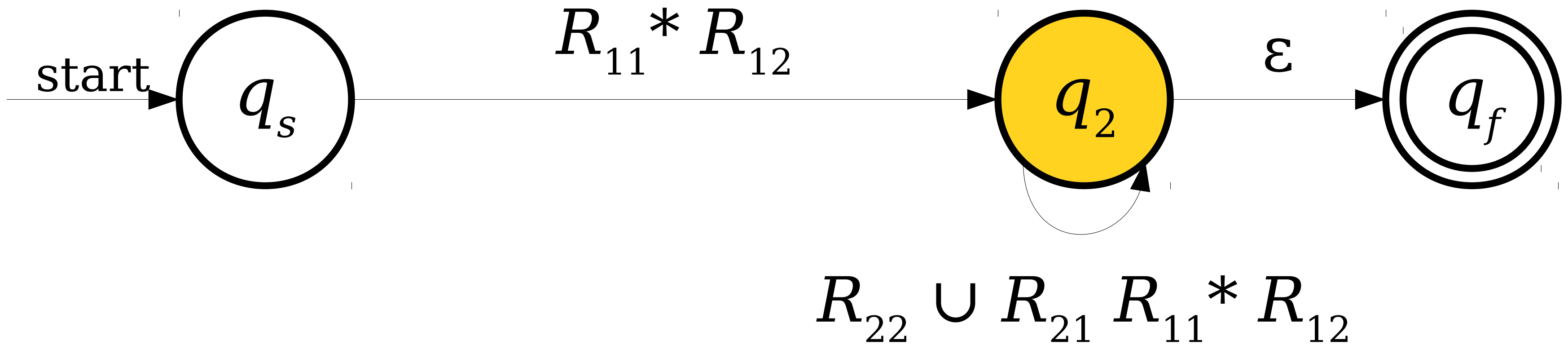

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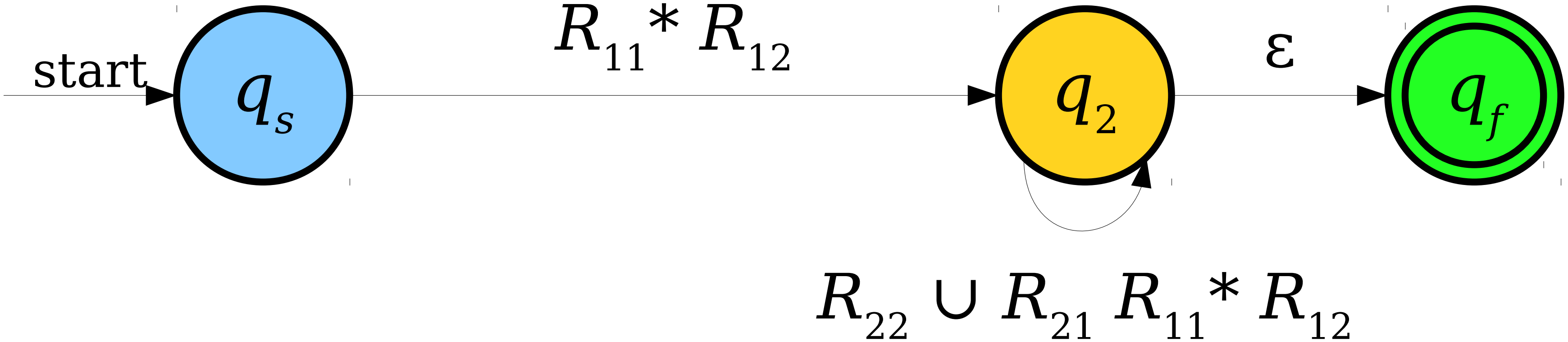

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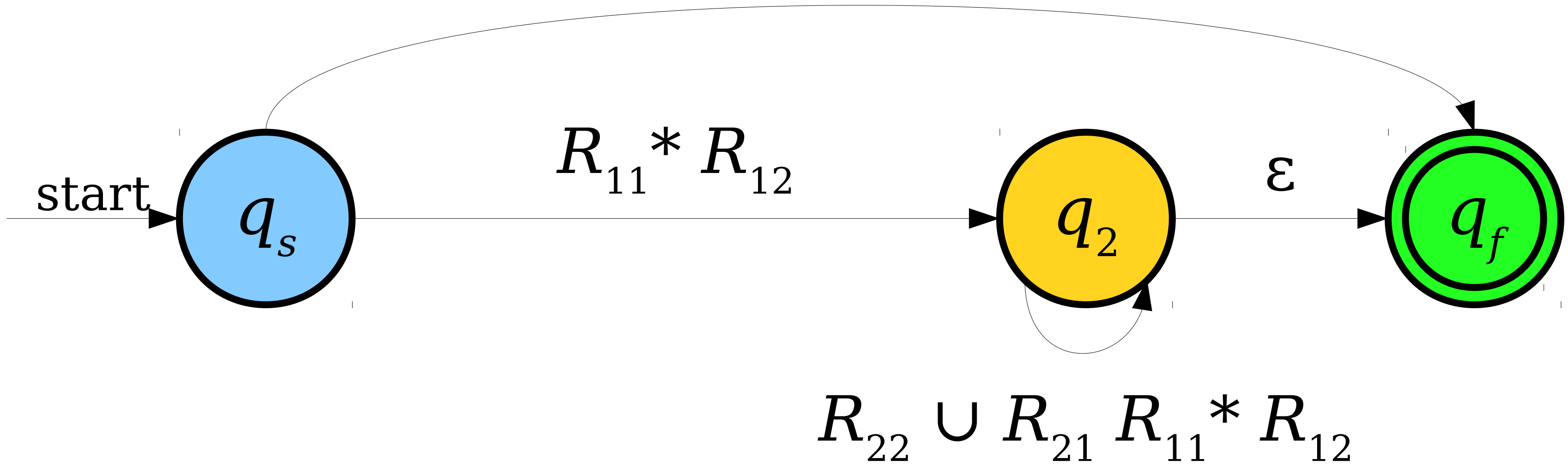

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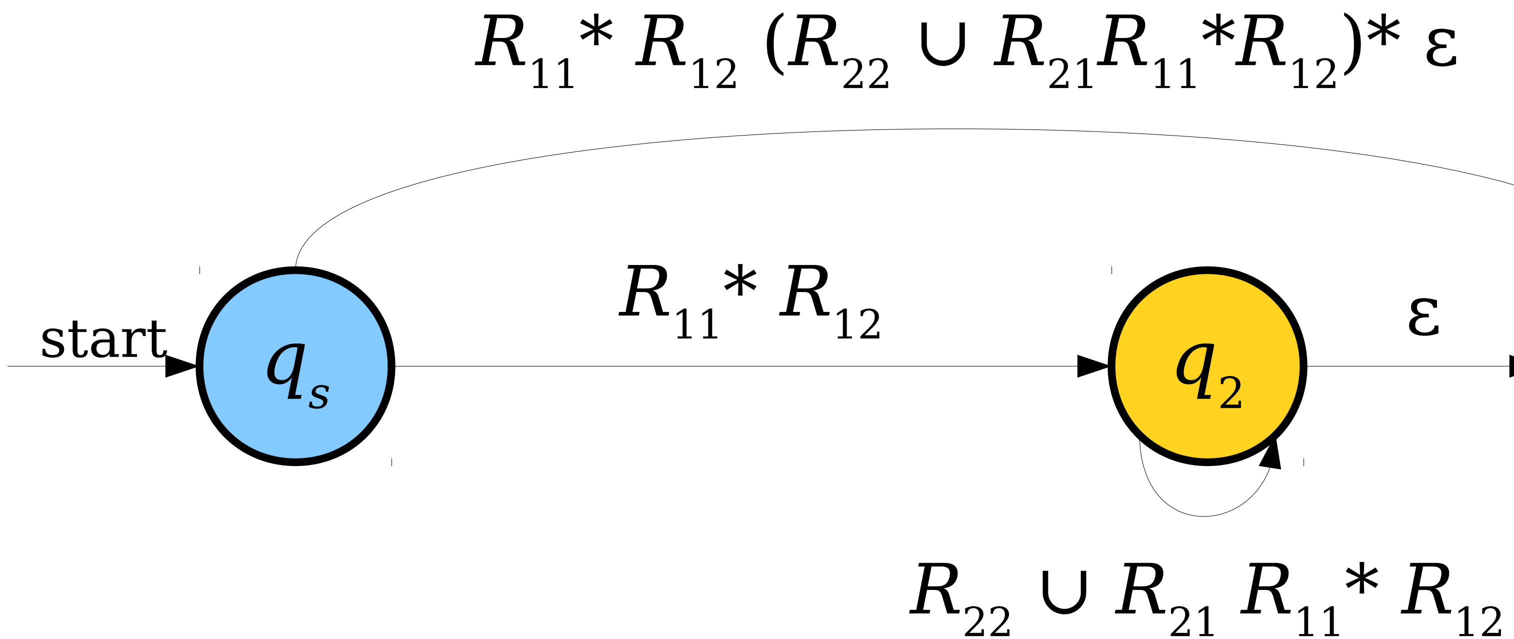

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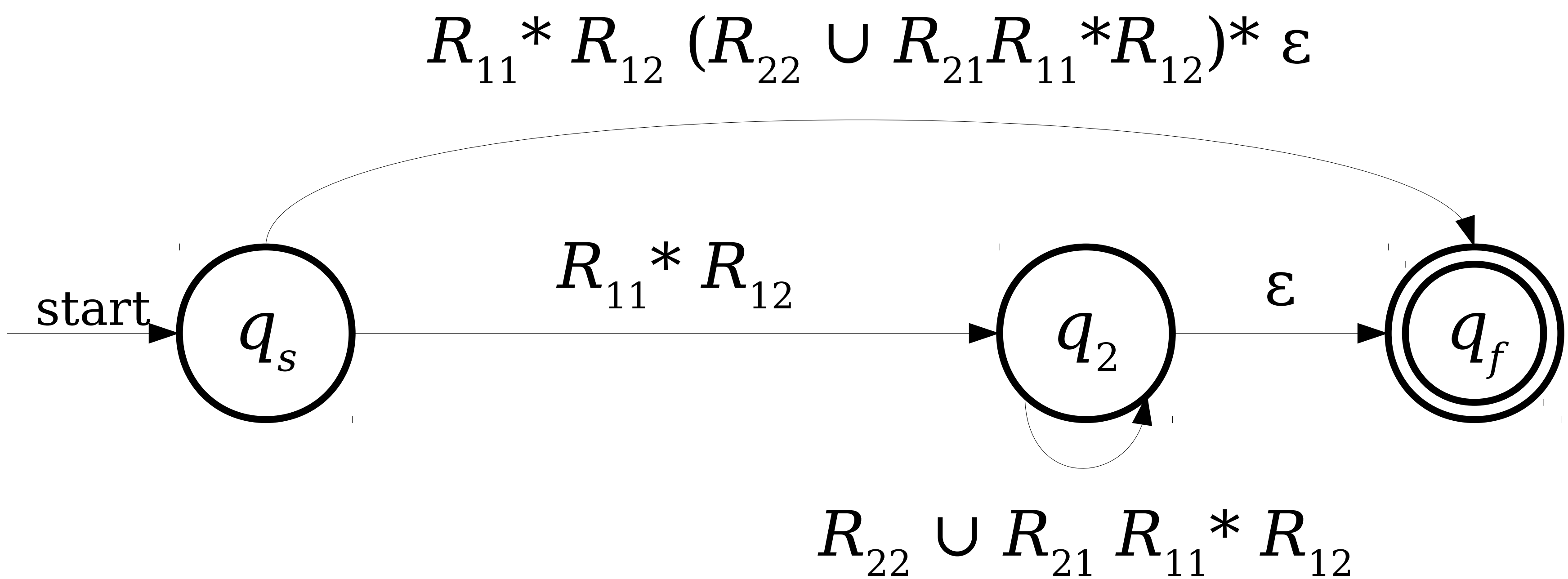

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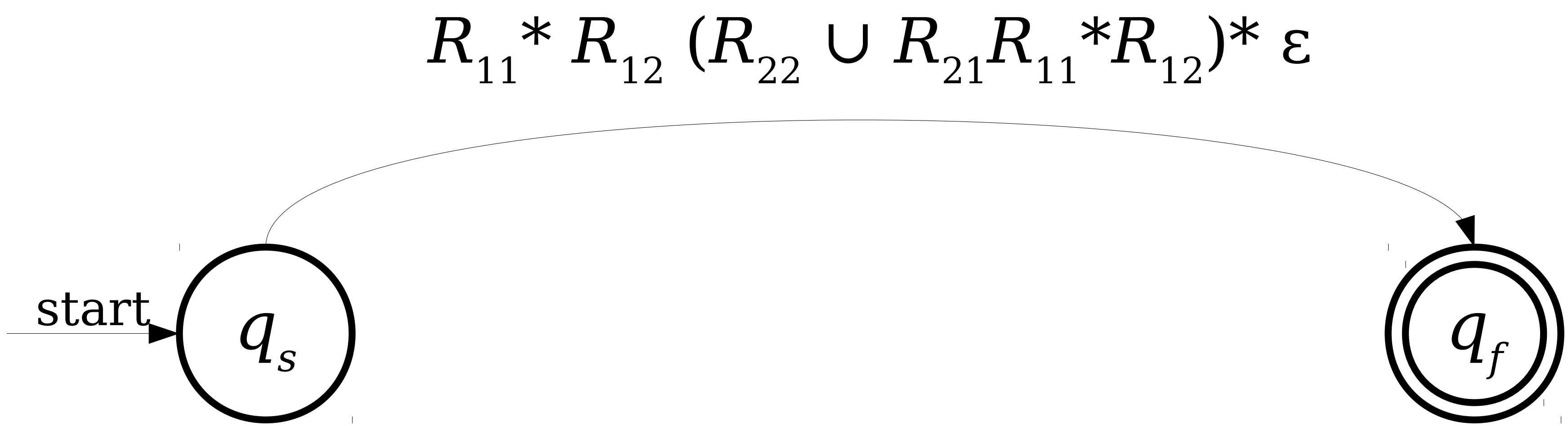

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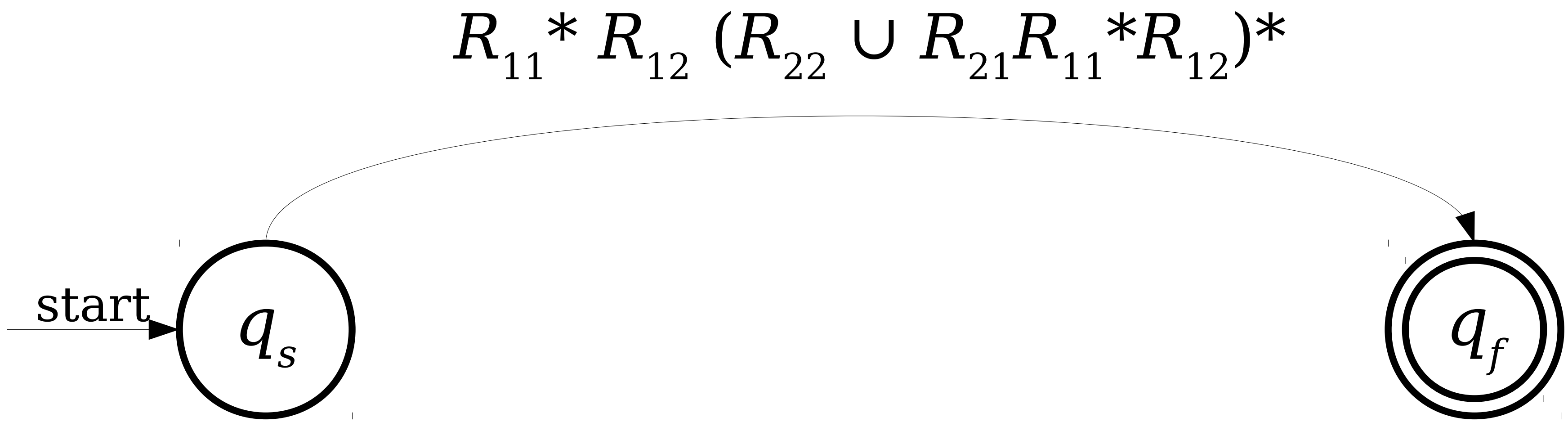

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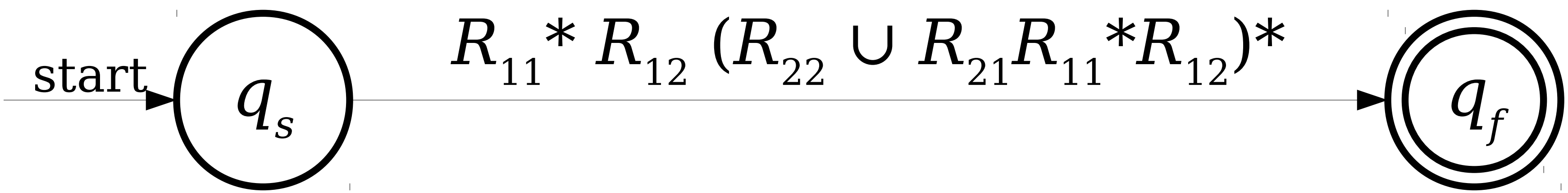

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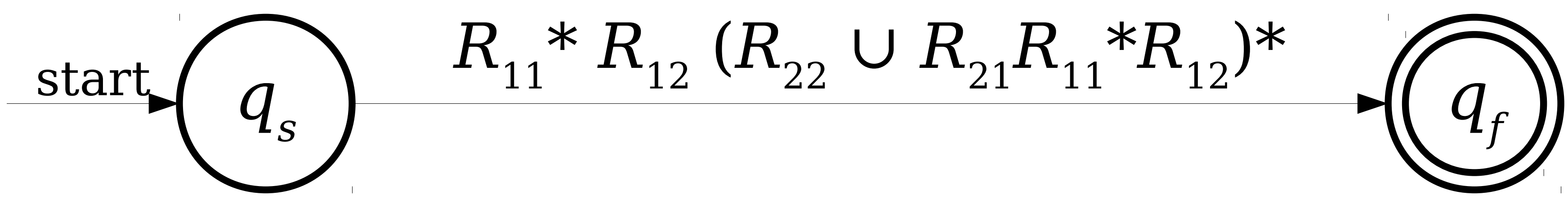
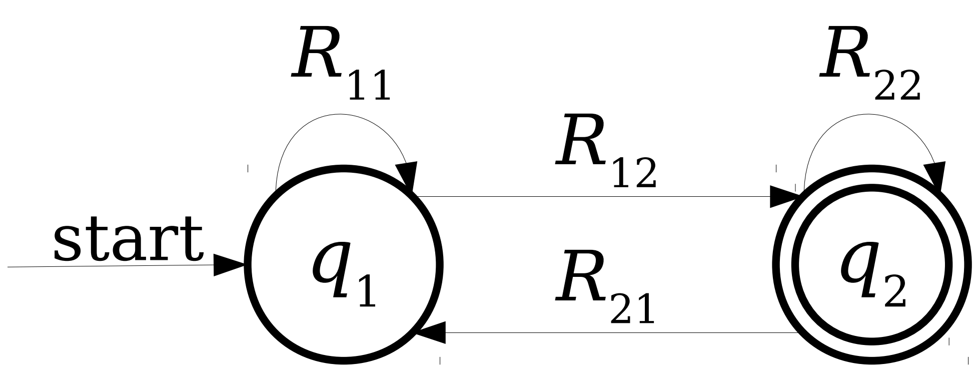




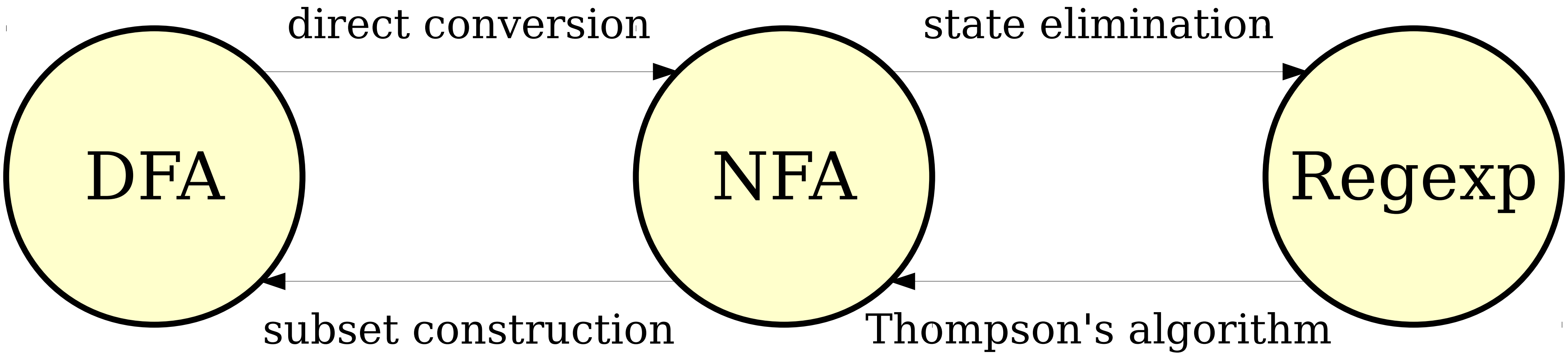
The Construction at a Glance

- Start with an NFA N for the language L.
- Add a new start state q_{s} and accept state q_{f} to the NFA.
  - Add an ε-transition from q_{s} to the old start state of N.
  - Add ε-transitions from each accepting state of N to q_{f}, then mark them as not accepting.
- Repeatedly remove states other than q_{s} and q_{f} from the NFA by "shortcutting" them until only two states remain: q_{s} and q_{f}.
- The transition from q_{s} to q_{f} is then a regular expression for the NFA.

Eliminating a State

- To eliminate a state q from the automaton, do the following for each pair of states q_0 and q_1, where there's a transition from q_0 into q and a transition from q into q_1:
  - Let R_{in} be the regex on the transition from q_0 to q.
  - Let R_{out} be the regex on the transition from q to q_1.
  - If there is a regular expression R_{stay} on a transition from q to itself, add a new transition from q_0 to q_1 labeled $((R_{in})(R_{stay})^*(R_{out}))$.
  - If there isn't, add a new transition from q_0 to q_1 labeled $((R_{in})(R_{out}))$
- If a pair of states has multiple transitions between them labeled $R_1, R_2, \ldots, R_k$, replace them with a single transition labeled $R_1 \cup R_2 \cup \ldots \cup R_k$.

Our Transformations



Theorem: The following are all equivalent:

- L is a regular language.
- There is a DFA D such that $\mathscr{L}(D) = L$.
- There is an NFA N such that $\mathscr{L}(N) = L$.
- There is a regular expression R such that $\mathscr{L}(R) = L$.

Why This Matters

- The equivalence of regular expressions and finite automata has practical relevance.
  - Tools like `grep` and `flex` that use regular expressions capture all the power available via DFAs and NFAs.
- This also is hugely theoretically significant: the regular languages can be assembled "from scratch" using a small number of operations!

Next Time

- ***Applications of Regular Languages***
  - Answering “so what?”
- ***Intuiting Regular Languages***
  - What makes a language regular?
- ***The Myhill-Nerode Theorem***
  - The limits of regular languages.