Complexity Theory

Part Two

Recap from Last Time

The Cobham-Edmonds Thesis

A language L can be ***decided efficiently*** if there is a TM that decides it in polynomial time.

Equivalently, L can be decided efficiently if it can be decided in time $O(n^k)$ for some $k \in \mathbb{N}$.

Like the Church-Turing thesis, this is ***not*** a theorem!

It's an assumption about the nature of efficient computation, and it is somewhat controversial.

The Complexity Class **P**

- The ***complexity class*** **P** (for ***p***olynomial time) contains all problems that can be solved in polynomial time.
- Formally:

  P = { *L* | There is a polynomial-time decider for *L* }

- Assuming the Cobham-Edmonds thesis, a language is in **P** if it can be decided efficiently.

Polynomial-Time Verifiers

- A ***polynomial-time verifier*** for L is a TM V such that
  - V halts on all inputs.
  - $w \in L \quad \text{iff} \quad \exists c \in \Sigma^*.\ V$ accepts $\langle w, c \rangle$.
  - V's runtime is a polynomial in $|w|$ (that is, V's runtime is $O(|w|^k)$ for some integer k)

The Complexity Class **NP**

- The complexity class **NP** (***nondeterministic polynomial time***) contains all problems that can be verified in polynomial time.
- Formally:

NP = { *L* | There is a polynomial-time verifier for *L* }

- The name **NP** comes from another way of characterizing **NP**. If you introduce *nondeterministic Turing machines* and appropriately define "polynomial time," then **NP** is the set of problems that an NTM can solve in polynomial time.

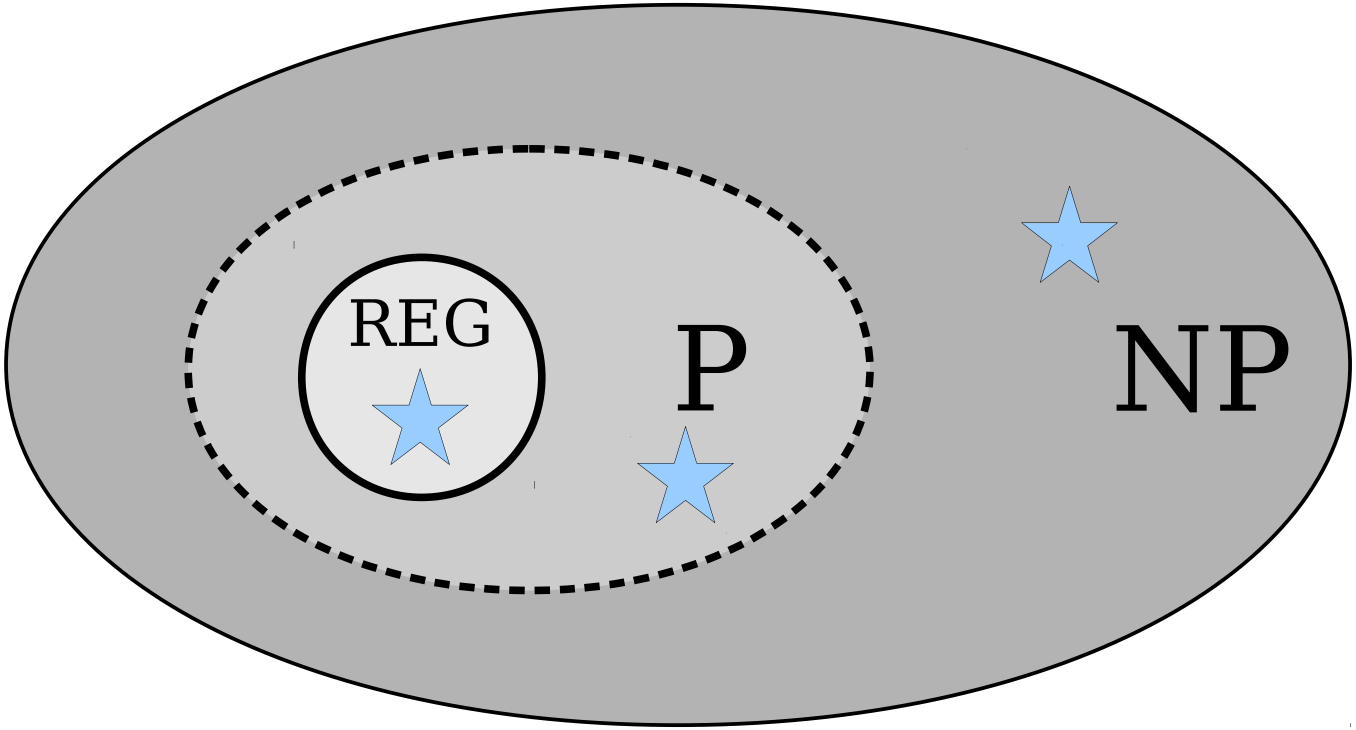
Theorem (Baker-Gill-Solovay): Any proof that purely relies on universality and self-reference cannot resolve $\mathbf{P} \overset{?}{=} \mathbf{NP}$.

Proof: Take CS154!

So how *are* we going to reason about **P** and **NP**?

New Stuff!

A Challenge



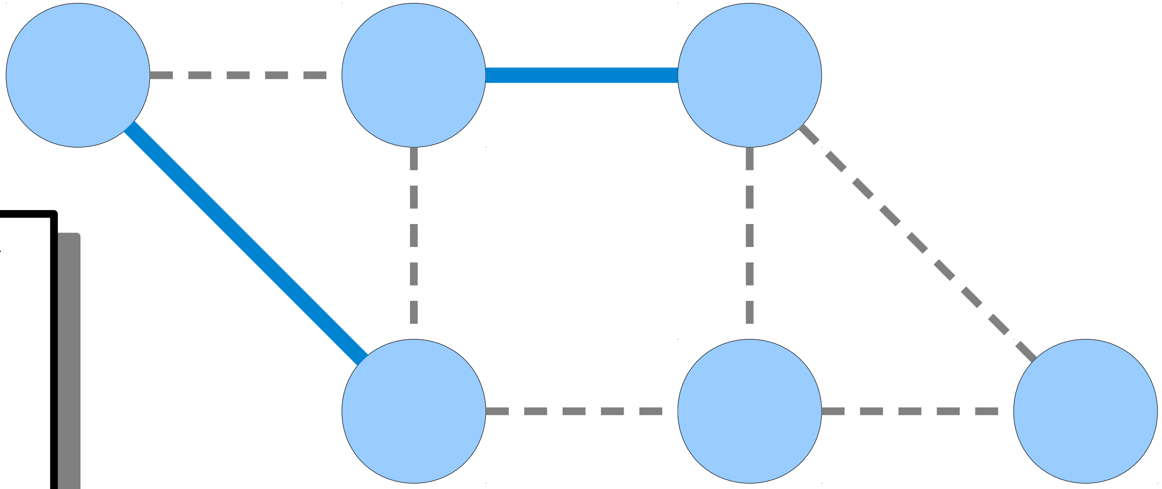
Problems in **NP** vary widely in their difficulty, even if **P** = **NP**.

How can we rank the relative difficulties of problems?

Reducibility

Maximum Matching

- Given an undirected graph *G*, a ***matching*** in *G* is a set of edges such that no two edges share an endpoint.
- A ***maximum matching*** is a matching with the largest number of edges.



A matching, but not a maximum matching.

Maximum Matching

- Given an undirected graph *G*, a ***matching*** in *G* is a set of edges such that no two edges share an endpoint.
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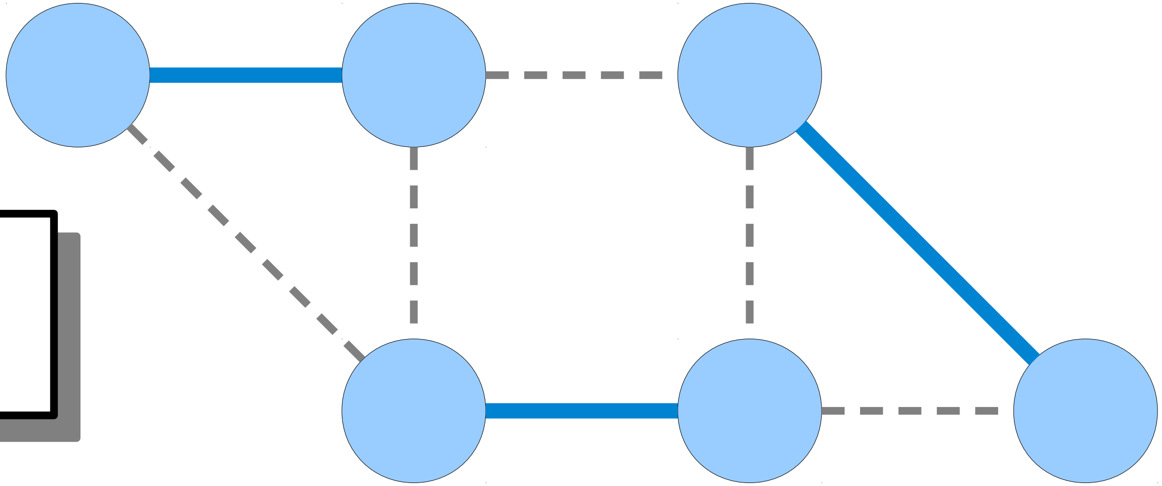

A maximum matching.

Maximum Matching

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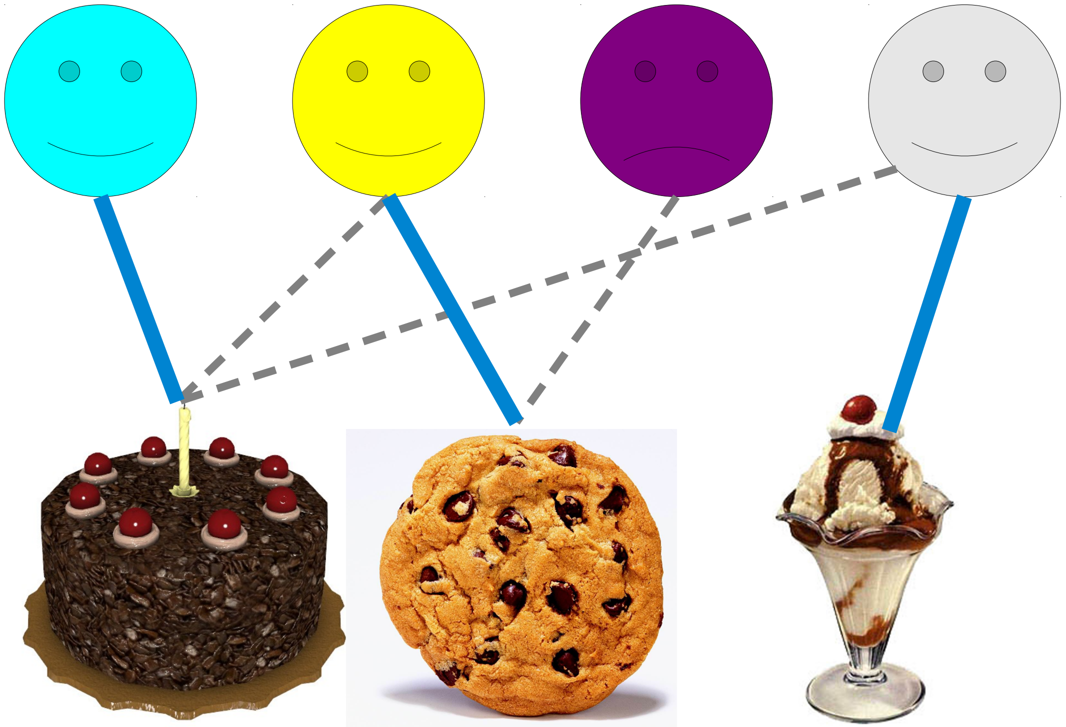
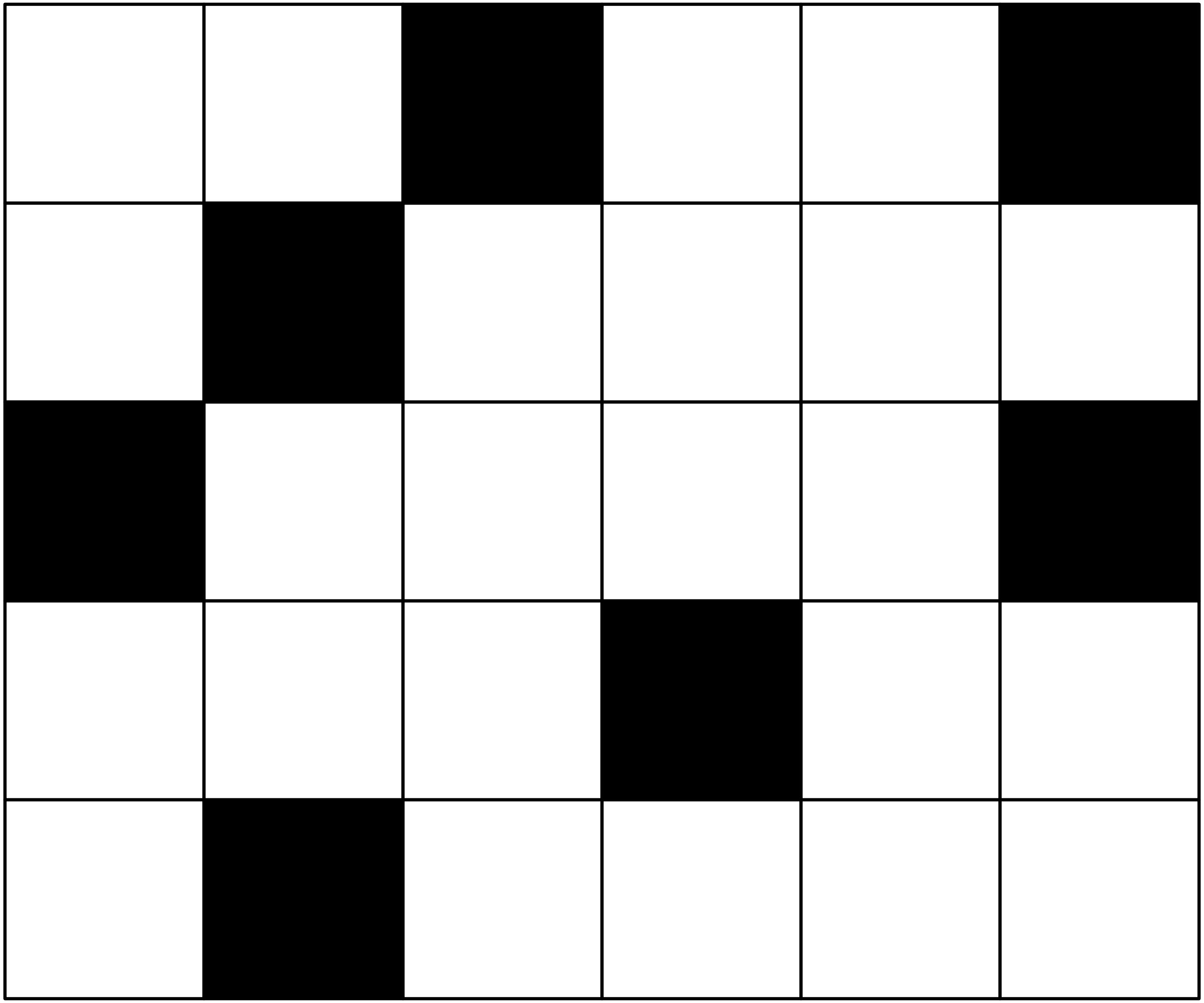


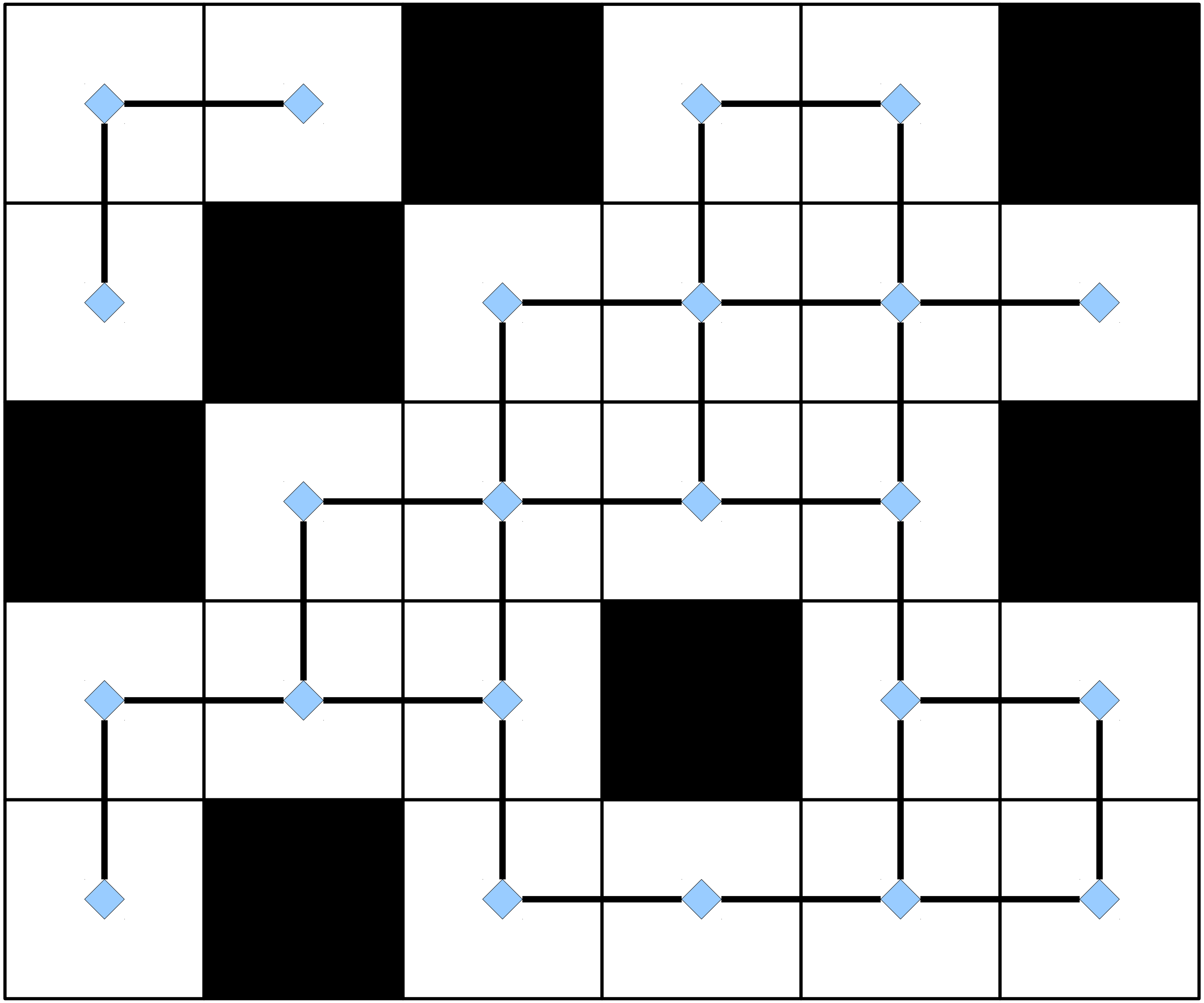
Maximum Matching

- Jack Edmonds' paper “Paths, Trees, and Flowers” gives a polynomial-time algorithm for finding maximum matchings.
  - (This is the same Edmonds as in “Cobham-Edmonds Thesis.”)
- Using this fact, what other problems can we solve?

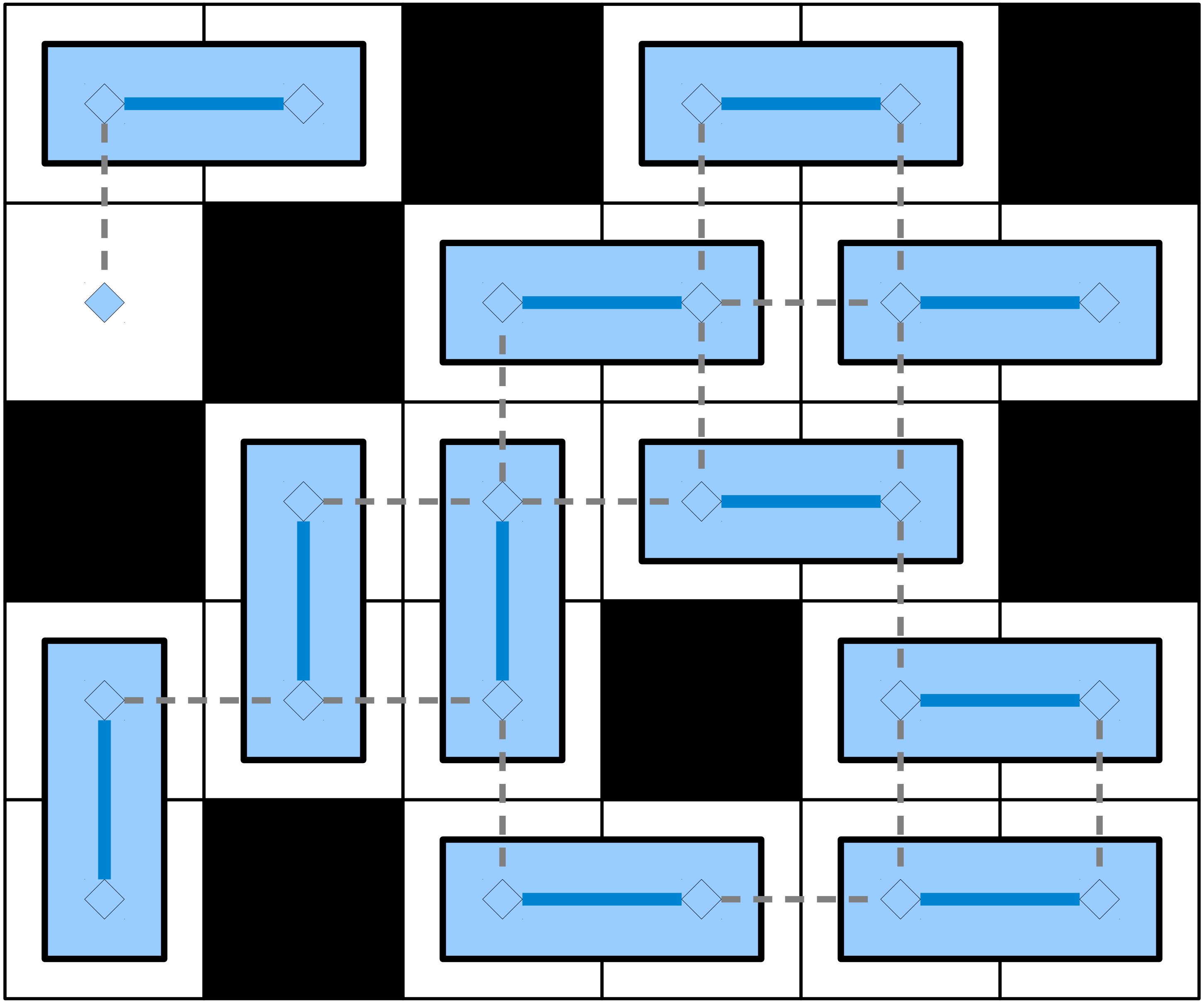
Solving Domino Tiling



Solving Domino Tiling



Solving Domino Tiling



In Pseudocode

```
boolean canPlaceDominos(Grid $G$, int $k$) {
    return hasMatching(gridToGraph($G$), $k$);
}
```

Based on this connection between maximum matching and domino tiling, which of the following statements would be more proper to conclude?

A. Finding a maximum matching isn't any more difficult than tiling a grid with dominoes.

B. Tiling a grid with dominoes isn't any more difficult than finding a maximum matching.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then ***A*** or ***B***.

Intuition:

Tiling a grid with dominoes can't be "harder" than solving maximum matching, because if we can solve maximum matching efficiently, we can solve domino tiling efficiently.

Another Example

Reachability

- Consider the following problem:

  Given an directed graph *G* and nodes *s* and *t* in *G*, is there a path from *s* to *t*?

- It's known that this problem can be solved in polynomial time (use DFS or BFS).
- Given that we can solve the reachability problem in polynomial time, what other problems can we solve in polynomial time?

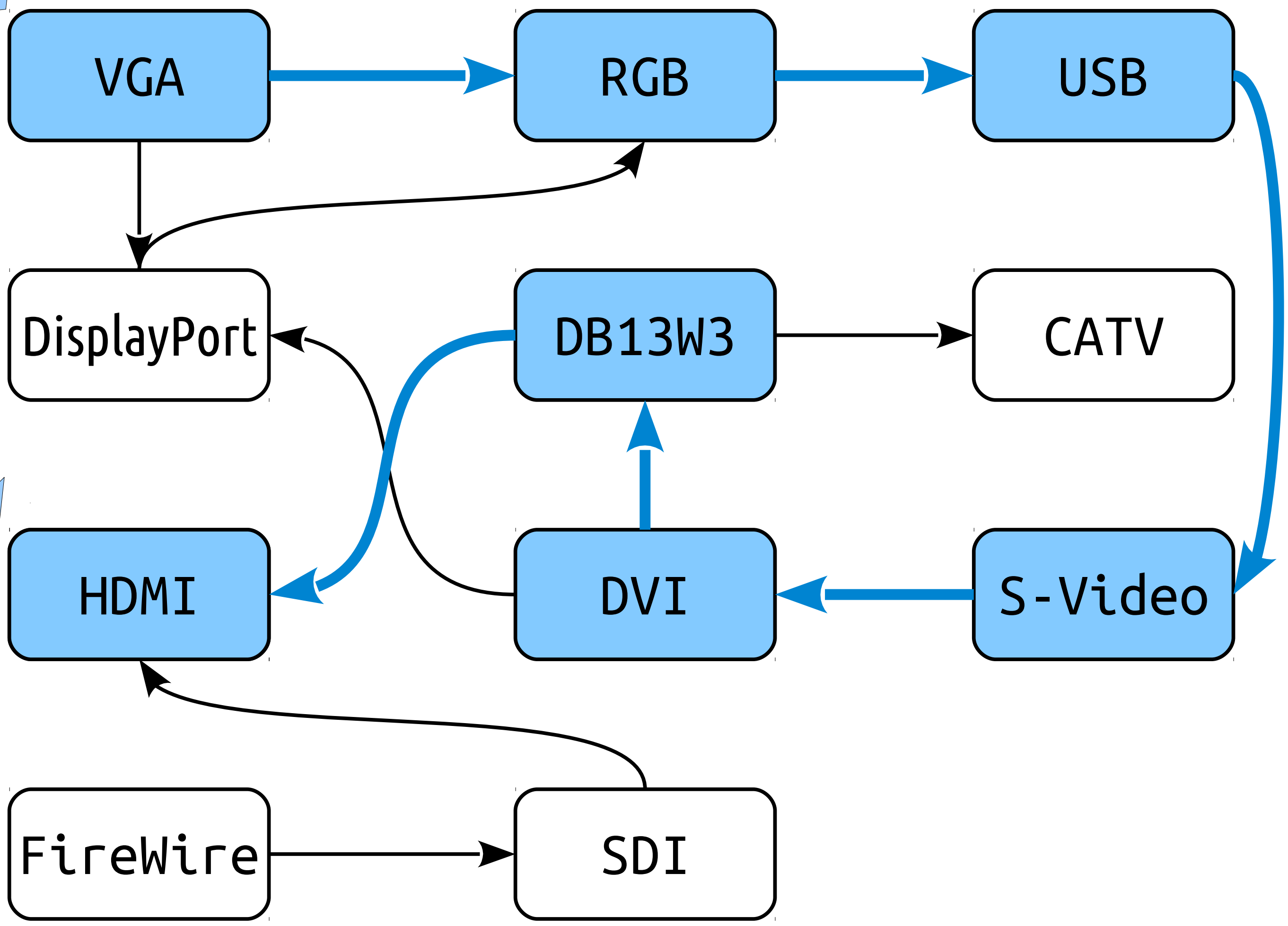
Converter Conundrums

- Suppose that you want to plug your laptop into a projector.
- Your laptop only has a VGA output, but the projector needs HDMI input.
- You have a box of connectors that convert various types of input into various types of output (for example, VGA to DVI, DVI to DisplayPort, etc.)
- ***Question:*** Can you plug your laptop into the projector?

Converter Conundrums

Connectors

RGB to USB
VGA to DisplayPort
DB13W3 to CATV
DisplayPort to RGB
DB13W3 to HDMI
DVI to DB13W3
S-Video to DVI
FireWire to SDI
VGA to RGB
DVI to DisplayPort
USB to S-Video
SDI to HDMI



VGA
RGB
USB
DisplayPort
DB13W3
CATV
HDMI
DVI
S-Video
FireWire
SDI

In Pseudocode

```
boolean canPlugIn(List<Plug> plugs) {
  return isReachable(plugsToGraph(plugs),
                     VGA, HDMI);
}
```

Based on this connection between plugging a laptop into a projector and determining reachability, which of the following statements would be more proper to conclude?

A. Plugging a laptop into a projector isn't any more difficult that computing reachability in a directed graph.

B. Computing reachability in a directed graph isn't any more difficult than plugging a laptop into a projector.

Answer at **PollEv.com/cs103** or
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Intuition:

Finding a way to plug a computer into a projector can't be “harder” than determining reachability in a graph, since if we can determine reachability in a graph, we can find a way to plug a computer into a projector.

```
bool solveProblemA(string input) {
    return solveProblemB(transform(input));
}
```

Intuition:

Problem *A* can't be “harder” than problem *B*, because solving problem *B* lets us solve problem *A*.

```
bool solveProblemA(string input) {
    return solveProblemB(transform(input));
}
```

- If A and B are problems where it's possible to solve problem A using the strategy shown above*, we write

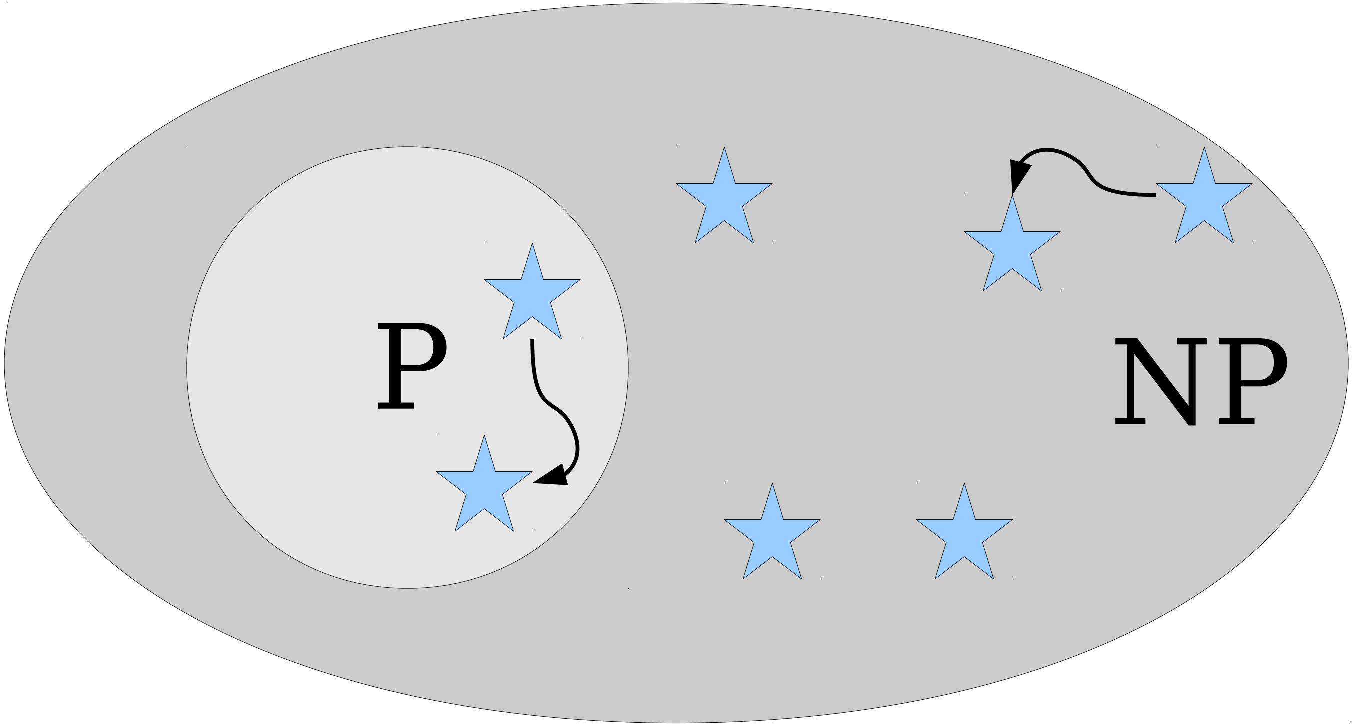
$$A \leq_p B.$$

- We say that ***A is polynomial-time reducible to B***.

* Assuming that `transform` runs in polynomial time.

Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.
- If $A \leq_p B$ and $B \in \mathbf{NP}$, then $A \in \mathbf{NP}$.



This $\leq_p$ relation lets us rank the relative difficulties of problems in **P** and **NP**.

What else can we do with it?

Time-Out for Announcements!

Please evaluate this course on Axess.

Your feedback makes a difference.

Problem Set Nine

- Problem Set Nine is due this Friday at 2:30PM.
    - ***No late submissions can be accepted***. This is university policy – sorry!
- As always, if you have questions, stop by office hours or ask on Piazza!

Final Exam Logistics

- Our final exam is Monday, March 19th from 3:30PM – 6:30PM, location Hewlett 200 & 201 (no special last name assignments).
  - Sorry about how soon that is – the registrar picked this time, not us. If we had a choice, it would be on the last day of finals week.
- The exam is cumulative. You’re responsible for topics from PS1 – PS9 and all of the lectures.
- As with the midterms, the exam is closed-book, closed-computer, and limited-note. You can bring one double-sided sheet of 8.5” × 11” notes with you to the exam, decorated any way you’d like.
- Students with OAE accommodations: if we don’t yet have your OAE letter, please send it to us ASAP.

Preparing for the Final

- On the course website you'll find
  - ***six*** practice final exams, which are all real exams with minor modifications, with solutions, and
  - a giant set of 46 practice problems (EPP3), with solutions.
- Our recommendation: Look back over the exams and problem sets and redo any problems that you didn't really get the first time around.
- Keep the TAs in the loop: stop by office hours to have them review your answers and offer feedback.

Practice Final Exam

- We will be holding a practice final exam in room 380-380X tonight from 7PM – 10PM.
- We’ll print out copies of a few of the different practice exams and you can pick whichever one you’d like!

Back to CS103!

NP-Hardness and **NP**-Completeness

Question: What makes a problem hard to solve?

Intuition: If $A \leq_{\mathrm{p}} B$, then problem B is at least as hard* as problem A.

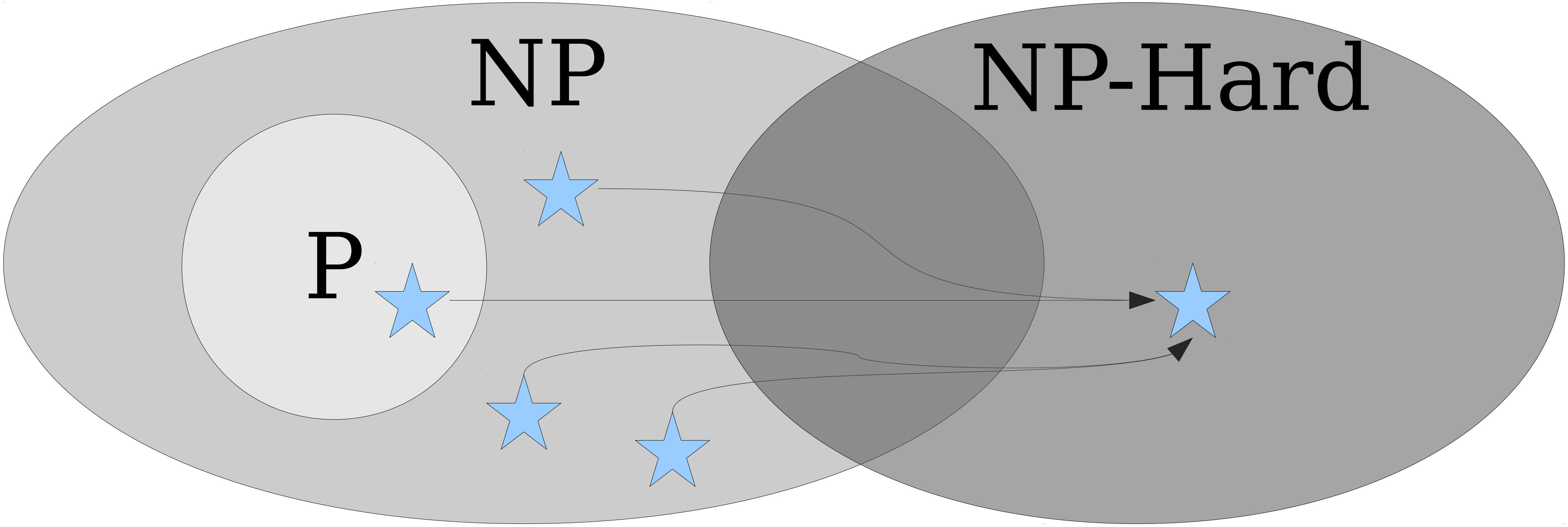
* for some definition of “at least as hard as.”

Intuition: To show that some problem is hard, show that lots of other problems reduce to it.

NP-Hardness

- A language L is called ***NP-hard*** if for *every* $A \in$ **NP**, we have $A \leq_P L$.

Intuitively: L has to be at least as hard as every problem in **NP**, since an algorithm for L can be used to decide all problems in **NP**.



NP-Hardness

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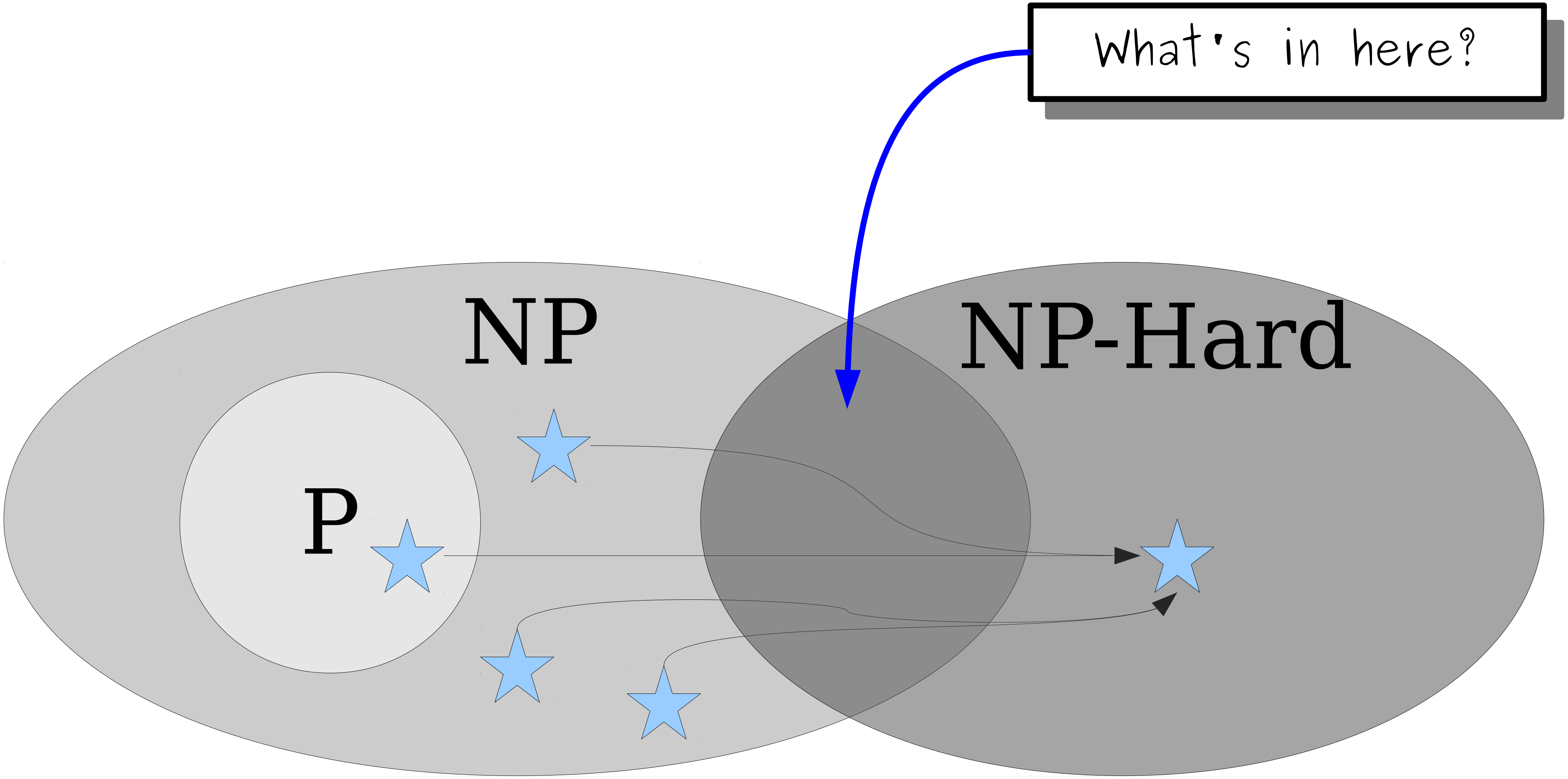
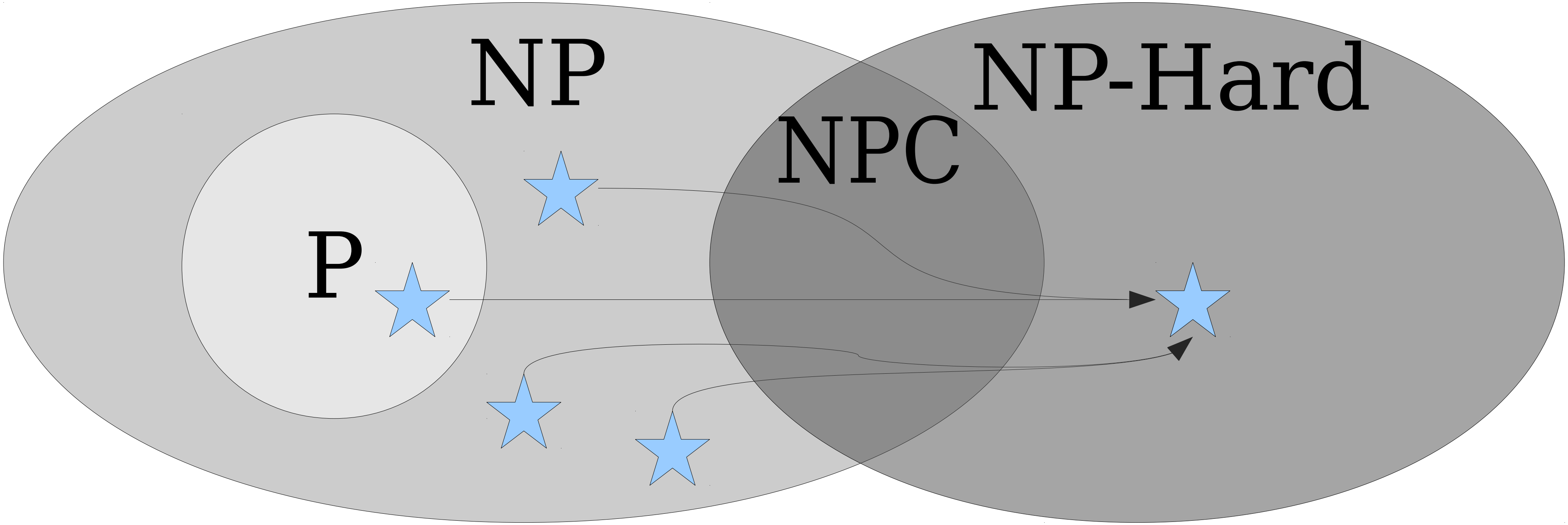


What's in here?

NP-Hardness

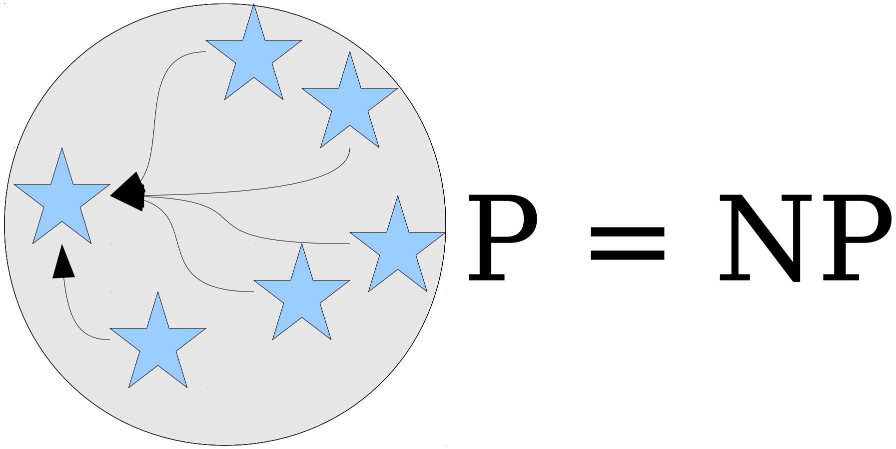
- A language L is called ***NP-hard*** if for *every* $A \in$ **NP**, we have $A \leq_{\mathrm{P}} L$.
- A language in L is called ***NP-complete*** if L is **NP**-hard and $L \in$ **NP**.
- The class ***NPC*** is the set of **NP**-complete problems.



The Tantalizing Truth

Theorem: If *any* **NP**-complete language is in **P**, then **P** = **NP**.

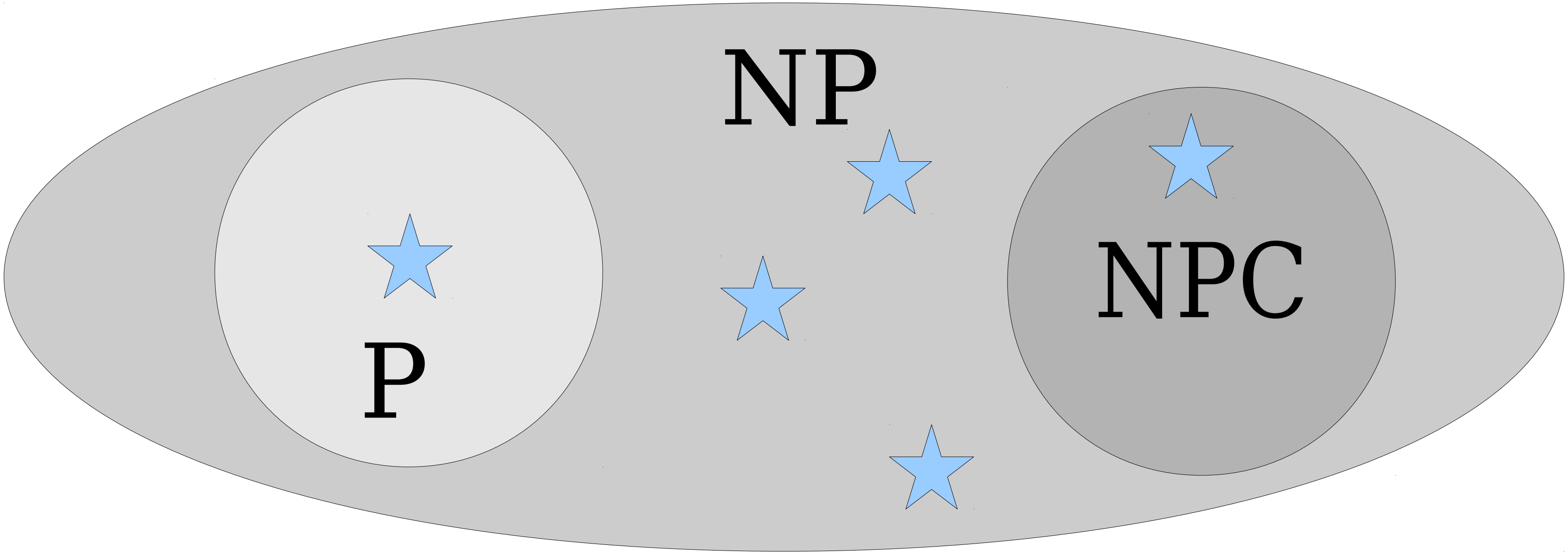
Proof: Suppose that L is **NP**-complete and $L \in$ **P**. Now consider any arbitrary **NP** problem A. Since L is **NP**-complete, we know that $A \leq_p L$. Since $L \in$ **P** and $A \leq_p L$, we see that $A \in$ **P**. Since our choice of A was arbitrary, this means that **NP** ⊆ **P**, so **P** = **NP**. ■



The Tantalizing Truth

Theorem: If *any* **NP**-complete language is not in **P**, then **P** ≠ **NP**.

Proof: Suppose that L is an **NP**-complete language not in **P**. Since L is **NP**-complete, we know that $L \in$ **NP**. Therefore, we know that $L \in$ **NP** and $L \notin$ **P**, so **P** ≠ **NP**. ■



How do we even know NP-complete problems exist in the first place?

Satisfiability

- A propositional logic formula φ is called ***satisfiable*** if there is some assignment to its variables that makes it evaluate to true.
    - $p \wedge q$ is satisfiable.
    - $p \wedge \neg p$ is unsatisfiable.
    - $p \rightarrow (q \wedge \neg q)$ is satisfiable.
- An assignment of true and false to the variables of φ that makes it evaluate to true is called a ***satisfying assignment***.

SAT

- The ***boolean satisfiability problem*** (***SAT***) is the following:

Given a propositional logic formula φ, is φ satisfiable?

- Formally:

***SAT* = { ⟨φ⟩ | φ is a satisfiable PL formula }**

SAT = { ⟨φ⟩ | φ is a satisfiable PL formula }

The language SAT happens to be in **NP**. Think about how a polynomial-time verifier for SAT might work. Which of the following would work as certificates for such a verifier, given that the input is a propositional formula φ?

A. The truth table of φ.
B. One possible variable assignment to φ.
C. A list of all possible variable assignments for φ.
D. None of the above, or two or more of the above.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then ***A***, ***B***, ***C***, or ***D***.

Theorem (Cook-Levin): SAT is **NP**-complete.

Proof Idea: To see that **SAT** ∈ **NP**, show how to make a polynomial-time verifier for it. Key idea: have the certificate be a satisfying assignment.

To show that **SAT** is **NP**-hard, given a polymomial-time verifier *V* for an arbitrary **NP** language *L*, for any string *w* you can construct a polynomially-sized formula φ(*w*) that says "there is a certificate *c* where *V* accepts ⟨*w*, *c*⟩." This formula is satisfiable if and only if *w* ∈ *L*, so deciding whether the formula is satisfiable decides whether *w* is in *L*.

Proof: Take CS154!

Why All This Matters

- Resolving $\mathbf{P} \overset{?}{=} \mathbf{NP}$ is equivalent to just figuring out how hard SAT is.
  - If SAT $\in$ **P**, then $\mathbf{P} = \mathbf{NP}$.

    If SAT $\notin$ **P**, then $\mathbf{P} \neq \mathbf{NP}$.
- We've turned a huge, abstract, theoretical problem about solving problems versus checking solutions into the concrete task of seeing how hard one problem is.
- You can get a sense for how little we know about algorithms and computation given that we can't yet answer this question!

Why All This Matters

- You will almost certainly encounter **NP**-hard problems in practice – they're everywhere!
- If a problem is **NP**-hard, then there is no known algorithm for that problem that
  - is efficient on all inputs,
  - always gives back the right answer, and
  - runs deterministically.
- ***Useful intuition:*** If you need to solve an **NP**-hard problem, you will either need to settle for an approximate answer, an answer that's likely but not necessarily right, or have to work on really small inputs.

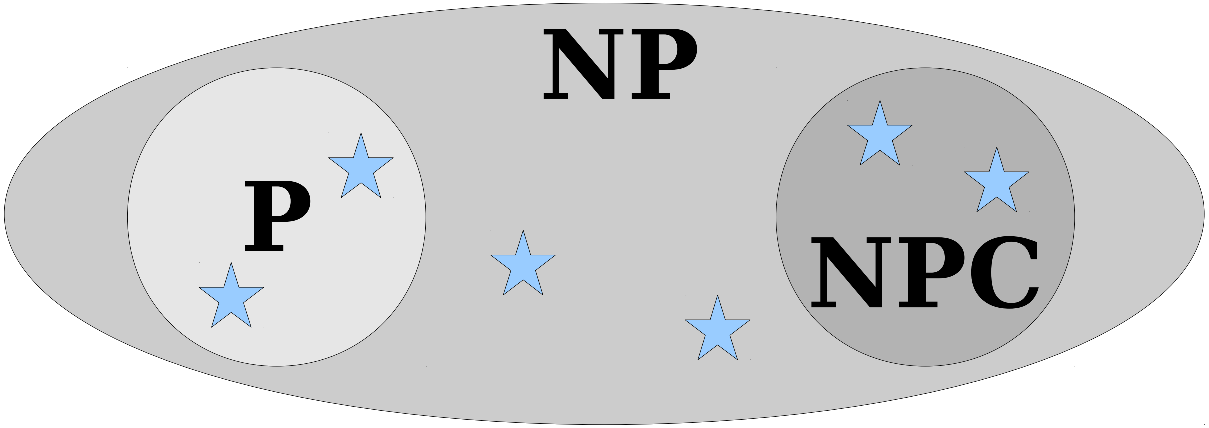
Sample **NP**-Hard Problems

- ***Computational biology:*** Given a set of genomes, what is the most probable evolutionary tree that would give rise to those genomes? *(Maximum parsimony problem)*
- ***Game theory:*** Given an arbitrary perfect-information, finite, twoplayer game, who wins? *(Generalized geography problem)*
- ***Operations research:*** Given a set of jobs and workers who can perform those tasks in parallel, can you complete all the jobs within some time bound? *(Job scheduling problem)*
- ***Machine learning:*** Given a set of data, find the simplest way of modeling the statistical patterns in that data *(Bayesian network inference problem)*
- ***Medicine:*** Given a group of people who need kidneys and a group of kidney donors, find the maximum number of people who can end up with kidneys *(Cycle cover problem)*
- ***Systems:*** Given a set of processes and a number of processors, find the optimal way to assign those tasks so that they complete as soon as possible *(Processor scheduling problem)*

Coda: What if $\mathbf{P} \overset{?}{=} \mathbf{NP}$ is resolved?

Intermediate Problems

- With few exceptions, every problem we've discovered in **NP** has either
  - definitely been proven to be in **P**, or
  - definitely been proven to be **NP**-complete.
- A problem that's **NP**, not in **P**, but not **NP**-complete is called ***NP-intermediate***.
- ***Theorem (Ladner):*** There are **NP**-intermediate problems if and only if **P** ≠ **NP**.



What if $\mathbf{P} \neq \mathbf{NP}$?

A Good Read:

“A Personal View of Average-Case Complexity” by Russell Impagliazzo

What if **P** = **NP**?

And a Dismal Third Option

Next Time

- ***The Big Picture***
- ***Where to Go from Here***
- ***A Final "Your Questions"***
- ***Parting Words!***